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GLOBAL ATTRACTORS FOR THE SUSPENSION BRIDGE EQUATIONS WITH NONLINEAR DAMPING

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Abstract. In this paper, we prove the existence of a global attractor for the suspension bridge equations with nonlinear damping.

1. Introduction. In this paper, we consider the asymptotic behavior of the solutions of the following initial-boundary value problem:

$$\begin{cases} u_{tt} + \Delta^2 u + ku^+ + a(x)g(u_t) + f(u) = h(x), & \text{in } \Omega \times \mathbb{R}^+, \\ u = \Delta u = 0, \text{ on } \Gamma, \\ u(\tau, x) = u_0(x), \ u_t(\tau, x) = u_1(x), \text{ in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^2$ with a smooth boundary Γ and $\tau \in \mathbb{R}^+$. $u(x,t)$ is an unknown function, which represents the deflection of the road bed in the vertical plane. k denotes the spring constant and $h(x) \in L^2(\Omega)$. The function

$$u^+ = \begin{cases} u, \text{ if } u > 0, \\ 0, \text{ if } u \le 0. \end{cases}$$

The function $a(x)$ satisfies:

$$a(x) \in L^\infty(\Omega), \ a(x) \ge \alpha_0 > 0 \ \text{ in } \Omega, \tag{1.2}$$

where α_0 is a constant.

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The nonlinear function $f \in C^2(\mathbb{R})$ satisfies the following assumptions: There exists a constant $C_1 > 0$ such that

$$\liminf_{|s|\to\infty} \frac{F(s)}{s^2} \geq 0, \quad F(s) = \int_0^s f(\tau)d\tau, \tag{1.3}$$

$$\limsup_{|s|\to\infty} \frac{|f'(s)|}{|s|^\gamma} = 0, \quad \text{where} \quad 0 \leq \gamma < \infty, \tag{1.4}$$

$$\liminf_{|s|\to\infty} \frac{sf(s) - C_1F(s)}{s^2} \geq 0. \tag{1.5}$$

The damping function $g \in C^1(\mathbb{R})$ satisfies

$$g(0) = 0, \ g \text{ strictly increasing, and } \liminf_{|s|\to\infty} g'(s) > 0, \tag{1.6}$$

$$|g(s)| \leq C_2(1 + |s|^q), \tag{1.7}$$

with $1 \leq q < \infty$.

The suspension bridge equations were presented by Lazer and McKenna as new problems in the field of nonlinear analysis [9]. For the problem corresponding to (1.1) without nonlinear damping $a(x)g(u_t)$, there are many classical results. We refer the reader to [1], [2], [9], [10], [11], [12], [13], [14], [16] and the references therein. An [2] obtained the existence and uniqueness of a weak solution for $k > -1$ and then showed decay estimates of the solution for the suspension problem. Ma and Zhong [10] investigated the existence of global attractors of the coupled system of a suspension bridge in the space $H_0^2(\Omega) \times L^2(\Omega)$. Zhong et al. [16] showed the existence of strong solutions and global attractors for the suspension bridge equations in the stronger space.

The long-time behavior of solutions for the equation with nonlinear damping has attracted much attention in recent years; we refer the reader to [3], [4], [5], [6], [7], [8], [15]. Chueshov and Lasiecka [4] studied the existence of weak attractors for von Karman equations with nonlinear dissipation. In that article the authors have proved the existence of a global attractor for large values of the damping parameter. Khanmamedov [5], [6] proved the global attractors for von Karman equations with nonlinear interior dissipation. Recently, Yang and Zhong [15] studied the existence of a global attractor for the plate equation without assuming large values for the damping parameter.

Motivated by the work in [16], we study the existence of the global attractor for suspension bridge equations with nonlinear damping. We use the methods provided by Yang and Zhong [15] to show the existence of the global attractor.

With the usual notation, we introduce the spaces $H = L^2(\Omega)$, $V = H_0^2(\Omega)$, and endow these spaces with the usual scalar products and norms, $(\cdot,\cdot), |\cdot|, ((\cdot,\cdot)), ||\cdot||$, where

$$(u,v) = \int_\Omega u(x)v(x)dx, \quad ((u,v)) = \int_\Omega \Delta u(x)\Delta v(x)dx.$$

From the Poincaré inequality, there exists a proper constant $C_\lambda > 0$ such that

$$||u||^2 \geq C_\lambda |u|^2, \ \forall u \in V. \tag{1.8}$$

The notation used in this paper is standard. The organization of this paper is as follows. In Section 2, we give some notation and prove some lemmas in order to show asymptotic compactness of $S(t)$. In Section 3, we establish the existence of a bounded absorbing set in $V \times H$. In Section 4, we prove the existence of a global attractor in $V \times H$.

2. Preliminaries and abstract results.

In this section, we recall some definitions and results concerning the attractor. It is known that under conditions (1.2)-(1.7) the solution operator $S(t) = (u(t), u_t(t)),\ t \geq 0$, of problems (1.1) generates a (C_0) semigroup on the energy space $V \times H$ (see [10], [15], [16]).

THEOREM 2.1 ([10], [15]). Let Ω be a bounded domain of $\mathbb{R}^2$ with smooth boundary, under assumptions (1.2)-(1.7). Then for any initial data $(u_0, u_1) \in V \times H$, problems (1.1) have a unique global solution $(u(t), u_t(t)) \in C([0,T]; V \times H)$ for any $T > 0$, and $(u(t), u_t(t))$ depends continuously on (u_0, u_1).

Next, we recall the simple compactness criterion stated as [3], [4], [15].

DEFINITION 2.1 ([4], [15]). Let X be a Banach space and B be a bounded subset of X. We call a function $\phi(\cdot, \cdot)$ which is defined on $X \times X$ a contractive function on $B \times B$ if for any sequence $\{x_n\}_{n=1}^{\infty} \subset B$, there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ such that

$$\lim_{k \to \infty} \lim_{l \to \infty} \phi(x_{n_k}, x_{n_l}) = 0.$$

Denote all such contractive functions on $B \times B$ by $C(B)$.

THEOREM 2.2 ([4], [15]). Let $\{S(t)\}_{t \geq 0}$ be a semigroup on a Banach space $(X, ||\cdot||)$ that has a bounded absorbing set B_0. Moreover, assume that for any $\epsilon \geq 0$ there exist $T = T(B_0, \epsilon)$ and $\phi_T(\cdot, \cdot) \in C(B)$ such that

$$||S(T)x - S(T)y|| \leq \epsilon + \phi_T(x, y) \quad \text{for all } x, y \in B_0,$$

where ϕ_T depends on T. Then $\{S(t)\}_{t \geq 0}$ is asymptotically compact in X; i.e., for any bounded sequence $\{y_n\}_{n=1}^{\infty} \subset X$ and $\{t_n\}$ with $t_n \to \infty$, $\{S(t_n)y_n\}_{n=1}^{\infty}$ is precompact in X.

LEMMA 2.1 ([5]). Let $g(\cdot)$ satisfy condition (1.6). Then for any $\delta > 0$ there exists $c(\delta) > 0$ such that

$$|u - v|^2 \leq \delta + c(\delta)(g(u) - g(v))(u - v) \text{ for } u, v \in \mathbb{R}. \tag{2.1}$$

3. Absorbing set in $V \times H$.

In this section, we prove the existence of the bounded absorbing set in $V \times H$.

LEMMA 3.1. Under assumptions (1.2)-(1.7), the semigroups $\{S(t)\}_{t \geq 0}$ corresponding to problems (1.1) have a bounded absorbing set in $V \times H$.

Proof. We set

$$E(t) = \frac{1}{2}(|u_t|^2 + ||u||^2 + k|u^+|^2) + \int_{\Omega} (F(u) - hu)dx.$$

Multiplying (1.1) by u_t and integrating over Ω, we get

$$\frac{d}{dt}E(t) + \int_\Omega a(x)g(u_t)u_t dx = 0, \tag{3.1}$$

so from (1.2) and (1.6) we have

$$E(t) \le E(0), \quad \forall\, t \ge 0. \tag{3.2}$$

It is obvious that (1.2) and (1.6) imply that there are $\delta > 0$ and $C_\delta > 0$ such that

$$\begin{aligned} &(a(x)g(u_t), u_t) \ge 2\delta|u_t|^2 - C_\delta \text{meas}(\Omega), \\ &(a(x)g(u_t) - \delta u_t, u_t) \ge \delta|u_t|^2 - C_\delta \text{meas}(\Omega), \end{aligned} \tag{3.3}$$

and from (1.3) and (1.5) we know that there are $C_\lambda > C'_\lambda > 0$ and $C_0 = C_1 - 2(C'_\lambda)^{-1}$ such that

$$(f(u), u) - C_1 \int_\Omega F(u)dx \ge -\frac{C'_\lambda}{4}|u|^2 - C_0 \text{meas}(\Omega), \tag{3.4}$$

$$\int_\Omega F(u)dx > -\frac{C'_\lambda}{8}|u|^2 - C_0 \text{meas}(\Omega), \tag{3.5}$$

for any $u \in V$. From (1.8), (3.2) and (3.5), we obtain

$$\begin{aligned} -C_1(\text{meas}(\Omega) + |h|^2) &\le -C_1(\text{meas}(\Omega) + |h|^2) + \frac{1}{2}(|u_t|^2 + k|u^+|^2) + \frac{1}{4}||u||^2 \\ &\le E(t) \le E(0). \end{aligned} \tag{3.6}$$

So from (3.1) and (3.6), we have

$$\int_0^t \int_\Omega a(x)g(u_t)u_t dx ds \le E(0) - E(t) \le E(0) + C_1(\text{meas}(\Omega) + |h|^2), \ \forall\, t \ge 0. \tag{3.7}$$

By assumptions (1.6) and (1.7), we have

$$|g(s)|^{\frac{q+1}{q}} = |g(s)|^{\frac{1}{q}} \cdot |g(s)| \le C(1 + |s|)|g(s)| \le \begin{cases} C, & |s| \le 1, \\ 2Cg(s)s, & |s| \ge 1, \end{cases} \tag{3.8}$$

where C is a constant which is independent of s. Then from (1.2), (3.8), using Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}
&|\int_\Omega a(x)g(u_t)udx| \\
&\quad\le \int_{\Omega(|u_t|\le 1)} |a(x)g(u_t)u|dx + \int_{\Omega(|u_t|\ge 1)} |a(x)g(u_t)u|dx \\
&\quad\le \int_{\Omega(|u_t|\le 1)} g(1)|a(x)u|dx \\
&\qquad + (\int_{\Omega(|u_t|\ge 1)} a(x)|g(u_t)|^{\frac{q+1}{q}}dx)^{\frac{q}{q+1}}(\int_{\Omega(|u_t|\ge 1)} a(x)|u|^{q+1}dx)^{\frac{1}{q+1}} \\
&\quad\le \int_{\Omega(|u_t|\le 1)} C|a(x)u|dx \\
&\qquad + (\int_{\Omega(|u_t|\ge 1)} 2Ca(x)g(u_t)u_tdx)^{\frac{q}{q+1}}(\int_{\Omega(|u_t|\ge 1)} a(x)|u|^{q+1}dx)^{\frac{1}{q+1}} \\
&\quad\le C\int_\Omega a(x)|u|dx + C_\eta||u||^{\frac{q-1}{q}}\int_\Omega a(x)g(u_t)u_tdx + \eta||u||^2, \qquad (3.9)
\end{aligned}$$

where η is a constant. We set $v = u_t + \delta u$ and rewrite the equation of (1.1) as follows:

$$v_t + \Delta^2 u + ku^+ - \delta u_t + a(x)g(u_t) + f(u) = h. \tag{3.10}$$

Taking the scalar product in $L^2(\Omega)$ of (3.10) with v and integrating over Ω, where δ comes from (3.3), we obtain

$$\begin{aligned}
&\frac{d}{dt}\big(\frac{1}{2}(|v|^2 + ||u||^2 + k|u^+|^2 - \delta^2|u|^2) + \int_\Omega (F(u) - hu)dx\big) + \delta||u||^2 + \delta k|u^+|^2 \\
&+(a(x)g(u_t) - \delta u_t, u_t) + \delta(a(x)g(u_t), u) + \delta(f(u), u) = \delta(h, u).
\end{aligned}$$

Set

$$E_\delta(t) = \frac{1}{2}|v|^2 + \frac{k}{2}|u^+|^2 - \frac{\delta^2}{2}|u|^2 + \frac{1}{2}||u||^2 + \int_\Omega (F(u) - hu)dx$$

and

$$\begin{aligned}
H(t) \quad = \quad & (a(x)g(u_t) - \delta u_t, u_t) + \delta(a(x)g(u_t), u) + \delta||u||^2 + \delta k|u^+|^2 \\
& +\delta(f(u), u) - \delta(h, u),
\end{aligned}$$

so we have

$$\frac{d}{dt}E_\delta(t) + H(t) = 0. \tag{3.11}$$

Now, using (3.5) and Young's inequality, we can choose δ small enough such that

$$E_\delta(t) \ge \frac{1}{2}|v|^2 + \frac{k}{2}|u^+|^2 + \frac{1}{4}||u||^2 - C_1(\text{meas}(\Omega) + |h|^2). \tag{3.12}$$

Similarly, from (1.8), (3.3)-(3.5) and (3.9), we get

$$\begin{aligned} H(t) \quad \geq \quad & \delta|u_t|^2 - C_\delta \text{meas}(\Omega) + \delta k|u^+|^2 + \delta||u||^2 \\ & -\delta(C\int_\Omega a(x)|u|dx + C_\eta||u||^{\frac{q-1}{q}}\int_\Omega a(x)g(u_t)u_t dx + \eta||u||^2) \\ & -\delta(\frac{C_1}{2}+1)\frac{C'_\lambda}{4}|u|^2 - \delta C_0\text{meas}(\Omega)(C_1+1) - \frac{\delta C'_\lambda}{4}|u|^2 - \frac{\delta}{C'_\lambda}|h|^2 \\ \geq \quad & \delta|u_t|^2 - C'_\delta(\text{meas}(\Omega)+|h|^2) + \delta k|u^+|^2 + \delta(\frac{1}{2} - \frac{C_1}{8} - \eta)||u||^2 \\ & -\delta(C\int_\Omega a(x)|u|dx + C_\eta||u||^{\frac{q-1}{q}}\int_\Omega a(x)g(u_t)u_t dx) \\ \geq \quad & C''_\delta(|u_t|^2 + k|u^+|^2 + ||u||^2) - C'_\delta(\text{meas}(\Omega)+|h|^2) \\ & -C_{E(0)}\int_\Omega a(x)g(u_t)u_t dx, \end{aligned} \tag{3.13}$$

where we can choose η so small that $\frac{1}{2} - \frac{C_1}{8} - \eta > 0$; $C_{E(0)}$ is a constant which depends on δ, C_η and $E(0)$; C''_δ and C'_δ are constants depending on δ and C_1. Note that

$$\begin{aligned} |u_t|^2 + ||u||^2 \quad &= \quad |u_t + \delta u - \delta u|^2 + ||u||^2 \\ &\leq \quad 2|v|^2 + (\frac{2\delta^2}{C_\lambda}+1)||u||^2 \leq c_0(|v|^2 + ||u||^2), \end{aligned} \tag{3.14}$$

where $c_0 = \max\{2, 1+2\delta^2 C_\lambda^{-1}\}$. Integrating (3.11), combining with (3.7), (3.12)-(3.14), we deduce

$$\begin{aligned} & |u_t|^2 + k|u^+|^2 + ||u||^2 - 2c_0C_1(\text{meas}(\Omega)+|h|^2) - 2c_0E_\delta(0) \\ & -2c_0C_{E(0)}(E(0) + C_1(\text{meas}(\Omega)+|h|^2)) \\ \leq \quad & -2c_0\int_0^t \big(C''_\delta(|u_t(s)|^2 + k|u^+(s)|^2 + ||u(s)||^2) - C'_\delta(\text{meas}(\Omega)+|h|^2)\big)ds. \end{aligned} \tag{3.15}$$

Therefore, for any $\rho > \frac{C'_\delta(\text{meas}(\Omega)+|h|^2)}{C''_\delta}$, there exists t_0 such that

$$||u(t_0)||^2 + |u_t(t_0)|^2 + k|u^+(t_0)|^2 \leq \rho.$$

Set

$$B_0 = \{(u_0, v_0) \in V \times H|\ ||u_0||^2 + |v_0|^2 + |u_0^+|^2 \leq \rho\},$$

so we have that B_0 is a bounded absorbing set. Define

$$B_1 = \bigcup_{t\geq 0} S(t)B_0;$$

therefore, B_1 is also a bounded absorbing set. □

4. Existence of global attractor in $V \times H$. In this section, we will first give some a priori estimates about the energy inequalities on account of the idea presented in [3], [4], [5], [15]. Then we use Theorem 4.1 to establish the asymptotic compactness in $V \times H$.

For convenience, we always denote by B_1 a bounded absorbing set obtained in Lemma 3.1. We will use the following notation:

$$E_w(t) = \frac{1}{2}|w_t(t)|^2 + \frac{k}{2}|w(t)^+|^2 + \frac{1}{2}||w(t)||^2.$$

4.1. A priori estimates

To obtain the asymptotic compactness, we establish a priori estimates. The following process is derived from the standard energy method given in [3], [4], [5], [15].

Let $(u_i(t), u_{i_t}(t))$ $(i = 1, 2)$ be the corresponding solution to $(u_0^i, v_0^i) \in B_1$ and let $w(t) = u_1(t) - u_2(t)$. Then $w(t)$ satisfies

$$w_{tt} + a(x)(g(u_{1_t}) - g(u_{2_t})) + \Delta^2 w + kw^+ + f(u_1) - f(u_2) = 0, \tag{4.1}$$

with the initial condition $(w(0), w_t(0)) = (u_0^1 - u_0^2, v_0^1 - v_0^2)$. At first, multiplying (4.1) by w and integrating over $[0,T] \times \Omega$, we get

$$\begin{aligned}
&\int_0^T ||w(s)||^2 ds + k\int_0^T |w(s)^+|^2 ds = \int_\Omega w_t(0)w(0)dx - \int_\Omega w_t(T)w(T)dx \\
&+ \int_0^T |w_t(s)|^2 ds - \int_0^T \int_\Omega (f(u_1(s)) - f(u_2(s)))w(s)dxds \\
&- \int_0^T \int_\Omega a(x)(g(u_{1_t}(s)) - g(u_{2_t}(s)))w(s)dxds.
\end{aligned} \tag{4.2}$$

Secondly, multiplying (4.1) by w_t and integrating over $[s,T] \times \Omega$, we obtain

$$\begin{aligned}
E_w(T) &+ \int_s^T \int_\Omega a(x)(g(u_{1_t}(\tau)) - g(u_{2_t}(\tau)))w_t(\tau)dxd\tau \\
&\leq E_w(s) - \int_s^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau)))w_t(\tau)dxd\tau,
\end{aligned} \tag{4.3}$$

where $0 \leq s \leq T$. Integrating (4.3) over $[0,T]$ with respect to s, we have that

$$TE_w(T) \leq \int_0^T E_w(s)ds - \int_0^T \int_s^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau)))w_t(\tau)dxd\tau ds. \tag{4.4}$$

Moreover from (1.2), (4.3) and Lemma 2.1, we obtain that, for any $\delta > 0$,

$$\begin{aligned}
\int_0^T |w_t(\tau)|^2 d\tau \quad &\leq \quad \delta T \text{meas}(\Omega) + C_2 E_w(0) \\
&\quad -C_2 \int_0^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau)))w_t(\tau)dxd\tau\ ,
\end{aligned} \tag{4.5}$$

where a constant $C_2 = C_\delta \alpha_0^{-1}$. Thus, from (4.2) and (4.5) we have

$$\begin{aligned}\int_0^T E_w(s)ds \quad \leq \quad & \delta T \text{meas}(\Omega) + C_2 E_w(0) + \int_\Omega w_t(0)w(0)dx - \int_\Omega w_t(T)w(T)dx \\ & -C_2 \int_0^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau)))w_t(\tau)dxd\tau \\ & - \int_0^T \int_\Omega (f(u_1(s)) - f(u_2(s)))w(s)dxds \\ & - \int_0^T \int_\Omega a(x)(g(u_{1_t}(s)) - g(u_{2_t}(s)))w(s)dxds. \end{aligned} \tag{4.6}$$

Now, we are going to estimate the last term in (4.6). Multiplying (1.1) by $u_{i_t}(t)$, we obtain

$$\frac{1}{2}\int_\Omega (|u_{i_t}|^2 + |\Delta u_i|^2 + k|u_i^+|^2 + 2F(u))dx + \int_\Omega a(x)g(u_{i_t})u_{i_t}dx = \int_\Omega h u_{i_t}dx,$$

which, combining with the existence of a bounded absorbing set, implies that

$$\int_0^T \int_\Omega a(x)g(u_{i_t})u_{i_t}dxds \leq C_\rho \quad (i = 1, 2), \tag{4.7}$$

where C_ρ is a constant which depends on meas(Ω), $|h|^2$ and the size of B_1. By the similar method of (3.9) and (4.7), we obtain

$$\begin{aligned} & |\int_0^T \int_\Omega a(x)g(u_{i_t}(s))w(s)dxds| \\ & \quad \leq \int_0^T \int_{\Omega(|u_{i_t}|\leq 1)} |a(x)g(u_{i_t}(s))w(s)|dxds + \int_0^T \int_{\Omega(|u_{i_t}|\geq 1)} |a(x)g(u_{i_t}(s))w(s)|dxds \\ & \quad \leq C\int_0^T \int_{\Omega(|u_{i_t}|\leq 1)} |a(x)w|dxds \\ & \qquad + (\int_0^T \int_{\Omega(|u_{i_t}|\geq 1)} a(x)|g(u_{i_t})|^{\frac{q+1}{q}}dxds)^{\frac{q}{q+1}}(\int_0^T \int_{\Omega(|u_{i_t}|\geq 1)} a(x)|w|^{q+1}dxds)^{\frac{1}{q+1}} \\ & \quad \leq C\int_0^T \int_{\Omega(|u_{i_t}|\leq 1)} |a(x)w|dxds \\ & \qquad + 2C(C_\rho)^{\frac{1}{q+1}}T^{\frac{1}{q+1}}(\int_0^T \int_{\Omega(|u_{i_t}|\geq 1)} a(x)g(u_{i_t})u_{i_t}dxds)^{\frac{q}{q+1}} \\ & \quad \leq C\int_0^T \int_\Omega a(x)|w|dx + 2CC_\rho T^{\frac{1}{q+1}} \ (i = 1, 2). \end{aligned} \tag{4.8}$$

Combining (4.4), (4.6) and (4.8), we have

$$\begin{aligned} TE_w(T) \leq\ & \delta T \mathrm{meas}(\Omega) + C_2 E_w(0) + \int_\Omega w_t(0)w(0)dx - \int_\Omega w_t(T)w(T)dx \\ & + C\int_0^T \int_\Omega a(x)|w|dx - \int_0^T \int_\Omega (f(u_1(s)) - f(u_2(s)))w(s)dxds \\ & + 2CC_\rho T^{\frac{1}{q+1}} - C_2 \int_0^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau)))w_t(\tau)dxd\tau \\ & - \int_0^T \int_s^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau)))w_t(\tau)dxd\tau ds. \end{aligned} \tag{4.9}$$

Set

$$\begin{aligned} C_B = & \delta T \mathrm{meas}(\Omega) + C_2 E_w(0) + \int_\Omega w_t(0)w(0)dx \\ & - \int_\Omega w_t(T)w(T)dx + 2CC_\rho T^{\frac{1}{q+1}}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} & \phi_T((u_0^1, v_0^1), (u_0^2, v_0^2)) \\ & = C\int_0^T \int_\Omega a(x)|w|dx \\ & \quad - \int_0^T \int_\Omega (f(u_1(s)) - f(u_2(s)))w(s)dxds \\ & \quad - C_2 \int_0^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau)))w_t(\tau)dxd\tau \\ & \quad - \int_0^T \int_s^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau)))w_t(\tau)dxd\tau ds. \end{aligned} \tag{4.11}$$

Then we have

$$E_w(T) \leq \frac{C_B}{T} + \frac{1}{T}\phi_T((u_0^1, v_0^1), (u_0^2, v_0^2)). \tag{4.12}$$

4.2. Asymptotic compactness

In this subsection, following similar arguments as in [5], [7], [15], we shall prove the asymptotic compactness of the semigroup $\{S(t)\}_{t\geq 0}$ in $V \times H$, which is given in the following theorem.

THEOREM 4.1. Under assumptions (1.2)-(1.7), then the semigroup $\{S(t)\}_{t\geq 0}$ corresponding to problems (1.1) is asymptotically compact in $V \times H$.

Proof. Since the semigroup $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set, for any fixed $\epsilon > 0$, we can first choose $\delta \leq \frac{\epsilon}{2\mathrm{meas}(\Omega)}$, and then let T be so large that

$$\frac{C_B}{T} \leq \epsilon.$$

Hence, thanks to Theorem 2.2, we only need to verify that the function $\phi_T(\cdot, \cdot)$ defined in (4.11) belongs to $C(B_1)$ for each fixed T. Let (u_n, u_{t_n}) be the corresponding solution of $(u_0^n, v_0^n) \in B_1,\ n = 1, 2, \ldots$. Then, since B_1 is a bounded positively invariant set in $V \times H$, without loss of generality, we assume that

$$u_n \to u \ \text{ weakly star in } \ L^\infty(0,T;H_0^2(\Omega)), \tag{4.13}$$

$$u_{n_t} \to u_t \ \text{ weakly star in } \ L^\infty(0,T;L^2(\Omega)), \tag{4.14}$$

$$u_n \to u \ \text{ strongly in } \ L^2(0,T;L^2(\Omega)), \tag{4.15}$$

$$u_n \to u \ \text{ strongly in } \ L^k(0,T;L^k(\Omega)), \tag{4.16}$$

for $k \le 2(\gamma+1)$, where we use the compact embedding $H_0^2 \hookrightarrow L^k$. Now, we will deal with each term corresponding to that in (4.11). At first, from (4.15), we have

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega (f(u_n(s)) - f(u_m(s)))(u_n(s) - u_m(s))dxds = 0. \tag{4.17}$$

Secondly, from (4.16) and (1.2), we obtain

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega a(x)|u_n(s) - u_m(s)|dxds = 0. \tag{4.18}$$

Finally, following the similar argument given in [7], we get

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_\Omega (f(u_n(s)) - f(u_m(s)))(u_{n_t}(s) - u_{m_t}(s))dxds = 0, \tag{4.19}$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T\int_s^T\int_\Omega (f(u_n(\tau)) - f(u_m(\tau)))(u_{n_t}(\tau) - u_{m_t}(\tau))dxd\tau ds = 0. \tag{4.20}$$

Hence, combining (4.17)-(4.20) we get $\phi_T(\cdot,\cdot) \in C(B_1)$ immediately.

□

4.3. Existence of global attractor

THEOREM 4.2. Under assumptions (1.2)-(1.7), then problems (1.1) have a global attractor in $V \times H$, which is invariant and compact.

Proof. Lemma 3.1 and Theorem 4.1 imply the existence of a global attractor. □

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