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GLOBAL ASYMPTOTIC STABILITY OF COHEN – GROSSBERG NEURAL NETWORKS OF NEUTRAL TYPE*

ГЛОБАЛЬНА АСИМПТОТИЧНА СТІЙКІСТЬ НЕЙРОННИХ МЕРЕЖ КОЕНА – ГРОССБЕРГА НЕЙТРАЛЬНОГО ТИПУ

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Sufficient conditions for existence and global asymptotic stability of a unique equilibrium point of a Cohen – Grossberg neural network of neutral type are obtained. An example is given.

Отримано достатні умови існування та глобальної стійкості єдиної точки рівноваги для нейронної мережі Коена – Гроссберга нейтрального типу.

1. Introduction. An artificial neural network is an information processing paradigm that is inspired by the way biological nervous systems, such as the brain, process information. The key element of this paradigm is the novel structure of the information processing system. It is composed of a large number of highly interconnected processing elements (neurons) working in unison to solve specific problems. Although the initial intent of artificial neural networks was to explore and reproduce human information processing tasks such as speech, vision, and knowledge processing, artificial neural networks also demonstrated their superior capability for classification and function approximation problems. This has great potential for solving complex problems such as systems control, data compression, optimization problems, pattern recognition, and system identification.

Cohen – Grossberg neural network [10] and its various generalizations with or without transmission delays and impulsive state displacements have been a subject of intense investigation recently [3, 6, 7, 13, 16, 17]. In a Cohen – Grossberg neural network model, the feedback terms consist of amplification and stabilizing functions which are generally nonlinear. These terms provide a model with a special kind of generalization wherein many neural network models that are capable for content addressable memory such as additive neural networks, cellular neural

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networks and bidirectional associative memory networks and also biological models such as Lotka – Volterra models of population dynamics are included as special cases.

In contrast to retarded systems, in neutral systems time delays appear explicitly in the state velocity vector. Neutral systems can be applied to describe more complicated nonlinear engineering and bioscience models, including those describing chemical reactors, transmission lines, partial element equivalent circuits in very large-scale integrated systems, and Lotka – Volterra systems [18, 14, 4, 1, 2, 9, 15]. Neural networks can be implemented using very large-scale integrated circuits. Therefore, both retarded-type delays and neutral type delays are inherent in the dynamics of neural networks.

In the present paper we consider a Cohen – Grossberg neural network of neutral type more general than in [8]. A discrete-time analogue of this system provided with impulse conditions was considered in our previous paper [11]. Sufficient conditions for global asymptotic stability of the unique equilibrium point of the system are obtained by exploiting an appropriate Lyapunov functional. The conditions obtained are much more precise than in [8]. An example is given.

2. Preliminaries. We consider a Cohen – Grossberg neural network of neutral type consisting of $m \geq 2$ elementary processing units (or neurons) whose state variables x_i ($i = \overline{1,m}$ which henceforth will stand for $i = 1, 2, \ldots, m$) are governed by the system

$$\dot{x}_i(t) + \sum_{j=1}^{m} e_{ij}\dot{x}_j(t-\tau_j) = a_i(x_i(t)) \left[-b_i(x_i(t)) + \sum_{j=1}^{m} c_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} d_{ij} g_j(x_j(t-\tau_j)) + I_i \right], \tag{1}$$

$$i = \overline{1,m}, \quad t > t_0 = 0,$$

with initial values prescribed by continuous functions $x_i(s) = \phi_i(s)$ for $s \in [-\tau, 0]$, $\tau = \max_{j=\overline{1,m}}\{\tau_j\}$. In (1), $a_i(x_i)$ denotes an amplification function; $b_i(x_i)$ denotes an appropriate function which supports the stabilizing (or negative) feedback term $-a_i(x_i)b_i(x_i)$ of the unit i; $f_j(x_j)$, $g_j(x_j)$ denote activation functions; the parameters c_{ij}, d_{ij} are real numbers that represent the weights (or strengths) of the synaptic connections between the jth unit and the ith unit, respectively without and with time delays τ_j; the real numbers e_{ij} show how the state velocities of the neurons are delay feed-forward connected in the network; the real constant I_i represents an input signal introduced from outside the network to the ith unit.

Let E be the unit $(m \times m)$-matrix. Denote by $\mathcal{E}$ and $|\mathcal{E}|$ the $(m \times m)$-matrices with entries e_{ij} and $|e_{ij}|$, respectively.

Definition 1 [5]. *A real matrix $A = (a_{ij})_{m\times m}$ is said to be an M-matrix if $a_{ij} \leq 0$ for $i, j = \overline{1,m}$, $i \neq j$, and all successive principal minors of A are positive.*

The assumptions that accompany the neural network (1) are given as follows:

$\mathbf{A}_1$. The amplification functions $a_i : \mathbb{R} \to \mathbb{R}^+$ are continuous and bounded in the sense that

$$0 < \underline{a}_i \leq a_i(x) \leq \overline{a}_i \quad \text{for} \quad x \in \mathbb{R}, \quad i = \overline{1,m},$$

for some constants $\underline{a}_i, \overline{a}_i$.

$\mathbf{A}_2$. The stabilizing functions $b_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous and monotone increasing, namely,

$$0 < \underline{b}_i \leq \frac{b_i(x) - b_i(y)}{x - y} \leq \overline{b}_i \quad \text{for} \quad x \neq y, \quad x, y \in \mathbb{R}, \quad i = \overline{1,m},$$

for some constants $\underline{b}_i$, $\overline{b}_i$.

$\mathbf{A}_3$. The activation functions $f_j, g_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous in the sense of

$$\sup_{x \neq y} \left| \frac{f_j(x) - f_j(y)}{x - y} \right| = F_j, \quad \sup_{x \neq y} \left| \frac{g_j(x) - g_j(y)}{x - y} \right| = G_j$$

for $x, y \in \mathbb{R}$, $j = \overline{1, m}$, where F_j, G_j denote positive constants.

$\mathbf{A}_4$. $\|\mathcal{E}\| < 1$, where $\| \cdot \|$ is the spectral matrix norm, and $E - |\mathcal{E}|$ is an M-matrix.

The "stability condition" $\|\mathcal{E}\| < 1$ guarantees the existence and uniqueness of the solution of the Cauchy problem. Since $E - |\mathcal{E}|$ is an M-matrix, it is nonsingular and its inverse has nonnegative entries only.

Under these assumptions and the given initial conditions, there is a unique solution of system (1). The solution is a vector $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$ in which $x_i(t)$ are continuously differentiable for $t \in (0, \beta)$, where β is some positive number, possibly ∞, An equilibrium point of system (1) is denoted by $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ where the components x_i^* are governed by the algebraic system

$$b_i(x_i^*) = \sum_{j=1}^{m} c_{ij} f_j(x_j^*) + \sum_{j=1}^{m} d_{ij} g_j(x_j^*) + I_i, \quad i = \overline{1, m}. \tag{2}$$

Definition 2. *The equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of system (1) is said to be globally asymptotically stable if any other solution $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$ of system (1) is defined for all $t > 0$ and satisfies*

$$\lim_{t \to \infty} x(t) = x^*.$$

3. Existence and global asymptotic stability of an equilibrium point. Sufficient conditions for existence and uniqueness of the solution x^* of the algebraic system (2) are given by the following theorem.

Theorem 1 ([11], Theorem 4.2). *Let the assumptions $\mathbf{A}_2$, $\mathbf{A}_3$ hold. Suppose, further, that the following inequalities are valid:*

$$\underline{b}_i - \frac{1}{2} \sum_{j=1}^{m} (|c_{ij}| F_j + |c_{ji}| F_i) - \frac{1}{2} \sum_{j=1}^{m} (|d_{ij}| G_j + |d_{ji}| G_i) > 0, \quad i = \overline{1, m}. \tag{3}$$

Then system (1) has a unique equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$.

Further on we give sufficient conditions for the global asymptotic stability of the equilibrium point x^* of system (1).

Theorem 2. *Let the assumptions* $\mathbf{A}_1 - \mathbf{A}_4$ *hold. Suppose, further, that the inequalities*

$$\underline{a}_i\underline{b}_i - \frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i|c_{ij}|F_j + \overline{a}_j|c_{ji}|F_i\right) - \frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i|d_{ij}|G_j + \overline{a}_j|d_{ji}|G_i\right) -$$

$$- \frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i\overline{b}_i|e_{ij}| + \overline{a}_j\overline{b}_j|e_{ji}|\right) - \frac{1}{2}\sum_{j=1}^{m}\overline{a}_j\sum_{k=1}^{m}\left(|c_{ji}|\,|e_{jk}|\,F_i + |c_{jk}|\,|e_{ji}|\,F_k\right) -$$

$$- \frac{1}{2}\sum_{j=1}^{m}\overline{a}_j\sum_{k=1}^{m}\left(|d_{ji}|\,|e_{jk}|\,G_i + |d_{jk}|\,|e_{ji}|\,G_k\right) > 0, \quad i = \overline{1,m}, \tag{4}$$

are valid and system (1) has an equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ *whose components satisfy (2). Then the equilibrium point* x^* *is globally asymptotically stable.*

Remark 1. Inequalities (3) can be deduced from (4) for $\underline{a}_i = \overline{a}_i = 1$, $e_{ij} = 0$ for $i, j = \overline{1,m}$. However, in general inequalities (4) do not imply (3).

Remark 2. Inequalities (4) were given in [11] (Theorem 4.3) as a part of the sufficient conditions for the global asymptotic stability of the equilibrium point of the discrete-time counterpart of system (1) provided with impulsive conditions, for small values of the discretization step h.

Remark 3. In [8] it is assumed that $g_j = f_j$, the functions $b_i(x_i)$ and $b_i^{-1}(x_i)$ are continuosly differentiable, and $b_i'(x_i)$ are bounded both below and above by positive constants. Instead of the m inequalities (4) a single inequality is presented, which in our notation can be written as

$$\min_{i=\overline{1,m}}(\underline{a}_i\underline{b}_i) - \max_{i=\overline{1,m}}(\overline{a}_i\overline{b}_i)\|\mathcal{E}\| - \max_{i=\overline{1,m}}\overline{a}_i\left(\max_{i=\overline{1,m}}F_i\|C\| + \max_{i=\overline{1,m}}G_i\|D\|\right)(1 + \|\mathcal{E}\|) > 0, \tag{5}$$

where C and D are $(m \times m)$-matrices with entries c_{ij} and d_{ij}, respectively.

Though condition (5) seems much simpler than (4), in our opinion it is much less precise since the individual lower and upper bounds, Lipschitz constants, and matrix entries are replaced by their minima or maxima, and matrix norms. Below we shall give an example of a system satisfying conditions (4) but not (5).

Proof. Upon introducing the translations

$$u_i(t) = x_i(t) - x_i^*, \quad \varphi_i(s) = \phi_i(s) - x_i^*$$

we derive the system

$$\dot{u}_i(t) + \sum_{j=1}^{m} e_{ij}\dot{u}_j(t - \tau_j) = \tilde{a}_i(u_i(t))\left[-\tilde{b}_i(u_i(t)) + \sum_{j=1}^{m} c_{ij}\tilde{f}_j(u_j(t)) + \right.$$

$$\left. + \sum_{j=1}^{m} d_{ij}\tilde{g}_j(u_j(t - \tau_j))\right], \quad t > t_0 = 0, \tag{6}$$

$$u_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], \quad i = \overline{1,m},$$

where

$$\tilde{a}_i(u_i) = a_i(u_i + x_i^*), \quad \tilde{b}_i(u_i) = b_i(u_i + x_i^*) - b_i(x_i^*),$$

$$\tilde{f}_j(u_j) = f_j(u_j + x_j^*) - f_j(x_j^*), \quad \tilde{g}_j(u_j) = g_j(u_j + x_j^*) - g_j(x_j^*).$$

This system inherits the assumptions $\mathbf{A}_1 - \mathbf{A}_4$ given before. It suffices to examine the stability characteristics of the trivial equilibrium point $u^* = 0$ of system (6).

We define a Lyapunov functional $V(t)$ by

$$V(t) = \frac{1}{2}\sum_{i=1}^{m}\left\{\left[u_i(t) + \sum_{j=1}^{m} e_{ij}u_j(t-\tau_j)\right]^2 + \omega_i \int\limits_{t-\tau_i}^{t} u_i^2(s)\,ds\right\},$$

where the positive constants ω_i, $i = \overline{1,m}$, will be determined later. First we notice that the value

$$V(0) = \frac{1}{2}\sum_{i=1}^{m}\left\{\left[\varphi_i(0) + \sum_{j=1}^{m} e_{ij}\varphi_j(-\tau_j)\right]^2 + \omega_i \int\limits_{-\tau_i}^{0} \varphi_i^2(s)\,ds\right\}$$

is completely determined from the initial values of the system. Then, calculating the rate of change of $V(t)$ along the solutions of (6), we successively find

$$\dot{V}(t) = \sum_{i=1}^{m}\left\{\left[u_i(t) + \sum_{j=1}^{m} e_{ij}u_j(t-\tau_j)\right]\left[\dot{u}_i(t) + \sum_{j=1}^{m} e_{ij}\dot{u}_j(t-\tau_j)\right] + \frac{\omega_i}{2}(u_i^2(t) - u_i^2(t-\tau_i))\right\} =$$

$$= \sum_{i=1}^{m}\left\{\left[u_i(t) + \sum_{j=1}^{m} e_{ij}u_j(t-\tau_j)\right]\tilde{a}_i(u_i(t))\left[-\tilde{b}_i(u_i(t)) + \sum_{j=1}^{m} c_{ij}\tilde{f}_j(u_j(t)) + \right.\right.$$

$$\left.\left. + \sum_{j=1}^{m} d_{ij}\tilde{g}_j(u_j(t-\tau_j))\right] + \frac{\omega_i}{2}\left(u_i^2(t) - u_i^2(t-\tau_i)\right)\right\} =$$

$$= \sum_{i=1}^{m}\left\{-\tilde{a}_i(u_i(t))\tilde{b}_i(u_i(t))u_i(t) + \tilde{a}_i(u_i(t))u_i(t)\left[\sum_{j=1}^{m} c_{ij}\tilde{f}_j(u_j(t)) + \sum_{j=1}^{m} d_{ij}\tilde{g}_j(u_j(t-\tau_j))\right] + \right.$$

$$+ \tilde{a}_i(u_i(t))\sum_{j=1}^{m} e_{ij}u_j(t-\tau_j)\left[-\tilde{b}_i(u_i(t)) + \sum_{j=1}^{m} c_{ij}\tilde{f}_j(u_j(t)) + \right.$$

$$\left.\left. + \sum_{j=1}^{m} d_{ij}\tilde{g}_j(u_j(t-\tau_j))\right] + \frac{\omega_i}{2}\left(u_i^2(t) - u_i^2(t-\tau_i)\right)\right\} \le$$

$$\le \sum_{i=1}^{m}\left\{-\underline{a}_i\underline{b}_i u_i^2(t) + \overline{a}_i|u_i(t)|\left[\sum_{j=1}^{m}|c_{ij}|\,F_j\,|u_j(t)| + \sum_{j=1}^{m}|d_{ij}|\,G_j\,|u_j(t-\tau_j)|\right] + \right.$$

$$+\,\overline{a}_i\sum_{j=1}^{m}|e_{ij}|\,|u_j(t-\tau_j)|\left[\overline{b}_i\,|u_i(t)|+\sum_{j=1}^{m}|c_{ij}|\,F_j\,|u_j(t)|+\sum_{j=1}^{m}|d_{ij}|\,G_j\,|u_j(t-\tau_j)|\right]+$$

$$\left.+\,\frac{\omega_i}{2}\left(u_i^2(t)-u_i^2(t-\tau_i)\right)\right\}\leq$$

$$\leq\sum_{i=1}^{m}\left\{-\underline{a}_i\underline{b}_iu_i^2(t)+\frac{\overline{a}_i}{2}\sum_{j=1}^{m}|c_{ij}|F_j\left(u_i^2(t)+u_j^2(t)\right)+\frac{\overline{a}_i}{2}\sum_{j=1}^{m}|d_{ij}|\,G_j\left(u_i^2(t)+u_j^2(t-\tau_j)\right)+\right.$$

$$+\,\frac{\overline{a}_i\overline{b}_i}{2}\sum_{j=1}^{m}|e_{ij}|\left(u_i^2(t)+u_j^2(t-\tau_j)\right)+\frac{\overline{a}_i}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}|e_{ij}|\,|c_{ik}|\,F_k\left(u_k^2(t)+u_j^2(t-\tau_j)\right)+$$

$$\left.+\,\frac{\overline{a}_i}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}|e_{ij}|\,|d_{ik}|\,G_k\left(u_j^2(t-\tau_j)+u_k^2(t-\tau_k)\right)+\frac{\omega_i}{2}\left(u_i^2(t)-u_i^2(t-\tau_i)\right)\right\}=$$

$$=\sum_{i=1}^{m}\left\{-\left[\underline{a}_i\underline{b}_i-\frac{1}{2}\left(\overline{a}_i\sum_{j=1}^{m}|c_{ij}|F_j+F_i\sum_{j=1}^{m}|c_{ji}|\overline{a}_j\right)-\frac{a_i}{2}\sum_{j=1}^{m}|d_{ij}|G_j\;-\right.\right.$$

$$\left.-\,\frac{\overline{a}_i\overline{b}_i}{2}\sum_{j=1}^{m}|e_{ij}|-\frac{F_i}{2}\,\sum_{j=1}^{m}\sum_{k=1}^{m}|c_{ki}|\,|e_{kj}|\overline{a}_k-\frac{\omega_i}{2}\right]u_i^2(t)+$$

$$+\,\frac{1}{2}\left[G_i\sum_{j=1}^{m}|d_{ji}|\overline{a}_j+\sum_{j=1}^{m}|e_{ji}|\overline{a}_j\overline{b}_j+\sum_{j=1}^{m}\sum_{k=1}^{m}|e_{ji}|\,|c_{jk}|\overline{a}_jF_k\;+\right.$$

$$\left.\left.+\,\sum_{j=1}^{m}\sum_{k=1}^{m}\left(|e_{ji}|\,|d_{jk}|\overline{a}_jG_k+|e_{kj}|\,|d_{ki}|\overline{a}_kG_i\right)-\omega_i\right]u_i^2(t-\tau_i)\right\}.$$

Choose

$$\omega_i\,=\,G_i\sum_{j=1}^{m}|d_{ji}|\overline{a}_j+\sum_{j=1}^{m}|e_{ji}|\overline{a}_j\overline{b}_j+\sum_{j=1}^{m}\sum_{k=1}^{m}|e_{ji}|\,|c_{jk}|\overline{a}_jF_k+$$

$$+\sum_{j=1}^{m}\sum_{k=1}^{m}\left(|e_{ji}|\,|d_{jk}|\overline{a}_jG_k+|e_{kj}|\,|d_{ki}|\overline{a}_kG_i\right)\,>\,0,$$

then after some simplifications we obtain

$$\dot{V}(t)\,\leq\,-\sum_{i=1}^{m}\left\{\underline{a}_i\underline{b}_i-\frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i|c_{ij}|F_j+\overline{a}_j|c_{ji}|F_i\right)-\right.$$

$$- \frac{1}{2}\sum_{j=1}^{m}(\overline{a}_i|d_{ij}|G_j + \overline{a}_j|d_{ji}|G_i) - \frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i\overline{b}_i|e_{ij}| + \overline{a}_j\overline{b}_j|e_{ji}|\right) -$$

$$\left. - \frac{1}{2}\sum_{j=1}^{m}\overline{a}_j\sum_{k=1}^{m}(|c_{ji}|\,|e_{jk}|\,F_i + |c_{jk}|\,|e_{ji}|\,F_k) - \frac{1}{2}\sum_{j=1}^{m}\overline{a}_j\sum_{k=1}^{m}(|d_{ji}|\,|e_{jk}|\,G_i + |d_{jk}|\,|e_{ji}|\,G_k)\right\} u_i^2(t).$$

According to inequalities (4) there exists $\mu > 0$ such that

$$\mu = \min_{i=\overline{1,m}}\left\{\underline{a}_i\underline{b}_i - \frac{1}{2}\sum_{j=1}^{m}(\overline{a}_i|c_{ij}|F_j + \overline{a}_j|c_{ji}|F_i) - \frac{1}{2}\sum_{j=1}^{m}(\overline{a}_i|d_{ij}|G_j + \overline{a}_j|d_{ji}|G_i) - \right.$$

$$- \frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i\overline{b}_i|e_{ij}| + \overline{a}_j\overline{b}_j|e_{ji}|\right) - \frac{1}{2}\sum_{j=1}^{m}\overline{a}_j\sum_{k=1}^{m}(|c_{ji}|\,|e_{jk}|\,F_i + |c_{jk}|\,|e_{ji}|\,F_k) -$$

$$\left. - \frac{1}{2}\sum_{j=1}^{m}\overline{a}_j\sum_{k=1}^{m}(|d_{ji}|\,|e_{jk}|\,G_i + |d_{jk}|\,|e_{ji}|\,G_k)\right\},$$

then

$$\dot{V}(t) \le -\mu\|u(t)\|^2, \quad t > 0, \tag{7}$$

where $\|v\| = \left(\sum_{i=1}^{m} v_i^2\right)^{1/2}$ is the Euclidean norm of the vector $v = (v_1, v_2, \ldots, v_m)^T \in \mathbb{R}^m$. Inequality (7) shows that for any solution $u(t)$ of system (6) the function $V(t)$ is monotone decreasing and it is bounded below by 0. Thus there exists the limit $L = \lim_{t\to\infty} V(t) \ge 0$.

Let us integrate inequality (7) from 0 to t,

$$V(t) - V(0) \le -\mu\int_0^t \|u(s)\|^2\,ds$$

for all $t > 0$, that is,

$$\int_0^t \|u(s)\|^2\,ds \le (V(0) - V(t))/\mu.$$

The last inequality and $L = \lim_{t\to\infty} V(t) \ge 0$ show that

$$\int_0^\infty \|u(t)\|^2\,dt < \infty. \tag{8}$$

Below we show that the zero solution of system (6) is stable and (8) implies $\lim_{t\to\infty}\|u(t)\| = 0$, that is, $\lim_{t\to\infty}\|x(t) - x^*\| = 0$. This means that the equilibrium point x^* of system (1) is globally asymptotically stable.

We complete the proof by arguments using fragments from the proofs of Theorems 1.1, 1.3 and 1.4 in [12] (Chapter 8). In the sequel for a vector $v = (v_1, v_2, \ldots, v_m)^T \in \mathbb{R}^m$ we shall also use the norm

$$|v| = \max_{i=\overline{1,m}} |v_i|.$$

First we shall prove that for any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that if

$$\left| u_i(t) + \sum_{i=1}^m e_{ij} u_j(t - \tau_{ij}) \right| \le \delta_1 \quad \text{for} \quad t \ge 0, \quad i = \overline{1,m}, \quad \text{and} \quad \sup_{s \in [-\tau, 0]} |\varphi(s)| \le \delta_1,$$

then $|u(t)| \le \varepsilon$ for $t \ge 0$.

Let T be an arbitrary positive number. For $0 \le t \le T$ we have

$$\begin{aligned}
|u_i(t)| &\le \left| u_i(t) + \sum_{i=1}^m e_{ij} u_j(t - \tau_{ij}) \right| + \left| \sum_{i=1}^m e_{ij} u_j(t - \tau_{ij}) \right| \le \\
&\le \delta_1 + \sum_{i=1}^m |e_{ij}|\, |u_j(t - \tau_{ij})| \le \\
&\le \delta_1 + \sum_{i=1}^m |e_{ij}| \sup_{-\tau \le t \le T} |u_j(t)| \le \\
&\le \delta_1 + \sum_{i=1}^m |e_{ij}| \left(\sup_{0 \le t \le T} |u_j(t)| + \sup_{-\tau \le s \le 0} |\varphi_j(s)| \right) \le \\
&\le \sum_{i=1}^m |e_{ij}| \sup_{0 \le t \le T} |u_j(t)| + \delta_1 \left(1 + \sum_{i=1}^m |e_{ij}| \right),
\end{aligned}$$

thus

$$\sup_{0 \le t \le T} |u_i(t)| \le \sum_{i=1}^m |e_{ij}| \sup_{0 \le t \le T} |u_j(t)| + \delta_1 \left(1 + \sum_{i=1}^m |e_{ij}| \right)$$

or

$$\sup_{0 \le t \le T} |u_i(t)| - \sum_{i=1}^m |e_{ij}| \sup_{0 \le t \le T} |u_j(t)| \le \delta_1 \left(1 + \sum_{i=1}^m |e_{ij}| \right) \quad \text{for} \quad i = \overline{1,m}.$$

If we introduce the vectors

$$U(T) = \left(\sup_{0 \le t \le T} |u_1(t)|, \sup_{0 \le t \le T} |u_2(t)|, \ldots, \sup_{0 \le t \le T} |u_m(t)| \right)^T \quad \text{and} \quad \mathbf{e} = (1, 1, \ldots, 1)^T,$$

we can write the last inequalities in a matrix form as

$$(E - |\mathcal{E}|) U(T) \le \delta_1 (E + |\mathcal{E}|) \mathbf{e},$$

meaning inequalities between the respective components of the vectors. Since, by condition $\mathbf{A}_4$, $E - |\mathcal{E}|$ is an M-matrix, we obtain

$$U(T) \le \delta_1 (E - |\mathcal{E}|)^{-1}(E + |\mathcal{E}|)\mathbf{e}.$$

We have

$$\sup_{0\le t\le T}|u(t)| = \sup_{0\le t\le T}\max_{i=\overline{1,m}}|u_i(t)| = \max_{i=\overline{1,m}}\sup_{0\le t\le T}|u_i(t)| = |U(T)| \le \delta_1|(E - |\mathcal{E}|)^{-1}(E + |\mathcal{E}|)\mathbf{e}|.$$

If we choose $\delta_1 > 0$ so small that $\delta_1|(E - |\mathcal{E}|)^{-1}(E + |\mathcal{E}|)\mathbf{e}| < \varepsilon$, then $|u(t)| \le \varepsilon$ for $0 \le t \le T$, where T was an arbitrary positive number. Thus, $|u(t)| \le \varepsilon$ for $t \ge 0$.

Next we shall show that the zero solution of system (6) is stable, that is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\sup_{s\in[-\tau,0]}|\varphi(s)| \le \delta$, then $|u(t)| \le \varepsilon$ for $t \ge 0$. For any $t \ge 0$ we get

$$\frac{1}{2}\sum_{i=1}^{m}\left[u_i(t) + \sum_{j=1}^{m} e_{ij}u_j(t-\tau_j)\right]^2 \le V(t) \le V(0) =$$

$$= \frac{1}{2}\sum_{i=1}^{m}\left\{\left[\varphi_i(0) + \sum_{j=1}^{m} e_{ij}\varphi_j(-\tau_j)\right]^2 + \omega_i\int_{-\tau_i}^{0}\varphi_i^2(s)\,ds\right\} \le$$

$$\le \frac{\delta^2}{2}\sum_{i=1}^{m}\left\{\left[1 + \sum_{j=1}^{m}|e_{ij}|\right]^2 + \omega_i\tau_i\right\}.$$

If we choose $\delta \in (0, \delta_1)$ so small that

$$\delta^2\sum_{i=1}^{m}\left\{\left[1 + \sum_{j=1}^{m}|e_{ij}|\right]^2 + \omega_i\tau_i\right\} \le \delta_1^2,$$

then

$$\sum_{i=1}^{m}\left[u_i(t) + \sum_{j=1}^{m} e_{ij}u_j(t-\tau_j)\right]^2 \le \delta_1^2,$$

which implies that

$$\left|u_i(t) + \sum_{i=1}^{m} e_{ij}u_j(t-\tau_{ij})\right| \le \delta_1 \quad \text{for} \quad t \ge 0, \quad i = \overline{1,m},$$

and, consequently, $|u(t)| \le \varepsilon$ for $t \ge 0$.

Because of stability of the zero solution of system (6) we can assume that $|u(t)| \le h$ for some positive constant h when $\sup_{s\in[-\tau,0]}|\varphi(s)| \le \delta$. Suppose that $\lim_{t\to\infty} u(t) = 0$ is not true.

In this case there exists a number $\nu > 0$ and an increasing sequence $\{t_k\}$ such that $t_k \to \infty$ and $|u(t_k)| \geq \nu$ for $k \in \mathbb{N}$. For the sake of brevity we write system (6) in the form

$$\frac{d}{dt}\left(u_i(t) + \sum_{j=1}^{m} e_{ij}u_j(t-\tau_j)\right) = \mathcal{F}_i(u(t), u(t-\tau)), \quad i = \overline{1,m}, \quad t > 0, \tag{9}$$

where

$$\mathcal{F}_i(u,\bar{u}) := \tilde{a}_i(u_i)\left[-\tilde{b}_i(u_i) + \sum_{j=1}^{m} c_{ij}\tilde{f}_j(u_j) + \sum_{j=1}^{m} d_{ij}\tilde{g}_j(\bar{u}_j)\right], \quad i = \overline{1,m}.$$

We denote

$$C_i = \sup_{|u|,|\bar{u}|\leq h} |\mathcal{F}_i|, \quad i = \overline{1,m}.$$

For $t \geq 0$ and $\Delta > 0$ integrate equation (9) from t to $t+\Delta$ to obtain

$$\begin{aligned} u_i(t+\Delta) - u_i(t) = \\ &= -\sum_{j=1}^{m} e_{ij}(u_j(t+\Delta-\tau_j) - u_j(t-\tau_j)) + \\ &+ \int\limits_t^{t+\Delta} \mathcal{F}_i(u(s), u(s-\tau))\,ds, \end{aligned}$$

hence

$$\begin{aligned} |u_i(t+\Delta) - u_i(t)| &\leq \sum_{j=1}^{m} |e_{ij}||u_j(t+\Delta-\tau_j) - u_j(t-\tau_j)| + \\ &+ \int\limits_t^{t+\Delta} |\mathcal{F}_i(u(s), u(s-\tau))|\,ds \leq \\ &\leq \sum_{j=1}^{m} |e_{ij}| \sup_{t\geq -\tau} |u_j(t+\Delta) - u_j(t)| + C_i\Delta \leq \\ &\leq \sum_{j=1}^{m} |e_{ij}| \left(\sup_{t\geq 0} |u_j(t+\Delta) - u_j(t)| + \sup_{s\in[-\tau,0]} |u_j(s+\Delta) - u_j(s)|\right) + C_i\Delta, \end{aligned}$$

thus

$$\sup_{t\geq 0}|u_i(t+\Delta)-u_i(t)| \leq$$

$$\leq \sum_{j=1}^{m}|e_{ij}|\left(\sup_{t\geq 0}|u_j(t+\Delta)-u_j(t)| + \sup_{s\in[-\tau,0]}|u_j(s+\Delta)-u_j(s)|\right) + C_i\Delta$$

or

$$\sup_{t\geq 0}|u_i(t+\Delta)-u_i(t)| - \sum_{j=1}^{m}|e_{ij}|\sup_{t\geq 0}|u_j(t+\Delta)-u_j(t)| \leq$$

$$\leq \sum_{j=1}^{m}|e_{ij}|\sup_{s\in[-\tau,0]}|u_j(s+\Delta)-u_j(s)| + C_i\Delta, \quad i=\overline{1,m}.$$

If we introduce the vectors

$$\rho(\Delta)=\left(\sup_{t\geq 0}|u_1(t+\Delta)-u_1(t)|, \sup_{t\geq 0}|u_2(t+\Delta)-u_2(t)|, \ldots, \sup_{t\geq 0}|u_m(t+\Delta)-u_m(t)|\right)^T,$$

$$\sigma(\Delta) =$$

$$= \left(\sup_{s\in[-\tau,0]}|u_1(s+\Delta)-u_1(s)|, \sup_{s\in[-\tau,0]}|u_2(s+\Delta)-u_2(s)|, \ldots, \sup_{s\in[-\tau,0]}|u_m(s+\Delta)-u_m(s)|\right)^T$$

and $\mathbf{C} = (C_1, C_2, \ldots, C_m)^T$, we can write the last inequalities in a matrix form as $(E-|\mathcal{E}|)\rho(\Delta) \leq \leq |\mathcal{E}|\sigma(\Delta)+\Delta\mathbf{C}$. From here as above we obtain $\rho(\Delta) \leq (E-|\mathcal{E}|)^{-1}(|\mathcal{E}|\sigma(\Delta)+\Delta\mathbf{C})$ and

$$\sup_{t\geq 0}|u(t+\Delta)-u(t)| \leq |(E-|\mathcal{E}|)^{-1}(|\mathcal{E}|\sigma(\Delta)+\Delta\mathbf{C})|. \tag{10}$$

Let $\eta > 0$ and $\Delta \leq \eta$. Since $u(t)$ is uniformly continuous on the interval $[-\tau,\eta]$, the right-hand side of (10) can be made arbitrarily small for sufficiently small values of Δ. Thus we can choose $\eta > 0$ so that $|u(t+\Delta)-u(t)| \leq \nu/2$ for all $t \geq 0$ and $\Delta \in [0,\eta]$. In particular,

$$|u(t_k+\Delta)| \geq |u(t_k)| - |u(t_k+\Delta)-u(t_k)| \geq \nu - \frac{\nu}{2} = \frac{\nu}{2}$$

or

$$|u(t)| \geq \frac{\nu}{2} \quad \text{and} \quad \|u(t)\|^2 \geq \frac{\nu^2}{4} \quad \text{for} \quad t\in[t_k, t_k+\eta], \quad k\in\mathbb{N}.$$

Without loss of generality we can assume that the intervals $[t_k, t_k+\eta]$ are disjoint (otherwise we choose a subsequence). Then

$$\int_0^{\infty}\|u(t)\|^2\,dt \geq \sum_{k=1}^{\infty}\int_{t_k}^{t_k+\eta}\|u(t)\|^2\,dt \geq \sum_{k=1}^{\infty}\eta\frac{\nu^2}{4} = \infty,$$

which contradicts (8). Thus $\lim_{t\to\infty} u(t) = 0$ is true and the proof is complete.

4. Example. Consider the system

$$\begin{aligned}
&\dot{x}_1(t) + 0.1\dot{x}_1(t-\tau_1) + 0.15\dot{x}_2(t-\tau_2) = (2 + 0.01\sin x_1(t))\,[-2x_1(t) + 0.1\arctan x_1(t) + \\
&\qquad + 0.15\arctan x_2(t) + 0.1\arctan x_1(t-\tau_1) + 0.15\arctan x_2(t-\tau_2) + 1]\,, \\
&\dot{x}_2(t) - 0.2\dot{x}_1(t-\tau_1) + 0.2\dot{x}_2(t-\tau_2) = (3 - 0.02\sin x_2(t))\,[-3x_2(t) + 0.15\arctan x_1(t) - \\
&\qquad - 0.2\arctan x_2(t) + 0.1\arctan x_1(t-\tau_1) - -0.2\arctan x_2(t-\tau_2) + 1]\,, \quad t > 0,
\end{aligned} \tag{11}$$

with arbitrary delays τ_1, τ_2 and initial conditions $x_i(s) = \phi(s)$, $i = 1, 2$, $s \in [-\max\{\tau_1, \tau_2\}, 0]$.

System (11) has the form (1). It satisfies assumptions $\mathbf{A}_1 - \mathbf{A}_4$ with $\underline{a}_1 = 1.99$, $\overline{a}_1 = 2.01$, $\underline{a}_2 = 2.98$, $\overline{a}_2 = 3.02$, $\underline{b}_1 = \overline{b}_1 = 2$, $\underline{b}_2 = \overline{b}_2 = 3$, $F_1 = F_2 = G_1 = G_2 = 1$, $\|\mathcal{E}\| = 0.2863903109$ and

$$E - |\mathcal{E}| = \begin{bmatrix} 0.9 & -0.15 \\ -0.2 & 0.8 \end{bmatrix}$$

is an M-matrix.

It is easy to see that system (11) satisfies inequalities (3). In fact, the left-hand sides of these inequalities are equal respectively to 1.525 and 2.325 for $i = 1$ and 2. Thus system (11) has a unique equilibrium point x^*. We can find that $x^* = (0.6027869379,\ 0.3353919007)^T$.

Furthermore, system (11) satisfies the assumptions of Theorem 2. In fact, the left-hand sides of inequalities (4) are equal respectively to 0.5415 and 4.449 for $i = 1$ and 2. Thus the equilibrium point x^* of system (11) is globally asymptotically stable.

On the other side, system (11) does not satisfy condition (5) since the left-hand side of the inequality equals -0.608775797.

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