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GLOBAL ASYMPTOTICS FOR MULTIPLE INTEGRALS WITH BOUNDARIES

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Dedicated to Frédéric Pham on the occasion of his 65th birthday.

Abstract
Under convenient geometric assumptions, the saddle-point method for multidimensional Laplace integrals is extended to the case where the contours of integration have boundaries. The asymptotics are studied in the case of nondegenerate and of degenerate isolated critical points. The incidence of the Stokes phenomenon is related to the monodromy of the homology via generalized Picard-Lefschetz formulae and is quantified in terms of geometric indices of intersection. Exact remainder terms and the hyperasymptotics are then derived. A direct consequence is a numerical algorithm to determine the Stokes constants and indices of intersections. Examples are provided.

1. Introduction

The asymptotic behaviour as $k \to \infty$ in the complex plane $\mathbb{C}$ of complex oscillatory integrals

$$I_\Gamma(k) = \int_\Gamma e^{-kf(z)} g(z)\, dz^{(1)} \wedge \cdots \wedge dz^{(n)}, \tag{1}$$

with $f, g : \mathbb{C}^n \to \mathbb{C}$ analytic functions of the variable $z = (z^{(1)}, \ldots, z^{(n)})$ and Γ a chain of real dimension n, has been the study of much work, both theoretical and practical. A discussion of the history of the problem can be found in V. Arnold, A. Varchenko, and S. Gussein-Zadè [3] and D. Kaminski and R. Paris [40]. Applications of these integrals in optics are detailed in [53] and the references therein. Much work has focused on obtaining the asymptotic expansions themselves. Here we focus on deriving "global asymptotics" in all sectors of the complex k-plane for these integrals when the contours of integration are finitely bounded. Interest in this area has been renewed recently following the ideas of R. Balian and C. Bloch [6] and F. Pham [63],

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[65], the development of the resurgence (see [23]) and hyperasymptotic theories (see [7]), and the work of Kaminski and Paris [40], [41], and C. Howls [36].

The main approach used in deriving the (global) asymptotics of such Laplace integrals is a generalisation of the Riemann-Debye saddle-point method, which can be reduced to the following algorithm (see [25]):

(1) the identification of all possible critical points;

(2) the topological operation of pushing the integration contour in $\mathbb{C}^n$ toward the directions of steepest descent, forming a chain of integration hypersurfaces;

(3) the local study near the critical points of the phase function and near the boundary of the hypersurfaces of integration Γ, and the computation of the relevant asymptotic expansions;

(4) the derivation of the exact remainder terms, their reexpansions in terms of distant critical points, and the calculation of the associated Stokes constants, thereby explicitly linking the contributions from all relevant critical points.

The first and third parts have been extensively discussed at leading order (see, e.g., Arnold, Varchenko, and Gussein-Zadè [3], V. Vassiliev [71], and B. Gaveau [26]) and for real variables (see R. Wong [76]). The second topic has been studied practically for real variables in terms of flows by Kaminski [39]. The third topic has been studied in great detail by Varchenko from the theoretical viewpoint, relating the characteristic exponents of the asymptotic expansions to mixed Hodge structure in vanishing cohomologies (see [69], [70]). Substantial practical progress in the derivation of asymptotic expansions with exact remainder terms for polynomial exponents using Mellin integral representations has been made by Kaminski and Paris [40], [41] and by G. Liakhovetski and Paris [43] using a Newton polygon to identify the appropriate contributions. The fourth point was studied for unbounded integration contours by Howls [36].

Our goal here is to combine the four points above so as to produce for the first time exact remainder terms and a self-consistent numerical algorithm to determine the Stokes constants when Γ is a bounded domain.

The asymptotic expansions are well known when f is a polynomial function (and g is "well behaved at infinity") and the contour of integration Γ is an unbounded n-chain of hypersurfaces of integration, satisfying a convergence criterion at infinity

$$\Re(kf) \to +\infty. \tag{2}$$

Under convenient geometric assumptions (effectively, that all critical points are isolated and that no critical points at infinity occur), Pham showed geometrically in [63] that the chain Γ (resp., integral $I_\Gamma(k)$) can be decomposed as a finite sum

$$\Gamma = \sum_{\alpha} \sum_{j=1}^{\mu_\alpha} N_{\alpha_j}(\Gamma)\Gamma_{\alpha_j} \qquad \Big(\text{resp., } I_\Gamma(k) = \sum_{\alpha} \sum_{j=1}^{\mu_\alpha} N_{\alpha_j}(\Gamma) I_{\Gamma_{\alpha_j}}(k)\Big). \tag{3}$$

Here $N_\alpha^j(\Gamma)$ are integers and the first sum runs over the finite set of critical points z_α of the phase function f, while the Γ_α^j denote a basis of μ_α (Milnor number) independent steepest-descent n-folds associated with the critical point z_α.

Such a steepest-descent n-fold is easy to describe locally when the critical point is a Morse singularity, yielding the notion of the *Lefschetz thimble* of geometers. In general, the local analysis near the (isolated) critical points z_λ of the phase function f reduces to a description of the topology of the generic fiber $f = t$, $t \in \mathbb{C}$ near z_λ, whose (reduced) homology is the vanishing homology of geometers.

From the above decomposition, the asymptotics follow from a local analysis near the critical points. Moreover, this geometrical viewpoint also yields any algebraic Stokes phenomena, that is, discontinuities of the decomposition (3) for special values of $\arg(k)$. These discontinuities are given by the Picard-Lefschetz formulae, and their use in deriving exact remainder terms, Stokes constants, and Riemann sheet structure was discussed in the hyperasymptotic study by Howls [36].

The purpose of this paper is to extend the results of [36] to consider integrals of type (1) when one adds a set $\{S_1, S_2, \ldots, S_m\}$ of $m \leq n$ smooth irreducible affine hypersurfaces as possible boundaries for the contour of integration.

Assuming that f is a polynomial function, we show (under some convenient geometrical assumptions) a hyperasymptotic extension of known results in real asymptotic analysis (see [76]): the chain Γ can be decomposed again as a finite sum

$$\Gamma = \sum_{\alpha} \sum_{j=1}^{\mu_\alpha} N_{\alpha_j}(\Gamma)\Gamma_{\alpha_j} \tag{4}$$

with integers $N_\alpha(\Gamma)$, where the sum runs over the (finite) set of critical values f_α of

(1) the critical points of the phase function f and

(2) the critical points of the restricted function $f|$ on the boundary.

The above description follows from a decomposition of the space of (relative) homology classes of n-chains satisfying the descent condition. Some discontinuities in decomposition (4) usually appear under variations of the phase of k. For a geometer, this phenomenon can be understood in terms of indices of intersection, described by generalized Picard-Lefschetz formulae. For the analyst, this is the Stokes phenomenon, and the previous indices of intersection are now seen as Stokes multipliers. The first viewpoint helps in understanding the quantized nature of these Stokes multipliers but, as a rule, fails at the stage of concrete computation. Understanding these indices of intersection as Stokes multipliers gives a numerical method of computation, using the tools and ideas of hyperasymptotic theory.

The structure of the paper is as follows. In Section 2 we briefly describe the space of contours of integration that will be considered and the decomposition property. This is done in an intuitive manner, saving the proofs and technicalities for Appendix A. In Section 3 we apply this decomposition to integrals of type (1), deriving the asymptotics. In Section 4 we discuss the Stokes phenomenon, describing the so-called resurgence properties. Here again, the technicalities of the proofs are left to Appendix B. In Section 5 we demonstrate how to derive the exact remainder term for a truncated asymptotic expansion, together with the calculation of the associated Stokes constants and thence the hyperasymptotic reexpansions. In Section 6 we provide an example. We conclude in Section 7 with a discussion of the hypotheses and some related topics; among other subjects, the problem of the "confluent" case is briefly described, and some links between oscillating integrals with boundaries and differential equations are suggested. We also mention some open problems.

By their very nature, the complex oscillatory integrals are crossroads for different scientific communities, from pure mathematicians to physicists and chemists (cf. [15] and the references therein). This paper has been written in such a way as to be readable by as large an audience as possible; consequently, although some comments in the text will appear trivial to geometers, they are intended to be helpful to the nonspecialist. Conversely, the procedure for deriving the new hyperasymptotic formulae associated with these integrals may be familiar to an (exponential) asymptotician, but this is explained in Section 5 for the convenience of others.

2. Contours of integration and assumptions

This section contains the assumptions we make about the types of integrals to be treated, together with the definitions of the technical geometrical terms that we use. For the benefit of the geometer, the technicalities of the explanations and proofs can be found in Appendix A.

We treat only the case when f is a polynomial. As in the $(n = 1)$-dimensional case, the $\mathbb{C}$-space of the values of f plays a central role and is called the *Borel* plane or the t-plane.

In what follows, we assume that k has a given phase $-\theta$:

$$k = |k| \exp(-i\theta) \in \mathbb{C}\backslash\{0\}. \tag{5}$$

2.1. Contours of integration with no boundary

In Howls [36] and Pham [63], integrals of type (1) are considered for unbounded (n-real)-dimensional contours of integration Γ, traveling between asymptotic valleys at infinity where $\Re(kf) \to +\infty$. This condition ensures the convergence of the integral and the validity of the Stokes theorem (at least when the growth of the function g at infinity remains small in comparison to the decay of the exponential involving f). It is assumed from now on that g is a polynomial function, although we believe the results to be more widely applicable.

As in [63], the set of these unbounded integration contours is denoted by $H_n^{\Psi}(\mathbb{C}^n)$.

The properties of $H_n^{\Psi}(\mathbb{C}^n)$ are essentially governed by the behavior of the function f near its critical points z_α, where the gradient ∇f vanishes. In principle, an integral I_Γ over one such contour can be reduced to the sum of integrals over a sequence, called a chain, of "steepest-descent" contours Γ_α, each of which encounters a single critical point z_α of f and follows the flow of the vector field $\nabla(\Re\, kf)$. This is true at least when the phase of k is "generic" in the sense that each Γ_α encounters no more than one critical point, so that no Stokes phenomenon occurs.

This contour decomposition follows from the work of Pham [63], under two main geometrical assumptions.

(1) *All critical points are isolated.* This excludes those f that contain a ridge of critical points (see, for instance, [76]).

(2) *There are no critical points at infinity.*

Since these two requirements are central to the paper, it is worth pausing to give an explanation.

To explore the space of integration contours, it is necessary to study the topology of the level hypersurfaces $f^{(-1)}(t)$ for $t \in \mathbb{C}$, and particularly for t near a critical value f_α for which the fiber $f^{(-1)}(f_\alpha)$ becomes singular. This amounts to studying the geometry near the corresponding critical values, which is completely understood (since J. Milnor [50]) only when isolated critical points are concerned. Hence we have the first assumption.

The definition of critical points at infinity is subtle but can be illustrated with an example (from [10]). A straightforward computation shows that the polynomial

$$f(z^{(1)}, z^{(2)}) = {z^{(1)}}^2 z^{(2)} + 2z^{(1)} \tag{6}$$

has no critical point, but, nevertheless, the fiber $f^{-1}(0)$ differs from the other level hypersurfaces $f^{(-1)}(t)$. This can be seen by deforming this polynomial into

$$f_k(z^{(1)}, z^{(2)}) = {z^{(1)}}^2 z^{(2)} + 2z^{(1)} + k^2 z^{(2)}. \tag{7}$$

For $k \in \mathbb{C}\backslash\{0\}$, the polynomials f_k have two nondegenerate critical points (resp., values) at $(z^{(1)}, z^{(2)}) = (\mathrm{i}k, \mathrm{i}/k)$ and $(-\mathrm{i}k, -\mathrm{i}/k)$ (resp., $2\mathrm{i}k$ and $-2\mathrm{i}k$). However, when $k \to 0$, the two Morse singularities evaporate to infinity while the two critical values converge to zero. This gives rise to a critical point at infinity, with zero for its corresponding bifurcation value. We see later in Section 7.1 how this bifurcation value coming from a critical point at infinity indeed affects the (hyper)asymptotics. Nevertheless, the general topology in the locality of such critical points at infinity is complicated and not yet well understood (apart from the dimension $n = 2$; see [32] and [22] for a survey of recent results), and this is the reason for the second assumption.

2.2. Contours of integration with boundaries

Returning to the general case of integrals of type (1), suppose that the contour Γ now encounters a sequence of boundaries $\{S_1, S_2, \ldots, S_m\}$, these being $m \le n$ smooth complex $(n-1)$-dimensional hypersurfaces. For instance, the contour may run from a boundary into an asymptotic valley at infinity where $\Re(kf) \to +\infty$ for convergence, or it may run between two finitely placed boundaries.

We assume that each boundary S_i is defined by a polynomial equation $P_i(z) = 0$ with $P_i \in \mathbb{C}[z]$. This prevents the boundaries from behaving too wildly at infinity, thus ensuring (1) the convergence of the integrals and (2) the validity of the Stokes theorem.

Notation. The set of these contours Γ, which we also refer to as *cycles*, is denoted by $H_n^{\Psi}(\mathbb{C}^n, S)$ with $S = S_1 \cup \cdots \cup S_m$.

The affine hypersurfaces $\{S_1, S_2, \ldots, S_m\}$ are assumed to satisfy the following hypotheses.

HYPOTHESIS H1

The m hypersurfaces $S_1, S_2, \ldots, S_m$ are in general position; that is, they may cross, but they are not tangential at the crossing points.

Notation. For brevity, the intersections of boundary surfaces $S_{i_1} \cap \cdots \cap S_{i_p}$ $(1 \le i_1 < \cdots < i_p \le m)$ are denoted by $S_{(i_1,\ldots,i_p)}$. These are smooth submanifolds of complex codimension p by Hypothesis H1.

From the polynomial nature of the phase function f and the Bertini-Sard theorem, it follows that the set $\Lambda_{()}$ of critical values of f is finite. Moreover, from the algebraic assumptions on the S_i, the restriction $f_{(i_1,\ldots,i_p)} := f \mid S_{(i_1,\ldots,i_p)}$ of f to the intersection of the boundaries $S_{(i_1,\ldots,i_p)}$ also has a *finite set* $\Lambda_{(i_1,\ldots,i_p)}$ *of critical values* (see [50]). We denote by $\Lambda = \bigcup \Lambda_{(i_1,\ldots,i_p)}$ the set of all these critical values. Notice here that if $m = n$, then the (finite) set of points $S_1 \cap \cdots \cap S_n$ are considered critical points. In one-dimensional integrals, this corresponds exactly to linear endpoints.

To avoid complications at infinity, we assume the following hypothesis.

HYPOTHESIS H2

We have that f, as well as each $f_{(i_1,\ldots,i_p)}$ $(1 \le i_1 < \cdots < i_p \le m)$, has no singularity at infinity.

We avoid nonisolated critical points.

HYPOTHESIS H3
All critical points of f, *as well as those of the restricted functions* $f_{(i_1,\ldots,i_p)}$ *from* $S_{(i_1,\ldots,i_p)}$ *to* $\mathbb{C}$, *are isolated.*

It is useful also to assume a one-to-one correspondence between critical points z_α in n dimensions and their critical values $f_\alpha = f(z_\alpha)$ in one dimension. This is the subject of the following hypothesis.

HYPOTHESIS H4
Each $f_\alpha \in \Lambda$ *is the image of a single critical point* z_α *of* f *or one of the* $f \mid S_{(i_1,\ldots,i_p)}$.

At this point it is helpful to introduce some definitions.

Definition 2.1
The *depth* of a (restricted) critical point z_α is the maximal number of boundaries p on which it lies.

Following the one-to-one equivalence between critical value and critical point (see Hypothesis H4), we can obviously extend the definition of depth to critical values. This definition allows us to classify the critical points according to their depth in the following way.*

Definition 2.2
Critical points of the *first type* are critical points of depth $p = 0$.

These are critical points of the unrestricted f and do not lie on a boundary.

Definition 2.3
Critical points of the *second type* are those of depth $p > 0$ and are critical only for f restricted to exactly p boundaries.

For such a critical point, and if $S_1, \ldots, S_p$ are the p boundaries (up to a reordering), it follows from Hypothesis H1 and Hadamard's lemma (see [2]) that there exist new local coordinates $(s^{(1)}, \ldots, s^{(p)}, s^{(p+1)}, \ldots, s^{(n)})$ such that we can write f as

$$f = f_\alpha + s^{(1)} + \cdots + s^{(p)} + F(s^{(p+1)}, \ldots, s^{(n)}) \tag{8}$$

with $s^{(i)} = 0$ as a local equation for the boundary S_i.

* Our nomenclature differs slightly from Wong's [76].

Definition 2.4
Critical points of the *third type* are all other cases (including when an actual critical point of the unrestricted f accidently lies on a boundary).

We discuss only critical points of the first two types due to the extra complications that can arise in third-type cases.

HYPOTHESIS H5
No critical value of the third type occurs.

2.3. The decomposition theorem

In the presence of the boundaries $S_1, S_2, \ldots, S_m$, it is by no means obvious that it is still possible to decompose the original integration contour into a sequence of steepest-descent contours, each passing over either a single critical point of f or its restriction $f_{(i_1,\ldots,i_p)}$. While this may be obvious for single-dimensional integrals, and while local results in real dimensions are obtainable by using "neutralisers" (in terms of Laplace integrals; see Wong [76]), until now it has not been demonstrated explicitly in higher complex dimensions.

In Section A.3, by extending ideas of Pham [63] and taking advantage of Hypotheses H1 and H2, it is shown that a given cycle of integration can be decomposed through a continuous deformation (isotopy) into a chain of cycles Γ_α such that their projections by f are straight half-lines $L_\alpha := (f_\alpha, \infty \exp(i\theta))$, where f_α belongs to the set of critical values Λ. This is true at least when the finite set of these L_α are two-by-two disjoint, and this is guaranteed for a "generic" phase of k.

Note that on each of these Γ_α the exponential factor $\exp(-kf(z))$ has the greatest exponential decay; that is, Γ_α satisfies the steepest-descent condition.

Under Hypotheses H3 and H4, we show in Section A.4 that the investigation of each cycle Γ_α essentially reduces to a local analysis near its relevant critical point z_α. Assuming these results, Sections 2.3.1 and 2.3.2 help us to understand the local geometry near a critical point, introducing objects such as "Lefschetz thimbles" and "vanishing cycles." These objects are most easily defined initially for the unbounded contour case. We do this in Section 2.3.1 before demonstrating how they are modified in the presence of boundaries.

2.3.1. The geometry near a first-type critical point

In the Borel plane, we consider restrictions to (sufficiently) small closed discs D_α centred on the critical values f_α, of radius r_0. This corresponds to local truncations of the steepest-descent contours Γ_α in neighbourhoods of the critical point z_α.

We first consider the case of a nondegenerate critical point z_α. The function f

has a Morse singularity there, and the Morse lemma ensures the existence of local coordinates $(s^{(1)}, \dots, s^{(n)})$ such that

$$\begin{aligned} f - f_\alpha &= {s^{(1)}}^2 + \cdots + {s^{(n)}}^2 \\ &= {x^{(1)}}^2 + \cdots + {x^{(n)}}^2 - ({y^{(1)}}^2 + \cdots + {y^{(n)}}^2) \\ &\quad + 2i(x^{(1)}y^{(1)} + \cdots + x^{(n)}y^{(n)}), \end{aligned} \tag{9}$$

where $x^{(i)} = \Re(s^{(i)})$ is the real part of $s^{(i)}$, and $y^{(i)} = \Im(s^{(i)})$ is the imaginary part. Assuming for simplicity that $\theta = 0$, and under the truncation condition $|f(s) - f_\alpha| \le r_0$, we see that the best realisation of the steepest-descent conditions is given by the so-called Lefschtez thimble,* defined by

$$\Gamma_\alpha^{t_0} = \left\{ (s^{(1)}, \dots, s^{(n)}) \in \mathbb{C}^n \,\middle|\, y^{(1)} = \cdots = y^{(n)} = 0\,,\ {x^{(1)}}^2 + \cdots + {x^{(n)}}^2 \le r_0 \right\}. \tag{10}$$

The Lefschtez thimble $\Gamma_\alpha^{t_0}$ is unique up to the choice of orientation. The standard one is that defined by the coordinates $(x^{(1)}, \dots, x^{(n)})$ (see [63]).

The case $\theta \neq 0$ is easily deduced from the case $k > 0$ by the linear mapping

$$(s^{(1)}, \dots, s^{(n)}) \mapsto (e^{i\theta/2}s^{(1)}, \dots, e^{i\theta/2}s^{(n)}). \tag{11}$$

Note that, from its very definition, a Lefschetz thimble $\Gamma_\alpha^{t_0}$ is mapped by f onto the closed segment $L_\alpha^{t_0} := (f_\alpha, t_0)$, where $t_0 = f_\alpha + r_0 \exp(i\theta)$.

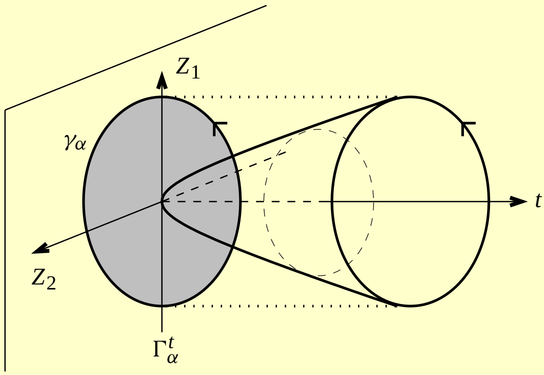

Figure 1. Lefschetz thimble $\Gamma_\alpha^{t_0}$ for $n = 2$ and its boundary $\gamma_\alpha^{t_0} = \partial[\Gamma_\alpha^{t_0}]$

An associated geometrical object is the *vanishing cycle*, $\gamma_\alpha^{t_0}$ (see Figure 1) being the $(n-1)$-real dimensional oriented boundary of the Lefschetz thimble, where $f =$

*In what follows, the term “absolute” is added to these Lefschtez thimbles to differentiate them from the “relative” Lefschtez thimbles introduced in Section 2.3.2.

t_0; when $\theta = 0$,

$$\gamma_\alpha^{t_0} = \left\{ (s^{(1)}, \ldots, s^{(n)}) \in \mathbb{C}^n \,\middle|\, y^{(1)} = \cdots = y^{(n)} = 0\,,\ {x^{(1)}}^2 + \cdots + {x^{(n)}}^2 = r_0 \right\}. \tag{12}$$

The orientation is determined by that of the Lefschetz thimble. It is convenient to refer to the vanishing cycle in terms of a boundary operator ∂,

$$\gamma_\alpha^{t_0} = \partial\, [\Gamma_\alpha^{t_0}]. \tag{13}$$

A Lefschetz thimble encountering a first-type nondegenerate critical point can thus be geometrically parametrised in that locality as the union of all vanishing cycles $\bigcup_t \gamma_\alpha^t$ as t runs along $L_\alpha^{t_0}$.

We note briefly that it is possible to extend the concepts of a Lefschetz thimble and a vanishing cycle to the case of an isolated degenerate critical point of the first type, that is, when the Hessian determinant of f vanishes. The idea is as follows. Under a small (generic) deformation, the degenerate critical point z_α splits into a finite number μ_α of nondegenerate critical points, μ_α being the multiplicity or Milnor number of the critical point.* We thus have a basis of μ_α Lefschetz thimbles $\Gamma_{\alpha_j}^{t_0}$, $j = 1, \ldots, \mu_\alpha$, and their corresponding vanishing cycles. Returning to the original unperturbed f, the basis of Lefschetz thimbles is deformed into a basis of μ_α "folded" Lefschetz thimbles.

Note that, starting with this local description of Lefschetz thimbles $\Gamma_\alpha^{t_0}$, we can extend them globally by following the flow of the vector field $\nabla(\Re\, kf)$ with the vanishing cycles as initial data. This defines what are called the *(absolute) steepest-descent n-folds* Γ_α. This is true at least when no Stokes phenomenon occurs, and this is guaranteed if the half-line L_α, onto which Γ_α is mapped by f, does not encounter any other critical value $f_\beta \in \Lambda_{()}$ of f.

2.3.2. The geometry near a second-type critical point

When boundaries are present, the definition of a Lefschetz thimble and a vanishing cycle needs clarification.

In the Borel plane, we again consider a restriction to a small enough closed disc D_α of radius r_0 centered on a critical value f_α of depth p. Localising near the corresponding critical point z_α, we use the local coordinates of (8).

If we assume that F has a nondegenerate critical point, then it may be written as

$$F = {s^{(p+1)}}^2 + \cdots + {s^{(n)}}^2 \tag{14}$$

* There are many equivalent definitions for the Milnor number. The nonspecialist may be referred to [58] or [2]. For instance, the Milnor number for the singularity ${z^{(1)}}^a + {z^{(2)}}^b$ $(a, b \in \mathbb{N}\backslash\{0\})$ is $(a-1)(b-1)$. Of course $\mu_\alpha = 1$ for a nondegenerate critical point.

by changing the $n-p$ local coordinates $(s^{(p+1)},\dots,s^{(n)})$ if necessary. This is a consequence of the Morse lemma applied on restriction to the $S_1\cap\cdots\cap S_p$, which of course does not affect the local equations $s^{(i)}=0$ for the boundary S_i, $i=1,\dots,p$. Assuming for a moment that $\theta=0$, we easily obtain

$$\Gamma_\alpha^{t_0}=\Big\{(s^{(1)},\dots,s^{(n)})\in\mathbb{C}^n\,\Big|\,y^{(1)}=\cdots=y^{(n)}=0\,,\ x^{(1)}\geq 0,\dots,x^{(p)}\geq 0,\\ x^{(1)}+\cdots+x^{(p)}+{x^{(p+1)}}^2+\cdots+{x^{(n)}}^2\leq r_0\Big\},\tag{15}$$

with $x^{(i)}=\Re(s^{(i)})$ and $y^{(i)}=\Im(s^{(i)})$ for the best realization of the steepest-descent conditions, under the truncation condition $|f(s)-f_\alpha|\leq r_0$.

We refer to $\Gamma_\alpha^{t_0}$ as a *relative Lefschetz thimble*. Again, the relative Lefschetz thimble is unique, up to the choice of the orientation. The orientation defined by the coordinates $(x^{(1)},\dots,x^{(n)})$ is called the *standard orientation*.

The case of general θ is deduced from the case $k>0$ by the simple linear mapping

$$(s^{(1)},\dots,s^{(p)},s^{(p+1)},s^{(n)})\mapsto(e^{i\theta}s^{(1)},\dots,e^{i\theta}s^{(p)},e^{i\theta/2}s^{(p+1)}e^{i\theta/2}s^{(n)}).\tag{16}$$

The relative Lefschetz thimble $\Gamma_\alpha^{t_0}$ is mapped by f onto the closed segment $L_\alpha^{t_0}:=(f_\alpha,t_0)$, where $t_0=f_\alpha+r_0\exp(i\theta)$.

The concept of a (relative) Lefschetz thimble now being clear, its companion vanishing cycle is defined in terms of a reduction process. This algorithm is important in the derivation of the asymptotics of Laplace integrals of type (1), and so we now explain it. This is done on the basis of Figure 2, where $n=3$ and $p=2$. We assume $\theta=0$ for simplicity. We first introduce $\partial\,[\Gamma_\alpha^{t_0}]$, where ∂ is the boundary operator that selects the part of the boundary of $\Gamma_\alpha^{t_0}$ where $f=t_0$:

$$\partial\,[\Gamma_\alpha^{t_0}]=\{x^{(1)}\geq 0,x^{(2)}\geq 0,\ x^{(1)}+x^{(2)}+{x^{(3)}}^2=r_0\}.\tag{17}$$

The relative Lefschetz thimble $\Gamma_\alpha^{t_0}$ can thus be locally parametrised as the union $\bigcup_t\partial\,[\Gamma_\alpha^t]$ with t running on $L_\alpha^{t_0}$.

Reading the equality in (17) as $x^{(2)}+{x^{(3)}}^2=r-x^{(1)}=r_1$ with $0\leq x^{(1)}\leq r$, we see that each $\partial\,[\Gamma_\alpha^t]$ can itself be parametrised by the union $\bigcup_{t_1}\partial_1\circ\partial\,[\Gamma_\alpha^{t_1}]$ for t_1 running on L_α^t (and $x_1=r-r_1$). Here ∂_1 is the boundary operator that selects the part of the boundary of $\partial\,[\Gamma_\alpha^{t_1}]$ lying on S_1, where $f|S_1=t_1$ and $t_1=f_\alpha+r_1\exp(i\theta)$; in other words,

$$\partial_1\circ\partial\,[\Gamma_\alpha^{t_1}]=\{x^{(2)}\geq 0,\ x^{(2)}+{x^{(3)}}^2=r_1\}.\tag{18}$$

Similarly introducing the boundary operator ∂_2, we get a parametrisation of each $\partial_1\circ\partial\,[\Gamma_\alpha^{t_1}]$ in terms of the sequence over $t_2\in L_\alpha^{t_1}$ of the boundaries

$$\partial_2\circ\partial_1\circ\partial\,[\Gamma_\alpha^{t_2}]=\{{x^{(3)}}^2=r_2\}\tag{19}$$

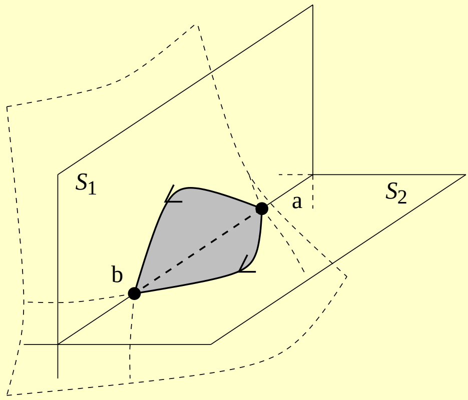

Figure 2. Relative Lefschetz thimble Γ_α^t for $n = 3$ and $p = 2$. The vanishing cycle is $\partial_2 \circ \partial_1 \circ \partial[\Gamma_\alpha^t] = [b] - [a]$.

(and $x_2 = r_1 - r_2$) lying on $S_2 \cap S_1$, where $f|S_2 \cap S_1 = t_2$ and $t_2 = f_\alpha + r_2 \exp(i\theta)$.

The boundary $\partial_2 \circ \partial_1 \circ \partial\,[\Gamma_\alpha^{t_0}]$ is precisely the *vanishing cycle* $\gamma_\alpha^{t_2}$ associated with the relative Lefschetz thimble $\Gamma_\alpha^{t_2}$. More generally, the vanishing cycle $\gamma_\alpha^{t_0}$ for a critical point of the second type of depth p is the boundary on the simultaneous intersection of (15) with the $p+1$ constraining boundaries, including the boundary condition $f = t_0$. In boundary operator notation, we have

$$\gamma_\alpha^{t_0} = \partial_{i_p} \circ \cdots \circ \partial_{i_1} \circ \partial[\Gamma_\alpha^{t_0}], \tag{20}$$

where the ∂_{i_p} is the operator that selects the part of the boundary lying on the boundary S_{i_p}. Its orientation is deduced from that prescribed to $\Gamma_\alpha^{t_0}$.

Note that the reduction process (20) is not canonical, in the sense that it is defined up to a permutation of the $p+1$ boundary operators. However, the resulting vanishing cycle does not depend on the order.

When F has a degeneracy of multiplicity μ_α, we obtain a basis of μ_α folded relative Lefschetz thimbles $\Gamma_{\alpha_j}^{t_0}$, $j = 1, \ldots, \mu_\alpha$, and the reduction process (20) to associate a (unique) vanishing cycle to a given (folded) relative fold Lefschetz thimble is unchanged.

To extend the local description of a relative Lefschetz thimble $\Gamma_\alpha^{t_0}$ into a global notion of a *relative steepest-descent n-fold* Γ_α is less obvious than in the "absolute case," where no boundary interferes (see [48]). This can be performed (cf. App. A), at least when no Stokes phenomenon occurs.

2.3.3. The corner case

In the above description of the behaviour of a second-type critical point, it is implicitly assumed that the depth p is less than n. The case where the depth is n, which corresponds to a corner, is actually simpler. As illustrated in Figure 3, the reduction process stops at the $(p = n)$th iteration, the boundary

$$\partial_{p-1} \circ \cdots \circ \partial_1 \circ \partial[\Gamma] \tag{21}$$

being just the end critical corner point, and no vanishing cycle needs to be defined.

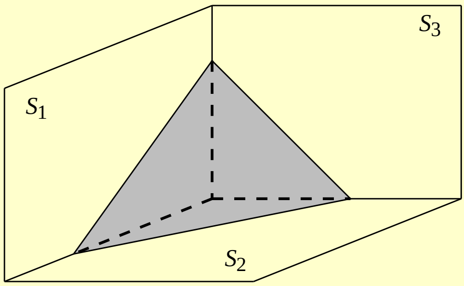

Figure 3. Relative Lefschetz thimble for $n = p = 3$. No vanishing cycle occurs.

2.4. Conclusion

Assuming that θ is *generic*, so that the set of half-lines $L_\alpha = f_\alpha + e^{i\theta}\mathbb{R}^+$ ($f_\alpha \in \Lambda$) are two-by-two disjoint, we saw in Section 2.3 how each critical point of multiplicity μ_α may be associated with μ_α independent (absolute or relative) steepest-descent n-folds. The set of all these steepest-descent n-folds then defines a basis for the contours of integration in the following sense.

THEOREM 2.1

In the presence of the boundaries $S_1, S_2, \ldots, S_m$, every contour of integration of integrals of type (1) can be decomposed uniquely as a chain (cf. (4)) of steepest-descent n-folds associated to the critical points, restricted or actual, of f.

This result is just a naïve formulation of Theorem A.2 proved in Appendix A. Note, moreover, that by adding the Milnor numbers of each critical point (with the convention $\mu = 1$ for a corner), we have a simple way of keeping an account of the number of possible independent steepest-descent contours.

Example
By way of an example, consider the function

$$f(z^{(1)}, z^{(2)}) = {z^{(1)}}^4 + z^{(1)}{z^{(2)}}^2 + {z^{(2)}}^3 \tag{22}$$

in the presence of a boundary $S_1 : z^{(1)} - z^{(2)} = 1$. The phase function f has no critical point at infinity. By dimensionality, the question of critical points at infinity is not relevant for the restricted functions on the boundary (there are no critical points at infinity in dimension 1).

The phase function f has two isolated critical points of the first type. One is nondegenerate; hence $\mu_1 = 1$ and corresponds to the critical value $f_1 = -1/19683$. The other critical point is $0 \in \mathbb{C}^2$; it is degenerate, with $f_2 = 0$ for its critical value. Theoretically, the computation of its Milnor number μ_2 follows from a result of V. Palamodov and H. Grauert (see, e.g., [58]) via an algorithm (see, e.g., [19], [2]). Here we can take advantage of the semi-quasi homogeneity of the polynomial f (see [2]); in the neighbourhood of the critical point, the singular locus $f = 0$ is isomorphic to $X(Y^2 + X^3) = 0$, that is, the union of a cusp and a smooth curve. In the language of catastrophe theory, this singularity is classified as the parabolic umbilic, D_5 (cf. [2]), and one finds that $\mu_2 = 5$.

The restricted function $f_{(1)}$ on S_1 has three nondegenerate critical points, and the critical values $f_{(1)1}, f_{(1)2}, f_{(1)3}$ are of the second type.

To conclude, we have a basis set of 9 $(1+5+3\times 1)$ possible independent steepest surfaces into which a general integration contour can be decomposed.*

Variation with θ. The above analysis has been carried out for fixed generic phase $\theta_0 = -\arg(k)$. When θ ranges from θ_0 to $\theta_0 + 2\pi$, a given cycle $\Gamma^{\theta_0} \in H_n^{\Psi(\theta_0)}(\mathbb{C}^n, S)$ is deformed continuously into a cycle $\Gamma^{\theta} \in H_n^{\Psi(\theta)}(\mathbb{C}^n, S)$, according to the continuous variation of the convergence criterion $\Re(kf) \to +\infty$ at infinity.† Note, however, the following.

- In general, the deformed cycle $\Gamma^{\theta_0+2\pi}$ differs from Γ^{θ_0}; therefore, as a rule, the integrals (1) are ramified at $k = 0$. (Just think, for instance, of the Airy function, where $f(z) = z - z^3/3$.)
- When nongeneric phases θ are crossed, the decomposition of a given cycle with respect to a basis of steepest-descent n-folds may encounter discontinui-

*In other words, $H_2^{\Psi}(\mathbb{C}^n, S_1)$ is a free $\mathbb{Z}$-module of rank 9.

†This translates in terms of sheaves; the family of groups of homology $H_n^{\Psi(\theta)}(\mathbb{C}^n, S)$ makes up a local system of free $\mathbb{Z}$-modules of finite rank on the circle $\mathbb{S}$ of directions. We denote this local system by $H_n^{\Psi}(\mathbb{C}^n, S)_{\mathbb{S}}$.

ties. This is the Stokes phenomenon. We return to this point later (see Section 4).

3. Integral representations and asymptotics

We are now ready to study the analytical and asymptotic properties of the Laplace-type integrals (1). We recall that $g = g(z)$ is a complex polynomial function, and we denote by $\omega = g(z)\, dz^{(1)} \wedge \cdots \wedge dz^{(n)}$ the corresponding holomorphic differential n-form.

It follows from the above analysis (see Section 2) that the mapping

$$\Gamma \mapsto I_\Gamma(k) = \int_\Gamma e^{-kf(z)}\omega \tag{23}$$

translates the geometrical properties of the family of spaces of integration contours $(H_n^{\Psi(\theta)}(\mathbb{C}^n, S))_{\theta\in\mathbb{S}}$ into analytic properties of integral functions defined on the universal covering of $\mathbb{C}\backslash\{0\}$.* Moreover, to give a complete description of this representation, it is enough

(1) to consider the action of (23) on each cycle of a basis of steepest-descent n-folds for generic θ (see Theorems 2.1 or A.2) and

(2) to analyse the possible discontinuities (Stokes phenomena) when nongeneric θ are crossed.

The second point is discussed in Section 4. Here we concentrate on the first point, the asymptotics with $|k| \to +\infty$. We thus consider a steepest-descent n-fold $\Gamma = \Gamma_\alpha(\theta)$ for a given generic θ. One can make two preliminary remarks.

(1) It follows from its very definition that f maps the steepest-descent n-fold Γ onto the half-line L_α. This ensures the exponential decay of $e^{-kf(z)}$ at infinity along Γ for k in all closed subsectors of

$$\Sigma_\theta = \left\{ \left|\arg(k) + \theta\right| < \frac{\pi}{2} \text{ and } |k| > 0 \right\}. \tag{24}$$

It thus follows that the Laplace integral

$$I_\alpha(k) = \int_\Gamma e^{-kf(z)}\omega \tag{25}$$

defines an analytic function in Σ_θ.

(2) Using the notation of Sections 2.3.1 and 2.3.2, let us "truncate" our chain Γ as Γ^{t_0} with $t_0 \in L_\alpha$ and close to f_α.† Then the difference integral $\int_{\Gamma-\Gamma^{t_0}} e^{-kf(z)}\omega$

*In other words, (23) gives a representation of the local system of homology $H_n^\Psi(\mathbb{C}^n, S)_\mathbb{S}$ in terms of the sheaf of holomorphic functions on $\mathbb{C}\backslash\{0\}$.

†With the notation of Sections A.4.1 and A.4.2, Γ^{t_0} is a relative cycle in $H_n(X_\alpha, X_\alpha^{t_0})$ if $\operatorname{depth}(f_\alpha) = 0$ and in $H_n(X_\alpha, X_\alpha^{t_0} \cup Y_1 \cup \cdots \cup Y_p)$ if $\operatorname{depth}(f_\alpha) > 0$.

defines an analytic function in k which is exponentially decreasing at infinity inside all closed subsectors of Σ_θ and is therefore “flat” (i.e., asymptotic to the zero function) at infinity. This means that $I_\alpha(k)$ and

$$I_\alpha^{t_0}(k) = \int_{\Gamma^{t_0}} e^{-kf(z)}\omega \tag{26}$$

have the same Poincaré asymptotics at infinity inside all subsectors of Σ_θ.

We can now compute these asymptotics by the saddle-point method. The result of course depends on the type of critical point we consider, and this is the purpose of Sections 3.1 and 3.2.

3.1. Absolute steepest-descent contours (no boundaries)

As an introduction to the bounded case, we first study the asymptotics associated with a critical value $f_\alpha = f(z_\alpha) \in \Lambda_{()}$ of the first type. This case is well known: the asymptotics are governed by the local behaviour of the integrand near the isolated critical point z_α. We briefly recall the ideas of B. Malgrange [45] (see also [63], [25], and [3, Section 11]), which relate the asymptotics directly to the geometry of the phase function. Furthermore, we demonstrate the Borel summability of these asymptotics, which can therefore be considered as an *exact* encoding of the function $I_\alpha(k)$ within a Stokes sector.

Writing

$$I_\Gamma(k) = e^{-kf_\alpha} \int_\Gamma e^{-k(f(z)-f_\alpha)}\omega, \tag{27}$$

one can assume that $f_\alpha = 0$ without loss of generality, and we furthermore assume that $\theta = 0$ for simplicity.

We now consider the truncated integral (26) and use the boundary operator ∂ (cf. (13) or (A.12)) to reduce the dimensionality. By the Stokes theorem (see [45]) and Leray residue theory (see [42]),

$$I_\alpha^{t_0}(k) = \int_{\Gamma^{t_0}} e^{-kf(z)}\omega = \int_0^{t_0} dt e^{-kt}\widehat{J}_\alpha(t) \quad \text{with } \widehat{J}_\alpha(t) = \int_{\partial[\Gamma^t]} \frac{\omega}{df_t}, \tag{28}$$

where ω/df_t denotes the Leray residue differential $(n-1)$-form of ω,* $\partial[\Gamma^t]$ being the vanishing cycle.†

It is known from [9] and [45] that $\widehat{J}_\alpha(t)$ defines an analytic function on the universal covering $\widetilde{D_\alpha\backslash\{0\}}$ of the punctured disc $D_\alpha\backslash\{0\}$. More precisely, $\widehat{J}_\alpha(t)$ is a solution

*That is, $\omega = df \wedge \omega/df$. This $(n-1)$-quotient form ω/df is holomorphic along each nonsingular fibre $X_\alpha^t = \{z \in f^{-1}(t), z \text{ near } z_\alpha \text{ and } t \text{ near } f_\alpha\}$.

†That is, $\partial[\Gamma^t] \in H_{n-1}(X_\alpha^t)$.

of a so-called Picard-Fuchs equation (or Gauss-Manin connection; see [61], for instance)

$$\frac{d^l \widehat{J_\alpha}}{dt^l} + q_1(t)\frac{d^{l-1}\widehat{J_\alpha}}{dt^{l-1}} + \cdots + q_l(t)\widehat{J_\alpha} = 0 \tag{29}$$

with $l \leq \mu_\alpha$ (the multiplicity of the singularity), which has (at most) a regular singular point at the origin. It follows from the theory of Fuchs (see [74]) that $\widehat{J_\alpha}(t)$ admits in each sector $a < \arg(t) < b$ a convergent series expansion of the form

$$\widehat{J_\alpha}(t) = \sum_{r,s} a_{r,s}(t)t^r(\ln t)^s, \tag{30}$$

where

- the $r \in \mathbb{Q}$, $r > -1$, belong to the finite set of distinct monodromy exponents of the classical monodromy operator in homology* (cf. [45]);
- to each r is associated a set of $s \in \mathbb{N}$ with $s \leq \inf\{\mu_\alpha - 1, n-1\}$ (see [3] for more details); of course, $s = 0$ for nondegenerate critical points;
- $a_{r,s}(t) = \sum_{j\geq 0} a_{r,s,j}t^j$ are convergent Taylor series.

It remains to use the standard integral

$$\int_0^{+\infty} e^{-kt}t^\lambda(\ln t)^l\,dt = \left(\frac{d}{d\lambda}\right)^l\left(\frac{\Gamma(\lambda+1)}{k^{(\lambda+1)}}\right) \tag{31}$$

to conclude from Watson's lemma that $I_\alpha^{t_0}(k)$; hence $I_\alpha(k)$ has

$$J_\alpha(k) = \sum_{r,s} T_{r,s}(k)\frac{(\ln k)^s}{k^{r+1}} \tag{32}$$

for its asymptotics when $|k| \to \infty$ in all closed subsectors of Σ_θ, where $T_{r,s}(k)$ are formal Gevrey-1 series expansions.†

*Define a base point t_0 near (but different from) the critical value, and consider (for t_0) a basis of vanishing cycles generating the vanishing homology. Consider the deformation of these cycles when t goes around the critical value, starting at and coming back to t_0. The resulting cycles still define a basis of the homology: comparing the two bases, one gets an invertible matrix with integer coefficients describing the monodromy of the vanishing homology. In the case of a nondegenerate critical point (only one vanishing cycle), the possible monodromy arises from the possible self-intersection of the vanishing cycle. In the case of a self-intersection, we get a square root singularity at the origin for $\widehat{J_\alpha}(t)$; otherwise, $\widehat{J_\alpha}(t)$ is analytic near t_0. In the case of a degenerate critical point, each of the deformed cycles is described in general as a linear combination (with integer coefficients) in terms of the preliminary basis. Write the eigenvalues of this matrix under the form $\exp(-2i\pi\lambda)$. Then these λ are rationals (see [45]), and, moreover, they are (up to an addition of an integer) the exponents r in the series expansion (30). It may happen that the monodromy matrix has multiple eigenvalues, resulting in possible logarithmic terms in (30). In Kaminski and Paris's scheme (see [40], [41]), this corresponds to multiple poles for the integrand for the associated Mellin-Barnes integral representation.

†That is, $T_{r,s}(k) = \sum_{j\geq 0} T_{r,s;j}/k^j$, and there exist $C_{r,s} > 0$ and $A_{r,s} > 0$ such that $|T_{r,l;j}| \leq C_{r,s}A_{r,s}^j\Gamma(j)$ (see Malgrange [47], J. P. Ramis [66], or M. Loday-Richaud [44]).

Now we can use our freedom to change θ slightly so that L_α still does not meet other critical values. The resulting integral is just the analytic continuation of the previous one, the asymptotics at infinity being preserved. This proves that the asymptotics (32) are valid inside a wider sector of aperture greater than π. Therefore formal expansion (32) is Borel resummable with $I_\alpha(k)$ as its Borel sum (Watson's theorem;* see [47]), and we thus obtain the following equivalent, one-dimensional, integral representation:

$$I_\alpha(k) = \int_0^{\infty e^{i\theta}} dt e^{-kt} \widehat{J_\alpha}(t). \tag{33}$$

In summary, for general value f_α, we have the following theorem.

THEOREM 3.1
Let f_α be a first-type critical value. The integral $I_\alpha(k)$ admits $e^{-kf_\alpha} J_\alpha(k)$ as its asymptotic series expansion for $k \to \infty$ in $\Sigma_\theta = \{|k| > 0,\ |\arg(k) + \theta| < \pi/2\}$; in other words, $I_\alpha(k) \sim e^{-kf_\alpha} J_\alpha(k)$ in Σ_θ, with

$$J_\alpha(k) = \sum_{r,s} T_{r,s}(k) \frac{(\ln k)^s}{k^{r+1}}, \tag{34}$$

where

$$T_{r,s}(k) = \sum_{j\geq 0} T_{r,s;j}/k^j \tag{35}$$

belongs to $\mathbb{C}[[k^{-1}]]_1$, the differential algebra of Gevrey-1 series expansions (see [66]). The $r \in \mathbb{Q}$ run over a finite spectral *set, and to each r is associated a set of $s \in \mathbb{N}$ satisfying $s \leq \inf\{\mu_\alpha - 1, n - 1\}$.*

Conversely, $I_\alpha(k)$ (resp., $\widehat{J_\alpha}(t)$) can be considered as the Borel sum (resp., the minor in the resurgence theory of J. Ecalle†) of $e^{-kf_\alpha} J_\alpha(k)$ in the direction of argument θ.

Remark. Note that we are here treating the $T_{r,s}$ and J_α as formal series expansions, although they can be interpreted also as (Poincaré) asymptotic expansions. The notational approach used here may be unfamiliar to some readers, but it is used to maintain consistency with the Borel and resurgence viewpoint (see [47], [23], [12]).

Note also that in Theorem 3.1 the direction of summation should be considered as a direction in $\widetilde{\mathbb{C}\backslash\{0\}}$, the universal covering of $\mathbb{C}\backslash\{0\}$.

*This is not to be confused with his lemma for standard integral expansions.
†See Ecalle [23], or [12], [18], or [16] for a short introduction to resurgence theory.

COROLLARY 3.1

If z_α is a nondegenerate quadratic critical point of f, then $I_\alpha(k) \sim e^{-kf_\alpha} J_\alpha(k)$ in Σ_θ, with

$$J_\alpha(k) = \frac{1}{k^{n/2}} T_\alpha(k) \qquad \text{and} \qquad T_\alpha(k) = \frac{g(0)(2\pi)^{n/2}}{\sqrt{\mathrm{Hess}(f)(0)}} + \sum_{j\geq 1} T_j^\alpha / k^j \in \mathbb{C}[[k^{-1}]], \tag{36}$$

where Hess(f) *is the hessian determinant. The choice of the root depends on the orientation of the vanishing cycle.*

This corollary is well known (cf. [25], [76]). One method for the practical computation of the T_j^α is described in [17].

3.2. Relative steepest-descent contours (bounded case)

This case corresponds to a critical value of the second type and is the main focus of this paper. Let $f_\alpha \in \Lambda_{(1,\ldots,p)}$ be a critical value of the restricted function $f_{(1,\ldots,p)}$ with $p > 0$ the depth. We apply the reduction process developed in Section 2.3.2 (resp., Section A.4.2) to analyse the asymptotics of our bounded integral. Again without loss of generality, we assume that $f_\alpha = 0$ and $\theta = 0$.

To compute the asymptotics, it is enough to consider the truncated integral. Since $f = 0$ is not a singular level, by following formula (17) and its comments, we can apply Fubini's theorem to write

$$I_\alpha^{t_0}(k) = \int_0^{t_0} dt e^{-kt} \int_{\partial[\Gamma^t]} \frac{\omega}{df_t}, \tag{37}$$

where ω/df_t is the (holomorphic) restriction of the differential quotient $(n-1)$-form ω/df to the level $f = t$, and $\partial[\Gamma^t]$ is that part of the boundary of Γ^t lying on the fibre $f^{-1}(t)$.* The same reduction can be repeated step by step. We use the local coordinates $(s^{(1)}, \ldots, s^{(p)}, s^{(p+1)}, \ldots, s^{(n)})$ so that f is given by formula (8), and we define the set of functions $f_{1,2,\ldots,i} : (s^{(1)}, \ldots, s^{(n)}) \mapsto f(0, \ldots, 0, s^{(i+1)}, \ldots, s^{(n)})$. At the second step (if $p \geq 2$), from Fubini's theorem we have

$$\widehat{J}_\alpha(t) = \int_{\partial[\Gamma^t]} \frac{\omega}{df} = \int_0^t dt_1 \int_{\partial_1 \circ \partial[\Gamma^{t_1}]} \frac{\omega}{df \wedge df_1}|_{f=t,\, f_1=t_1}, \tag{38}$$

where $\omega/df \wedge df_1|_{f=t,\, f_1=t_1}$ denotes the (holomorphic) restriction of the differential quotient $(n-2)$-form $\omega/df \wedge df_1$ along the nonsingular level $f = t$, $f_1 = t_1$: here again, (38) is a simple translation of formula (18) with its comments. At the pth step

*Precisely, the class of homology of $[\Gamma^t]$ belongs to $H_n(X_\alpha, X_\alpha^t \cup Y_1 \cup \cdots \cup Y_p)$, and we select that part of its boundary lying on X_α^t.

we obtain

$$I_\alpha^{t_0}(k) = \int_0^{t_0} dt e^{-kt} \int_0^t dt_1 \cdots \int_0^{t_{p-2}} dt_{p-1} \int_{\partial_{(p-1,\dots,1,0)}[\Gamma^{t_{p-1}}]} \omega_{(p-1,\dots,1,0)}, \tag{39}$$

where

$$\partial_{(p-1,\dots,1,0)} = \partial_{p-1} \circ \cdots \circ \partial_1 \circ \partial, \tag{40}$$

while

$$\omega_{(p-1,\dots,1,0)} = \frac{\omega}{df \wedge df_1 \wedge \cdots \wedge df_{1,\dots,(p-1)}} |_{f=t,\, f_1=t_1,\dots,\, f_{1,\dots,(p-1)}=t_{p-1}} \tag{41}$$

is the corresponding Leray quotient $(n-p)$–differential form.

If $p = n$ (corner critical point), then $\omega_{(p-1,\dots,1,0)}$ is just a holomorphic function of $(t, t_1, \dots, t_{p-1})$ and the reduction process stops here. If $p \leq n-1$, it then remains to use the same argument as in Section 3.1 to obtain

$$I_\alpha^{t_0}(k) = \int_0^{t_0} dt e^{-kt} \int_0^t dt_1 \cdots \int_0^{t_{p-1}} dt_p \int_{\partial_{(p,\dots,1,0)}[\Gamma^{t_p}]} \omega_{(p,\dots,1,0)}, \tag{42}$$

where $\partial_p \circ \cdots \partial_1 \circ \partial[\Gamma^{t_p}]$ is the vanishing cycle of the critical point.

In the case $p = n$, the integral

$$\widehat{J_\alpha}(t) = \int_0^t dt_1 \cdots \int_0^{t_{p-2}} dt_{p-1} \int_{\partial_{(p-1,\dots,1,0)}[\Gamma^{t_{p-1}}]} \omega_{(p-1,\dots,1,0)} \tag{43}$$

defines an analytic function on D_α, with $\sum_{j\geq 0} a_j t^{j+p-1}$ as its convergent Taylor expansion. In the case $p \leq n-1$, by using the results from Section 3.2, the function

$$h_\alpha(t, t_1, \dots, t_p) = \int_{\partial_{(p,\dots,1,0)}[\Gamma^{t_p}]} \omega_{(p,\dots,1,0)}, \tag{44}$$

considered as a function of t_p with $(t, t_1, \dots, t_{p-1})$ as a parameter, is defined as an analytic function on the universal covering $\widetilde{D_\alpha \setminus \{f_\alpha\}}$ and admits in each sector $a < \arg(t) < b$ a convergent series expansion of the form

$$\sum_{r,s} a_{r,s}(t, t_1, \dots, t_{p-1}, t_p) t_p^r (\ln t_p)^s, \tag{45}$$

where

- the r belong to a finite set of distinct monodromy exponents of the classical monodromy operator in homology, and the $s \in \mathbb{N}$ satisfy $s \leq \inf\{\mu_\alpha - 1, n - p - 1\}$;
- $a_{r,s}(t, t_1, \dots, t_{p-1}, t_p) = \sum_{j\geq 0} a_{r,s,j}(t, t_1, \dots, t_{p-1}) t_p^j$ are convergent Taylor series with analytic dependence with respect to $(t, t_1, \dots, t_{p-1})$.

It thus follows that the function

$$\widehat{J_\alpha}(t) = \int_0^t dt_1 \cdots \int_0^{t_{p-1}} dt_p \int_{\partial_{(p,\dots,1,0)}[\Gamma^{t_p}]} \omega_{(p,\dots,1,0)} \tag{46}$$

has the same properties, with

$$\widehat{J_\alpha}(t) = \sum_{r,s} b_{r,s}(t) t^{r+p} (\ln t)^s \tag{47}$$

for its convergent series expansion, where $b_{r,s}(t) = \sum_{j\geq 0} b_{r,s,j} t_p^j$ are convergent Taylor series.

As in Section 3.1, we finally recast $I_\alpha(k)$ as a one-dimensional integral representation

$$I_\alpha(k) = \int_0^{\infty e^{i\theta}} dt e^{-kt} \widehat{J_\alpha}(t), \tag{48}$$

and we obtain the following theorem.

THEOREM 3.2
If f_α is a second-type critical value of depth p, the integral $I_\alpha(k)$ admits $e^{-kf_\alpha} J_\alpha(k)$ as its asymptotic series expansion for $k \to \infty$ in Σ_θ (i.e., $I_\alpha(k) \sim e^{-kf_\alpha} J_\alpha(k)$ in Σ_θ), where

$$\textit{for}\ \ p = n, \quad J_\alpha(k) = \frac{1}{k^p} \sum_{j\geq 0} T_j / k^j \in \mathbb{C}[[k^{-1}]]_1, \tag{49}$$

$$\textit{for}\ \ p \leq n-1, \quad J_\alpha(k) = \sum_{r,s} T_{r,s}(k) \frac{(\ln k)^s}{k^{r+p+1}}, \tag{50}$$

where $T_{r,s}(k) = \sum_{j\geq 0} T_{r,s;j}/k^j$ belongs to $\mathbb{C}[[k^{-1}]]_1$. The rational r's run over a finite spectral set associated with the critical point, and to each r is associated a set of $s \in \mathbb{N}$ satisfying $s \leq \inf\{\mu_\alpha - 1, n-p-1\}$.

Conversely, $I_\alpha(k)$ (resp., $\widehat{J_\alpha}(t)$) is the Borel sum (resp., the minor) of $e^{-kf_\alpha} J_\alpha(k)$ in the direction of argument θ.

The asymptotics are simpler in the case of a nondegenerate critical point (see, for instance, [17], [76], [3]), arising from p boundaries.

COROLLARY 3.2
When the singular point is quadratic, $I_\alpha(k) \sim e^{-kf_\alpha} J_\alpha(k)$ in Σ_θ, with

$$J_\alpha(k) = \frac{1}{k^{(n+p)/2}} T_\alpha(k) \qquad \textit{and} \qquad T_\alpha(k) = \sum_{j=0}^{\infty} \frac{T_j^\alpha}{k^j} \tag{51}$$

for the asymptotic series expansion.

4. Stokes phenomenon

We described in the previous section the asymptotics of the multiple Laplace integrals of type (1), as well as their representation as Borel sums for generic summation directions (θ). To obtain the global asymptotics, we must analyse the Stokes phenomena. This amounts to analysing the singularities in the Borel plane of the analytic continuations of (the minors) $\widehat{J}_\alpha(t)$. How this can be obtained from generalised Picard-Lefschetz formulae (detailed in Section B.1) is discussed in Section 4.1. Against this Borel viewpoint, we demonstrate in Section 4.2 how the Stokes phenomenon can be understood directly in terms of steepest-descent contours, thus keeping the geometric ideas of the saddle-point method as in the one-dimensional case.

In what follows, it is convenient to assume the following hypothesis.

HYPOTHESIS H6

All singular points are nondegenerate.

4.1. Ramifications

We return to the $\widehat{J}_\alpha(t)$ of Theorems 3.1 and 3.2, together with their corollaries. From (36) and (51), if f_α has depth p_α, then locally near the origin,

$$\widehat{J}_\alpha(t) = t^{(n+p_\alpha-2)/2} H_\alpha(t), \tag{52}$$

where H_α is holomorphic near zero. Thus when $n + p_\alpha$ is even, $\widehat{J}_\alpha(t)$ is an analytic function at the origin, but when $n + p_\alpha$ is odd, $\widehat{J}_\alpha(t)$ has a square-root singularity there.*

Note that the $\widehat{J}_\alpha(t)$ have been defined as (germs of ramified) analytic functions at the origin. If $f_\alpha \neq 0$, it is necessary to take into account translations by the complex numbers

$$f_{\alpha\beta} = f_\beta - f_\alpha. \tag{53}$$

The generalised Picard-Lefschetz formulae described in Section B.1 allow us to identify the type of singularity in the analytic continuations of each $\widehat{J}_\alpha(t)$. Denoting by $\mathrm{Var} = \rho - \mathbb{I}$ the "variation" operator, where $\mathbb{I}$ is the identity operator and ρ is the analytic continuation around $f_{\alpha\beta}$ anticlockwise, it follows from formula (B.3) that for t near $f_{\alpha\beta}$,

$$\mathrm{Var}\, \widehat{J}_\alpha(t) = \kappa_{\alpha\beta} \widehat{J}_\beta(t - f_{\alpha\beta}), \tag{54}$$

where $\kappa_{\alpha\beta}$ is a (positive or negative) integer.

* This is consistent with the geometric argument (B.5). The vanishing homology $H_{n-p_\alpha-1}(S_p \cap \cdots \cap S_1 \cap X^t_\alpha)$ is a trivial covering on the circle of directions $\mathbb{S}$ when $n + p_\alpha$ is even, but it is a two-fold covering when $n + p_\alpha$ is odd.

PROPOSITION 4.1
If p_β is the depth of the critical value f_β, then

- *when $n - p_\beta$ is even (so that $\widehat{J_\beta}(t)$ is not ramified around $t = 0$),*

$$\widehat{J_\alpha}(t) = \kappa_{\alpha\beta}\widehat{J_\beta}(t - f_{\alpha\beta})\frac{\ln(t - f_{\alpha\beta})}{2i\pi} + \mathrm{Hol}(t - f_{\alpha\beta}) \tag{55}$$

with Hol *a holomorphic function near zero,* * *and*

- *when $n - p_\beta$ is odd,*

$$\widehat{J_\alpha}(t) = -\frac{\kappa_{\alpha\beta}}{2}\widehat{J_\beta}(t - f_{\alpha\beta}) + \mathrm{Hol}(t - f_{\alpha\beta}). \tag{56}$$

Arguments developed in Malgrange [45] show that $\widehat{J_\alpha}(t)$ remains bounded (when $n \geq 2$, the analysis being obvious when $n = 1$) near $f_{\alpha\beta}$. Proposition 4.1 now results from the variation formulae (54) and Riemann's removable singularity theorem (see [28]).

We show in Section 4.2 how the above properties can be derived independently.

4.2. Stokes phenomenon

We consider a singular direction (θ) and assume that the closed half-line $L_\alpha = L_\alpha(\theta)$ meets a singular value f_β; this (possibly) gives rise to a Stokes phenomenon, which can be described as follows. Assume for simplicity that L_α meets no singular point other than f_β. We define the half-lines $L_\alpha^- = L_\alpha(\theta^-)$ and L_β^- (resp., $L_\alpha^+ = L_\alpha(\theta^+)$ and L_β^+) by slightly rotating L_α and L_β clockwise (resp., anticlockwise; see Figure 4). We perform the same rotations for the other half-lines L_λ.

For fixed θ^- (resp., θ^+), define a basis of steepest-descent n-folds $(\Gamma_{\alpha^-}, \Gamma_{\beta^-}, \ldots, \Gamma_{\lambda^-}, \ldots)$ (resp., $(\Gamma_{\alpha^+}, \Gamma_{\beta^+}, \ldots, \Gamma_{\lambda^+}, \ldots)$). From our hypothesis, we can assume that $(\Gamma_{\beta^+}, \ldots, \Gamma_{\lambda^+}, \ldots)$ is deduced from $(\Gamma_{\beta^-}, \ldots, \Gamma_{\lambda^-}, \ldots)$ by an isotopy (continuous deformation) when the argument runs from θ^- to θ^+, so that we can remove the upper scripts $\pm$ in the notation.

Concerning the $\Gamma_{\alpha^\pm}$'s, we can assume *only* that the one is deduced from the other by a local deformation near the critical point z_α. This may not work globally due to the critical value f_β. We thus get the decomposition

$$\Gamma_{\alpha^-} = \Gamma_{\alpha^+} + \Gamma_{\alpha^\pm}, \tag{57}$$

where necessarily

$$\Gamma_{\alpha^\pm} = \Gamma_{\alpha^-} - \Gamma_{\alpha^+} = \kappa_{\alpha\beta}\Gamma_\beta. \tag{58}$$

*In other words, $\widehat{J_\alpha}(t)$ and $\kappa_{\alpha\beta}\widehat{J_\beta}(t - f_{\alpha\beta})\ln(t - f_{\alpha\beta})/2i\pi$ are equal when one considers them as microfunctions.

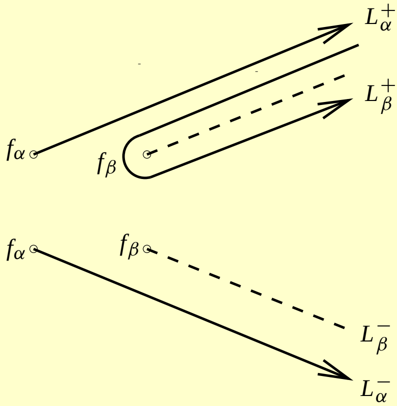

Figure 4. Generic Stokes phenomenon: Below, before the Stokes phenomenon; above, after the Stokes phenomenon

In this formula, $\kappa_{\alpha\beta}$ can be understood geometrically as an *index of intersection*,* and therefore it is a positive or negative integer, the sign depending on the orientations of Γ_α and Γ_β.

We now apply (58) to our integral representation. We start with the integral

$$I_\alpha(k) = \int_{\Gamma_\alpha^-} e^{-kf(z)}\omega = e^{-kf_\alpha} \int_0^{\infty e^{i\theta^-}} e^{-kt}\,\widehat{J}_\alpha(t)\,dt, \tag{59}$$

which defines an analytic function within the sector $\Sigma_{\theta^-} = \{|k| > 0,\ |\arg(k)+\theta^-| < \pi/2\}$. We know that $I_\alpha(k)$ extends analytically over $\mathbb{C}\backslash\{0\}$ as a multivalued function, and from (58) we see that

$$I_\alpha(k) = \int_{\Gamma_\alpha^+} e^{-kf(z)}\omega + \kappa_{\alpha\beta} \int_{\Gamma_\beta^+} e^{-kf(z)}\omega \tag{60}$$

for $k \in \Sigma_{\theta^+}$, that is, that

$$I_\alpha(k) = e^{-kf_\alpha} \int_0^{\infty e^{i\theta^+}} e^{-kt}\,\widehat{J}_\alpha(t)\,dt + \kappa_{\alpha\beta} e^{-kf_\beta} \int_0^{\infty e^{i\theta^+}} e^{-kt}\,\widehat{J}_\beta(t)\,dt. \tag{61}$$

Formula (61) provides a complete description of the Stokes phenomenon. We can recast these results in the framework of resurgence theory, using the notation of Section 3. From (59), $I_\alpha(k)$ is the left (lateral) Borel sum in the direction of argument θ of the

*Compare with Section B.2. With the convention of formula (B.8), $\kappa_{\alpha\beta}$ is given by the equality $\kappa_{\alpha\beta} = \langle \Gamma_\beta^\star, \Gamma_{\alpha^\pm} \rangle$.

resurgent symbol $e^{-kf_\alpha} J_\alpha(k)$,

$$I_\alpha(k) = e^{-kf_\alpha} \mathrm{s}_{(\theta^-)}\, J_\alpha(k), \tag{62}$$

and from (61) the action on $J_\alpha(k)$ of the Stokes automorphism $\mathfrak{S}_{(\theta)}$ is given by

$$\mathfrak{S}_{(\theta)} J_\alpha(k) = J_\alpha(k) + \kappa_{\alpha\beta} e^{-kf_{\alpha\beta}} J_\beta(k), \tag{63}$$

which finally yields

$$\Delta_{f_{\alpha\beta}}\, J_\alpha(k) = \kappa_{\alpha\beta} J_\beta(k), \tag{64}$$

where $\Delta_{f_{\alpha\beta}}$ is the “alien derivative operator” (see [23], [12], [18], [16]) at $f_{\alpha\beta}$.

It is important to notice that the alien derivatives (as well as the directions of summations) have to be indexed over the two-fold covering of $\mathbb{C}\backslash\{0\}$ (i.e., the Riemann surface of the square root) when $n + p_\alpha$ is odd. The sign of $\kappa_{\alpha\beta}$ depends on which sheet of this covering is under consideration.

4.3. Conclusion

In (64), the index of intersection $\kappa_{\alpha\beta}$ now appears as a so-called Stokes multiplier. Note that (64) can be understood in terms of singularity in the Borel plane. The analytic continuation of the minor $\widehat{J_\alpha}(t)$ of $J_\alpha(k)$ along a straight half-line $(0, f_{\alpha\beta}\infty)$ encounters a singularity at $t = f_{\alpha\beta}$ if $\kappa_{\alpha\beta} \neq 0$, and its variation there (as defined in Section 4.1) is $\kappa_{\alpha\beta}\widehat{J_\beta}(t - f_{\alpha\beta})$. This is nothing but (54).

To compute the constants $\kappa_{\alpha\beta}$ and thus to obtain the complete resurgence structure, it is helpful to keep in mind their two interpretations. The geometric description in terms of indices of intersection has already demonstrated their quantised nature. This helps again to show that some of them necessarily vanish. This stems from the fact that the set of critical values $\Lambda = \bigcup \Lambda_{(i_1,\ldots,i_p)}$ has a natural hierarchy that follows directly from the stratification of S. This means that a relative steepest-descent n-fold Γ_α having $S_1, \ldots, S_p$ (say) as its boundary can be eventually affected *only* by critical values f_β belonging to the subset $\Lambda_\alpha \subset \Lambda$ of critical values of f or of the restricted $f|$ to one of the S_i, $i = 1, \ldots, p$, or their different intersections. Otherwise, $\kappa_{\alpha\beta} = 0$ necessarily.*

This is almost all that we can learn from the geometry. To get quantitative information about the remaining indices of intersections $\kappa_{\alpha\beta}$, we have to turn to the hyperasymptotic analysis, interpreting this time the $\kappa_{\alpha\beta}$ as Stokes multipliers. This is the aim of the next section. It is convenient for this purpose to represent the results of (64) in terms of the series expansions T_α introduced in formulae (36)–(51). Hereafter, formula (65) is simply derived from formula (64) by the Leibniz rule (see [23], [16]).

*See also the second remark of Section B.1.

THEOREM 4.1
The series expansions $T_\alpha(k) \in \mathbb{C}[[k^{-1}]]_1$ are Gevrey-1 resurgent resummable. If f_α is a critical value of depth p_α, the adjacent singularities * *of the minor $\widehat{T_\alpha}(t)$ of $T_\alpha(k)$ are (at most) the $f_{\alpha\beta} = f_\beta - f_\alpha$, with $f_\beta \in \Lambda_\alpha$. Moreover, in the (generic) case where these relevant $f_{\alpha\beta}$ have distinct phases,*

$$\Delta_{f_{\alpha\beta}} T_\alpha(k) = \kappa_{\alpha\beta} k^{(p_\alpha - p_\beta)/2} T_\beta(k), \tag{65}$$

where p_β ($\leq p_\alpha$) is the depth of f_β, whereas the $\kappa_{\alpha\beta}$ are integers.

Remark. In (65), the alien derivatives can now be indexed over $\mathbb{C}\backslash\{0\}$. This does not make (65) ambiguous. One has to keep in mind that the sign of the index of intersection $\kappa_{\alpha\beta}$ depends on the orientations of the steepest-descent n-folds Γ_α and Γ_β, on which the determination of the square root $k^{(p_\alpha - p_\beta)/2}$ also depends.

5. Hyperasymptotic analysis: Calculation of Stokes constants

We assume here for simplicity that the values $f_{\alpha\beta}$ have distinct phases. Theorem 4.1 thus applies, allowing us to identify the singularity types.

The results from the previous section can now be used to deduce the exact remainder term for a truncated asymptotic expansion about any of the singularities. As we have converted integral (1) into a one-dimensional Laplace integral (Borel sum), the procedure follows closely that of Howls [36] and Olde Daalhuis [55], allowing us to be brief.

It is important to stress that at no point in the hyperasymptotic procedure detailed in this section are the full infinite asymptotic/formal power series used. All the expansions are finite and exactly terminated by the appropriate (hyperasymptotic) remainder term.

We now describe the hyperasymptotic analysis for the (slowly varying) related analytic function

$$\mathscr{T}_\alpha(k) = k^{(n+p_\alpha)/2} e^{kf_\alpha} I_\alpha(k) = \mathrm{s}_{(\theta)}\, T_\alpha(k) = T_0^\alpha + \int_0^{\infty e^{i\theta}} dt\, e^{-kt}\, \widehat{T_\alpha}(t). \tag{66}$$

Here $\mathrm{s}_{(\theta)}\, T_\alpha(k)$ is the Borel sum of the formal series expansions $T_\alpha(k)$ in the nonsingular direction of argument θ, and therefore $\mathscr{T}_\alpha(k) \sim T_\alpha(k)$ for $k \to \infty$ in Σ_θ. We represent the local behaviour of the function $\widehat{T_\alpha}(t)$ in terms of a Cauchy integral representation

$$\widehat{T_\alpha}(t) = \frac{1}{2i\pi} \oint_{u=t} du \frac{\widehat{T_\alpha}(u)}{u-t}. \tag{67}$$

* The *adjacent singularities* of a germ of analytic functions at the origin are the singularities of the analytic continutions of this germ along half-lines emanating from the origin.

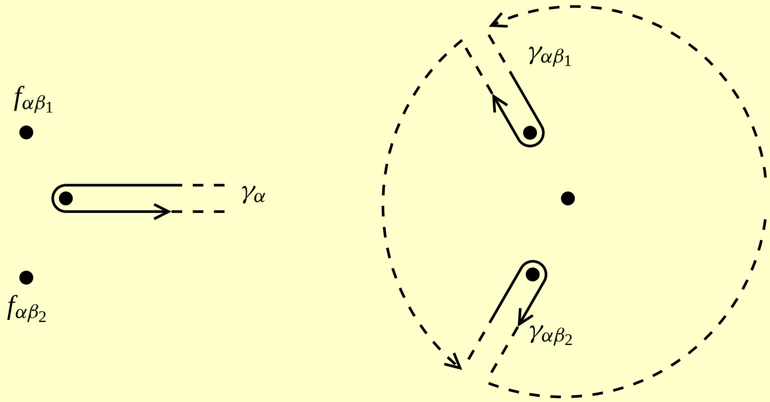

Figure 5. The path γ_α on the left. Its deformation is on the right with the contributions $\gamma_{\alpha\beta}$ from the adjacent singularities.

After a binomial expansion to the order of truncation N required, we have exactly

$$\mathscr{T}_\alpha(k) = \sum_{r=0}^{N-1} \frac{T_r^\alpha}{k^r} + R_\alpha(k, N), \tag{68}$$

where

$$R_\alpha(k, N) = \frac{1}{2i\pi k^N} \int_0^{+\infty} dw e^{-w} w^{N-1} \int_{\gamma_\alpha} du \frac{\widehat{T_\alpha}(u)}{(1 - w/ku)u^N}. \tag{69}$$

The contour γ_α encircles the positive real axis as in Figure 5. We then deform γ_α (see Figure 5) to encounter the other singularities $f_{\alpha\beta}$ that are adjacent (see [36]).

By a suitable restriction of the type of integrand functions we consider, or by going to sufficiently high truncation N, the arcs at infinity (see Figure 5) make no contribution (see [36]).* We thus have

$$R_\alpha(k, N) = \frac{1}{2i\pi k^N} \sum_{\text{adjacent } f_{\alpha\beta}} \int_0^{+\infty} dw e^{-w} w^{N-1} \int_{\gamma_{\alpha\beta}} du \frac{\widehat{T_\alpha}(u)}{(1 - w/ku)u^N}. \tag{70}$$

*The general case can be treated as in Olde Daalhuis [55]: instead of working with full Borel sums, one uses truncated Laplace integrals (essentially changing summation to presummation in resurgence-speak; cf. [12], [18]). Pushing the circular arcs far enough away, their contributions can be bounded away to an exponential level smaller than the one to which the hyperasymptotics are eventually taken, that is, less than $\exp(-M|k|)$ for any chosen $M > 0$.

At each of the singularities β, we make a change of variables and collapse the contour $\gamma_{\alpha\beta}$ onto the associated cut. The results of the Picard-Lefschetz analysis, embodied in (65), then guarantee that the discontinuities generate self-similar integrands, with contours now over the critical point β. The final result is

$$\mathscr{T}_\alpha(k) = \sum_{r=0}^{N-1} \frac{T_r^\alpha}{k^r} + R_\alpha(k, N) \tag{71}$$

with

$$R_\alpha(k, N) = -\frac{1}{2i\pi} \sum_{\kappa_{\alpha\beta}\neq 0} \frac{\kappa_{\alpha\beta}}{k^{N-1}} \int_0^{[-\theta_{\alpha\beta}]} dt_0\, e^{-t_0 f_{\alpha\beta}} \frac{t_0^{N-1+(p_\alpha - p_\beta)/2}}{k - t_0} \mathscr{T}_\beta(t_0), \tag{72}$$

where we have used the notation $\int_0^{[\eta]} = \int_0^{\infty e^{i\eta}}$ (see [55]), with $\theta_{\alpha\beta}$ denoting the phase of $f_{\alpha\beta}$.*

The indices p_α and p_β are the depths of (number of boundaries associated with) α and β, respectively. The minus sign comes from the fact that the orientation of the contours $\gamma_{\alpha\beta}$ is in the opposite sense to the convention used to define the index of intersection $\kappa_{\alpha\beta}$ (compare the $\gamma_{\alpha\beta}$ in Figure 5 with the path L_β^+ in Figure 4, for instance). The (quantised and, as yet, unknown) Stokes constants associated with α and β are nonzero integers if the singularity is adjacent, defined up to a sign depending on the branch of $t_0^{(p_\alpha - p_\beta)/2}$ (see the remark following Theorem 4.1).

We determine the Stokes constants by resorting to a resurgence formula for the coefficients in the expansions themselves. Using the fact that

$$\begin{aligned} T_N^\alpha &= k^N \big(R_\alpha(k, N) - R_\alpha(k, N+1)\big) \\ &= -\frac{1}{2i\pi} \sum_{\kappa_{\alpha\beta}\neq 0} \frac{\kappa_{\alpha\beta}}{f_{\alpha\beta}^{N+(p_\alpha - p_\beta)/2}} \int_0^{+\infty} dv\, e^{-v} v^{N-1+(p_\alpha - p_\beta)/2} \mathscr{T}_\beta\Big(\frac{v}{f_{\alpha\beta}}\Big), \end{aligned} \tag{73}$$

we substitute the corresponding exactly terminated asymptotic expansions of all the adjacent $\mathscr{T}_\beta$ of type (71) with $v/f_{\alpha\beta}$ playing the role of k. If the next-nearest singularity is some distance further away from α than the nearest, we obtain the usual (Dingle type; see [21]) leading-order approximation to the late terms:

$$T_N^\alpha = -\frac{\kappa_{\alpha\beta_{\min}}}{2i\pi} \frac{\Gamma(N + (p_\alpha - p_{\beta_{\min}})/2)}{f_{\alpha\beta_{\min}}^{N+(p_\alpha - p_{\beta_{\min}})/2)}} T_0^{\beta_{\min}} \bigg(1 + O\Big(\frac{N}{|k f_{\alpha\beta_{\min}}|}\Big)^{-N}\bigg) \quad \text{as } N \to \infty. \tag{74}$$

*The analytic function $\mathscr{T}_\beta$ can be thought of as the Borel sum of the series expansion T_β in the direction of $\arg(f_{\alpha\beta})$.

Here $f_{\alpha\beta_{\min}}$ is the distance in the Borel plane to the nearest potential singularity (which may or may not be adjacent).

As the Stokes constants are quantised integers, we only need to determine $\kappa_{\alpha\beta}$ to an accuracy of within 1/2 to infer its value. The large parameter in (74) is now N. An appropriately high value of N will give that accuracy if we know the Nth term, $f_{\alpha\beta_{\min}}$, and the number of boundaries that α, β sit on (p_α, p_β). We can then move to determine whether the next-nearest neighbour is adjacent using the same level of approximation. The calculations may be checked by including further terms in the series expansion taking larger values of N and/or higher $|k|$.

Note that this procedure is a simplification of the method in [36] and [55] and reduces the work required to determine the full set of Stokes constants. However, in this form it only works for integrals due to the quantised nature of the Stokes constants.

Even this new method cannot determine all the $\kappa_{\alpha\beta}$ at this stage. Stokes constants from next-nearest neighbours β_1 can be determined, provided that $|f_{\beta_{\min}\beta_1}| < |f_{\alpha\beta_{\min}}|$. By summing the nearest neighbour contributions to the least term and including a single term of the next-nearest contribution, we may again shorten the method of [36] and [55] and use the following to determine $\kappa_{\alpha\beta_1}$:

$$\begin{aligned} T^{\alpha}_{N_0} &+ \frac{\kappa_{\alpha\beta_{\min}}}{2\pi i} \sum_{r_1=0}^{N_1-1} \frac{\Gamma(N_0 + (p_\alpha - p_{\beta_{\min}})/2 - r_1)}{f_{\alpha\beta_{\min}}^{N_0+(p_\alpha - p_{\beta_{\min}})/2 - r_1}} T^{\beta_{\min}}_{r_1} \\ &= -\frac{\kappa_{\alpha\beta_1}}{2\pi i} \frac{\Gamma(N_0 + (p_\alpha - p_{\beta_1})/2)}{f_{\alpha\beta_1}^{N_0+(p_\alpha - p_{\beta_1})/2}} T^{\beta_1}_0 \left(1 + O\left(\frac{N_0}{|kf_{\alpha\beta_1}|}\right)^{-N_0}\right), \\ N_0 &= |k|\big(|f_{\alpha\beta_{\min}}| + |f_{\beta_{\min}\beta_1}|\big), \qquad \text{and} \qquad N_1 = N_0 - |kf_{\alpha\beta_1}|. \end{aligned} \tag{75}$$

This can be demonstrated by an intricate use of Stirling's approximation. If this is not sufficient to determine all the $\kappa_{\alpha\beta}$, one must resort to hyperasymptotic approximations and successively reexpand the remainders for each β. The result of this incestuous iterative reexpansion and substitution of the exact remainder term into itself is the treelike hyperasymptotic expansion (see Howls [36])

$$\begin{aligned} T^{\alpha}_{N^\alpha_0} = &\sum_{\kappa_{\alpha\beta_1}\neq 0} \frac{\kappa_{\alpha\beta_1}}{2i\pi} \sum_{r=0}^{N_1^{\alpha\beta_1}-1} T^{\beta_1}_r K^{(1)}(0; \alpha, \beta_1, N^\alpha_0 + 1, r) \\ &- \sum_{\kappa_{\alpha\beta_1}\neq 0} \sum_{\kappa_{\beta_1\beta_2}\neq 0} \frac{\kappa_{\alpha\beta_1}\kappa_{\beta_1\beta_2}}{(2i\pi)^2} \sum_{r=0}^{N_2^{\alpha\beta_1\beta_2}-1} T^{\beta_2}_r K^{(2)}(0; \alpha, \beta_1, \beta_2, N^\alpha_0 + 1, N^{\alpha\beta_1}_1, r) \end{aligned}$$

$$+\cdots+(-1)^{m-1}\sum_{\kappa_{\alpha\beta_1}\neq 0}\cdots\sum_{\kappa_{\beta_{m-1}\beta_m}\neq 0}\frac{\kappa_{\alpha\beta_1}\cdots\kappa_{\beta_{m-1}\beta_m}}{(2i\pi)^m}\sum_{r=0}^{N_m^{\alpha\beta_1\cdots\beta_m}-1}T_r^{\beta_m}$$
$$\times K^{(m)}(0;\alpha,\beta_1,\ldots,\beta_m,N_0^{\alpha}+1,N_1^{\alpha\beta_1},\ldots,N_{m-1}^{\alpha\cdots\beta_{m-1}},r)$$
$$+\sum_{\kappa_{\alpha\beta_1}\neq 0}\cdots\sum_{\kappa_{\beta_{m-1}\beta_m}\neq 0}\frac{\kappa_{\alpha\beta_1}\cdots\kappa_{\beta_{m-1}\beta_m}}{(2i\pi)^m}r_{\alpha\beta_1\cdots\beta_m}(N_0^{\alpha},N_1^{\alpha\beta_1},\ldots,N_m^{\alpha\beta_1\cdots\beta_m}), \tag{76}$$

where

$$\begin{aligned}
&K^{(1)}\left(k;\alpha,\beta_1,N_0^{\alpha},r\right)\\
&\quad= F^{(1)}\left(k;\begin{array}{c} N_0^{\alpha}+(p_\alpha-p_{\beta_1})/2-r\\ f_{\alpha\beta_1}\end{array}\right),\\
&K^{(2)}(k;\alpha,\beta_1,\beta_2,N_0^{\alpha},N_1^{\alpha\beta_1},r)\\
&\quad= F^{(2)}\left(k;\begin{array}{cc} N_0^{\alpha}-N_1^{\alpha\beta_1}+1+(p_\alpha-p_{\beta_1})/2, & N_1^{\alpha\beta_1}+(p_{\beta_1}-p_{\beta_2})/2-r\\ f_{\alpha\beta_1}, & f_{\beta_1\beta_2}\end{array}\right),
\end{aligned} \tag{77}$$

and, more generally,

$$\begin{aligned}
&K^{(m)}\left(k;\alpha,\beta_1,\ldots,\beta_m,N_0^{\alpha},N_1^{\alpha\beta_1},\ldots,N_{m-1}^{\alpha\ldots\beta_{m-1}},r\right)\\
&\quad= F^{(m)}\left(k;\begin{array}{cc} N_0^{\alpha}-N_1^{\alpha\beta_1}+1+(p_\alpha-p_{\beta_1})/2, & \ldots,\\ f_{\alpha\beta_1}, & \ldots,\end{array}\right.\\
&\left.\begin{array}{cc} N_{m-2}^{\alpha\cdots\beta_{m-2}}-N_{m-1}^{\alpha\cdots\beta_{m-1}}+(p_{\beta_{m-2}}-p_{\beta_{m-1}})/2, & N_{m-1}^{\alpha\cdots\beta_{m-1}}+(p_{\beta_{m-1}}-p_{\beta_m})/2-r\\ f_{\beta_{m-2}\beta_{m-1}}, & f_{\beta_{m-1}\beta_m}\end{array}\right).
\end{aligned} \tag{78}$$

The $F_r^{\alpha\beta_1\cdots\beta_m}$ are the canonical hyperterminants (see [7], [35], [36], [55])

$$\begin{cases}
F^{(0)}(z)=1,\\
F^{(\alpha)}\left(z;\begin{array}{c}M_0\\ \sigma_0\end{array}\right)=\int_0^{[-\theta_0]}e^{-t_0\sigma_0}\frac{t_0^{M_0-1}}{z-t_0}\,dt_0,\\
F^{(l+1)}\left(z;\begin{array}{ccc}M_0, & \ldots, & M_l\\ \sigma_0, & \ldots, & \sigma_l\end{array}\right)\\
\qquad=\int_0^{[-\theta_0]}\cdots\int_0^{[-\theta_l]}e^{-(t_0\sigma_0+\cdots+t_l\sigma_l)}\frac{t_0^{M_0-1}\cdots t_l^{M_l-1}}{(z-t_0)(t_0-t_1)\cdots(t_{l-1}-t_l)}\,dt_0\cdots dt_l,
\end{cases} \tag{79}$$

where θ_i is the phase of $\sigma_i\in\mathbb{C}\backslash\{0\}$, $\Re M_i>1$. When $\mathrm{ph}\,\sigma_j=\mathrm{ph}\,\sigma_{j+1}\,(\mathrm{mod}\,2\pi)$, the t_j-path of integration is deformed to the left or the right as in [55], yielding lateral canonical hyperterminants, intrinsically linked to the lateral summations of the resurgence theory (see [12], [16], [18]). These multiple integrals converge when $-\theta_0 < \operatorname{ph} z < 2\pi - \theta_0$ and can be evaluated easily by the methods of Olde Daalhuis [54], [56] with the convention that

$$\mu_{\alpha\beta}|_{\text{Olde Daalhuis}} = \frac{(p_\alpha - p_\beta)}{2}. \tag{80}$$

Thus we can employ his truncations and error estimates directly to minimise the overall remainder terms. After m stages of hyperasymptotics, the optimal truncations that globally minimise the remainder term are

$$\begin{aligned} N_0^{\alpha} &= |k| \times \min_{\substack{\kappa_{\alpha\beta_1}\neq 0 \\ \kappa_{\beta_l\beta_{l+1}}\neq 0}} \Big(| f_{\alpha\beta_1}| + \sum_{l=1\cdots m} | f_{\beta_l\beta_{l+1}}| \Big), \\ N_1^{\alpha\beta_1} &= \max\big(0,\ N_0^{\alpha} - |k f_{\alpha\beta_1}|\big), \\ N_2^{\alpha\beta_1\beta_2} &= \max\big(0,\ N_1^{\alpha\beta_1} - |k f_{\beta_1\beta_2}|\big), \\ &\ \ \vdots \\ N_m^{\alpha\beta_1\cdots\beta_m} &= \max\big(0,\ N_{m-1}^{\alpha\beta_1\cdots\beta_{m-1}} - |k f_{\beta_{m-1}\beta_m}|\big). \end{aligned} \tag{81}$$

At each level of hyperasymptotics, if a $\kappa_{\alpha\beta}$ is not yet known, then it is initially included, regardless of whether it subsequently turns out to be zero (at which point all branches containing this constant may be immediately pruned). Knowing only the relevant $f_{\alpha\beta}$, p_α, p_β along each of the branches and using the truncations (81) allows each $\kappa_{\alpha\beta}$ to be determined from an algebraic system of equations as outlined in [36] and [55].

It might appear that hyperasymptotic expansions are an unnecessary technical numerical detail. However, we know of no other general and systematic analytical or numerical method that is a practical tool for calculating the Stokes constants of these types of integrals.

Following Theorem 4.1, there is one qualitative difference between the unbounded and bounded integral case. In the unbounded integrals, every distant quadratic critical point β that could be seen by the initial one α could, in turn, see α itself. In the bounded case, some of the α (and even β) arise only because of the presence of the boundaries. If α lies on a boundary $S_1 \cap S_2$, then α can see β only if β lies on one of the strata $S_1 \cap S_2$, S_1, or S_2, or if it arises from the phase function f itself. If β is a quadratic critical point arising from the phase function f, since this is a fundamental property of the integrand, β is a likely candidate for adjacency to all the boundary α.

There is thus a hierarchy that can be inferred and can be used to simplify the hyperasymptotic analysis and deduction of Stokes constants. If $p_\beta > p_\alpha$, we may deduce that $\kappa_{\alpha\beta} = 0$ immediately since β exists only because of the presence of an extra boundary that α knows nothing about. Note that the opposite inference cannot be made; unlike the unbounded case, $\kappa_{\beta\alpha}$ may differ from $\kappa_{\alpha\beta}$, and so care must be taken in calculations.

Once the Stokes constants have been determined, it is possible to obtain a (hyper)exponential accurate approximation to (1) via the full expansion

$$\begin{aligned}
\mathscr{T}_\alpha(k) = {} & \sum_{r=0}^{N_0^\alpha-1} \frac{T_r^\alpha}{k^r} - \sum_{\kappa_{\alpha\beta_1}\neq 0} \frac{\kappa_{\alpha\beta_1}}{2i\pi k^{N_0^\alpha-1}} \sum_{r=0}^{N_1^{\alpha\beta_1}-1} T_r^{\beta_1} K^{(1)}(k;\alpha,\beta_1,N_0^\alpha,r) \\
& + \sum_{\kappa_{\alpha\beta_1}\neq 0} \sum_{\kappa_{\beta_1\beta_2}\neq 0} \frac{\kappa_{\alpha\beta_1}\kappa_{\beta_1\beta_2}}{(2i\pi)^2 k^{N_0^\alpha-1}} \sum_{r=0}^{N_2^{\alpha\beta_1\beta_2}-1} T_r^{\beta_2} K^{(2)}(k;\alpha,\beta_1,\beta_2,N_0^\alpha,N_1^{\alpha\beta_1},r) \\
& + \cdots + (-1)^m \sum_{\kappa_{\alpha\beta_1}\neq 0} \cdots \sum_{\kappa_{\beta_{m-1}\beta_m}\neq 0} \frac{\kappa_{\alpha\beta_1}\cdots\kappa_{\beta_{m-1}\beta_m}}{(2i\pi)^m k^{N_0^\alpha-1}} \sum_{r=0}^{N_m^{\alpha\beta_1\cdots\beta_m}-1} T_r^{\beta_m} \\
& \times K^{(m)}(k;\alpha,\beta_1,\ldots,\beta_m,N_0^\alpha,N_1^{\alpha\beta_1},\ldots,N_{m-1}^{\alpha\cdots\beta_{m-1}},r) \\
& + \sum_{\kappa_{\alpha\beta_1}\neq 0} \cdots \sum_{\kappa_{\beta_{m-1}\beta_m}\neq 0} \frac{\kappa_{\alpha\beta_1}\cdots\kappa_{\beta_{m-1}\beta_m}}{(2i\pi)^m k^{N_0^\alpha-1}} R_{\alpha\beta_1\cdots\beta_m} \\
& \times (k, N_0^\alpha, N_1^{\alpha\beta_1},\ldots,N_m^{\alpha\beta_1\cdots\beta_m}).
\end{aligned} \tag{82}$$

Using truncations (81), this yields an accuracy of $O(e^{-M|k|})$ at the Mth iteration (see [55]), leaving an unevaluated remainder $R_{\alpha\beta_1\cdots\beta_m}(k, N_0^\alpha, \ldots, N_m^{\alpha\beta_1\cdots\beta_m})$. This expression also widens the domain of validity of the original asymptotics and automatically and exactly incorporates any Stokes phenomenon through the hyperterminants, as explained in [36] and [55].

6. Example

We illustrate the theory with the following example, for which explicit Borel transforms can be deduced as benchmarks against which to test the hyperasymptotic analysis.

We take

$$f(z) = z^{(1)} + 2z^{(2)} + 3z^{(3)} + z^{(1)}z^{(2)} + z^{(2)}z^{(4)} + z^{(3)}z^{(4)} + {z^{(2)}}^2 + {z^{(4)}}^2 \tag{83}$$

and $g(z) = 1$. The boundary is $S = S_1 \cup S_2 \cup S_3$, where S_1, S_2, S_3 are hypersurfaces defined by the equations $S_1 : z^{(1)} = 2\mathrm{i}$, $S_2 : z^{(2)} = 1 + 2\mathrm{i}$, and $S_3 : z^{(3)} = 2 - 3\mathrm{i}$. The set of isolated critical values on the different strata is thus

- $\Lambda_{()} = \{f_0 = 11\}$ and $z_0 = (3, -1, 7, -3)$,

- $\Lambda_{(1)} = \{ f_{(1)} = 39/4 + 3\mathrm{i}\}$ and $z_{(1)} = (2\mathrm{i}, 1/2 - \mathrm{i}, 11/2 + \mathrm{i}, -3)$,
- $\Lambda_{(3)} = \{ f_{(3)} = 7 - 15\mathrm{i}/2\}$ and $z_{(3)} = (1/2 - 3\mathrm{i}/2, -1, 2 - 3\mathrm{i}, -1/2 + 3\mathrm{i}/2)$,
- $\Lambda_{(1,2)} = \{ f_{(1,2)} = 1 + 6\mathrm{i}\}$ and $z_{(12)} = (2\mathrm{i}, 1 + 2\mathrm{i}, 5 - 2\mathrm{i}, -3)$,
- $\Lambda_{(1,3)} = \{ f_{(1,3)} = 11 - 19\mathrm{i}/3\}$ and $z_{(13)} = (2\mathrm{i}, -2/3 - 7\mathrm{i}/3, 2 - 3\mathrm{i}, -2/3 + 8\mathrm{i}/3)$,
- $\Lambda_{(1,2,3)} = \{ f_{(1,2,3)} = -1 + 9\mathrm{i}/2\}$ and $z_{(123)} = (2\mathrm{i}, 1 + 2\mathrm{i}, 2 - 3\mathrm{i}, -3/2 + \mathrm{i}/2)$.

We choose $\theta = 0$ for the (generic) direction of summation. To the family of closed half-lines drawn on Figure 6 corresponds a basis of six steepest-descent contours $([\Gamma_0], \dots, [\Gamma_{(1,2,3)}])$ generating the space $H_4^{\Psi}(\mathbb{C}^4, S)$ of allowed cycles of integration (cf. Theorem 2.1).

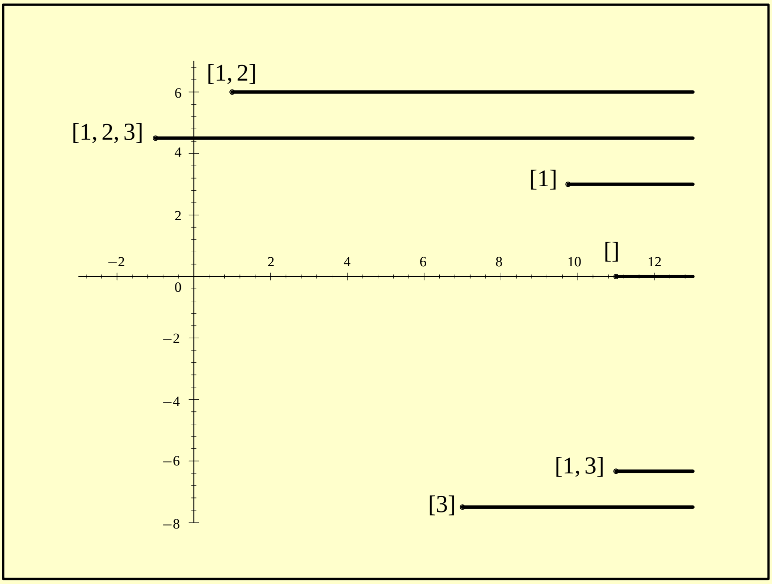

Figure 6. The set of singularities and the family of half-lines L_α in the complex Borel t-plane

6.1. Asymptotics

We now describe the representation of the integration contours space $H_4^{\Psi}(\mathbb{C}^4, S)$ via Laplace integrals. From Theorem 2.1, it is enough to consider the Laplace integrals over each of the steepest-descent contours $[\Gamma_0], \dots, [\Gamma_{(1,2,3)}]$.

We recall that the Lefschetz thimble, in principle, determines completely the homology of the steepest-descent contour, and thus we do not provide a detailed global description of $[\Gamma_0], \dots, [\Gamma_{(1,2,3)}]$. We have just to prescribe the orientations, which we take to be those of the standard Lefschetz thimble (cf. Sections 2.3.1 and 2.3.2).

In what follows, the square root $\sqrt{k}$ refers to the usual determination (real positive for positive real k).

• Straightforward computations give

$$I_{\Gamma_0}(k) = \int_{\Gamma_0} e^{-kf(z)}\, dz^{(1)} \wedge \cdots \wedge dz^{(4)} = \frac{4\pi^2}{k^2} e^{-11k}. \tag{84}$$

• Convenient reductions show that the integral

$$I_{\Gamma_{(1)}}(k) = \int_{\Gamma_{(1)}} e^{-kf(z)}\, dz^{(1)} \wedge \cdots \wedge dz^{(4)} \tag{85}$$

can be written as

$$I_{\Gamma_{(1)}}(k) = \frac{4\mathrm{i}\pi^{3/2}}{k^{3/2}} e^{-(39/4+3\mathrm{i})k} \int_0^{+\infty} e^{-kt} \frac{dt}{\sqrt{5-12\mathrm{i}-4t}}. \tag{86}$$

It follows that $I_{\Gamma_{(1)}}(k)$ is the Borel sum of $T_{(1)}(k)k^{-(4+1)/2}e^{-(39/4+3\mathrm{i})k}$, where

$$T_{(1)}(k) = 4\mathrm{i}\pi^{3/2} \frac{3+2\mathrm{i}}{13} \sum_{j=0}^{\infty} \left(\frac{20+48\mathrm{i}}{169}\right)^j \frac{\Gamma(j+1/2)}{\Gamma(1/2)} k^{-j} \in \mathbb{C}[[k^{-1}]]_1. \tag{87}$$

$I_{\Gamma_{(1)}}(k)$ can be analytically continued by rotating the direction of summation, but a Stokes phenomenon occurs due to the adjacent singularity at $t = 5/4 - 3\mathrm{i} = f_0 - f_{(1)} = f_{(1),0}$.

• Similarly, $I_{\Gamma_{(3)}}(k) = \int_{\Gamma_{(3)}} e^{-kf(z)}\, dz^{(1)} \wedge \cdots \wedge dz^{(4)}$ yields

$$I_{\Gamma_{(3)}}(k) = \frac{4\mathrm{i}\pi^{3/2}}{k^{3/2}} e^{-(7-15\mathrm{i}/2)k} \int_0^{+\infty} e^{-kt} \frac{dt}{\sqrt{16+30\mathrm{i}-4t}}. \tag{88}$$

This is the Borel sum of $T_{(3)}(k)k^{-5/2}e^{-(7-15\mathrm{i}/2)k}$, where

$$T_{(3)}(k) = 2\mathrm{i}\pi^{3/2} \frac{5-3\mathrm{i}}{17} \sum_{j=0}^{\infty} \left(\frac{16-30\mathrm{i}}{289}\right)^j \frac{\Gamma(j+1/2)}{\Gamma(1/2)} k^{-j} \in \mathbb{C}[[k^{-1}]]_1. \tag{89}$$

A Stokes phenomenon occurs due to the adjacent singularity at $t = 4 + 15\mathrm{i}/2 = f_0 - f_{(3)} = f_{(3),0}$.

• In the same way as above, the integral $I_{\Gamma_{(12)}}(k) = \int_{\Gamma_{(12)}} e^{-kf(z)}\, dz^{(1)} \wedge \cdots \wedge dz^{(4)}$ is conveniently reduced to

$$I_{\Gamma_{(12)}}(k) = \frac{\pi}{k^2} e^{-(1+6\mathrm{i})k} \int_0^{+\infty} e^{-kt} \frac{dt}{\sqrt{35/4-3\mathrm{i}-t}\,(3/2-\mathrm{i}+\mathrm{i}\sqrt{35/4-3\mathrm{i}-t})}. \tag{90}$$

This is the Borel sum of $T_{(12)}(k)k^{-(4+2)/2}e^{-(1+6\mathrm{i})k}$, where

$$T_{(12)}(k) = -2\mathrm{i}\pi \sum_{j=0}^{\infty}\Big(\frac{5+7\mathrm{i}}{148}\Big)^{j+1}\Big(\sum_{l=0}^{j}\Big(\frac{14-10\mathrm{i}}{37}\Big)^{l}\frac{\Gamma(j+l+1)}{\Gamma(l+1)}\Big)k^{-j} \in \mathbb{C}[[k^{-1}]]_1. \tag{91}$$

Stokes phenomena occur due to adjacent singularities at $t = 35/4 - 3\mathrm{i} = f_{(1)} - f_{(12)} = f_{(12),(1)}$, and at $t = 10 - 6\mathrm{i} = f_0 - f_{(12)} = f_{(12),0}$ for the Borel transform $\widehat{T_{(12)}}(t)$.

• Continuing, we have $I_{\Gamma_{(13)}}(k) = \int_{\Gamma_{(13)}} e^{-kf(z)}\, dz^{(1)} \wedge \cdots \wedge dz^{(4)}$, which can be written as

$$I_{\Gamma_{(13)}}(k) = \frac{3\sqrt{3}\pi}{2k} e^{-(11-19\mathrm{i}/3)k} \int_0^{+\infty} dt e^{-kt} \int_0^t dt_1 \times \frac{1}{\sqrt{15/4-28\mathrm{i}+3t_1}\sqrt{3(t-t_1)+(3/2+9\mathrm{i}/4+\sqrt{15/4-28\mathrm{i}+3t_1}/2)^2}}. \tag{92}$$

This gives $I_{\Gamma_{(13)}}(k)$ as the Borel sum of $T_{(13)}(k)k^{-3}e^{-(11-19\mathrm{i}/3)k}$, with

$$T_{(13)}(k) = \frac{2\pi}{\sqrt{3}} \sum_{j=0}^{\infty} \frac{(-1)^j}{3^j}\Big(\frac{567+369\mathrm{i}}{5650}\Big)^{j+1} \times \Big(\sum_{l_1=0}^{j}\sum_{l_2=0}^{j-l_1}\Big(\frac{49+57\mathrm{i}}{113}\Big)^{l_1-l_2}\frac{\Gamma(j-l_1+l_2+1)\Gamma(j+l_1-l_2+1)}{\Gamma(l_1+1)\Gamma(l_2+1)\Gamma(j-l_1-l_2+1)}\Big)k^{-j} \in \mathbb{C}[[k^{-1}]]_1. \tag{93}$$

From (92), a Stokes phenomenon occurs due to an adjacent singularity at $t = -5/4 + 28\mathrm{i}/3 = f_{(1)} - f_{(13)} = f_{(13),(1)}$ (to see this, set $t_1 = t$ in the integrand) and also at $t = -4 - 7\mathrm{i}/6 = f_{(3)} - f_{(13)} = f_{(13),(3)}$ ($t_1 = 0$). Referring to Section 4.1, a singularity at $t = 19\mathrm{i}/3 = f_0 - f_{(13)} = f_{(13),0}$ is also expected when one continues onto another sheet.

• The last integral $I_{\Gamma_{(123)}}(k) = \int_{\Gamma_{(123)}} e^{-kf(z)}\, dz^{(1)} \wedge \cdots \wedge dz^{(4)}$ may be written in the form

$$I_{\Gamma_{(123)}}(k) = \frac{12\mathrm{i}\sqrt{\pi}}{k^{3/2}} \int_0^{+\infty} dt e^{-kt} \int_0^t dt_1 \times \frac{1}{\sqrt{8+6\mathrm{i}-4t_1}(\sqrt{16t_1-12t+132-54\mathrm{i}-(4+24\mathrm{i})\sqrt{8+6\mathrm{i}-4t_1}})} \times \frac{1}{(4-6\mathrm{i}-\sqrt{8+6\mathrm{i}-4t_1}+\mathrm{i}\sqrt{16t_1-12t+132-54\mathrm{i}-(4+24\mathrm{i})\sqrt{8+6\mathrm{i}-4t_1}})}; \tag{94}$$

$I_{\Gamma_{(123)}}(k)$ is then the Borel sum of $T_{(123)}(k)k^{-(4+3)/2}$ with $T_{(123)}(k) \in \mathbb{C}[[k^{-1}]]_1$,

$$\begin{aligned} T_{(123)}(k) \\ &= \frac{\sqrt{\pi}}{8} \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j}} \left(\sum_{l_1=0}^{j} \sum_{l_2=0}^{j-l_1} \sum_{l_3=0}^{j-l_1-l_2} (2\mathrm{i})^{l_2+1} 3^{l_3} (1-i)^{l_1+1} \left(\frac{5-13\mathrm{i}}{97}\right)^{l_1+l_2+2l_3+1} \right. \\ &\quad \times \left. \left(\frac{1+3\mathrm{i}}{5}\right)^{2j-2l_1-l_2-2l_3+1} \frac{\Gamma(l_1+l_2+2l_3+1)\Gamma(2j-2l_1-l_2-2l_3+1)}{\Gamma(l_2+1)\Gamma(l_3+1)\Gamma(j-l_1-l_2-l_3+1)} \right) k^{-j}. \end{aligned} \tag{95}$$

Moreover, one can deduce from the integral representation above that a Stokes phenomenon occurs due to an adjacent singularity at $t = 2 + 3\mathrm{i}/2 = f_{(12)} - f_{(123)} = f_{(123),(12)}$ ($t_1 = t$) and also at $t = 12 - 65\mathrm{i}/6 = f_{(13)} - f_{(123)} = f_{(123),(13)}$ ($t_1 = 0$). For reasons explained in Section 4.1, singularities can be expected at $f_{(123),(1)}$, $f_{(123),(3)}$, and $f_{(123),()}$ for some analytic continuations.

6.2. Resurgence formulae

The sheet structure in this example can be derived from knowledge of explicit formulae for the Borel transforms of the multiple integrals. In general, the situation is not so simple. Such a study can nevertheless be done numerically with only the asymptotic expansions $T_{(\dots)}$ as inputs by appealing to hyperasymptotics. Since the values $f_{\alpha\beta}$ have distinct phases, formulae (74), (75), (76), (81), (82) can thus be applied.

Note that from the “one way” adjacency property and the fact that there is only one critical point on each stratum, the hyperasymptotic expansion (82) (or (76)) terminates at most (when $T_{(123)}$ is concerned) at the $m = 3$ level.

By comparison with Figure 6, part of the first sheet structure can be recovered from the leading order formula (74), which yields information about the indices of intersections corresponding to the nearest adjacent singularities for each critical point in turn. One obtains

$$\begin{gathered} \kappa_{(1,2,3),(1,2)} = +1, \qquad \kappa_{(1,3),(3)} = -1, \qquad \kappa_{(1,2),(1)} = -1, \\ \kappa_{(3),()} = +1, \qquad \kappa_{(1),()} = +1. \end{gathered} \tag{96}$$

Now one turns to hyperasymptotics to gain more information. We know that $\alpha = (1, 2)$ has $\beta_{\min}(1)$ for its nearest adjacent singularity, while its next-nearest singularity is $\beta_1 = ()$. Figure 6 shows that expansion (75) may be applied. With $k = 1$, (75) yields $N_0 = 13$, $N_1 = 4$, and $\kappa_{(1,2),()} \simeq -0.924 + 0.04\mathrm{i}$. From the arguments of Section 5, this is quite enough to conclude that $\kappa_{(1,2),()} = -1$.

Formula (76) at the first level ($m = 1$) yields

$$\kappa_{(1,2,3),(1)} = 0, \qquad \kappa_{(1,3),()} = 0, \qquad \kappa_{(1,3),(1)} = -1, \tag{97}$$

Table 1. The Stokes multipliers

$\kappa_{\alpha\beta}$	β=()	(1)	(3)	(1,2)	(1,3)	(1,2,3)
α= ()	0	0	0	0	0	0
(1)	+1	0	0	0	0	0
(3)	+1	0	0	0	0	0
(1,2)	−1	−1	0	0	0	0
(1,3)	0	−1	−1	0	0	0
(1,2,3)	0	0	−1	+1	+1	0

but it may also be used to recover our previous results (cf. [36], [55]). For instance, replacing $k = 5$ in (81) gives $N_0^{(1,3)} = 63$, $N_1^{(1,3),(1)} = 16$, $N_1^{(1,3),(3)} = 42$, and $N_1^{(1,3),()} = 31$ as optimal truncations. On replacing N_0^{α} by $N_0^{(1,3)}$, $N_0^{(1,3)} - 1$, and $N_0^{(1,3)} - 2$ in formula (76), ignoring the error term, and explicitly substituting $T^{(1,3)}_{N_0^{(1,3)}}$, $T^{(1,3)}_{N_0^{(1,3)}-1}$, and $T^{(1,3)}_{N_0^{(1,3)}-2}$, we get a system of three linear equations. Solving these equations yields $\kappa_{(1,3),(1)}$, $\kappa_{(1,3),(3)}$, and $\kappa_{(1,3),()}$. Using some higher hyperasymptotic levels in the same spirit, one can completely determine the sheet structure. The level-two expansion yields

$$\kappa_{(1,2,3),(3)} = -1, \qquad \kappa_{(1,2,3),()} = 0, \tag{98}$$

while the level-three expansion gives the last unknown index of intersection, $\kappa_{(1,2,3),(1,3)} = +1$ (see Table 1). The results of the adjacency calculations are displayed in Table 2.

The sheet structure is now in its complete form.

6.3. Hyperasymptotic computations

The Stokes multipliers being known, (82) gives a (hyper)exponential accurate approximation for the integrals, yielding the results in Table 2.

Based on the adjacency calculations, we have calculated hyperasymptotic approximations to the integrals $\mathscr{T}_{(1)}$, $\mathscr{T}_{(3)}$, $\mathscr{T}_{(12)}$, $\mathscr{T}_{(13)}$ for various values of large parameter k. "Exact" values of the integrals were obtained from numerical integration schemes. The accuracy obtainable for $\mathscr{T}_{(123)}$ by this approach was sufficient only for comparison with the first hyper-level, and so we have not included this. Nevertheless, the agreement was as expected and consistent with the adjacency of (3), (12), and (13) to (123).

Table 2. Hyperasymptotic levels, truncations, approximations, and achieved accuracies for $\mathscr{T}_{(1)}$ and $k = 4$

Level	*Truncations*	*Approximation*	$\lvert 1 - \text{approx./exact} \rvert$
Lowest	$N(1) = 1$	$-3.42666338266567+$ $5.13999507399850i$	3.9×10^{-2}
Level 0 (Super)	$N(1) = 13$	$-3.66008976352172+$ $5.06575794497829i$	2.7×10^{-6}
Level 1	$N(1) = 26$, $N(1, 0) = 1$	$-3.66007435340317+$ $5.06575062056987i$	0
Exact	$k = 4$	$-3.66007435340317+$ $5.06575062056987i$	0

Table 3. Hyperasymptotic levels, truncations, approximations, and achieved accuracies for $\mathscr{T}_{(3)}$ and $k = 1$

Level	*Truncations*	*Approximation*	$\lvert 1 - \text{approx./exact} \rvert$
Lowest	$N(3) = 1$	$1.96529223417590+$ $3.27548705695983i$	6.1×10^{-2}
Level 0 (Super)	$N(3) = 9$	$2.19659657698560+$ $3.22186580822527i$	2.5×10^{-4}
Level 1	$N(3) = 17$, $N(3, 0) = 1$	$2.19566048409404+$ $3.22216423713061i$	0
Exact	$k = 1$	$2.19566048409404+$ $3.22216423713061i$	0

Table 4. Hyperasymptotic levels, truncations, approximations, and achieved accuracies for $\mathscr{T}_{(12)}$ and $k = 1$

Level	*Truncations*	*Approximation*	$\lvert 1 - \text{approx./exact} \rvert$
Lowest	$N(12) = 1$	0.29717768344768 $-0.21226977389120i$	1.2×10^{-1}
Level 0 (Super)	$N(12) = 9$	0.34290482693409 $-0.20268045850289i$	6.2×10^{-4}
Level 1	$N(12) = 13$, $N(12, 1) = 3$, $N(12, 0) = 1$	0.34300280953350 $-0.20290401468035i$	2.4×10^{-6}
Level 2	$N(12) = 15$, $N(12, 1) = 5$, $N(12, 0) = 1$, $N(12, 1, 0) = 1$	0.34299544259126 $-0.20290986661454i$	$< 5.1 \times 10^{-14}$
Exact	$k = 1$	0.34299544259128 $-0.20290986661454i$	0

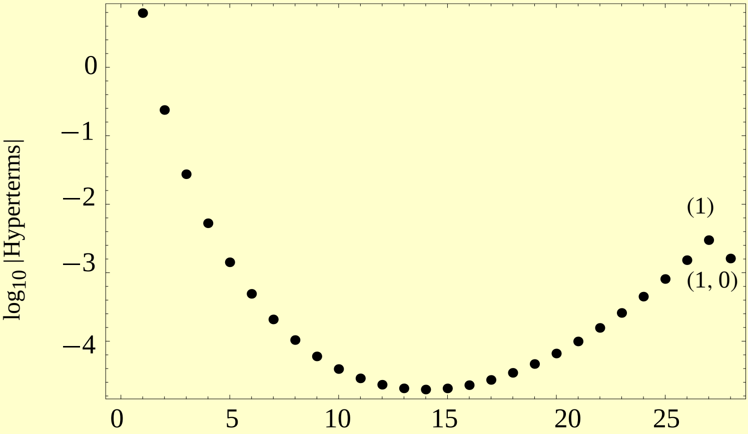

Figure 7. The size of the terms in the hyperasymptotic expansion of $\mathscr{T}_{(1)}$, from the critical points (1) (ordinary asymptotics) and (0) (hyperasymptotics along path (1,0)), with $k = 4$. From the truncations (see Table 2 and (81)) and adjacency of the singularities, only the (0) critical point contributes at the first hyperlevel. As this contributes an exact exponential, only one term is generated, and the hyperasymptotics terminates, generating the exact result.

7. Discussion

We first discuss our hypotheses, suggesting how the above results may be generalised.

7.1. Critical points at infinity (Hypothesis H2)

Throughout this article, we have assumed that no critical point at infinity occurred (see Hypothesis H2). This assumption was a consequence partly of our own ignorance and partly of a lack of general results about critical points at infinity, although this subject is a matter of an intensive current research (see [59], [31], [33], [67], [13]). However, the following simple example may suggest that Hypothesis H2 can be removed in some instances.

We go back to the family of polynomials (7), and we study the behaviour near $k = 0$ of the integral

$$I(k) = \int_\Gamma e^{-f(z^{(1)},z^{(2)};k)}\, dz^{(1)} \wedge dz^{(2)}. \tag{99}$$

For nonzero k, this integral (99) fits into our frame, and we may take for Γ an unbounded chain of integration. By a simple argument of quasi-homogeneity, we first

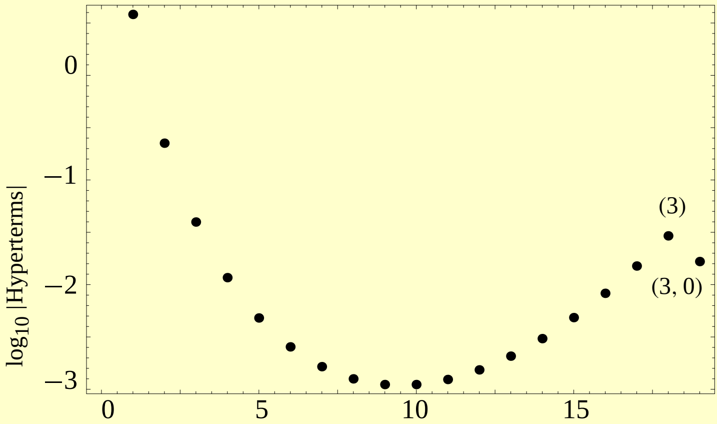

Figure 8. As for Figure 7, but for $\mathscr{T}_{(3)}$, with $k = 1$. Again only critical point (0) is adjacent, so the hyperasymptotics terminates at the first reexpansion, generating the exact result.

notice that

$$I(k) = \int_\Gamma e^{-kf(z^{(1)},z^{(2)};1)}\, dz^{(1)} \wedge dz^{(2)}. \tag{100}$$

(We keep the same notation for the chain Γ.) Instead of analysing the behaviour near $k = 0$, we first compute the asymptotics when $k \to \infty$. We consider the critical value $+2\mathrm{i}$ (resp., $-2\mathrm{i}$), choose the nonsingular direction ($\theta = 0$), define the associated Lefschetz thimble $\Gamma_{2\mathrm{i}}$ (resp., $\Gamma_{-2\mathrm{i}}$), and consider the asymptotics inside the sector $|\arg(k)| < \pi/2$. With a convenient orientation of $\Gamma_{2\mathrm{i}}$ and $\Gamma_{-2\mathrm{i}}$, computations (thanks to the algorithm of [17]) suggest that $I_{2\mathrm{i}}(k) = (\pi/k + o(k^{-N}))e^{-2\mathrm{i}k}$ and $I_{-2\mathrm{i}}(k) = (\pi/k + o(k^{-N}))e^{2\mathrm{i}k}$ for all integers $N > 0$; hence $I_{2\mathrm{i}}(k) = \pi e^{-2\mathrm{i}k}/k$ and $I_{-2\mathrm{i}}(k) = \pi e^{+2\mathrm{i}k}/k$ exactly by a Borel-summability argument (Watson theorem; see [47]).* This would mean that

$$I(k) = \langle \Gamma^\star_{2\mathrm{i}}, \Gamma\rangle I_{2\mathrm{i}}(k) + \langle \Gamma^\star_{-2\mathrm{i}}, \Gamma\rangle I_{-2\mathrm{i}}(k) \tag{101}$$

*More generally, we suspect (by direct computations) that for every polynomial function $g(z^{(1)}, z^{(2)})$, we have the equality

$$\int_\Gamma e^{-kf(z^{(1)},z^{(2)};1)} g(z^{(1)}, z^{(2)})\, dz^{(1)} \wedge dz^{(2)} = \Big(\sum_{j=1}^{6} \frac{a_j}{k^j}\Big) e^{-2\mathrm{i}k} + \Big(\sum_{j=1}^{6} \frac{b_j}{k^j}\Big) e^{2\mathrm{i}k},$$

where the a_j, b_j are complex coefficients depending on g.

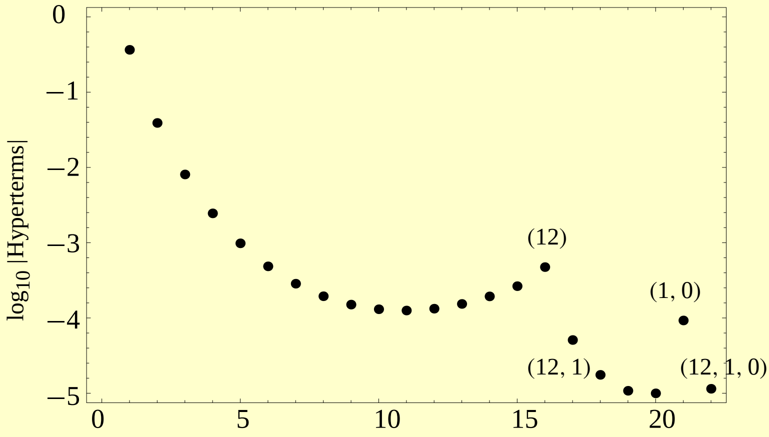

Figure 9. As for Figure 7, but for $\mathscr{T}_{(12)}$, with $k = 1$ showing the various sequence of critical points that contribute to the hyperasymptotics. Here critical points (1) and (0) are adjacent to (12) and so generate first-level hyperasymptotic corrections. A second level is generated by (0) as this is adjacent to the (first-level) contribution (1).

satisfies the differential equation

$$\mathscr{L}(k, \partial_k)I = 0 \quad \text{with } \mathscr{L}(k, \partial_k) = \frac{d^2}{dk^2} + \frac{2}{k}\frac{d}{dk} + 4 \tag{102}$$

with a regular singular point at the origin. This property is satisfied if the differential 2-form $\omega_0 = (4k^2 z^{(2)^2} - 6z^{(2)} + 4)e^{-f(z^{(1)},z^{(2)};k)}\, dz^{(1)} \wedge dz^{(2)}$ is exact (Stokes theorem), and actually,

$$\omega_0 = d\big(4z^{(2)^2} e^{-f(z^{(1)},z^{(2)};k)}\, dz^{(1)} + (-2 + 2z^{(1)}z^{(2)})e^{-f(z^{(1)},z^{(2)};k)}\, dz^{(2)}\big).$$

Thus $I(k)$ defined by (100) has a simple pole at $k = 0$, except when $\langle\Gamma^\star_{2\mathrm{i}}, \Gamma\rangle = -\langle\Gamma^\star_{-2\mathrm{i}}, \Gamma\rangle$ where $I(k)$ can be extended analytically at the origin. This suggests the existence of convenient valleys at infinity such that the integral (99) still converges for $k = 0$.*

*We formulate a conjecture. The integral

$$\Phi(k) = \int_\Gamma e^{-kf(z^{(1)},z^{(2)};0)} g(z^{(1)}, z^{(2)})\, dz^{(1)} \wedge dz^{(2)}$$

might be defined along a (steepest-descent) chain around the connected component $z^{(1)} = 0$ of the special fiber $f(\cdot; 0) = 0$, whose image in f would be a path λ starting at infinity along the half-line L_0, running around the origin, and returning to infinity. This integral could be recast as $\Phi(k) = \int_\lambda e^{-kt} \overset{\vee}{\varphi}(t)\, dt$, where

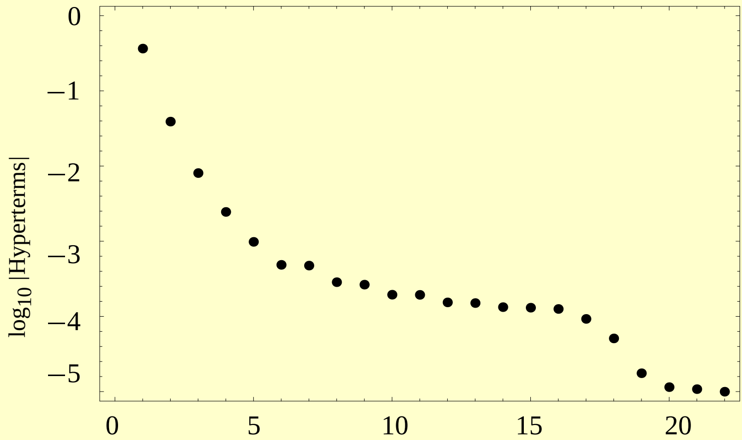

Figure 10. As for Figure 9, but with the terms ordered according to absolute size

Taking for granted that our oscillating integrals can be defined without Hypothesis H2, can we expect an analogy of Theorem 4.1, at least when the critical points at infinity are isolated (see [59], [67])? Consider the integral with boundaries

$$I(k) = \int_\Gamma e^{-kf(z^{(1)},z^{(2)};0)}\, dz^{(1)} \wedge dz^{(2)}, \tag{103}$$

where, for instance, $\Gamma = \{(z^{(1)}, z^{(2)}) \in [0, +\infty[\times[1, +\infty[\}$. For $\Re(k) > 0$, integral (103) defines an analytic function in k and can be recast as

$$I(k) = e^{-2k} \int_0^\infty dt\, e^{-kt}\Big(1 - \frac{2}{t+2}\Big). \tag{104}$$

Integral (104) has obvious resurgence properties: $t = -2$ is the sole singularity in the Borel plane, and, taking into account the translation in the Borel plane induced by the exponential factor $e^{-2k} = e^{-kf(1,0;0)}$, we see that the location of the singularity arises from the sole bifurcation value corresponding to the critical point at infinity of $f(z^{(1)}, z^{(2)}; 0)$. Nevertheless, comparing this result with Theorem 4.1, we note a major novelty: the singularity is nonintegrable. Removing Hypothesis H2 thus generates a challenging problem for a geometer—to relate these resurgence properties

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$\overset{\vee}{\varphi}(t)$ is an endlessly continuable major (see [18]), whose singularity $\overset{\triangledown}{\varphi}$ (corresponding microfunction) at the origin could be represented by

$$\overset{\triangledown}{\varphi}(t) \equiv \int_{\gamma(t)} \frac{g(z^{(1)}, z^{(2)})\, dz^{(1)} \wedge dz^{(2)}}{df},$$

where $\gamma(t)$ is a semicycle vanishing at infinity (see [29]).

Table 5. Hyperasymptotic levels, truncations, approximations, and achieved accuracies for $\mathscr{T}_{(13)}$ and $k = 1$

Level	*Truncations*	*Approximation*	$\lvert 1 - \text{approx./exact}\rvert$
Lowest	$N(13) = 1$	0.36404397859143+ 0.23691750987697i	1.5×10^{-1}
Level 0 (Super)	$N(13) = 4$	0.32105083513787+ 0.19423741497334i	1.5×10^{-2}
Level 1	$N(13) = 12$, $N(13,1) = 2$, $N(13,3) = 7$	0.32676405233855+ 0.19354284669953i	5.9×10^{-6}
Level 2	$N(13) = 15$, $N(13,1) = 5$, $N(13,3) = 10$, $N(13,1,0) = 1$, $N(13,3,0) = 1$	0.32676199784979+ 0.19354375274768i	$< 10^{-14}$
Exact	$k = 1$	0.32676199784980+ 0.19354375274769i	0

to the geometry of the phase function $f(z^{(1)}, z^{(2)}; 0)$ through, for instance, extended Picard-Lefschetz formulae (see [31]).

7.2. Nonisolated critical points (Hypothesis H3)

Although the asymptotics of oscillating integrals have been calculated for a class of nonisolated critical points in [76], Hypothesis H3 plays an essential role in all parts of our study, from the proof of Theorem A.2 and the description of the asymptotics (see Theorems 3.1 and 3.2) to the Picard-Lefschetz formulae and the resurgence properties (see Theorem 4.1) and hence the hyperasymptotics. However, we believe it is still possible to generate exact remainder terms by other means.

7.3. "Multiple" critical values (Hypothesis H4)

We discuss condition H4 using the following example.* For $q \in \mathbb{C}\{0\}$,

$$I(q,k,s) = \int_\Gamma e^{-kf(z^{(1)},z^{(2)};q)}\, g(z^{(1)}, z^{(2)})\, dz^{(1)} \wedge dz^{(2)} \tag{105}$$

with

$$f(z^{(1)}, z^{(2)}; q) = qz^{(1)}z^{(2)} - 4{z^{(1)}}^3/3 - {z^{(2)}}^3/24 - q^3/3$$

and Γ an unbounded chain of integration of real dimension 2. The phase function f has four (isolated) nondegenerate critical points, and a little thought shows that the

* When $g = {z^{(1)}}^s$ with $s \in \mathbb{C}$, (105) is the solution of the differential equation $(\partial_q^2 - k^2q^4 + 2ksq)I = 0$. For $s = 0$, (105) can be reduced to a Hardy integral (see [34]). Note that, for noninteger s, the ramification of the amplitude function around the complex curve $z^{(1)} = 0$ enlarges the space of independent contours (see [64] for a similar one-dimensional feature).

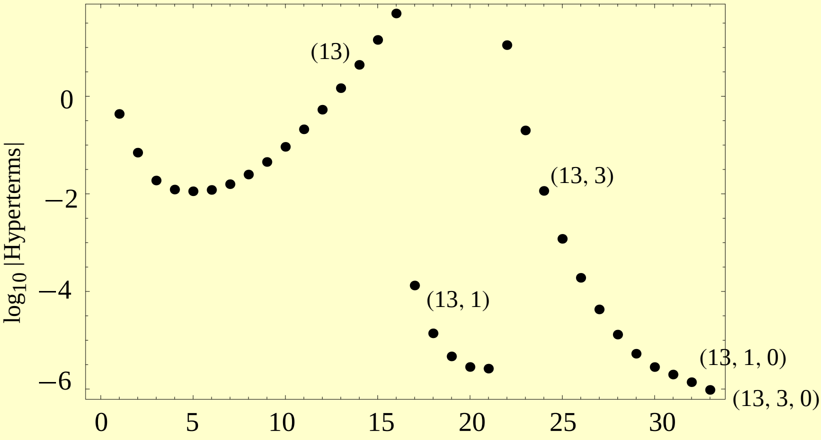

Figure 11. As for Figure 9, but for $\mathscr{T}_{(13)}$, with $k = 1$

space of allowed contours of integration is generated by four independent cycles.* Nevertheless, Hypothesis H4 is here violated. While one of the critical points (the origin) corresponds to the critical value $-q^3/3$, the three others have the same critical value $+q^3/3$ for their image by f. However, we can still define four Lefschetz thimbles properly, say, Γ_- and Γ_+^1, Γ_+^2, Γ_+^3, and compute the asymptotics of the corresponding I_- and I_+^1, I_+^2, I_+^3.

This suggests that Hypothesis H4 is essentially a technicality. It remains to extend the generalised Picard-Lefschetz formulae to understand the Stokes phenomenon. The contour Γ_- may intersect (the duals of) Γ_+^1, Γ_+^2, and Γ_+^3 *simultaneously* when a Stokes phenomenon occurs. To compute the indices of intersection, one can introduce a suitable deformation of f so that the "multiple critical value" splits into distinct critical values.† For ϵ near but different from zero, the family $f_\epsilon = f + \epsilon z^{(1)}$ can be used: Γ_-, Γ_+^1, Γ_+^2, and Γ_+^3 are deformed into $\Gamma_{\epsilon-}$, $\Gamma_{\epsilon+}^1$, $\Gamma_{\epsilon+}^2$, and $\Gamma_{\epsilon+}^3$, and the indices $\langle \Gamma_{\epsilon+}^{l\,\star}, \Gamma_{\epsilon-}\rangle$ $(l = 1, 2, 3)$ are now computable; hence $\langle \Gamma_+^{l\,\star}, \Gamma_-\rangle$ is also computable by continuity. Note in this case that the indices of intersection $\langle \Gamma_+^{i\,\star}, \Gamma_+^j\rangle$, $i \neq j$, are necessarily zero for topological reasons. (The dependence in ϵ is regular in the sense of [18].)

*The phase function is governed by the monomials ${z^{(1)}}^3$ and ${z^{(3)}}^3$ near infinity; therefore each variable has three possible asymptotic valleys. This gives a basis of $((3 - 1) \times (3 - 1))$-cycles.

†Note here that Hypothesis H2 remains true under the deformation. In general, under our hypotheses, this is not guaranteed. For instance, although the set of tame polynomials is a dense (constructible) set in the set of polynomials of a given degree (see [11]), it is not open in dimension 3 (see [13]).

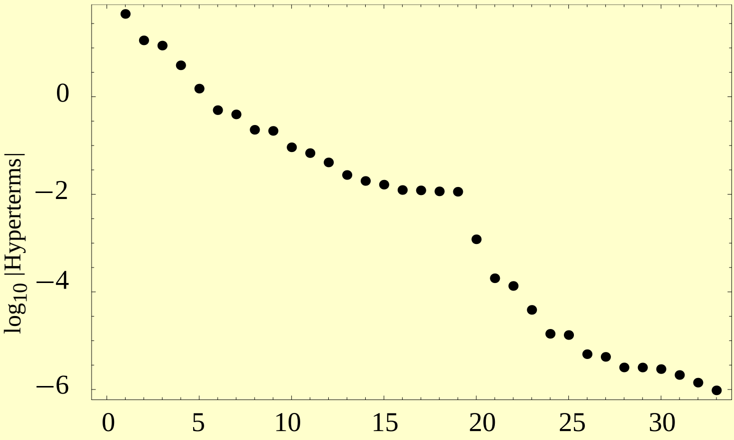

Figure 12. As for Figure 11, but with the terms ordered according to absolute size. Note that here the final exact result is $0.32676\ldots + i0.19354\ldots$, although there are several terms larger than this. Cancellations occur between these larger numbers.

7.4. Confluent cases (Hypothesis H5)

We give here a flavour of the difficulty arising from third-type singularities.

The main difference between third-type and first- or second-type critical values is primarily topological in nature. When a first- or a second-type critical value is considered, we have seen (see Section 2) that the corresponding Milnor fiber has the homotopy of a bouquet of μ spheres, defining a *single* group of vanishing homology. The situation is quite different for a third-type critical point, where different groups of vanishing homologies can be defined, depending on which strata are considered, with each of these groups playing a role. Figure 13 exemplifies this situation for $n = 3$. Here z_α is a nondegenerate critical point for f ($f_\alpha \in \Lambda_{()}$), but the same z_α is also a nondegenerate critical point for $f|S_1$ and $f|S_2$. To such a geometry correspond four vanishing homologies: the vanishing cycle γ (the sphere) is a generator of the vanishing homology group $H_2(X^t_\alpha)$, the vanishing cycle γ_1 (resp., γ_2) generates the vanishing homology $H_1(X^t_\alpha \cap S_1)$ (resp., $H_1(X^t_\alpha \cap S_2)$), and γ_{12} is a basis for the remaining vanishing (reduced) homology group $H_0(X^t_\alpha \cap S_1 \cap S_2)$.

These four groups can be understood in terms of our "allowed" cycles of integration. In Figure 13, consider the ball B_1, bounded by the vanishing sphere, as a representation of the Lefschetz thimble. Consider similarly one of the two half-balls $B^1_{1/2}$ (resp., $B^2_{1/2}$), bounded by the vanishing sphere and S_1 (resp., S_2), and finally one of the four quarter-balls $B_{1/4}$, bounded by the sphere, S_1, and S_2. Then, obviously, all possible allowed (localized) chains of integration can be described as combinations (with integer coefficients) of B_1, $B^1_{1/2}$, $B^2_{1/2}$, and $B_{1/4}$. Hence we have the following lemma.

LEMMA 7.1
The local homology is a free $\mathbb{Z}$-module of finite rank, isomorphic to the direct sum of $H_2(X^t_\alpha)$, $H_1(X^t_\alpha \cap S_1)$, $H_1(X^t_\alpha \cap S_2)$, and $H_0(X^t_\alpha \cap S_1 \cap S_2)$.

With this kind of decomposition in hand, one can then concentrate on the integral representation. With the notation of Section 3.2 and in convenient local coordinates, the phase function f reads

$$f = s^{(1)} + \cdots + s^{(\min_\alpha -1)} + F(s^{(\min_\alpha)}, \ldots, s^{(n)}), \tag{106}$$

where F has an (isolated) critical point at the origin, while the (considered) boundary is given by the set of local equations $s^{(1)} = 0, \ldots, s^{(p)} = 0$, with $\min_\alpha \le p$. Section 3.2 suggests a way to derive the asymptotics by reducing step by step the multiple integral into a one-dimensional Laplace integral. But a new difficulty arises in the confluent case: the reduction process means considering integrals of differential quotient forms where the quotients are differential forms $df_{(1,\ldots,q)}$ that *vanish* at the critical point as soon as $q \ge \min_\alpha$. This makes the analysis of the analytic behaviour of the Borel transform $\widehat{J_\alpha}(t)$ (analogous to (46)) more complicated. It seems that a result similar to Theorem 3.2 works also in this case.

(1) The asymptotics are essentially described by Gevrey-1 resurgent (see [23], [18]) asymptotic expansions. Generically, its leading term is $\mathrm{Const}/k^{\min_\alpha + 1/d}$, where d is the distance to the Newton diagram of F (cf. Figure 14; see [71]).
(2) The asymptotics are essentially governed by the geometry of the singularity (monodromy of the vanishing homologies).

The first assertion boils down to proving that the integral may be written as a Laplace transform of a solution of a Picard-Fuchs differential equation with (at most) a regular singular point at the origin; similar results have been proved in [52]. To us, to provide a precise statement of the second assertion seems to be much harder.

It is worthwhile noting here that the class of integrals considered by Kaminski and Paris [40], [41] (see also [49]) enters into the framework of our third-type critical values (when the phase function is a polynomial function). This class of integrals can thus be used to experiment. Consider, for instance, the function of the example in Section 2.4,

$$f(z^{(1)}, z^{(2)}) = {z^{(1)}}^4 + z^{(1)}{z^{(2)}}^2 + {z^{(2)}}^3. \tag{107}$$

We have seen that the origin in $\mathbb{C}^2$ is an isolated critical point with $\mu = 5$ for its Milnor number. We concentrate on

$$I(k) = \int_\Gamma e^{-kf(z^{(1)}, z^{(2)})}\, dz^{(1)}\, dz^{(2)}, \tag{108}$$

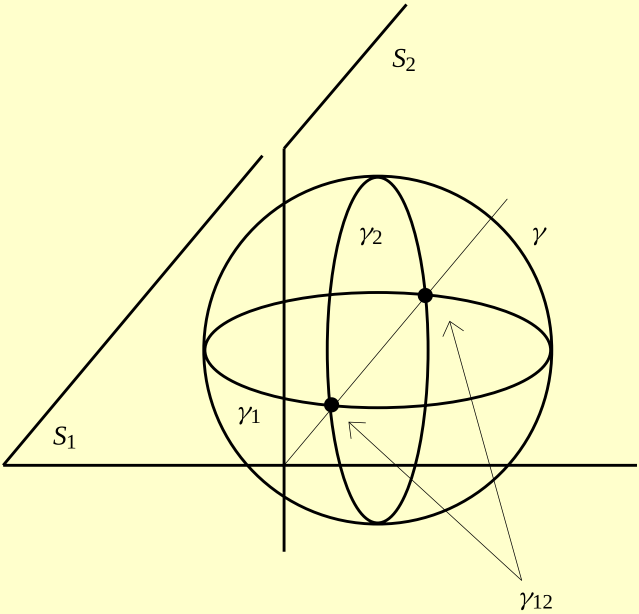

Figure 13. Confluent case

with $\Gamma = [0, +\infty] \times [0, +\infty]$, so that the origin is a critical point of type three. As shown in [40], the asymptotics of $I(k)$ when $k \to +\infty$ depend heavily on the Newton diagram of the singularity f. In the terminology of Kaminski and Paris, we are here in the one-internal-point case, and the point $(1, 2)$ lies behind the back face of the Newton diagram. Following the key idea of Kaminski and Paris, we transform our integral representation into a new Mellin-Barnes-type integral representation

$$I(k) = \frac{k^{-7/12}}{24i\pi} \int_{-i\infty}^{i\infty} \Gamma(t)\Gamma\Big(\frac{1-t}{4}\Big)\Gamma\Big(\frac{1-2t}{3}\Big)k^{-t/12}\,dt, \tag{109}$$

where the path of integration avoids the origin to the right. The asymptotics are now simply obtained by taking into account the right-hand poles of the integrand.* They occur at the points $t = 4j + 1$ and $t = (3j + 1)/2$, with a sequence of double poles at $t = 12j + 5$, $j \in \mathbb{N}$. This yields $J = I_1 + I_2 + I_3$ for the asymptotic expansion,

*The Mellin-Barnes representation can also be used to compute the behaviour of $I(k)$ near the origin: $I(k)$ is analytic on the twelve-fold covering of $\mathbb{C}\backslash\{0\}$, that is, the Riemann surface of $k^{1/12}$.

where

$$I_1 := \frac{k^{-2/3}}{3} \sum_{j\in\mathbb{N},\, j-1\notin 3\mathbb{N}} \frac{(-1)^j}{\Gamma(j+1)} \Gamma(1+4j)\Gamma\Big(-\frac{1+8j}{3}\Big) k^{-j/3},$$

$$I_2 := \frac{k^{-5/8}}{8} \sum_{j\in\mathbb{N},\, j-3\notin 8\mathbb{N}} \frac{(-1)^j}{\Gamma(j+1)} \Gamma(\frac{1+3j}{2})\Gamma\Big(\frac{1-3j}{8}\Big) k^{-j/8},$$

$$I_3 := \frac{k^{-1}}{24} \sum_{j\in\mathbb{N}} \frac{\Gamma(5+12j)}{\Gamma(2+3j)\Gamma(4+8j)} \times \big(\ln(k) - 12\Psi(5+12j) + 8\Psi(4+8j) + 3\Psi(2+3j)\big) k^{-j} \tag{110}$$

(Ψ denotes the digamma function (see [57])). It is easy to check that each of the series expansions in J is Gevrey-1, as expected (see assertion (1) above), but more can be said. While the asymptotics are governed by the local properties of f near the critical point, the reduction to a Mellin-Barnes integral representation proceeds from global information, and, as shown in [40], the asymptotics remain valid in a whole open sectorial neighbourhood at infinity with opening $|\arg k| < 23\pi/2$. We thus deduce (Watson theorem; see [47]) that J is actually Borel resummable for an argument of summation running over $]-11\pi, 11\pi[$. One may guess here that when $|\arg k| = 11\pi$, then the deformed (steepest-descent) chain of integration Γ encounters the other critical point, so that the minor of J has the corresponding negative critical value $-1/19683$ as singularity for its analytic continuations, thus inducing a Stokes phenomenon.

To illustrate assertion (2) above, it is interesting to compare asymptotics (110) to what we would get in the absence of a boundary. The so-called spectral set of r (see [3]) of Theorem 3.1 can be computed in various ways,* but the best references here are certainly the works of A. Varchenko [69], [70]. Taking into account that only the quasi-homogeneous part ${z^{(1)}}^4 + z^{(1)}{z^{(2)}}^2$ (with type $(1/4, 3/8)$ and weight 1) of f plays a role, one can directly apply known results from [3, Section 13, Theorem 5], which gives $\{-3/8, -1/8, 0, 1/8, 3/8\}$ for the spectral set. Moreover, all the s of Theorem 3.1 are less than or equal to $n-1=1$. We thus get $I(k) \sim J(k)$ when $|k| \to +\infty$ with

$$J(k) = \sum_{s=0}^{1} \sum_{r\in\{-3/8,-1/8,0,1/8,3/8\}} T_{r,s}(k) \frac{(\ln k)^s}{k^{r+1}} \tag{111}$$

*It is known, for instance (cf. [46]), that the eigenvalues of the monodromy are $\exp(-2i\pi\lambda)$, where the λ are the zeros of the polynomial function $\widetilde{b}(\lambda)$ related to the Bernstein polynomial $b(t)$ by $b(t) = (t+1)\widetilde{b}(t)$. This polynomial can be computed directly from the Newton diagram (cf. [8]), giving the exponents up to translations.

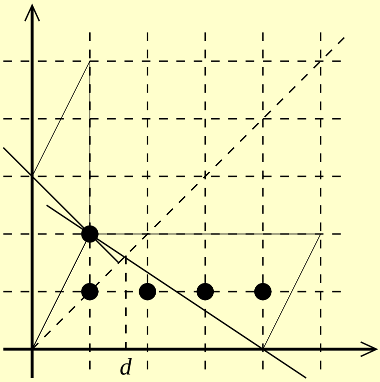

Figure 14. Newton diagram of the singularity. The dots correspond to the cosets of the monomials $1, z^{(2)}, z^{(1)}, {z^{(1)}}^2, {z^{(1)}}^3$ after multiplication by $z^{(1)}z^{(2)}$, which may be chosen as a $\mathbb{C}$-basis for the Milnor algebra $\mathscr{O}/(\partial f)$. The distance to the Newton diagram is $d = 8/5$.

and $T_{r,s}(k) \in \mathbb{C}[[k^{-1}]]$. Comparing now (110) with (111), one can remark that the presence of a boundary has enriched the spectral set; while only the quasi-homogeneous part of the phase function plays a role in the unbounded case, in the bounded case the two faces of the Newton diagram must be taken into account.

7.5. Integrals and differential equations

When unbounded integration contours are considered, integrals (1) with polynomials f and g belong to a class of functions known as "Bernstein functions"; that is, they satisfy a system of holonomic differential equations in $\mathbb{C}[k]\langle\partial_k\rangle$ (cf. [63]). It is beyond our scope to study in detail how this property extends to our integrals with boundaries. We discuss our example here (see Section 6) only from the viewpoint of differential equations. This suggests new links between our results and those developed in [55] or [44] and gives a new insight into the hierarchy property.

The basic integral (84) obviously satisfies

$$\mathscr{L}_0(k, \partial_k)\; I_{\Gamma_0}(k) = 0, \quad \text{with } \mathscr{L}_0(k, \partial_k) = \frac{d}{dk} + \left(11 + \frac{2}{k}\right), \tag{112}$$

while (86) is not only a solution of the second-order differential equation

$$\mathscr{L}_1(k, \partial_k)\mathscr{L}_0(k, \partial_k)\; I_{\Gamma_{(1)}}(k) = 0, \quad \text{with } \mathscr{L}_1(k, \partial_k) = \frac{d}{dk} + \left(39/4 + 3\mathrm{i} + \frac{5}{2k}\right) \tag{113}$$

but also, more precisely, a solution of the inhomogeneous equation

$$\mathscr{L}_0(k, \partial_k)\, I_{\Gamma_{(1)}}(k) = \frac{\mathrm{i}\pi^{3/2}\sqrt{5-12\mathrm{i}}}{k^{5/2}} e^{-(39/4+3\mathrm{i})k}. \tag{114}$$

It is important to note that in (114) the homogeneous part is a *property of the phase function*, while the inhomogeneous part is a *consequence of the boundary*. This inhomogeneity provides a new insight into the hierarchy property as stated in Theorem 4.1. The characteristic equation associated with the irregular singular point at infinity of (113) shows that $5/4 - 3\mathrm{i}$ is the sole (possible) singularity for the Borel transform of $I_{\Gamma_{(1)}}$. Appealing now to resurgence and using the fact that the dotted alien derivative $\dot{\Delta}_\omega$ commutes with the usual differentiation (cf. [23], [12], [16]), (114) yields*

$$\mathscr{L}_0(k, \partial_k)(\dot{\Delta}_{(5/4-3\mathrm{i})}\, I_{\Gamma_{(1)}}) = \dot{\Delta}_{(5/4-3\mathrm{i})} \left(\frac{\mathrm{i}\pi^{3/2}\sqrt{5-12\mathrm{i}}}{k^{5/2}} e^{-(39/4+3\mathrm{i})k}\right) = 0. \tag{115}$$

The inhomogeneous part thus disappears, and one obtains

$$\dot{\Delta}_{(5/4-3\mathrm{i})}\, I_{\Gamma_{(1)}} = \kappa_{(1),()} I_{\Gamma_0}, \tag{116}$$

where $\kappa_{(1),()} \in \mathbb{C}$ is the Stokes multiplier.

More interestingly, (90) satisfies

$$\mathscr{L}_0(k, \partial_k)\, I_{\Gamma_{(12)}}(k) = \frac{\pi}{k^2} e^{-(1+6\mathrm{i})k} \left(-\frac{\mathrm{i}}{k} + \left(\frac{3}{2} - \mathrm{i}\right) \int_0^{+\infty} e^{-kt} dt \frac{1}{\sqrt{35/4 - 3\mathrm{i} - t}} \right), \tag{117}$$

$$\mathscr{L}_1(k, \partial_k)\mathscr{L}_0(k, \partial_k)\, I_{\Gamma_{(12)}}(k) = \pi e^{-(1+6\mathrm{i})k} \left(\frac{1 - 25\mathrm{i}/2}{k^3} + \frac{\mathrm{i}}{2k^2} \right), \tag{118}$$

$$\mathscr{L}_2(k, \partial_k)\mathscr{L}_1(k, \partial_k)\mathscr{L}_0(k, \partial_k)\, I_{\Gamma_{(12)}}(k) = 0, \tag{119}$$

$$\text{with} \quad \mathscr{L}_2(k, \partial_k) = \frac{d}{dk} + \left(1 + 6\mathrm{i} + \frac{3}{k} + \frac{1}{(1-(25+2\mathrm{i})k)k}\right). \tag{120}$$

Starting with (119), a third-order linear differential equation with a singularity of rank one at infinity, we see that $(35/4 - 3\mathrm{i})$ and $(10 - 6\mathrm{i})$ are the two possible adjacent singularities. The one-way adjacency (hierarchy property), which is hidden in (119), clearly appears in the lower-order differential equations (117) and (118). One can analyse each Stokes phenomenon as before. We see, for instance, that the right-hand part of (117) encounters a Stokes phenomenon due to the singularity at $t = 35/4 - 3\mathrm{i}$; replacing the functions by their asymptotic series expansion and applying the alien derivatives $\dot{\Delta}_{(35/4-3\mathrm{i})}$ to (117), we obtain an equation similar to (114).

*With an abuse of notation, in (115) and (116), $I_{\Gamma_{(1)}}$ and I_{Γ_0} stand for their asymptotics.

Appendices

A. The space of homology classes

We detail here the proofs for Section 2. We freely appeal to homology and geometric integration theories. We refer readers unfamiliar with these topics to, for instance, [75], [68], [60], and [14] (see also [37] for a physicist's approach).

We recall that $S_1, S_2, \ldots, S_m$ are $m \leq n$ smooth irreducible affine hypersurfaces.

A.1. Chains of integration and their homology classes

First we describe the space of allowed endless contours of integration running between valleys at infinity where $\Re(kf) \to +\infty$ in the absence of boundaries. Following Pham [62], [63], we introduce the half-planes: for all $c > 0$,

$$S_c^+ = S_c^+(\theta) = \left\{t \in \mathbb{C}, \Re(te^{-i\theta}) \geq c\right\}, \qquad S_c^- = S_c^-(\theta) = \left\{t \in \mathbb{C}, \Re(te^{-i\theta}) \leq c\right\}. \tag{A.1}$$

Let $\Psi = \Psi(\theta)$ be the family of closed subsets $A \in \mathbb{C}^n$ such that, for all $c > 0$, $A \cap f^{-1}(S_c^-)$ is compact. As shown in [63], Ψ is actually a "family of supports" in $\mathbb{C}^n$ in the sense of homology theory, which allows us to define a complex of chains $C_\star^\Psi(\mathbb{C}^n)$ such that, for all nonzero k, the integrals $\int e^{-kf(z)} g(z)\, dz^{(1)} \wedge \cdots \wedge dz^{(n)}$ along the elements of $C_\star^\Psi(\mathbb{C}^n)$ are convergent and satisfy the Stokes theorem. We recall that the function g is assumed to be polynomial, so that the growth of g at infinity remains small compared with the exponential's decay.

In order to deal with chains bounded by $S = S_1 \cup \cdots \cup S_m$, it is natural to introduce the relative chain complex

$$C_\star^\Psi(\mathbb{C}^n, S) := C_\star^\Psi(\mathbb{C}^n)/\mathscr{I}_\star C_\star^\Psi(S), \tag{A.2}$$

where $\mathscr{I} : S \hookrightarrow \mathbb{C}^n$ is the natural mapping. The space of contours of integration $H_\star^\Psi(\mathbb{C}^n, S)$ defined in Section 2.2 is precisely the homology of this complex of chains.

The fact that integrals of type (1) are still convergent along the elements of the complex of chains (A.2) and satisfy the Stokes theorem follows from arguments used in the appendix of [63]. From the fact that the S_i's are the zero level of (irreducible) polynomials P_i, the "semialgebraic" nature of the subchain complex $C_\star^{[\Psi]}$ introduced by Pham is preserved under the mapping $\mathscr{I}$.

Throughout this article, integrals (1) are defined on n-cycles belonging to the homology group $H_n^\Psi(\mathbb{C}^n, S)$.

A.2. Global fibration and homology

We recall that $S_1, S_2, \ldots, S_m$ are assumed to be in general position (see Hypothesis H1), so that the space $\mathbb{C}^n$ has a natural (Whitney) stratification.*

It is also assumed that f and its restriction on each stratum have no singularity at infinity (see Hypothesis H2). When f is concerned, this could mean in practice that it is tame, for instance; that is, there exists $\delta > 0$ such that the set $\{z \in \mathbb{C}^n, \| \operatorname{grad} f(z) \| \leq \delta\}$ is compact, or even $\mathscr{M}$-tame (see [51]). Of course, the sole tameness condition on f does not prevent critical points at infinity for the restricted function $f|$ on the strata (just consider $f(z^{(1)}, z^{(2)}, z^{(3)}) = {z^{(1)}}^2 z^{(2)} - z^{(1)} z^{(3)} + z^{(2)}(z^{(3)} - 1)$ with the boundary $z^{(3)} = 1$), and this has to be checked otherwise.†

Under Hypothesis H2, f realises a topological trivial stratified fibration outside an open ball $B \subset \mathbb{C}^n$ of large enough radius (topological triviality at infinity). The following theorem is then a direct consequence of the Thom-Mather first isotopy lemma (see, e.g., [27], [60], [20]).

THEOREM A.1
We consider S with its natural stratification; then the mapping $\mathbb{C}^n \setminus f^{-1}(\Lambda) \xrightarrow{f|} \mathbb{C} \setminus \Lambda$ is a topological locally trivial stratified fibration.

Thus, for all $t_0 \in \mathbb{C} \setminus \Lambda$, there exists an open neighbourhood $U \subset \mathbb{C} \setminus \Lambda$ of t_0, a stratified set V, and a homeomorphism $\varphi : f^{-1}(U) \to V \times U$ which maps any stratum of $f^{-1}(U)$ onto the product of a stratum of V by U. This result is central to our analysis of the homology through its main consequence, the lifting property.

As a first consequence, if $c > c' > 0$ are large enough that $f^{-1}(S^+_{c'})$ does not contain any element of Λ, then the natural mapping

$$\big(\mathbb{C}^n, f^{-1}(S^+_c)\big) \hookrightarrow \big(\mathbb{C}^n, f^{-1}(S^+_{c'})\big) \tag{A.3}$$

as well as

$$\big(S, S \cap f^{-1}(S^+_c)\big) \hookrightarrow \big(S, S \cap f^{-1}(S^+_{c'})\big) \tag{A.4}$$

can be considered as equivalences of homotopy. Thus the complex of chains $C^\Psi_\star(\mathbb{C}^n)$ (resp., $C^\Psi_\star(S)$) can be identified with the projective limit

$$C^\Psi_\star(\mathbb{C}^n) = \lim_{c \to +\infty} \operatorname{proj} \; C_\star\big(\mathbb{C}^n, f^{-1}(S^+_c)\big),$$

respectively,

$$C^\Psi_\star(S) = \lim_{c \to +\infty} \operatorname{proj} \; C_\star\big(S, S \cap f^{-1}(S^+_c)\big). \tag{A.5}$$

*The strata are $\mathbb{C}^n \setminus S$, $S_i \setminus \bigcup_{j \neq i} S_i \cap S_j$, $S_i \cap S_j \setminus \bigcup_{k \neq i,j} S_i \cap S_j \cap S_k$, and so on (see [20]).

† We have no general practical way of doing this, the gradient tool not being available, as a rule.

We therefore get the following projective limit of isomorphisms:

$$H_n^{\Psi}(\mathbb{C}^n, S) = H_n\big(\mathbb{C}^n; S,\ f^{-1}(S_c^+)\big) = H_n\big(\mathbb{C}^n, S \cup f^{-1}(S_c^+)\big) \tag{A.6}$$

for all c large enough. Now $\mathbb{C}^n$ being contractible (and working with a reduced homology), we finally get the isomorphism

$$H_n^{\Psi}\big(\mathbb{C}^n, S\big) \xrightarrow{\partial} H_{n-1}\big(S \cup f^{-1}(t)\big) \tag{A.7}$$

for $t \in S_c^+$.

In the absence of boundaries and if f has only isolated critical points, it is well known that the (reduced) homology $H_{n-1}(f^{-1}(t))$ (hence $H_n^{\Psi}(\mathbb{C}^n)$) of the generic fiber $f^{-1}(t)$ is a free $\mathbb{Z}$-module of finite rank, its rank μ_f* being the sum of the Milnor numbers at the critical points of f in $\mathbb{C}^n$. In the language of the saddle-point method, this translates to being able to decompose the unbounded chain of integration into a chain of steepest-descent n-folds.

Following (A.7) and from an algebraic viewpoint, analysing $H_n^{\Psi}(\mathbb{C}^n, S)$ reduces to computing the total Milnor number of the (generic) fiber $P = 0$, where P is the polynomial $(f - t)\prod_{i=1}^{n} P_i$ (each S_i being defined by the polynomial P_i). We, however, follow another way for two reasons: (1) our hypotheses (especially Hypothesis H2) do not translate easily in terms of fibration properties of P; (2) having the saddle-point method in mind, it is useful to understand $H_n^{\Psi}(\mathbb{C}^n, S)$ directly in terms of steepest-descent cycles. We do this in Section A.3.

A.3. Localization at the target

Following ideas developed in [63], in the complex plane we draw the family (L_α) of closed half-lines $L_\alpha = f_\alpha + e^{i\theta}\mathbb{R}^+$ for all f_α belonging to Λ. Assume also that θ has been chosen generically so that no Stokes phenomenon is currently occurring; that is, all these half-lines are two-by-two disjoint. For all $f_\alpha \in \Lambda$, let T_α be a closed neighbourhood of L_α, retractable by deformation onto L_α. It is assumed that all these T_α are disjoint from one another, as shown in Figure A.1.

The reduction process hereafter draws heavily on the work of Pham [63], and so the discussion is brief. We start with (A.6), construct a deformation-retraction of $\mathbb{C}$ onto $S_c^+ \cup_{f_\alpha} T_\alpha$, and lift it by f (by virtue of Theorem A.1). This gives

$$H_n\big(\mathbb{C}^n, S \cup f^{-1}(S_c^+)\big) = H_n\Big(\bigcup_{f_\alpha} f^{-1}(T_\alpha \cup S_c^+), \bigcup_{f_\alpha}(f|S)^{-1}(T_\alpha) \cup f^{-1}(S_c^+)\Big). \tag{A.8}$$

*This is the total Milnor number, usually defined as $\mu_f = \dim_{\mathbb{C}}\mathbb{C}[z]/(\partial f)$.

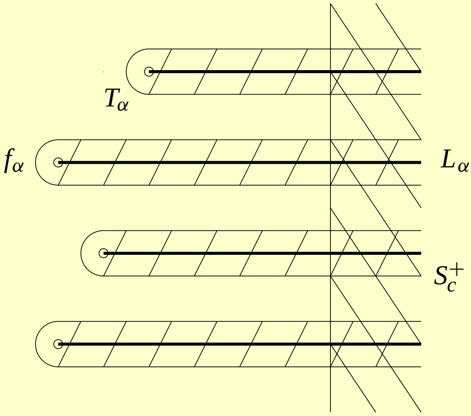

Figure A.1. The family of half-lines L_α and their closed neighbourhoods T_α for $\theta = 0$

By excision and deformation-retraction (using Theorem A.1), we have

$$\begin{aligned} H_n\big(\mathbb{C}^n, S \cup f^{-1}(S_c^+)\big) &= H_n\Big(\bigcup_{f_\alpha} f^{-1}(T_\alpha), \bigcup_{f_\alpha}\big((f|S)^{-1}(T_\alpha) \cup f^{-1}(T_\alpha \cap S_c^+)\big)\Big) \\ &= \bigoplus_{f_\alpha \in \Lambda} H_n\big(f^{-1}(T_\alpha), (f|S)^{-1}(T_\alpha) \cup f^{-1}(T_\alpha \cap S_c^+)\big). \end{aligned} \tag{A.9}$$

For each $f_\alpha \in \Lambda$, let D_α be an open disc centered at f_α with a very small radius r and $D_\alpha^+(\theta) = D_\alpha \cap \{\Re((t - f_\alpha)e^{-i\theta}) \geq r/2\}$ (cf. Figure A.2). Then, again by excision and deformation-retraction,

$$H_n\big(\mathbb{C}^n, S \cup f^{-1}(S_c^+)\big) = \bigoplus_{f_\alpha \in \Lambda} H_n\big(f^{-1}(D_\alpha), (f|S)^{-1}(D_\alpha) \cup f^{-1}(D_\alpha^+(\theta))\big). \tag{A.10}$$

This result shows that the homology group of our integration chains can be decomposed into a direct sum of subgroups by localisation near each of the target critical values. In order to categorise all these subgroups, we must perform a detailed analysis at the source in $\mathbb{C}^n$, under suitable hypotheses about the critical points. This is the purpose of Section A.4, which draws heavily on existing work from many authors (e.g., [60], [72]; see also [4], [5]), but the latter half contains the new results necessary to extend the saddle-point method.

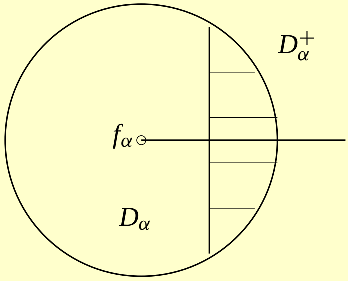

Figure A.2. Localisation near f_α for $\theta = 0$

A.4. Localisation at the source

We have reduced the problem to a local analysis above small open discs centered on the critical values. Up until now, nothing was assumed about the corresponding critical points. It is now time to add Hypotheses H2 and H3.

Note. In what follows, it is assumed that one works with a reduced homology.

A.4.1. First-type case

We localise near an (isolated) first-type singular point z_α of depth $p = 0$; hence $f_\alpha \in \Lambda_{()}$ and z_α does not belong to S.

It is well known since Milnor (see [50]; see also [58], [20]) that one can choose an open ball B_α centered on z_α and with radius ϵ small enough such that the level set $X_\alpha^t = \{z \in B_\alpha,\ f(z) = t\}$ intersects transversely the boundary of B_α for all $t \in D_\alpha$ (for the radius r of D_α such that $\epsilon \gg r > 0$). This means that the restricted function $f \mid f^{-1}(D_\alpha) \cap B_\alpha$ from $f^{-1}(D_\alpha) \cap B_\alpha$ to D_α is a trivial fibration, thus allowing us to analyse the homology group by localisation at the source.

We use the notation $X_\alpha = B_\alpha \cap f^{-1}(D_\alpha)$, $X_\alpha^+ = B_\alpha \cap f^{-1}(D_\alpha^+(\theta))$.

It follows from our hypotheses that

$$H_n\big(f^{-1}(D_\alpha), (f|S)^{-1}(D_\alpha) \cup f^{-1}(D_\alpha^+(\theta))\big) = H_n\big(f^{-1}(D_\alpha),\ f^{-1}(D_\alpha^+(\theta))\big) \tag{A.11}$$

by an easy argument of deformation-retraction. Moreover, from [63] or [3, Section 11], one has the following isomorphisms:

$$H_n\big(f^{-1}(D_\alpha),\ f^{-1}(D_\alpha^+(\theta))\big) = H_n(X_\alpha, X_\alpha^+) \xrightarrow{\partial} H_{n-1}(X_\alpha^t) \tag{A.12}$$

for all $t \in D_\alpha^+(\theta)$.

From Milnor [50] again, we obtain that the fibre X_α^t has the homotopy type of a bouquet (wedge) of μ spheres, where $\mu = \mu_\alpha$ is the Milnor number of the critical point z_α; hence $H_{n-1}(X_\alpha^t) = \mathbb{Z}^\mu$, the so-called vanishing homology of the critical point. When $\mu = 1$, the relative homology $H_n(X_\alpha, X_\alpha^+)$ is generated by the Lefschetz thimble Γ_α^t, as shown in Figure 1.

A.4.2. Critical values from the boundary: Second-type case

We concentrate now on the case where f_α is of the second type, with depth $p > 0$. Up to a reordering, we can assume that f_α belongs to $\Lambda_{(1,\ldots,p)}$.

We introduce as above the sets $X_\alpha = B_\alpha \cap f^{-1}(D_\alpha)$, $X_\alpha^+ = B_\alpha \cap f^{-1}(D_\alpha^+(\theta))$, and $X_\alpha^t = B_\alpha \cap f^{-1}(t)$ for $t \in D_\alpha(\theta)$. We write also $Y_i = B_\alpha \cap (f|S_i)^{-1}(D_\alpha) = X_\alpha \cap S_i$. Note that $Y_i = \emptyset$ for all $i \notin \{1, \ldots, p\}$ as a consequence of our hypothesis.

The restricted function $f \mid f^{-1}(D_\alpha) \cap B_\alpha$ now realises a trivial fibration of the stratified set. Analysing the homology thus reduces to a local analysis near the critical point z_α, and copying arguments in [63, Section I.3], we obtain

$$H_n\big(f^{-1}(D_\alpha), (f|S)^{-1}(D_\alpha) \cup f^{-1}(D_\alpha^+(\theta))\big) = H_n(X_\alpha, X_\alpha^+ \cup Y_1 \cup \cdots \cup Y_p) \quad \text{(A.13)}$$

as a preliminary result.

Following ideas developed in [60, Section 5.2], we now reduce the homology step by step so as to reach the vanishing homology. We first use the fact that X_α and $Y_1 \cup \cdots \cup Y_p$ are contractible;* hence the exact sequence of a triple and a deformation-contraction argument yield the isomorphism

$$\begin{aligned} H_n(X_\alpha, X_\alpha^+ \cup Y_1 \cup \cdots \cup Y_p) &\overset{\partial}{\to} H_{n-1}\big(X_\alpha^t, X_\alpha^t \cap (Y_1 \cup \cdots \cup Y_p)\big), \\ [\Gamma] &\mapsto \partial[\Gamma] \end{aligned} \quad \text{(A.14)}$$

for $t \in D_\alpha^+(\theta)$, where ∂ is the boundary operator that selects the part of the boundary lying on X_α^t. Second, we observe that both X_α^t and $X_\alpha^t \cap (Y_2 \cup \cdots \cup Y_p)$ are contractible, so using the exact sequence of a triple again gives the isomorphism

$$H_{n-1}\big(X_\alpha^t, X_\alpha^t \cap (Y_1 \cup \cdots \cup Y_p)\big) \overset{\partial_1}{\to} H_{n-2}\big(Y_1 \cap X_\alpha^t, Y_1 \cap X_\alpha^t \cap (Y_2 \cup \cdots \cup Y_p)\big), \quad \text{(A.15)}$$

where ∂_1 is the boundary operator that takes the part of the boundary lying on Y_1.

The same argument can be used p times, yielding the following sequence of isomorphisms:

$$\begin{aligned} &H_n(X_\alpha, X_\alpha^+ \cup Y_1 \cup \cdots \cup Y_p) && \text{(A.16)} \\ &\overset{\partial}{\to} H_{n-1}\big(X_\alpha^t, X_\alpha^t \cap (Y_1 \cup \cdots \cup Y_p)\big) \\ &\overset{\partial_1}{\to} H_{n-2}\big(Y_1 \cap X_\alpha^t, Y_1 \cap X_\alpha^t \cap (Y_2 \cup \cdots \cup Y_p)\big) \\ &\overset{\partial_2}{\to} \cdots \overset{\partial_i}{\to} H_{n-i-1}\big(Y_i \cap \cdots \cap Y_1 \cap X_\alpha^t, Y_i \cap \cdots Y_1 \cap X_\alpha^t \cap (Y_{i+1} \cup \cdots \cup Y_p)\big) \\ &\overset{\partial_{i+1}}{\to} \cdots \overset{\partial_{p-1}}{\to} H_{n-p}(Y_{p-1} \cap \cdots \cap Y_1 \cap X_\alpha^t, Y_{p-1} \cap \cdots Y_1 \cap X_\alpha^t \cap Y_p), && \text{(A.17)} \end{aligned}$$

* Just use the local representation (8).

where ∂_i is the boundary operator that takes the part of the boundary lying on Y_i.

Now assuming that $p \leq n-1$ and taking into account that $Y_{p-1} \cap \cdots \cap Y_1 \cap X^t_\alpha$ is still contractible by virtue of Hypothesis H3, it follows from the exact sequence of a pair that

$$H_{n-p}(Y_{p-1} \cap \cdots \cap Y_1 \cap X^t_\alpha, Y_{p-1} \cap \cdots Y_1 \cap X^t_\alpha \cap Y_p) \xrightarrow{\partial_p} H_{n-p-1}(Y_p \cap \cdots \cap Y_1 \cap X^t_\alpha) \tag{A.18}$$

is again an isomorphism. We finally get the isomorphism

$$H_n(X_\alpha, X^+_\alpha \cup Y_1 \cup \cdots \cup Y_p) \xrightarrow{\partial_p \circ \cdots \circ \partial_1 \circ \partial} H_{n-p-1}(Y_p \cap \cdots \cap Y_1 \cap X^t_\alpha). \tag{A.19}$$

Of course, this isomorphism depends on the chosen ordering of $Y_1, \ldots, Y_p$ (cf. [5, Chapter 4, Section 1.14]).

The homology group $H_{n-p-1}(Y_p \cap \cdots \cap Y_1 \cap X^t_\alpha) = H_{n-p-1}(S_p \cap \cdots \cap S_1 \cap X^t_\alpha)$ is the vanishing homology of the critical point; it is isomorphic to the free $\mathbb{Z}$-module $\mathbb{Z}^{\mu_\alpha}$, where μ_α is the Milnor number of the critical point z_α of $f_{(1,\ldots,p)}$. When $\mu_\alpha = 1$, the relative homology $H_n(X_\alpha, X^+_\alpha \cup Y_1 \cup \cdots \cup Y_p)$ is generated by the relative Lefschetz thimble Γ^t_α, as described in Figure 2.

The case of a corner, where $p = n$, corresponds to the so-called linear pinching case (see [60]) where $\partial_p \circ \cdots \circ \check{\partial_i} \circ \cdots \partial_1 \circ \partial[\Gamma]$ is just reduced to a point: no vanishing cycle exists, and the local homology $H_n(X_\alpha, X^+_\alpha \cup Y_1 \cup \cdots \cup Y_p)$ is generated by a single relative Lefschetz thimble (see Figure 3).

A.4.3. Concluding theorem

Putting the pieces together, we have shown how the homology group $H^\Psi_n(\mathbb{C}^n, S)$ of the allowed contours of integration in the presence of boundaries can be decomposed into a direct sum (see Section A.3) of free $\mathbb{Z}$-modules of finite rank (see Section A.4), at least under Hypothesis H5 (no critical value of the third type). We have also demonstrated a natural way to define the rank by reduction to the vanishing homology (if defined) through localisation on each stratum. Moreover, when the closed half-lines $L_\alpha = f_\alpha + e^{i\theta}\mathbb{R}^+$ ($f_\alpha \in \Lambda$) are two-by-two disjoint (θ is “generic in the Stokes sense”), then a basis is given by the set of (relative) steepest-descent n-folds $(\Gamma_\alpha)_{\alpha\in\Lambda}$, where each of the Γ_α projects by f onto $L_\alpha \in \mathbb{C}$. We have thus obtained the following theorem.

THEOREM A.2

The space $H^\Psi_n(\mathbb{C}^n, S)$ of relative homology classes is a free $\mathbb{Z}$-module of finite rank. Moreover, if all the half-lines L_α are two-by-two disjoint, then every cycle can be decomposed into a chain of (relative) steepest-descent n-folds $(\Gamma_\alpha)_{\alpha\in\Lambda}$ of $H^\Psi_n(\mathbb{C}^n, S)$.

B. Duality and Stokes phenomenon

The aim of this appendix is to provide the geometrical tools to understand the Stokes phenomenon, as discussed in Section 4. We discuss the now more or less classical Picard-Lefschetz formulae, which help in understanding the Stokes phenomena as singularities in the Borel plane. Then we see how a suitable duality provides a direct insight into the Stokes phenomenon, in the spirit of the usual saddle-point method.

We here make use of Hypothesis H6, namely, that all singular points are nondegenerate, with the following comment.

It is possible to extend the results hereafter by allowing degenerate critical points. The generalised Picard-Lefschetz formulae, as well as the nondegeneracy properties of the Kronecker index (see B.6), follow from a local analysis near the critical points, and this can be studied in the degenerate case by local generic deformations.* However, the information we would obtain does not translate easily into the language of resurgence of asymptotic expansions.

B.1. Generalised Picard-Lefschetz

We assume that θ has been chosen generically so that no Stokes phenomenon is occurring. We denote by $(\Gamma_\alpha)_{f_\alpha}$ a basis of relative steepest-descent n-cycles (or relative Lefschetz thimbles) of $H_n^{\Psi(\theta)}(\mathbb{C}^n, S)$ considered in Section 2 (with a given orientation). We introduce

$$\gamma_\alpha(t) = \partial\,\Gamma_\alpha(t) \;\in H_{n-1}\big(X_\alpha^t, X_\alpha^t \cap (S_1 \cup \cdots \cup S_m)\big) \tag{B.1}$$

as well as its corresponding vanishing cycle. If f_α belongs to $\Lambda_{(i_1,\ldots,i_q)}$, then

$$e_\alpha(t) = \partial_{i_q} \circ \cdots \circ \partial_{i_1}\,\gamma_\alpha(t) \;\in H_{n-q-1}(S_{i_q} \cap \cdots \cap S_{i_1} \cap X_\alpha^t), \tag{B.2}$$

which vanishes when $t \to f_\alpha$ along $L_\alpha(\theta)$.

Let $t^\star$ be a fixed regular point in the half-complex plane $S_c^+(\theta)$ for $c > 0$ large enough. We denote by (l_α) a system of paths, where l_α starts from $t^\star$ and travels to f_α along a straight line. This system (l_α) is a so-called distinguished system of paths (see [3]), where the cycle $e_\alpha(t)$ vanishes when $t \to f_\alpha$ along l_α. For each l_α we associate a closed path ℓ_α starting from $t^\star$, following l_α, running around f_α in the positive sense, and returning to $t^\star$ along l_α (see Figure B.1). This defines a basis $(\ell_\alpha)_{f_\alpha}$ of the free fundamental group $\pi_1(\mathbb{C}\backslash\Lambda, t^\star)$. Hence the variation of the homology when t runs along a loop in $\mathbb{C}\backslash\Lambda$ with $t^\star$ as base point reduces to a description of what happens for each of the ℓ_α.

We now apply the generalised Picard-Lefschetz formula described in [60] (see also [72], [5]). Starting with the cycle $\gamma_\alpha(t)$, we follow its deformation when t runs

*Such local Morsification does not bring into play any global deformation of f, which may destroy Hypothesis H2 (cf. [13]).

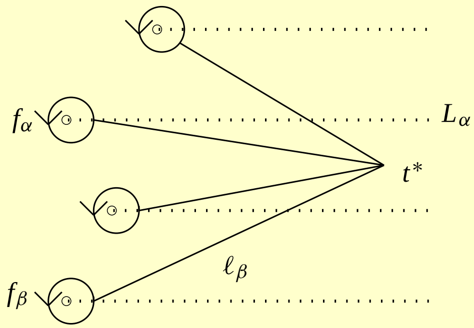

Figure B.1. The basis $(\ell_\alpha)_{f_\alpha}$

along the path l_β. We consider the trace $\gamma_\alpha^\star$ of $\gamma_\alpha(t)$ in the Milnor ball B_β as belonging to $H^F_{n-1}(X^t_\beta \backslash X^t_\beta \cap (S_1 \cup \cdots \cup S_m))$, the closed homology group dual from $H_{n-1}(X^t_\beta, X^t_\beta \cap (S_1 \cup \cdots \cup S_m))$ by the Poincaré duality (covanishing homology). We can now define the variation operator Var when t goes around f_β along ℓ_β. Assuming that f_β belongs to $\Lambda_{(1,\ldots,p)}$, [60] now yields

$$\begin{cases} \operatorname{Var} \gamma_\alpha^\star = \kappa_{\alpha\beta}\, \gamma_\beta, \\ \kappa_{\alpha\beta} = (-1)^{(n-p)(n-p+1)/2} \langle e_\beta, \epsilon_\alpha \rangle, \end{cases} \tag{B.3}$$

where $\langle\ ,\ \rangle$ is the (Kronecker) index of intersection. The cycle $\epsilon_\alpha \in H_{n-p-1}(S_1 \cap \cdots \cap S_p \cap X^t_\beta)$ is deduced from $\gamma_\alpha^\star$ by

$$\gamma_\alpha^\star = \delta_1 \circ \cdots \circ \delta_p\, \epsilon_\alpha, \tag{B.4}$$

where δ_i denotes the Leray coboundary operator (cf. [42]) with respect to S_i. Moreover, the index of self-intersection is given by

$$\langle e_\beta, e_\beta \rangle = \begin{cases} 2(-1)^{(n-p)(n-p-1)/2} & \text{if } n-p \text{ is odd}, \\ 0 & \text{if } n-p \text{ is even}. \end{cases} \tag{B.5}$$

Remarks. When $m = n$ (corner critical point) and f_β belongs to $\Lambda_{(1,\ldots,n)}$, then the vanishing cycle e_β does not exist. We can give a meaning to the previous equations with the convention $e_\beta = 0$; hence $\langle e_\beta, \epsilon_\alpha \rangle - = 0$.

One can remark also that if f_α belongs to $\Lambda_{(i_1,\ldots,i_q)}$, equalities (B.3) and (B.4) show that when $\{1, \ldots, p\}$ is not a subset of the set $\{i_1, \ldots, i_q\}$, then $\gamma_\alpha^\star$ has a trivial variation around f_β (i.e., $\kappa_{\alpha\beta} = 0$).

B.2. Duality from the viewpoint of Laplace integrals

The generalised Picard-Lefschetz formulae recalled in Section B.1 follow from the duality between vanishing homology and covanishing homology. Here we realise this duality directly from the viewpoint of Laplace integrals by describing the dual space of the homology group $H_n^{\Psi}(\mathbb{C}^n, S)$ with which we started.

We consider a relative n-cycle $\Gamma \in H_n^{\Psi(\theta)}(\mathbb{C}^n, S)$ with support in $\Psi(\theta)$ and an n-cycle $\Gamma' \in H_n^{\Psi(\theta+\pi)}(\mathbb{C}^n \backslash S)$ with support in $\Psi(\theta+\pi)$. Up to a deformation by isotopy, these two cycles can be assumed to be in a general position. It follows now by definition that the intersection $\Psi(\theta) \cap \Psi(\theta+\pi)$ of the families of supports $\Psi(\theta)$ and $\Psi(\theta+\pi)$ is the family of compact subsets of $\mathbb{C}^n$. This implies that the intersection in $\mathbb{C}^n$ of Γ with Γ' defines a 0-cycle with compact support and hence an integer because $H_0(\mathbb{C}^n) = \mathbb{Z}$. This allows us to define the index of intersection $\langle \Gamma', \Gamma \rangle$ by the bilinear map

$$\langle\, ,\, \rangle : H_n^{\Psi(\theta+\pi)}(\mathbb{C}^n \backslash S) \otimes H_n^{\Psi(\theta)}(\mathbb{C}^n, S) \to \mathbb{Z}. \tag{B.6}$$

This is a direct generalisation of the Kronecker index introduced in [63]. Moreover, arguments based on [63] show the following lemma.

LEMMA B.1
The bilinear map (B.6) is nondegenerate.

This induces a duality between $H_n^{\Psi(\theta)}(\mathbb{C}^n, S)$ and $H_n^{\Psi(\theta+\pi)}(\mathbb{C}^n \backslash S)$.

We recall here that θ has been chosen generically so that the lines $L_\alpha(\theta) \cup L_\alpha(\theta+\pi)$ are disjoint. We first remark that our bilinear map is diagonal for the decomposition of the homology shown in Appendix A. This allows us to localise the study. Assuming that f_α belongs to $\Lambda_{(1,\ldots,q)}$, we focus on $H_n(X_\alpha, S_q \cup \cdots \cup S_1 \cup X_\alpha^+)$. Following Section A.4.2, this group is isomorphic to $H_{n-q}(S_q \cap \cdots \cap S_1 \cap X_\alpha, S_q \cap \cdots \cap S_1 \cap X_\alpha^+)$ via the isomorphism $\partial_q \circ \cdots \circ \partial_1$. Returning to the notation of Section A.4.2, we now define

$$D_\alpha^- = D_\alpha^+(\theta+\pi) = D_\alpha \cap \left\{\Re\left((t - f_\alpha)e^{-i(\theta+\pi)}\right) \geq r/2\right\} \tag{B.7}$$

and, respectively, X_α^-. From [63, Section I.5] (see also the remark hereafter), we know that the spaces of homology $H_{n-q}(S_q \cap \cdots \cap S_1 \cap X_\alpha, S_q \cap \cdots \cap S_1 \cap X_\alpha^+)$ and $H_{n-q}(S_q \cap \cdots \cap S_1 \cap X_\alpha, S_q \cap \cdots \cap S_1 \cap X_\alpha^-)$ are dual.

As a consequence, each n-cycle $\Gamma \in H_n^{\Psi(\theta)}(\mathbb{C}^n, S)$ can be decomposed as

$$\Gamma = \sum_{f_\alpha} \langle \Gamma_\alpha^\star, \Gamma \rangle \Gamma_\alpha \tag{B.8}$$

with respect to the basis $(\Gamma_\alpha)_{f_\alpha}$ of (oriented) relative Lefschetz thimbles, where $(\Gamma_\alpha^\star)_{f_\alpha}$ is the dual basis (hence $\langle \Gamma_\alpha^\star, \Gamma_\alpha \rangle = +1$). Figure B.2 describes this duality for the Airy pattern.

Remark. A "concrete dual basis" can be built as follows. We start with a basis of relative Lefschetz thimbles $(\Gamma_\alpha)_{f_\alpha}$ of $H_n^{\Psi(\theta)}(\mathbb{C}^n, S)$ with the standard orientation. Assuming that f_α belongs to $\Lambda_{(1,\dots,p)}$, we introduce

$$\chi_\alpha = \partial_p \circ \cdots \circ \partial_1 \Gamma_\alpha \in H_{n-p}(S_p \cap \cdots \cap S_1 \cap X_\alpha, S_p \cap \cdots \cap S_1 \cap X_\alpha^+). \quad \text{(B.9)}$$

We define $\chi_\alpha^\star \in H_{n-p}(S_p \cap \cdots \cap S_1 \cap X_\alpha, S_p \cap \cdots \cap S_1 \cap X_\alpha^-)$ as this cycle deduced from χ_α by rotating the direction from $e^{i\theta}$ to $e^{i(\theta+\pi)}$ on $\mathbb{S}$ in the positive sense; then χ_α (resp., $\chi_\alpha^\star$) can be identified with the ascent (resp., descent) gradient surfaces of a Morse function $F = x^{(p+1)^2} + \cdots + x^{(n)^2} - y^{(p+1)^2} - \cdots - y^{(n)^2}$ of index $n-p$ in $\mathbb{R}^{2(n-p)} = \mathbb{C}^{(n-p)}$. We compare the orientation $(\partial/\partial x^{(p+1)}, \dots, \partial/\partial x^{(n)}, \partial/\partial y^{(p+1)}, \dots, \partial/\partial y^{(n)})$ with the canonical orientation $(\partial/\partial x^{(p+1)}, \partial/\partial y^{(p+1)}, \dots, \partial/\partial x^{(n)}, \partial/\partial y^{(n)})$ of $\mathbb{C}^{(n-p)}$. This yields

$$\langle \chi_\alpha^\star, \chi_\alpha \rangle = (-1)^{(n-p)(n-p-1)/2}. \quad \text{(B.10)}$$

Using now the Leray coboundary isomorphisms, we can define

$$\Gamma_\alpha^\star = \delta_1 \circ \cdots \circ \delta_p \, \chi_\alpha^\star, \quad \text{(B.11)}$$

which extends as an element of $H_n^{\Psi(\theta+\pi)}(\mathbb{C}^n \backslash S)$. Using (B.10), we thus obtain a basis $(\Gamma_\alpha^\star)_{f_\alpha}$ of n-cycles of $H_n^{\Psi(\theta+\pi)}(\mathbb{C}^n \backslash S)$ dual to $(\Gamma_\alpha)_{f_\alpha}$.

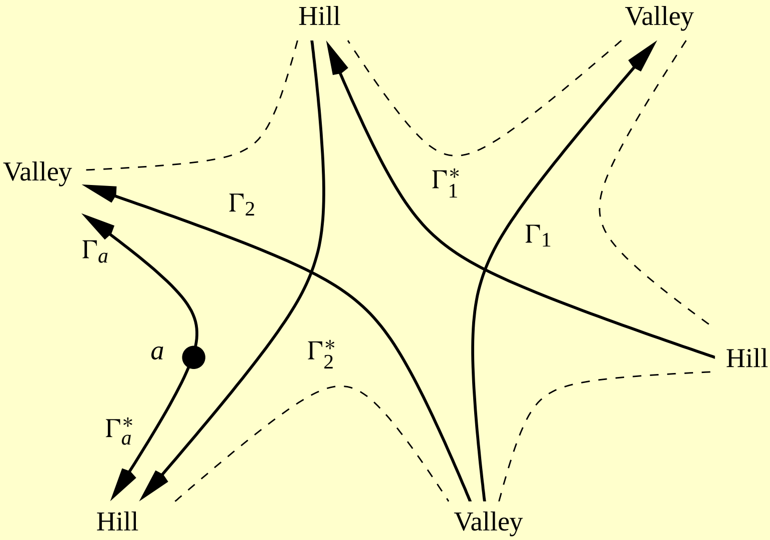

Figure B.2. The 1-cycles $(\Gamma_1, \Gamma_{-1}, \Gamma_a)$ as a basis of $H_1^\Psi(\mathbb{C}, a)$ and its dual basis $(\Gamma_1^\star, \Gamma_{-1}^\star, \Gamma_a^\star)$ for the Airy pattern

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