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GLOBAL BOUNDS FOR THE CUBIC NONLINEAR SCHRÖDINGER EQUATION (NLS) IN ONE SPACE DIMENSION

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Abstract. This article is concerned with the small data problem for the cubic nonlinear Schrödinger equation (NLS) in one space dimension, and short range modifications of it. We provide a new, simpler approach in order to prove that global solutions exist for data which is small in $H^{0,1}$. In the same setting we also discuss the related problems of obtaining a modified scattering expansion for the solution, as well as asymptotic completeness.

1. Introduction

We consider the cubic nonlinear Schrödinger equation (NLS) problem in one space dimension

$$iu_t + \frac{1}{2}u_{xx} = \lambda u|u|^2, \qquad u(0) = u_0, \tag{1.1}$$

where u is a complex valued function, $u : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$, and $\lambda = 1$ or -1 corresponding to the defocusing, respectively the focusing case.

Our results and proofs apply equally to short range modifications of it

$$iu_t + \frac{1}{2}u_{xx} = \lambda u|u|^2 + uF(|u|^2), \qquad u(0) = u_0, \tag{1.2}$$

where F satisfies

$$|F(r)| \lesssim |r|^{1+\delta}, \qquad |F'(r)| \lesssim |r|^{\delta}, \qquad \delta > 0.$$

A common feature of these two equations is that they exhibit Galilean invariance as well as the phase rotation symmetry, both of which are used in our arguments.

The question at hand is that of establishing global existence and asymptotics for solutions to (1.1) and then (1.2), provided that the initial data is small and spatially localized. Traditionally this is done in Sobolev spaces of the form $H^{m,k}$, whose norms are defined by

$$\|u\|^2_{H^{m,k}} := \|(1-\partial_x^2)^{\frac{m}{2}}u\|^2_{L^2} + \|(1+|x|^2)^{\frac{k}{2}}u\|^2_{L^2}, \qquad m,k \geq 0.$$

The problem (1.1) is completely integrable, which allows one to use very precise techniques, i.e., the inverse scattering method, to obtain accurate long range asymptotics, even for large data in the defocusing case. These have the form

$$u(t,x) \approx t^{-\frac{1}{2}} e^{\frac{ix^2}{2t} + \lambda|W(x/t)|^2 \log t} W(x/t).$$

One notes that this is not linear scattering, but rather a modified linear scattering. Indeed, in work of Deift and Zhou [2], the inverse scattering method is used to show that the above

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asymptotics hold for data in $H^{1,1}$, not only for (1.1), but also for (1.2) with a restricted range of powers.

In the meantime, two alternate approaches have emerged, which do not depend on the complete integrability of the problem. The first, initiated by Hayashi and Naumkin [3], and refined by Kato-Pusateri [7], is based on deriving an asymptotic equation for the Fourier transform of the solutions,

$$\frac{d}{dt}\hat{u}(t,\xi) = \lambda i t^{-1}\, \hat{u}(t,\xi)|\hat{u}(t,\xi)|^2 + O_{L^\infty}(t^{-1-\epsilon}).$$

This gives a result for data in $H^{1,1}$.

A second approach, introduced by Lindblad-Soffer [9], is based on deriving an asymptotic equation in the physical space along rays

$$(t\partial_t + x\partial_x)u(t,x) = \lambda i t^{-1} u(t,x)|u(t,x)|^2 + O_{L^\infty}(t^{-1-\epsilon}).$$

This argument yields a similar result, though the precise regularity at which this works is not computed, and is likely higher.

The method in the present paper in some sense interpolates between the two ideas above. Instead of localizing sharply on either the Fourier or the physical side, we use a mixed wave packet style phase space localization, loosely inspired from the analysis in [10]. The idea is that using wave packets one can produce a more accurate approximate solution to the linear Schrödinger equation, and use that to test for the long range behavior in the nonlinear equation.

Our interest in this problem arose from working on two dimensional water waves, where a similar situation occurs. There, a global result was independently proved by Ionescu-Pusateri [5] and Alazard-Delort [1] using methods based on the two ideas above. However, implementing either of these strategies brings on considerable difficulties. Many of these difficulties are bypassed by the authors in [4], where a simpler proof of the global result is given.

The present paper contains the implementation of the ideas in [4] for the simpler problems (1.1), (1.2). Our goal is two-fold, namely to provide a simpler proof of the global result with fewer assumptions, and also to give a more transparent introduction to the work in [4]. Our main result is Galilean invariant:

Theorem 1. *a) (Global well-posedness and decay) Consider either the equation* (1.1) *or* (1.2)*, with initial data u_0 which is small in $H^{0,1}$,*

$$\|u_0\|_{H^{0,1}} \leq \epsilon \ll 1. \tag{1.3}$$

Then there exists a unique global solution u with regularity $e^{-\frac{it}{2}\partial_x^2}u \in C(\mathbb{R}; H^{0,1}(\mathbb{R}))$ which satisfies the pointwise estimate

$$\|u\|_{L^\infty} \lesssim \epsilon |t|^{-\frac{1}{2}}, \tag{1.4}$$

as well as the energy bound

$$\|e^{\frac{-it}{2}\partial_x^2}u\|_{H^{0,1}} \lesssim \epsilon(1+t)^{C\epsilon^2}. \tag{1.5}$$

b) (Asymptotic behavior) Let u be a solution to either (1.1) *or* (1.2) *as in part (a). Then there exists a function $W \in H^{1-C\epsilon^2}(\mathbb{R})$ such that*

$$u(x,t) = \frac{1}{\sqrt{t}} e^{\frac{ix^2}{2t}} W(x/t) e^{i \log t |W(x/t)|^2} + err_x, \tag{1.6}$$

$$\hat{u}(\xi,t) = e^{-\frac{it\xi^2}{2}} W(\xi) e^{i \log t |W(\xi)|^2} + err_\xi, \tag{1.7}$$

where

$$err_x \in \epsilon \left(O_{L^\infty}((1+t)^{-\frac{3}{4}+C\epsilon^2}) \cap O_{L^2_x}((1+t)^{-1+C\epsilon^2}) \right),$$

$$err_\xi \in \epsilon \left(O_{L^\infty}((1+t)^{-\frac{1}{4}+C\epsilon^2}) \cap O_{L^2_\xi}((1+t)^{-\frac{1}{2}+C\epsilon^2}) \right).$$

c) (Asymptotic completeness for small data) Let C be a large universal constant. For each W satisfying

$$\|W\|_{H^{1+C\epsilon^2}(\mathbb{R})} \ll \epsilon \ll 1$$

there exists $u_0 \in H^{0,1}$ satisfying (1.3) *so that* (1.6) *and* (1.7) *hold for the corresponding solution u to* (1.1) *or* (1.2).

The next section contains the proof of the theorem. We begin with the proof of part (a), which is a self contained argument. The argument for part (b) is based on a more careful analysis of the outcome of (1a). Finally, the proof of the asymptotic completeness is again a self contained argument, which is a simpler lower regularity version of the original result in [6]. Several remarks may be of interest:

Remark 1.1. *Since one goal of this article is to present a clear and simple statement, the result and the proofs are done in the setting of $H^{0,1}$ data. However, with some extra work, the same method will also work for data in $H^{0,s}$ with $\frac{1}{2} < s \leq 1$.*

Remark 1.2. *One may ask whether one does not have $W \in H^1$, with a smooth one to one correspondence between $u_0 \in H^{0,1}$ and W. The work* [2] *of Deift and Zhou shows that this is not the case, and that there is necessarily some logarithmic type correction to such a property. We leave open the question of providing a direct proof of such a correspondence in a suitable functional setting.*

2. Proof of the Theorem 1.3

2.1. **Local well-posedness.** While the equation (1.1) is locally well-posed for data in L^2, working with data in $H^{0,1}$ requires a brief discussion. The initial data space has norm

$$\|u_0\|^2_{H^{0,1}} = \|u_0\|^2_{L^2} + \|xu_0\|^2_{L^2}.$$

However, we cannot use this same space at later times since the weight x does not commute with the linear Schrödinger flow. Instead, we introduce the vector field $L = x + it\partial_x$, which is the conjugate of x with respect to the linear flow, $e^{\frac{it}{2}\partial_x^2} x = L e^{\frac{it}{2}\partial_x^2}$, as well as the generator for the Galilean group of symmetries. Naturally we have

$$\left[i\partial_t + \frac{1}{2}\partial_x^2 \, , L \right] = 0, \qquad L(\lambda u |u|^2) = 2\lambda |u|^2 Lu - \lambda u^2 \overline{Lu}.$$

Next, we state and prove a preliminary global result:

Proposition 2.1. *The equation (1.1) is (globally) well-posed for initial data in $H^{0,1}$, in the sense that it admits a unique solution $u \in C(\mathbb{R}, L^2)$ such that $Lu \in C(\mathbb{R}, L^2)$. Further, such a solution is continuous away from $t = 0$, and satisfies $u \in C(\mathbb{R} \setminus \{0\}, L^\infty)$. Furthermore, near $t = 0$ we have*

$$|u(t,x)| \lesssim t^{-\frac{1}{2}} \|u_0\|_{H^{0,1}}. \tag{2.1}$$

Proof. We start with the L^2 well-posedness, which is based on the Strichartz estimate for the linear inhomogeneous problem

$$(i\partial_t + \frac{1}{2}\partial_x^2)u = f, \qquad u(0) = u_0,$$

which has the form

$$\|u\|_{L_t^\infty L_x^2} + \|u\|_{L_t^4 L_x^\infty} \lesssim \|u_0\|_{L^2} + \|f\|_{L_t^1 L_x^2}. \tag{2.2}$$

This allows us to treat the nonlinearity perturbatively and obtain the unique local solution via the contraction principle in the space $L_t^\infty(0,T; L_x^2) \cap L_t^4(0,T; L_x^\infty)$ provided that T is small enough[1], $T \ll \|u_0\|_{L^2}^4$. The local well-posedness in L^2 implies global well-posedness due to the conservation of the mass $\|u\|_{L^2}^2$.

To switch to the $H^{0,1}$ data we need to write the equation for Lu, which has the form

$$(i\partial_t + \frac{1}{2}\partial_x^2)Lu = 2\lambda|u|^2 Lu - \lambda u^2 \overline{Lu}. \tag{2.3}$$

We remark that this is exactly the linearization of the equation (1.1). The L^2 well-posedness of this problem also follows from the Strichartz estimate (2.2).

Finally, we consider pointwise bounds. Denoting $w = ue^{-\frac{ix^2}{2t}}$, we have $ie^{-\frac{ix^2}{2t}} Lu = it\partial_x w$. Hence, away from $t = 0$ we have $w \in C(\mathbb{R}\backslash\{0\}; H^1)$, and the continuity property of w, namely $w \in C_{loc}(\mathbb{R} \setminus \{0\}; C_0(\mathbb{R}))$, follows from the Sobolev embedding $H^1(\mathbb{R}) \subset C_0(\mathbb{R})$. Since w has limit zero at infinity, the similar property for u also follows. Finally, the pointwise bound (2.1) is a consequence of the Gagliardo-Nirenberg type estimate

$$\|w\|_{L^\infty} \lesssim \|w\|_{L^2}^{\frac{1}{2}} \|\partial_x w\|_{L^2}^{\frac{1}{2}}.$$

□

2.2. **Wave packets and the asymptotic equation.** To study the global decay properties of solutions to (1.1) and (1.2), we introduce a new idea, which is to test the solution u with wave packets which travel along the Hamilton flow. A wave packet, in the context here, is an approximate solution to the linear system, with $O(1/t)$ errors. Precisely, for each trajectory $\Gamma_v := \{x = vt\}$, traveling with velocity v, we establish decay along this ray by testing with a wave packet moving along the ray.

To motivate the definition of this packet we recall some useful facts. First, this ray is associated with waves which have spatial frequency

$$\xi_v := v = \frac{x}{t}.$$

This is associated with the phase function

$$\phi(t,x) := \frac{x^2}{2t}.$$

[1]This is exactly the scaling relation.

Then it is natural to use as test functions wave packets of the form

$$\Psi_v(t,x) := \chi\left(\frac{x-vt}{\sqrt{t}}\right) e^{i\phi(t,x)}.$$

Here we take χ to be a Schwartz function. In other related problems it might be more convenient to take χ with compact support. For normalization purposes we assume that

$$\int \chi(y)dy = 1.$$

The $t^{\frac{1}{2}}$ localization scale is exactly the scale of wave packets which are required to stay coherent on the time scale t. To see that these are reasonable approximate solutions we observe that we can compute

$$(i\partial_t + \frac{1}{2}\partial_x^2)\Psi_v = \frac{1}{2t} e^{i\phi}\partial_x \left[t^{\frac{1}{2}}\chi'\left(\frac{x-vt}{\sqrt{t}}\right) + i(x-vt)\chi\left(\frac{x-vt}{\sqrt{t}}\right)\right], \tag{2.4}$$

and observe that the right hand side has the same localization as Ψ_v and size smaller by a factor t^{-1}. Thus one can think of Ψ_v as good approximate solutions for the linear Schrödinger equation only on dyadic time scales $\Delta t \ll t$.

If one compares Ψ_v with the fundamental solution to the linear Schrödinger equation, conspicuously the $t^{-\frac{1}{2}}$ factor is missing. Adding this factor does not improve the error in the interpretation of Ψ_v as a good approximate solution, so we have preferred instead a normalization which provides simpler ode dynamics for the function γ defined below.

As a measure of the decay of u along Γ_v we use the function

$$\gamma(t,v) := \int u\bar{\Psi}_v \, dx.$$

For the purpose of proving part (a) of the theorem we only need to consider γ along a single ray. However, in order to obtain the more precise asymptoptics in part (b) we will think of γ as a function $\gamma(t,v)$.

We can also express $\gamma(t,v)$ in terms of the Fourier transform of u,

$$\gamma(t,v) = \int \hat{u}(t,\xi)\bar{\hat{\Psi}}(t,\xi)\, d\xi.$$

Here a direct computation yields

$$\begin{aligned}
\hat{\Psi}(t,\xi) =& \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi} e^{i\frac{x^2}{2t}} \chi(t^{-\frac{1}{2}}(x-vt))\, dx \\
=& \frac{1}{\sqrt{2\pi}} e^{-i\frac{t\xi^2}{2}} e^{i\frac{t(\xi-v)^2}{2}} \int e^{-i(x-vt)(\xi-v)} e^{i\frac{(x-tv)^2}{2t}} \chi(t^{-\frac{1}{2}}(x-vt))\, dx \\
=& t^{\frac{1}{2}} e^{-i\frac{t\xi^2}{2}} \chi_1(t^{\frac{1}{2}}(\xi-v)),
\end{aligned}$$

where $\chi_1 = e^{i\frac{\xi^2}{2}} \widehat{e^{\frac{ix^2}{2}}\chi}$ is a Schwartz function with the additional property that

$$\int \chi_1(\xi)\, d\xi = \int \chi(x)\, dx = 1.$$

Then we can write

$$\gamma(t,\xi) = e^{\frac{it\xi^2}{2}} \hat{u}(t,\xi) *_\xi t^{\frac{1}{2}} \chi_1(t^{\frac{1}{2}}\xi). \tag{2.5}$$

Both the solution u of (1.1) along the ray Γ_v and its Fourier transform evaluated at v are compared to $\gamma(t,v)$ as follows:

Lemma 2.2. *The function γ satisfies the bounds*

$$\|\gamma\|_{L^\infty} \lesssim t^{\frac{1}{2}}\|u\|_{L^\infty}, \qquad \|\gamma\|_{L^2_v} \lesssim \|u\|_{L^2_x}, \qquad \|\partial_v\gamma\|_{L^2_v} \lesssim \|Lu\|_{L^2_x}. \tag{2.6}$$

We have the physical space bounds

$$\begin{aligned} &\|u(t,vt) - t^{-\frac{1}{2}}e^{i\phi(t,vt)}\gamma(t,v)\|_{L^2_v} \lesssim t^{-1}\|Lu\|_{L^2_x}, \\ &\|u(t,vt) - t^{-\frac{1}{2}}e^{i\phi(t,vt)}\gamma(t,v)\|_{L^\infty} \lesssim t^{-\frac{3}{4}}\|Lu\|_{L^2_x}, \end{aligned} \tag{2.7}$$

and the Fourier space bounds

$$\begin{aligned} &\|\hat{u}(t,\xi) - e^{-i\frac{t\xi^2}{2}}\gamma(t,\xi)\|_{L^2_\xi} \lesssim t^{-\frac{1}{2}}\|Lu\|_{L^2_x}, \\ &\|\hat{u}(t,\xi) - e^{-i\frac{t\xi^2}{2}}\gamma(t,\xi)\|_{L^\infty} \lesssim t^{-\frac{1}{4}}\|Lu\|_{L^2_x}. \end{aligned} \tag{2.8}$$

Proof. Denote $w := e^{-i\phi}u$. Then $\partial_v w = it\partial_x w = it\partial_x(e^{-i\phi}u) = ie^{-i\phi}Lu$, and we can express γ in terms of w as a convolution with respect to the v variable,

$$t^{-\frac{1}{2}}\gamma(t,v) = w(t,vt) *_v t^{\frac{1}{2}}\chi(t^{\frac{1}{2}}v), \tag{2.9}$$

where the kernel on the right has unit integral. In other words, $t^{-\frac{1}{2}}\gamma(t,v)$ is a regularization of $w(t,vt)$ on the $t^{-\frac{1}{2}}$ scale in v, or equivalently, a localization of $w(t,vt)$ to frequencies less than $t^{-\frac{1}{2}}$. Hence, via Young's inequality, we have the straightforward convolution bounds

$$\|\gamma(t,v)\|_{L^\infty} \lesssim t^{\frac{1}{2}}\|w(t,vt)\|_{L^\infty} = t^{\frac{1}{2}}\|u\|_{L^\infty},$$

$$\|\gamma(t,v)\|_{L^2_v} \lesssim t^{\frac{1}{2}}\|w(t,vt)\|_{L^2_v} = \|u\|_{L^2_x},$$

as well as

$$\|\partial_v\gamma(t,v)\|_{L^2_v} \lesssim t^{\frac{1}{2}}\|\partial_v w(t,vt)\|_{L^2_v} = \|Lu\|_{L^2_x}.$$

Here we have used the fact that the L^2_v, L^2_x norms are related by

$$\|f\|_{L^2_x} = t^{\frac{1}{2}}\|f\|_{L^2_v}.$$

To bound the difference $t^{-\frac{1}{2}}\gamma(t,v) - w(t,vt)$ we use the fact that the above kernel has unit integral to write

$$\begin{aligned} |t^{-\frac{1}{2}}\gamma(t,v) - w(t,vt)| &= \left|\int (w(t,(v-z)t) - w(t,vt))\chi(t^{\frac{1}{2}}z)t^{\frac{1}{2}}\,dz\right| \\ &\leq \int |w(t,(v-z)t) - w(t,vt)||\chi(t^{\frac{1}{2}}z)|t^{\frac{1}{2}}\,dz. \end{aligned} \tag{2.10}$$

To prove the pointwise bound in (2.7) we use Hölder's inequality to obtain

$$|w(t,vt) - w(t,(v-z)t)| \lesssim |z|^{\frac{1}{2}}\|\partial_v w\|_{L^2_v},$$

which by (2.10) leads to

$$|e^{-i\phi}u(t,vt) - t^{-\frac{1}{2}}\gamma(t,v)| \lesssim \|\partial_v w(t,vt)\|_{L^2_v}\int |z|^{\frac{1}{2}}t^{\frac{1}{2}}|\chi(t^{\frac{1}{2}}z)|\,dz \approx t^{-\frac{1}{4}}\|\partial_v w(t,vt)\|_{L^2_v} = t^{-\frac{3}{4}}\|Lu\|_{L^2_x}.$$

To prove the L_v^2 bound in (2.7) we express the right hand side in the last integrand in (2.10) in terms of the derivative of w to obtain

$$|t^{-\frac{1}{2}}\gamma(t,v) - w(t,vt)| \lesssim \int_0^1 \int |z||\partial_v w(t,(v-hz)t)|t^{\frac{1}{2}}\chi(t^{\frac{1}{2}}z)t^{\frac{1}{2}}\,dz\,dh.$$

Hence we can evaluate the L^2 norm as follows:

$$\begin{aligned}\|e^{-i\phi}u(t,vt) - t^{-\frac{1}{2}}\gamma(t,v)\|_{L^2} &\lesssim \|\partial_v w(t,vt)\|_{L_v^2} \int |z|t^{\frac{1}{2}}|\chi(t^{\frac{1}{2}}z)|\,dz \\ &\approx t^{-\frac{1}{2}}\|\partial_v w(t,vt)\|_{L_v^2} = t^{-1}\|Lu\|_{L_x^2}.\end{aligned}$$

This concludes the proof of the bound (2.7). The estimate (2.8) is obtained in a similar manner, but using (2.5) instead of (2.9), as well as the relation

$$\|\partial_\xi(e^{\frac{it\xi^2}{2}}\hat{u}(t,\xi))\|_{L_\xi^2} = \|Lu\|_{L_x^2}.$$

□

By the previous Lemma 2.2 we can conclude that γ is indeed a good approximation of u along a ray, but no information on the rate of decay of γ was established. Hence, the crucial next step is to obtain an approximate ode dynamics for $\gamma(t,v)$:

Lemma 2.3. *If u solves* (1.1) *then we have*

$$\dot{\gamma}(t,v) = -it^{-1}\lambda|\gamma(t,v)|^2\gamma(t,v) - R(t,v), \tag{2.11}$$

where the remainder R satisfies

$$\|R\|_{L_x^\infty} \lesssim t^{-\frac{1}{4}}\|Lu\|_{L_x^2}\left(t^{-1} + \|u\|_{L_x^\infty}^2\right), \qquad \|R\|_{L_v^2} \lesssim t^{-\frac{1}{2}}\|Lu\|_{L_x^2}\left(t^{-1} + \|u\|_{L_x^\infty}^2\right). \tag{2.12}$$

Proof. A direct computation yields

$$\begin{aligned}\dot{\gamma}(t) &= \int u_t\bar{\Psi}_v + u\bar{\Psi}_{vt}\,dx = \int i(\frac{1}{2}u_{xx} - \lambda u|u|^2)\bar{\Psi}_v + u\bar{\Psi}_{vt}\,dx \\ &= \int -iu\overline{(i\partial_t + \frac{1}{2}\partial_x^2)\Psi_v} - i\lambda u|u|^2\bar{\Psi}_v\,dx.\end{aligned}$$

Using the relation (2.4) and integrating by parts we obtain

$$\begin{aligned}\dot{\gamma}(t) &= \int i\frac{1}{2t}\partial_x(t^{\frac{1}{2}}\chi' + i(x-vt)\chi)e^{-i\phi}u\,dx - \int i\lambda u|u|^2\bar{\Psi}_v\,dx \\ &= -\int \frac{1}{2t^2}(t^{\frac{1}{2}}\chi' + i(x-vt)\chi)e^{-i\phi}Lu\,dx - \int i\lambda u|u|^2\bar{\Psi}_v\,dx.\end{aligned}$$

Hence we can write an evolution equation for $\gamma(t)$ of the form

$$\dot{\gamma}(t,v) = -i\lambda t^{-1}|\gamma(t,v)|^2\gamma(t) - R(t,v),$$

where $R(t,v)$ contains error terms which are the contributions arising from using Ψ_v as a good approximation of the solution of the linear Schödinger equation, and also from substituting

u by γ in the cubic nonlinearity. We write the remainder $R(t,v)$ as a sum of three quantities which can be easily bounded:

$$\begin{aligned}R(t,v) :=& -\int \frac{1}{2t^2}(t^{\frac{1}{2}}\chi' + i(x-vt)\chi)e^{-i\phi}Lu\,dx - i\lambda \int u\bar{\Psi}_v(|u|^2 - |u(t,vt)|^2)\,dx \\ &+ i\lambda\gamma(|u(t,vt)|^2 - t^{-1}|\gamma(t,v)|^2) \\ :=& R_1 + R_2 + R_3.\end{aligned}$$

The integral R_1 is expressed as a convolution in v,

$$R_1 = -\frac{1}{t}(t^{\frac{1}{2}}\chi'(t^{\frac{1}{2}}v) + itv\chi(t^{\frac{1}{2}}v)) *_v (\partial_v w(t,vt)).$$

Hence, by Hölder's inequality we obtain the pointwise bound

$$|R_1| \lesssim t^{-\frac{3}{4}}\|\partial_v w(t,vt)\|_{L^2_v} = t^{-\frac{5}{4}}\|Lu\|_{L^2_x},$$

while estimating the convolution kernel in L^1_v yields the L^2_x bound

$$\|R_1\|_{L^2_v} \lesssim t^{-1}\|\partial_v w(t,vt)\|_{L^2_v} = t^{-\frac{3}{2}}\|Lu\|_{L^2_x}.$$

Since $|u| = |w|$, the second term $R_2 := -i\lambda\int u\bar{\Psi}_v(|u|^2 - |u(t,vt)|^2)\,dx$ is bounded by

$$\begin{aligned}|R_2(t,v)| &\lesssim \|u\|^2_{L^\infty}\int |\chi(t^{-\frac{1}{2}}(x-vt))|(|w(t,x) - w(t,vt)|)\,dx \\ &= t^{\frac{1}{2}}\|u\|^2_{L^\infty}\int |\chi(t^{\frac{1}{2}}z)|(|w(t,(v-z)t) - w(t,vt)|)t^{\frac{1}{2}}\,dz,\end{aligned}$$

where the last integrand is the same as in (2.10). Then R_2 is estimated exactly as in the proof of (2.7) following (2.10).

Finally, for R_3 it suffices to combine the estimates (2.6) and (2.7).

□

2.3. **Proof of the global well-posedness result.** From Proposition 2.1 we know that a global solution exists, so it remains to establish the bounds (1.5) and (1.4). Proposition 2.1 also shows that $\|u(t)\|_{L^\infty}$ is continuous in time away from $t = 0$. Then a continuity argument implies that it suffices to prove these bounds under the additional bootstrap assumption:

$$\|u\|_{L^\infty} \le D\epsilon|t|^{-\frac{1}{2}}, \tag{2.13}$$

where D is a large constant such that $1 \ll D \ll \epsilon^{-1}$. Then we want to prove the energy bound (1.5), and then show that (1.4) holds with an implicit constant which does not depend on D.

The energy estimate for Lu: To advance frome time 0 to time 1 we use the local well-posedness result above. This gives

$$\|Lu(1)\|_{L^2} \lesssim \|xu(0)\|_{L^2} \le \epsilon.$$

To move forward in time past time 1 we use energy estimates in (2.3) and then (2.13) to obtain

$$\begin{aligned}\|Lu(t)\|_{L^2} &\le \|Lu(1)\|_{L^2} + \int_1^t \|u(s)\|^2_\infty\|Lu(s)\|_{L^2}\,ds \\ &\le \|Lu(1)\|_{L^2} + D^2\epsilon^2\int_1^t s^{-1}\|Lu(s)\|_{L^2}\,ds.\end{aligned}$$

Applying Gronwall's inequality gives

$$\|Lu(t)\|_{L^2} \lesssim \epsilon(1+t)^{D^2\epsilon^2}, \tag{2.14}$$

which, combined with the conservation of mass, leads to

$$\|e^{-\frac{it}{2}\partial_x^2} u(t)\|_{H^{0,1}} \lesssim \epsilon(1+t)^{D^2\epsilon^2}.$$

The pointwise decay bound: From the bound (2.7) in Lemma 2.2 and (2.14) we get

$$\|e^{-i\phi}u - t^{-\frac{1}{2}}\gamma\|_{L_x^\infty} \lesssim t^{-\frac{3}{4}}\|Lu\|_{L_x^2} \lesssim \epsilon(1+t)^{-\frac{3}{4}+D^2\epsilon^2},$$

so it remains to estimate γ. At time $t = 1$ we can use (2.1) and the pointwise part of (2.6) to conclude that

$$\|\gamma(1, v)\|_{L^\infty} \lesssim \epsilon.$$

On the other hand, using our bootstrap assumption (2.13) and the L^2 bound (2.14) in Lemma 2.3 we obtain a good bound for $R(t, v)$, namely

$$\|R(t, v)\|_{L^\infty} \lesssim \epsilon(1 + D^2\epsilon^2)t^{-\frac{5}{4}+D^2\epsilon^2}.$$

Then integrating in (2.11) we obtain

$$|\gamma(t, v)| \leq |\gamma(1, v)| + \int_1^t |R(s, v)| ds \lesssim \epsilon(1 + D^2\epsilon^2),$$

which leads to

$$|u| \lesssim (\epsilon + D^2\epsilon^3)|t|^{-\frac{1}{2}}.$$

Under the constraint $1 \ll D \ll \epsilon^{-1}$ we obtain (1.4), and conclude the bootstrap argument.

2.4. **The asymptotic expansion of the solution.** To construct the asymptotic profile W we use the ode in Lemma 2.3 for $\gamma(t, v)$. The inhomogeneous term $R(t, v)$ is estimated in L^∞ and L_v^2 by combining (2.12) with (1.4) and (2.14) to obtain

$$\|R(t, v)\|_{L^\infty} \lesssim \epsilon t^{-\frac{5}{4}+D^2\epsilon^2}, \qquad \|R(t, v)\|_{L_v^2} \lesssim \epsilon t^{-\frac{3}{2}+D^2\epsilon^2}. \tag{2.15}$$

The ODE for γ, namely

$$\dot{\gamma}(t) = -\frac{i}{t}|\gamma(t)|^2\gamma(t) - R(t, v)$$

can be explicitly solved in polar coordinates. Since $R(t, v)$ in uniformly integrable in time, it follows that for each v, $\gamma(t, v)$ is well approximated at infinity by a solution to the unperturbed ODE corresponding to $R_1 = 0$, in the sense that

$$\gamma(t, v) = W(v)e^{i|W(v)|^2 \log t} + O_{L_v^\infty}(\epsilon t^{-\frac{1}{4}+D^2\epsilon^2}). \tag{2.16}$$

Integrating the L_v^2 part of (2.15) leads to a similar L_v^2 bound

$$\gamma(t, v) = W(v)e^{i|W(v)|^2 \log t} + O_{L_v^2}(\epsilon t^{-\frac{1}{2}+\epsilon^2 D^2}). \tag{2.17}$$

Then the asymptotic expansions in (1.6), (1.7) follow directly from (2.7) and (2.8), where $\|Lu\|_{L^2}$ is bounded as in (2.14).

It remains to establish the regularity of W. By conservation of mass we have

$$\|u(0)\|_{L_x^2} = \|u(t)\|_{L_x^2} = t^{\frac{1}{2}}\|w(t, vt)\|_{L_v^2}.$$

Hence by (2.7) and (2.17) we obtain

$$\|W\|_{L_x^2} = \|u\|_{L_x^2}.$$

On the other hand, from (2.16) and (2.17) we get

$$\|W(v) - \gamma(t,v)e^{-i|\gamma(t,v)|^2\log t}\|_{L^2_v} \lesssim \epsilon t^{-\frac{1}{2}+D^2\epsilon^2}\log t,$$

while by (2.6) and (2.14) we have

$$\|\partial_v[\gamma(t,v)e^{-i|\gamma(t,v)|^2\log t}]\|_{L^2_v} \lesssim \epsilon t^{D^2\epsilon^2}\log t.$$

It follows that for all large t we have

$$W(v) = O_{H^1_v}(\epsilon t^{D^2\epsilon^2}\log t) + O_{L^2_v}(\epsilon t^{-\frac{1}{2}+D^2\epsilon^2}\log t),$$

so by interpolation we obtain for large enough C the regularity

$$\|W\|_{H^{1-C\epsilon^2}_v} \lesssim \epsilon.$$

2.5. **The asymptotic completeness problem.** Here we solve the problem from infinity. For convenience, throughout this section, we set $\lambda = 1$. The naive idea would be to start with the asymptotic profile

$$u_{asymptotic} = \frac{1}{\sqrt{t}}e^{\frac{ix^2}{2t}}W(x/t)e^{i|W(x/t)|^2\log t},$$

and correct this to an exact solution u to the cubic NLS (1.1), by perturbatively solving the equation for the difference from infinity. However, as defined above, the function $u_{asymptotic}$ does not have enough regularity in order for it to be a good approximate solution. To remedy this, we replace W in the above formula with a regularization of W on the time dependent scale, namely

$$\mathcal{W}(t,v) := W_{<t^{\frac{1}{2}}}(v),$$

which selects the frequencies less than $t^{\frac{1}{2}}$ in W. This is the analogue of the function γ defined for forward problem, with the same time dependent regularization scale. Then our approximate solution is

$$u_{app} = \frac{1}{\sqrt{t}}e^{\frac{ix^2}{2t}}\mathcal{W}(t,x/t)e^{i|\mathcal{W}(t,x/t)|^2\log t}.$$

To start with we make the more general assumption that

$$\|W\|_{H^{1+2\delta}_v} \leq M, \qquad M,\delta > 0, \qquad \delta \gg M^2. \tag{2.18}$$

Then by Bernstein's inequality we have the bounds

$$\|\mathcal{W}(t,v) - W(v)\|_{L^2_v} \lesssim Mt^{-\frac{1}{2}-\delta}, \qquad \|\mathcal{W}(t,v) - W(v)\|_{L^\infty} \lesssim Mt^{-\frac{1}{4}-\delta},$$

which imply that the functions $u_{asymptotic}$ and u_{app} are equally good as asymptotic profiles,

$$\|u_{asymptotic} - u_{app}\|_{L^2_x} \lesssim Mt^{-\frac{1}{2}-\delta}, \qquad \|u_{asymptotic} - u_{app}\|_{L^\infty} \lesssim \epsilon t^{-\frac{1}{4}-\delta}.$$

To find the exact solution u matching u_{app} at infinity we denote by f the error

$$f = (i\partial_t + \frac{1}{2}\partial_x^2)u_{app} - u_{app}|u_{app}|^2, \tag{2.19}$$

and then solve for the diffrence $v = u - u_{app}$

$$(i\partial_t + \frac{1}{2}\partial_x^2)v = (u_{app} + v)\,|u_{app} + v|^2 - u_{app}|u_{app}|^2 - f.$$

The u_{app}-cubic term cancels, and we are left with

$$(i\partial_t + \frac{1}{2}\partial_x^2)v = N(v, u_{app}) - f, \qquad v(\infty) = 0, \tag{2.20}$$

where

$$N(v, u_{app}) = v|v|^2 + v^2\bar{u}_{app} + 2|v|^2 u_{app} + 2v|u_{app}|^2 + \bar{v}u_{app}^2.$$

The solution operator for the inhomogeneous Schrödinger equation with zero Cauchy data at infinity

$$(i\partial_t + \frac{1}{2}\partial_x^2)v = f, \qquad u(\infty) = 0,$$

is given by

$$v(t) = i\lambda \int_t^\infty e^{\frac{(t-s)\partial_x^2}{2}} f(s)\, ds := \Phi f.$$

Hence the equation (2.20) is rewritten in the form

$$v = \Phi N(v, u_{app}) - \Phi f. \tag{2.21}$$

We will solve this via the contraction principle, using the energy/Strichartz type bound (2.2)

$$\|\Phi f\|_{L_t^\infty(T,\infty;L_x^2)} + \|\Phi f\|_{L_t^4(T,\infty;L_x^\infty)} \lesssim \|f\|_{L_t^1(T,\infty;L_x^2)}. \tag{2.22}$$

The equation for v will be solved in a function space X defined using the above $L_t^\infty L_x^2$ and $L_t^4 L_x^\infty$ norms, with appropriate time decay. Precisely, we set

$$\|v\|_X := \sup_{T\geq 1} \frac{T^{\frac{1}{2}+\delta}}{(1+M^2\log t)^2} \left(\|v\|_{L_t^\infty(T,2T;L_x^2)} + \|v\|_{L_t^4(T,2T;L_x^\infty)} \right).$$

We also want a bound for Lv, for which we need to use the larger space $\tilde{X}$, whose norm carries a different time decay weight,

$$\|w\|_{\tilde{X}} := \sup_{T\geq 1} \frac{T^{\delta}}{(1+M^2\log t)^3} \left(\|w\|_{L_t^\infty(T,2T;L_x^2)} + \|w\|_{L_t^4(T,2T;L_x^\infty)} \right).$$

The first task at hand is to estimate the contribution of the inhomogeneous term f. This is done in the following

Lemma 2.4. *Assume that* (2.18) *holds with* $\delta \gtrsim M^2$*. Then* f *defined by* (2.19) *satisfies the following estimates:*

$$\|\Phi f\|_X + \|\Phi L f\|_{\tilde{X}} \lesssim M. \tag{2.23}$$

We postpone the proof of the lemma in order to conclude first the proof of the main result. We succesively consider the equation for v and the equation for Lv.

(i) The equation for v in L^2. In view of (2.23), in order to solve the equation (2.21) in X using the contraction principle we need to show that the map $v \to \Phi N(v, u_{app})$ maps X into X with a small Lipschitz constant for v in a ball of radius CM, where $1 \ll C \ll M^{-1}$. Then we obtain a solution v satisfying

$$\|v\|_X \lesssim M. \tag{2.24}$$

Using the linear bound (2.2), it suffices to show that

$$\|N(v_1, u_{app}) - N(v_2, u_{app})\|_{L_t^1(T,\infty;L_x^2)} \lesssim \|v_1 - v_2\|_X (M + \|v_1\|_X^2 + \|v_2\|_X^2). \tag{2.25}$$

For simplicity we consider the case $v_2 = 0$ and show that

$$\|N(v, u_{app})\|_{L^1_t(T,\infty;L^2_x)} \lesssim M\|v\|_X + \|v\|^3_X. \tag{2.26}$$

The general case is identical. To bound $\|N(v, u_{app})\|_{L^1_t(T,\infty;L^2_x)}$ we we divide $[T, \infty)$ into dyadic subintervals, estimate $N(u_{app}, v; f)$ in each such interval, and then sum up. For the terms in N we succesively compute

$$\|v|u_{app}|^2\|_{L^1_t(T,2T;L^2_x)} \lesssim T\|u_{app}\|^2_{L^\infty([T,2T]\times\mathbb{R})}\|v\|_{L^\infty_t(T,2T;L^2_x)} \lesssim M^2T^{-\frac{1}{2}-\delta}(1 + M^2\log t)^2\|v\|_X, \tag{2.27}$$

$$\begin{aligned} \||v|^2u_{app}\|_{L^1_t(T,2T;L^2_x)} &\lesssim T^{\frac{3}{4}}\|u_{app}\|_{L^\infty([T,2T]\times\mathbb{R})}\|v\|_{L^\infty_t(T,2T;L^2_x)}\|v\|_{L^4_t(T,2T;L^\infty_x)} \\ &\lesssim MT^{-\frac{3}{4}+2\delta}(1 + M^2\log t)^4\|v\|^2_X, \end{aligned} \tag{2.28}$$

respectively

$$\|v|v|^2\|_{L^1_t(T,2T;L^2_x)} \lesssim T^{\frac{1}{2}}\|v\|_{L^\infty_t(T,2T;L^2_x)}\|v\|^2_{L^4_t(T,2T;L^\infty_x)} \lesssim T^{-1-3\delta}(1 + M^2\log t)^6\|v\|^3_X. \tag{2.29}$$

Thus, (2.26) follows.

(ii) The equation for Lv in L^2. Applying L to (2.20) we obtain

$$(i\partial_t + \frac{1}{2}\partial^2_x)Lv = LN(v, u_{app}) - Lf.$$

Then for Lv we seek to solve the linear problem

$$Lv = \Phi(LN(v, u_{app})) - \Phi Lf$$

in the space $\tilde{X}$. The bound for ΦLf is provided by (2.23). We expand $LN(v, u_{app})$ as

$$\begin{aligned} LN(v, u_{app}\,; f) :&= L(v|v|^2) + L(v^2\bar{u}_{app}) + 2L(|v|^2u_{app}) + 2L(v|u_{app}|^2) + L(\bar{v}u^2_{app}) - Lf \\ &= Q(Lv) + g - Lf, \end{aligned}$$

where the linear part $Q(Lv)$, respectively the inhomogeneous term g are given by

$$\begin{aligned} Q(Lv) &:= 2|v|^2Lv - v^2\overline{Lv} + 2\bar{u}_{app}vLv - u^2_{app}\overline{Lv} + \bar{v}u_{app}Lv - vu_{app}\overline{Lv} + |u_{app}|^2Lv, \\ g &:= 2u_{app}\bar{v}Lu_{app} - v^2\overline{Lu}_{app} + |v|^2Lu_{app} + v\bar{u}_{app}Lu_{app} - vu_{app}\overline{Lu}_{app}. \end{aligned}$$

We can use again (2.2), so it remains to estimate $Q(Lv)$ and g in $L^1_t(T, \infty; L^2_x)$. For u_{app} we make use only of the pointwise bound $\|u_{app}\|_{L^\infty} \lesssim Mt^{-\frac{1}{2}}$ and the L^2_x bound for Lu_{app}

$$\|Lu_{app}\|_{L^2_x} \lesssim M(1 + M^2\log t),$$

while for v we use the X norm bound (2.24). The same type of analysis as in the proof of (2.27)-(2.29) leads to the estimate

$$\|Q(Lv)\|_{L^1_t(T,2T;L^2_x)} \lesssim M^2T^{-\delta}(1 + M^2\log T)^3\|Lv\|_{\tilde{X}},$$

where the worst term in $Q(Lu)$ is the last one. After dyadic summation this yields

$$\|Q(Lv)\|_{L^1_t(T,\infty;L^2_x)} \lesssim \delta^{-1}M^2T^{-\delta}(1 + M^2\log T)^3\|Lv\|_{\tilde{X}}.$$

This is where we need the condition $\delta \gg M^2$ both in order to have a good dyadic summation, and in order to gain a small Lipschitz constant. Next we bound g in $L^1_t(T,\infty;L^2_x)$; this is better since we use at least one v norm, and we obtain

$$\|g\|_{L^1_t(T,\infty;L^2_x)} \lesssim M^3 T^{-\frac{1}{4}-\delta}(1+M^2\log T)^3.$$

The proof of the theorem is concluded, modulo the proof of Lemma 2.4, which follows.

Proof of Lemma 2.4. We first compute f. For that we need the time derivative of $\mathcal{W}$,

$$\partial_t \mathcal{W}(t,v) = t^{-1} W_{t^{\frac{1}{2}}}(v),$$

where $W_{t^{\frac{1}{2}}}$ is obtained from W via a zero order multiplier which is localized exactly at dyadic frequency $t^{\frac{1}{2}}$. Then we can write

$$\begin{aligned} f = \frac{1}{t^{\frac{1}{2}}} e^{\frac{ix^2}{4t}} e^{i\log t|\mathcal{W}|^2} \Big\{ & \frac{1}{t}\left[W_{t^{\frac{1}{2}}} + 2i\mathcal{W}\log t\Re(W_{t^{\frac{1}{2}}}\bar{\mathcal{W}})\right] \\ & + \frac{1}{t^2}\left[\mathcal{W}'' + 2i\mathcal{W}\log t\Re(\mathcal{W}''\bar{\mathcal{W}}) - 4\mathcal{W}\left(\log t\Re(\mathcal{W}'\bar{\mathcal{W}})\right)^2\right] \\ & + \frac{1}{t^2}\left[2i\mathcal{W}'\log t\Re(\mathcal{W}'\bar{\mathcal{W}}) + 2i\mathcal{W}\log t|\mathcal{W}'|^2\right] \Big\}, \end{aligned}$$

where $\mathcal{W}'$ and $\mathcal{W}''$ denote the first and the second derivative with respect to v. The expression for Lf is computed from this using the observation that $L(e^{\frac{ix^2}{2t}} g(x/t)) = ie^{\frac{ix^2}{2t}}\partial_v g(x/t)$. From (2.18) we have the L^∞ and L^2_v and bounds

$$\begin{aligned} & \|\mathcal{W}\|_{L^\infty} \lesssim M, \quad \|\mathcal{W}'\|_{L^\infty} \lesssim Mt^{\frac{1}{4}-\delta}, \\ & \|\mathcal{W}'\|_{L^2_v} \lesssim M, \qquad \|\mathcal{W}''\|_{L^2_v} \lesssim Mt^{\frac{1}{2}-\delta}, \qquad \|\mathcal{W}'''\|_{L^2_v} \lesssim Mt^{1-\delta}, \\ & \|W_{t^{\frac{1}{2}}}\|_{L^2_v} \lesssim Mt^{-\frac{1}{2}-\delta}, \quad \|W'_{t^{\frac{1}{2}}}\|_{L^2_v} \lesssim Mt^{-\delta}. \end{aligned} \tag{2.30}$$

Using these bounds it is easy to see that the following estimates hold

$$\|f\|_{L^2_x} \lesssim Mt^{-\frac{3}{2}-\delta}(1+M^2\log t)^2, \quad \|Lf\|_{L^2_x} \lesssim Mt^{-1-\delta}(1+M^2\log t)^3. \tag{2.31}$$

Then the bound for Φf in (2.23) follows easily by time integration and (2.2). Unfortunately, a direct integration in the bound for Lf in (2.31) yields an extra δ^{-1} factor,

$$\|Lf\|_{L^1_t(T,\infty;L^2_x)} \lesssim \delta^{-1}Mt^{-\delta}(1+M^2\log t)^3,$$

so the bound for ΦLf in (2.23) cannot be obtained directly.

To improve on this, we first peel off the better part of Lf, which includes all terms which do not contain either of the factors $W_{t^{\frac{1}{2}}}$, $\mathcal{W}''$. Precisely, we set

$$h := \frac{1}{t^{\frac{3}{2}}} e^{\frac{ix^2}{2t}} \partial_v Z,$$

where the expression of Z is given by

$$Z(t,v) = e^{i\log t|\mathcal{W}|^2}\left(W_{t^{\frac{1}{2}}} + 2i\mathcal{W}\log t\Re(W_{t^{\frac{1}{2}}}\bar{\mathcal{W}}) + \frac{1}{t}\mathcal{W}'' + 2i\frac{1}{t}\mathcal{W}\log t\Re(\mathcal{W}''\bar{\mathcal{W}})\right).$$

The function Z is essentially localized around frequency $t^{\frac{1}{2}}$; this is seen in the estimates below for Z, which are computed in terms of the L^2_v-norm of $W_{\leq t^{\frac{1}{2}}}$:

$$\|\partial_v^j Z\|_{L^2_v} \lesssim t^{-1+\frac{j}{2}}(1+M^2\log t)^{j+1}\|W''_{\leq t^{\frac{1}{2}}}\|_{L^2_v}, \quad j=0,1,2. \tag{2.32}$$

Since the regularity of W is $H^{1+\delta}$, this shows that the map from W to Z is mostly diagonal with respect to frequencies, with rapidly decaying off-diagonal tails.

The difference $Lf - h$ can be shown to have better time decay,

$$\|Lf - h\|_{L^2_x} \lesssim M^3 t^{-\frac{5}{4}-\delta}(1+M^2\log t)^3,$$

which is stronger than needed. It remains to consider the output of h, for which it is no longer enough to obtain a fixed time L^2 bound and then integrate it in time. Instead, we consider Φh directly.

To estimate Φh we first compute the Fourier transform of h,

$$\hat{h}(\xi) = \frac{1}{t^{\frac{3}{2}}}\int e^{-ix\xi}e^{\frac{ix^2}{2t}}\partial_v Z(t,x/t)\,dx = \frac{1}{t^{\frac{1}{2}}}e^{\frac{it\xi^2}{2}}\int e^{\frac{it(\xi-v)^2}{2}}\partial_v Z(t,v)\,dv.$$

Interpreting the last integral as a convolution, we compute its pullback to time zero,

$$(e^{\frac{it\partial_x^2}{2}}h)(t,x) = t^{-1}e^{\frac{ix^2}{2t}}\widehat{(\partial_v Z)}(t,x) = t^{-1}e^{\frac{ix^2}{2t}}ix\hat{Z}(t,x),$$

which, in view of (2.32), is mainly concentrated in the dyadic region $x \approx t^{\frac{1}{2}}$. Then the solution to the backward Schrödinger equation is

$$\Phi h(t) = e^{-\frac{it\partial_x^2}{2}}z(t), \qquad z(t,x) = ix\int_t^\infty s^{-1}e^{\frac{ix^2}{2s}}\hat{Z}(s,x)\,ds$$

Now we take advantage of the fact that, in the above integral, dyadic regions in t essentially contribute to different dyadic regions in x. This shows that

$$\|t^{-1}e^{\frac{ix^2}{2s}}x\hat{Z}(s,x)\|_{l^2L^1_t(T,\infty;L^2_x)} \lesssim \|W'_{\geq T^{\frac{1}{2}}}\|_{L^2_v} + T^{-\frac{1}{2}}\|W''_{\leq T^{\frac{1}{2}}}\|_{L^2_v} \lesssim MT^{-\delta}(1+M^2\log T)^3,$$

where the l^2 norm is taken with respect to dyadic regions in frequency. After time integration this implies that

$$\|z(t)\|_{l^2\dot{W}^{1,1}(T,\infty;L^2_x)} \lesssim MT^{-\delta}(1+M^2\log T).$$

Here we cannot interchange the l^2 and the $\dot{W}^{1,1}$ norm. However, we can do it if we relax $\dot{W}^{1,1}$ to the space V^2 of functions with bounded 2 variation,

$$l^2\dot{W}^{1,1}(T,\infty;L^2_x) \subset l^2V^2(T,\infty;L^2_x) \subset V^2(T,\infty;l^2L^2_x) = V^2(T,\infty;L^2_x).$$

Thus we obtain

$$\|z(t)\|_{V^2(T,\infty;L^2_x)} \lesssim MT^{-\delta}(1+M^2\log T)^3.$$

Then the desired conclusion

$$\|\Phi h(t)\|_{L^\infty(0,T;L^2)} + \|\Phi h(t)\|_{L^4(0,T;L^\infty)} \lesssim Mt^{-\delta}(1+M^2\log t)^3$$

follows in view of the Strichartz embeddings for V^2 spaces,

$$\|e^{-\frac{it\partial_x^2}{2}}z(t)\|_{L^\infty L^2} + \|\Phi h(t)\|_{L^4L^\infty} \lesssim \|z\|_{V^2L^2},$$

see Section 4 in [8].

□

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