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Global classification of isolated singularities in dimensions (4,3) and (8,5)

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Abstract. We characterize those closed $2k$-manifolds admitting smooth maps into $(k+1)$-manifolds with only finitely many critical points, for $k \in \{2, 4\}$. We compute then the minimal number of critical points of such smooth maps for $k = 2$ and, under some fundamental group restrictions, also for $k = 4$. The main ingredients are King's local classification of isolated singularities, decomposition theory, low dimensional cobordisms of spherical fibrations and 3-manifolds topology.

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1. Introduction and statements

Let $\varphi(M^m, N^n)$ denote the minimal number of critical points (not necessarily non-degenerate) of smooth proper (*i.e.* such that the inverse image of the boundary is the boundary) maps between the smooth manifolds M^m and N^n. When superscripts are specified they denote the dimension of the respective manifolds.

When we take $N^n = \mathbb{R}$ one obtains a classical homotopy invariant, namely the F-category of the manifold M^m. Using Morse-Smale theory Takens proved that the F-category of a closed manifold is bounded by $m + 1$ (see [57] for a refined upper bound in terms of the connectedness, the homology groups and the dimension). The F-category is bounded from below by the Lusternik-Schnirelman category. Related results for h-cobordisms were obtained by Pushkar and Rudyak (see [46]).

We are interested below in the case when $m \geq n \geq 2$ and the manifolds under consideration are compact and orientable. The invariant φ is not anymore a homotopy invariant of the pair (M^m, N^n) and actually it is sensitive to the smooth structures. There are actually only a few examples where this invariant can be calculated. The main problem in this area is to characterize those pairs of manifolds for which φ is finite non-zero and then to compute its value (see [12, page 617]). The problem of finding non-trivial local isolated singularities was also stated by Milnor ([44, Section 11, page 100]).

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In [1] the authors found that, in all codimension $0 \leq m-n \leq 2$, if $\varphi(M^m, N^n)$ is finite then $\varphi(M^m, N^n) \in \{0, 1\}$, except for the exceptional pairs of dimensions $(m, n) \in \{(2, 2), (4, 3), (4, 2)\}$. Further, if $m - n = 3$ and there exists a smooth function $M^m \to N^n$ with finitely many critical points, all of them cone-like, then $\varphi(M^m, N^n) \in \{0, 1\}$ except for the exceptional pairs of dimensions $(m, n) \in \{(5, 2), (6, 3), (8, 5)\}$. Moreover, under the finiteness hypothesis, $\varphi(M^m, N^n) = 1$ if and only if M^m is the connected sum of a smooth fibration over N^n with an exotic sphere and not a fibration itself. A short outline of this and related results for Lefschetz fibrations can be found in [2]. These results comfort the idea that, in general, φ is infinite. There is no enough room to plug inside a singular fiber enough of the topology of the manifold, at least when the codimension is much smaller with respect to the dimension. This might be compared to the Conjecture stated by Milnor in ([44, page 100]).

There are two essential ingredients in this result. First, the germs of smooth maps $\mathbb{R}^m \to \mathbb{R}^n$, in the given range of dimensions, having an isolated singularity at origin are locally topologically equivalent to a projection. Apparently the first who noticed that a smooth map of the m-disk in itself with a single interior critical point should actually be a homeomorphism for $m \geq 3$ was Heinz Hopf in the early thirties. The next step was taken by Antonelli, Church, Hemmingsen and Timourian who analyzed in the seventies the local structure of smooth maps around a critical point, in codimension at most 2. Except a few exceptional dimensions these maps are locally topologically equivalent to projections and thus they are topological fibrations. The second point is that each critical point can be smoothed by removing a disk and gluing it back, namely by adjoining a homotopy sphere. Thus all but possibly one critical points merge together.

The simplest exceptional case is that of pairs of surfaces, which admits an elementary treatment and is completely understood (see [2] for explicit computations). In [22] the authors described a series of examples in dimensions $(4, 3)$, $(8, 5)$ and $(16, 9)$. In these dimensions one knew about the existence of smooth functions with genuine singular points since the papers of Antonelli ([3, 4]). These are particular Montgomery-Samelson fibrations ([45]) obtained from a spherical fibration by pinching a number of fibers to points. Around such a critical point the map is locally topologically equivalent to the suspension over a Hopf fibration of spheres. This also explains the occurrences of dimensions $(4, 3)$, $(8, 5)$ and $(16, 9)$. The novelty brought out in [22] is that the invariant φ can be explicitly computed. There are obstructions to the clustering of genuine singular points, whose number is determined by the algebraic topology of the manifold. In particular, one founds that φ can take now arbitrarily large even values, contrasting with the situation treated before in [1].

The first aim of this paper is to explore the codimension 3 situation by 3-manifold topology methods. When the singularities are cone-like the general methods from [1] permit to settle the problem, at least for $n \geq 4$. However, Takens (see [56]) provided examples of wild (*i.e.* not cone-like) codimension 3 isolated singularities. In order to understand wild singularities we revisit King's ([36, 37]) local classification of isolated singularities in terms of topological data consisting of a local fibration and a vanishing compactum. We first obtain that there are actually uncountably many non-equivalent germs of wild isolated singularities in codimension 3, so previous techniques cannot work. The equivalence classes here correspond to right composition by (germs of) homeomorphisms. We introduce then the notion of removable singularity, where one enlarges the orbit of a singularity germ by allowing right compositions with limits of homeomorphisms. The key technical result is that codimension 3 singularities are removable, as soon as $n \geq 4$. In particular, we obtain:

Theorem 1.1. *Let $M^{n+3} \to N^n$ be a smooth map with finitely many critical points. Then M^{n+3} is diffeomorphic to $\Sigma^{n+3} \# Q^{n+3}$ where*:

(1) *Σ^{n+3} is a homotopy sphere*;
(2) *there is a smooth* (*induced*) *map $Q^{n+3} \to N^n$ having only finitely many critical points and fulfilling*:
   (a) *if $n \geq 6$ or $n = 4$ then there are no critical points*;
   (b) *if $n = 5$ then around each critical point the map is smoothly equivalent to the cone over the corresponding Hopf fibration.*

We do not know whether wild singularities show up for general (m, n). A more tractable problem in higher dimensions is to understand whether there exist smooth maps $M^m \to N^n$ with only cone-like isolated singularities. The local version of this problem consists of constructions of non-trivial germs of smooth (or even polynomial) maps and it was actually reduced by Looijenga, Church and Lamotke (see [11, 40]) to the existence of high dimensional fibered links. This was further pursued by Rudolph, Kauffman and Neumann ([35, 49]) who considered general open book decompositions and more recently in [5]. The global question is how can we piece together conical extensions of open book decompositions to get well-defined maps $M^m \to N^n$ and whether critical points could be cancelled.

Milnor asked in ([44, page 100]) about the right meaning of being a *non-trivial* local isolated singularity and gave as an example the requirement that the local generic fiber be a disk. It was shown in [40] that an isolated singularity is trivial iff its germ is right equivalent to a topological projection (at least when the codimension is different from 4 and 5). In this paper we consider a weaker non-triviality condition, which we call removability, whose meaning is that we can change the map in a small neighborhood of the critical point so that the new function has no (topological) critical points anymore. In other words the critical point does not contribute to φ. Notice that the non-trivial examples from [11] are actually removable isolated singularities. The search for non-removable global examples is therefore rather subtle.

Remark 1.2. If the group of homeomorphisms of a contractible manifold F (of dimension $m - n \geq 5$) which are identity on the boundary were contractible, then we could prove by the present methods that $\varphi(M^m, N^n) \in \{0, 1\}$ whenever $n \geq m - n + 1$, $n \geq 2$ and $(m, n) \notin \{(4, 3), (8, 5), (16, 9)\}$. However, the connectivity results from [9] are not sufficient for obtaining the triviality of the fibrations with fiber F when $n \geq m - n + 1$. This is related to Milnor's Conjecture from ([44, page 100]) stating the triviality (here in the sense of removability) of local isolated cone-like singularities in this range of dimensions. This is similar to the constraints given in [47] for the existence of even non-trivial cone-like isolated singularities.

The second aim of this paper is to use the local description of singularities in order to analyze the global topology of the involved manifolds in the case of exceptional dimensions. We provide here a complete solution for this problem in dimensions $(4, 3)$ and $(8, 5)$. We will show that the examples obtained in [22] are essentially all examples with finite φ, up to a certain operation on manifold fibrations called *fiber sum*.

The case of dimensions $(4, 3)$ still relies on the local classification of singularities due to Church and Timourian but the case of dimensions $(8, 5)$ needs Theorem 1.1, which enables us to work with local models which are cones over Hopf fibrations. The main difficulty is then to determine which are the smooth manifolds obtained by piecing together these local models.

It appears that the gluing process is surprisingly rigid and we obtain always connected sums of $S^2 \times S^2$ and respectively of $S^4 \times S^4$. Eventually, we can also glue along the fibers a multi-spherical fibration which does not contribute to the critical points. The result of such a construction is abstracted below:

Definition 1.3. Assume given the following data:

(1) an F-bundle $g : E^m \to Q^{k+1}$, and a smooth map $h : X^m \to D^{k+1}$ with finitely many critical points such that the generic fiber of h is F and
(2) an isomorphism between the F-fibrations $g|_{\partial E} : \partial E \to \partial Q$ and $h|_{\partial X} : \partial X \to \partial D^{k+1}$, (thus $\partial Q^{k+1} = S^k$).

Glue together g and h to obtain a smooth map $f : E^m \cup X^m \to Q^{k+1} \cup D^{k+1}$, with finitely many critical points, by identifying the boundary fibrations by means of the fixed bundle isomorphism. We call f the *generalized fiber sum* of g and h and denote it by $g \oplus h$; sometimes we mention also the isomorphism type α of the boundary F-fibration $h|_{\partial X}$ over S^k and write $g \oplus_\alpha h$. Notice that this fibration might well be non-trivial, in contrast to the usual fiber sum construction used in [22]. One says that h is the *non-trivial summand* and g the *fibrewise summand* of the fiber sum.

When $g : E^m \to Q^{k+1}$ and $h : X^m \to D^{k+1}$ are obtained from the smooth maps $G : \widehat{E}^m \to \widehat{Q}^{k+1}$, $H : \widehat{X}^m \to S^{k+1}$ between closed manifolds, by removing trivial fibrations over disks, then $g \oplus h$ is the *fiber sum* of the maps G and H.

The main result of this paper is the following structure theorem:

Theorem 1.4. *Let M^{2k} and N^{k+1} be closed orientable manifolds having finite $\varphi(M^{2k}, N^{k+1})$, where $k \in \{2, 4\}$. Then*

(1) *either M^{2k} is diffeomorphic to $W^{2k} \# \Sigma^{2k}$, where W^{2k} is a fibration over N^{k+1} and Σ^{2k} is a homotopy $2k$-sphere. In this case $\varphi(M^{2k}, N^{k+1}) \in \{0, 1\}$.*

(2) *or else, M^{2k} is diffeomorphic to the iterated fiber sum $W^{2k} \oplus_{j=1}^{D} \#_{r_j} S^k \times S^k \# \Sigma^{2k}$, with $r_i \geq -1$, but at least one $r_i \geq 0$, where*:

(a) *the fibrewise summand W^{2k} is a S^{k-1}-fibration over some closed manifold $\widehat{N^{k+1}}$, where $\widehat{N^{k+1}} \to N^{k+1}$ is a non-ramified covering of finite degree D.*

(b) *the fiber sum $W^{2k} \oplus_{j=1}^{D} \#_{r_j} S^k \times S^k$ is the fiber sum along the fiber of the fibration $W^{2k} \to N^{k+1}$ (which consists of D disjoint copies of S^{k-1}) with the disjoint union $\sqcup_{j=1}^{D} \#_{r_j} S^k \times S^k$. Each connected component of the later is endowed with a smooth map $\#_{r_j} S^k \times S^k \to S^{k+1}$ whose generic fiber is S^{k-1} (to be defined below).*

(c) *$\#_{-1} S^k \times S^k$ denotes $S^{k-1} \times S^{k+1}$, $\#_0 S^k \times S^k$ denotes S^{2k} and Σ^{2k} a homotopy $2k$-sphere.*

Moreover, in this case we have:

$$\varphi(M^{2k}, N^{k+1}) \leq 2r_1 + \cdots + 2r_D + 2D.$$

Remark 1.5. The fiber sum of W^{2k} with $\#_{-1} S^k \times S^k = S^{k-1} \times S^{k+1}$ has no effect on the topology of W^{2k} since it is diffeomorphic to W^{2k}. Thus, if all $r_i = -1$ then we recover the manifold W^{2k} and thus M^{2k} fibers over N^{k+1}.

Remark 1.6. There is one extra piece in the fiber sum data, namely the choice of a bundle isomorphism between each pair of trivial S^{k-1} fibrations over S^k. Such bundle isomorphisms up to isotopy correspond to the set of homotopy classes of maps $S^k \to \mathrm{Homeo}^+(S^{k-1})$.

If $k = 2$ this set is in bijection with $\pi_2(SO(2)) = 0$ and thus the fiber sum construction is unambiguously defined.

When $k = 4$, using Hatcher's solution to the Smale Conjecture, this set is in bijection with $\pi_4(\mathrm{Homeo}^+(S^3)) = \pi_4(SO(4)) = \mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$ and the fiber sum depends on the choice of D elements of this group, which will be called gluing parameters in the sequel.

When M^{2k} satisfies the first alternative of Theorem 1.4 it is a topological fibration over N^{k+1}. If M^{2k} is not a smooth fibration then $\varphi(M^{2k}, N^{k+1}) = 1$.

Remark 1.7. The manifold $M^m = \Sigma^m \# S^{m-n-1} \times S^{n+1}$ is not diffeomorphic to $S^{m-n-1} \times S^{n+1}$ if Σ^m is an exotic sphere (see [50]). This yields effective examples where $\varphi = 1$. For instance, if Σ^8 is the exotic 8-sphere which generates the group $\Gamma_8 = \mathbb{Z}/2\mathbb{Z}$ of homotopy spheres of dimension 8 then $\varphi(\Sigma^8 \# S^3 \times S^5, S^5) = 1$ (see *e.g.* [22]).

When the second alternative of Theorem 1.4 holds, M^{2k} will be called *non-fibered* (over N^{k+1}). Of course these furnish the most interesting examples with non-trivial φ. Moreover, Theorem 1.4 leads to effective topological obstructions to the finiteness of φ, as follows:

Corollary 1.8.

(1) *If $\varphi(M^8, N^5)$ is finite and M^8 is non-fibered over N^5 then $\pi_1(M^8)$ is a finite index subgroup of $\pi_1(N^5)$.*
(2) *If $\varphi(M^4, N^3)$ is finite and M^4 is non-fibered over N^3 then $\pi_1(M^4) \cong \pi_1(N^3)$.*

Further, there are global constraints of topological nature to the clustering of genuine critical points. This situation seems rather exceptional and specific to the dimensions (4, 3), (8, 5) and (16, 9). Moreover, the number of genuine critical points is independent on the smooth function considered and coincides with φ, which is therefore controlled by the topology, as follows:

Theorem 1.9. *Let M^{2k} and N^{k+1} be closed orientable manifolds with finite $\varphi(M^{2k}, N^{k+1})$, with $k \in \{2, 4\}$, and M^{2k} non-fibered over N^{k+1}. Let $r = r_1 + r_2 + \cdots + r_D$, where r_j, D are those furnished by Theorem* 1.4.

(1) *If $k = 4$, assume that $\pi_1(M^8) \cong \pi_1(N^5)$ is a co-Hopfian group or a finitely generated free non-Abelian group. Then $\varphi(M^8, N^5) = 2r + 2D$.*
(2) *If $k = 2$, then $\varphi(M^4, N^3) = 2r + 2D$.*

One might conjecture that the assumptions on $\pi_1(M^8)$, when $k = 4$, are superfluous and that we have $\varphi(M^8, N^5) = 2r + 2D$ for any non-fibered M^8, so that the right hand side is independent on the particular choices we made for writing the non-fibered M^8 as a fiber sum.

Remark 1.10. If the fibration of W^{2k} is the connected sum of S^1-products of Hopf fibrations, namely it is the fibration $\#_c S^1 \times S^{2k-1} \to \#_c S^1 \times S^k$ then the manifold M^{2k} is diffeomorphic to $\Sigma^{2k} \#_{r+c} S^k \times S^k \#_c S^1 \times S^{2k-1}$ and the value for φ given above is consistent with that obtained in the main theorem of [22].

Remark 1.11. For $k \in \{2, 4\}$, the isomorphism classes of smooth S^{k-1}-fibrations over $\#_c S^1 \times S^k$ are classified by elements of $\pi_{k-1}(SO(k))^c$ which are either $(\mathbb{Z} \oplus \mathbb{Z})^c$ (for $k = 4$) or else $\mathbb{Z}^c$ (for $k = 2$).

As a consequence of the present paper and [1] we obtain the global classification of isolated singularities in codimension at most 3 except for the dimensions (4, 2), (5, 2) and (6, 3).

The local classification of isolated cone-like singularities in dimensions (6, 3) corresponds to that of fibered links $\sqcup_{j=1}^{k} S^2$ into S^5 for $k \geq 2$, which is given by the linking matrix. However, new methods are necessary for obtaining the global classification of isolated singularities in dimensions (6, 3). For instance, we don't know whether in the process of pasting together local singularities we could have any cancellation of critical points.

The case of dimensions (4, 2) offers another interesting perspective, because of the abundance of non-equivalent fibered links in S^3. Their suspensions yield examples in dimensions (5, 2).

The plan of the paper is as follows. The first section describes the local classification of isolated singularities of codimension at most 2, following Church and Timourian and of cone-like isolated singularities of codimension 3 and dimension at least 4. The next section deals with wild isolated singularities. In order to understand them better we revisit King's classification of cone-like and general singularities. Each local singularity corresponds to some topological data consisting of a fibration E (like in a open book structure) over a sphere, a vanishing compactum $\mathcal{A} \subset E$ whose complementary is a trivial fibration over the sphere and a retraction r of E onto the singular fiber V obtained from the regular fiber by crushing the vanishing compactum. King's classification of isolated singularities states that this data up to a certain equivalence relation determines the right equivalence class of the germ. In the cone-like case the statement is more precise as one showed that all such data with tame vanishing compactum $\mathcal{A}$ are realized by a cone-like singularity. Our contribution is to explain that in general, what we require is that the mapping cylinder of the retraction r be a topological manifold. We intend further to use this to the classification of wild isolated singularities of codimension 3. It appears that fibrations whose fiber is a manifold with boundary of dimension at most 3 and whose boundary is already a product (over a high dimensional sphere) tend to be trivial. This is a consequence of the contractibility of the group of homeomorphisms, due to Earle, Eells and Schatz for large surfaces and to Hatcher for irreducible 3-manifolds. Some work is needed to understand that the reducible case would lead to non-trivial homotopy of the singular fiber, which cannot happen in large dimensions. When the associated fibration is trivial the mapping cylinder of the retraction r is the quotient of a product by the vanishing compactum. Decomposition theory will tell us that such a quotient space is a manifold only when the crushed compactum is cellular and then the quotient surjection is a near-homeomorphism. This actually explains how the wild singularities can be constructed (by using suitable Artin -Fox wild arcs in S^3) and why their local structure is almost-trivial: although there are uncountably many wild singularities non-equivalent by right composition with homeomorphisms, they are all equivalent by right composition with a near-homeomorphism. The near-homeomorphisms are precisely the cell-like maps, as it was proved by Siebenmann and thus they are limits of homeomorphism. This leads to Theorem 1.1. We further want to describe a global structure result for manifolds M^{2k} admitting smooth maps with finitely many critical points into N^{k+1}, for $k \in \{2, 4\}$, where we know how the local germs look like. We split off a fibration factor to obtain a sub-manifold X^{2k} having finite $\varphi(X^{2k}, D^{k+1})$, which is called a *disk block*. If $\varphi(X^{2k}, S^{k+1})$ is finite then X^{2k} is called a *spherical block*. We can first use general classification results of manifolds to find the diffeomorphism type of spherical blocks in dimension 8 and of the homeomorphism type of spherical blocks in dimension 4. These manifolds are actually Montgomery-Samelson fibrations with finite branch set and their algebraic invariants are known since the seventies. In order to pursue further we need some ad-hoc techniques. We remark that disk blocks are obtained by gluing together the cones over the Hopf fibrations. Constructing such a disk block amounts to find a cobordism between the associated fibrations. However the theory of cobordisms of spherical fibrations is quite simple in these dimensions. This shows that such disk blocks are connected sums of spherical blocks from which one delete a neighborhood of a generic fiber. This gives the structure Theorem 1.4 in dimension 8. In order to obtain the 4-dimensional situation we have to understand the spherical blocks with 2 critical points. As a consequence of Cerf's theorem $\Gamma_4 = 0$ these are smooth S^4. We can further show that all spherical blocks are connected sums of $S^2 \times S^2$ up to adding some homotopy 4-sphere. We obtain also an upper bound for the number φ of critical points, which we are able in the last section to prove that it is sharp. In fact, when the fundamental group is co-Hopfian we can show that φ is determined by the homology of the manifolds involved. Using Thurston's geometrization Conjecture in dimension 3 (as settled by Perelman) and the results of Wang, Yu and Wu which characterize all 3-manifolds whose fundamental group is non co-Hopfian we can settle completely the case of dimensions $(4, 3)$, by showing that the previous value of φ is independent on the choices of intermediary coverings.

Proviso. We will make use through out the present paper of the validity of the Thurston geometrization Conjecture and, in particular, of the Poincaré conjecture in dimension 3, as settled by Perelman (see [7] for a detailed proof), but we will specify its use each time when applied.

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2. Local structure of isolated singularities of small codimension

2.1. Local classification of isolated singularities of codimension at most 2

Let $f : M^m \to N^n$ be a smooth map with finitely many critical points throughout this section.

Church and Timourian (see [12, 14]) proved the following:

Proposition 2.1. *If $m-n \leq 2$ and $x \in M$ then f is locally topologically equivalent to one of the following local models*:

1. *a projection*;
2. *the map $g : \mathbb{C} \to \mathbb{C}$ given by $g(z) = z^d$, $d \in \mathbb{Z}_+$, when $m = n = 2$*;
3. *the Kuiper map $\tau_{2,1} : \mathbb{C}\times\mathbb{C} \to \mathbb{C}\times\mathbb{R}$ given by $\tau_{2,1}(z, w) = (2z\bar{w}, |w|^2-|z|^2)$, when $m = 4$, $n = 3$*;
4. *a map $\rho : \mathbb{R}^4 \to \mathbb{R}^2$ which is locally topologically equivalent to the projection except at one point, when $m = 4$ and $n = 2$.*

Remark 2.2. There are more general Kuiper maps (see [44, page 103]) $\tau_{a,k}$: $\mathbb{R}^{2ak} \to \mathbb{R}^{a+1}$, where $a \in \{1, 2, 4\}$ and $k \in \mathbb{Z}_+$, defined as

$$\tau_{a,k} : A^k \times A^k \to A \times \mathbb{R},\ \tau_{a,k}(x, y) = (\langle x, y\rangle, |x|^2 - |y|^2).$$

Here $A \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$, where $\mathbb{H}$ denotes the quaternions, $\mathbb{O}$ denotes the Cayley numbers, $a \in \{2, 4, 8\}$ corresponds to the dimension of A as a real vector space and $\langle *, *\rangle$ denotes a Hermitian inner product. The map $\tau_{a,1}$ restricts to the Hopf fibration between the corresponding unit spheres $S^{2a-1} \to S^a$ and thus it is equivalent to the suspension of the Hopf fibration. For any $k \in \mathbb{Z}_+$ the map $\tau_{a,k}$ has an isolated singularity at the origin whose local fiber is a disk bundle over S^{ak-1}.

One might wonder whether there are only finitely many non-equivalent germs of isolated singularities, when $m - n = 3$. First we should add one more germ appearing when $m = 8$ and $k = 5$, namely the Kuiper map $\tau_{4,1}$, which is the cone of the Hopf fibration $S^7 \to S^4$. However, it is still not enough (see the next section) since there exist infinitely many non-equivalent isolated germs, in general. But there are some slightly weaker statements which are true. We could restrict ourselves to cone-like isolated singularities, as those arising from functions which are locally real analytic around the critical points. On the other hand we will define the concept of removability and show that those singularities which are not cone-like are actually removable.

The aim of the next two sections is to give a more conceptual and self-contained proof of Proposition 2.1 and its various extensions to codimension 3.

Definition 2.3. Let $V = f^{-1}(f(x))$, where x is a critical point. Following King (see [36]) the singular point x is called *cone-like* if it admits a cone neighborhood in V, *i.e.* there exists some closed manifold $L \subset V \setminus \{x\}$ and a neighborhood N of x in V which is homeomorphic to the cone $C(L)$ over L. Recall that the cone is defined as the quotient $C(L) = L \times (0, 1]/L \times \{1\}$. Then the manifold L is called the local link at x. If x is not cone-like then x (and also V) are called *wild*. An isolated point will be considered to be cone-like, as being the cone over the empty set.

The first examples of smooth maps with isolated wild singularities were obtained by Takens (see [56]) in codimension 3. From [12, 14] such examples cannot occur in smaller codimension because of the following:

Proposition 2.4. *Isolated singularities of smooth functions in codimension at most 2 are cone-like.*

Proof. This is clear from Proposition 2.1, except when $m = 4$ and $k = 2$. This case is settled in ([14, Lemmas 2.1 and 2.4]). We will give another proof in the next section. □

2.2. Local classification of cone-like isolated singularities in codimension 3

Denote by $S(f)$ the *topological branch locus* of the map f. Namely $x \notin S(f)$ if f is locally topologically equivalent to a projection around x.

Proposition 2.5. *Let $f : M^{n+3} \to D^n$, $n \geq 4$ be a smooth proper map from the compact manifold M^{n+3} onto the n-disk having a single critical point $x \in M$ and suppose that $f(x) = 0$. Then one of the following holds*:

(1) *$x \notin S(f)$;*
(2) *x is an isolated point of $V = f^{-1}(0)$;*
(3) *or else $x \in S(f)$, V is wild at x.*

In the proof of this Proposition we use the following lemma stated also in [22]:

Lemma 2.6. *If M^{n+q} and N^n are smooth manifolds and $f : M^{n+q} \to N^n$ is a smooth map with finitely many critical points, then the inclusions $M \setminus V \hookrightarrow M$ and $N \setminus B \hookrightarrow N$ are $(n-1)$-connected.*

Proof. For the sake of completeness we give here the proof. The result is obvious for $N \setminus B \hookrightarrow N$. It remains to prove that $\pi_j(M, M \setminus V) \cong 0$, for $j \leq n-1$. Take $\alpha : (D^{j+1}, S^j) \to (M, M \setminus V)$ to be an arbitrary smooth map of pairs. Since the critical set $C(f)$ of f is finite and included in V, there exists a small homotopy of α relative to the boundary such that the image $\alpha(D^{j+1})$ avoids $C(f)$. By compactness there exists a neighborhood U of $C(f)$ consisting of disjoint balls centered at the critical points such that $\alpha(D^{j+1}) \subset M \setminus U$. We can arrange by a small isotopy that V become transversal to ∂U.

Observe further that $V \setminus U$ consists of regular points of f and thus it is a properly embedded sub-manifold of $M \setminus U$. General transversality arguments show that α can be made transverse to $V \setminus U$ by a small homotopy. By dimension counting this means that $\alpha(D^{j+1}) \subset M \setminus U$ is disjoint from V and thus the class of α in $\pi_j(M, M \setminus V)$ vanishes. □

Lemma 2.7. *If M^{n+3} is contractible and $n \geq 4$ then the fiber F^3 of the fibration $f : M^{n+3} \setminus V \to D^n \setminus \{0\}$ is diffeomorphic to a 3-disk.*

Proof. Lemma 2.6 states that $\pi_j(M, M \setminus V) = 0$, for $j \leq n-1$ and thus $\pi_j(M \setminus V) = 0$, for $j \leq n-2$. The long exact sequence in homotopy of the fibration $f : M^{n+3} \setminus V \to D^n \setminus \{0\}$ implies that the generic fiber F^3 has $\pi_j(F) = 0$, for $j \leq n-3$. If $n \geq 5$ then F^3 is contractible and thus, by the Poincaré conjecture, diffeomorphic to a 3-disk.

If $n = 4$ then F^3 is a simply connected 3-manifold with boundary and hence its boundary consists of several 2-spheres. Thus $H_2(F)$ is free Abelian. The base space of the fibration $f : M^{n+3} \setminus V \to D^n \setminus \{0\}$ is homotopy equivalent to S^{n-1} and thus there is a Wang exact sequence for this fibration which reads:

$$0 = H_3(F) \to H_3(M^{n+3} \setminus V) \to H_0(F) \to H_2(F) \to H_2(M^{n+3} \setminus V) = 0.$$

Assume that $\pi_3(M \setminus V) \neq 0$. Then, using Hurewicz the group $H_3(M \setminus V) \neq 0$ is a non-trivial subgroup of $H_0(F) = \mathbb{Z}$ and hence it is isomorphic to $\mathbb{Z}$. This implies that the rank of $H_2(F)$ over $\mathbb{Q}$ is zero and hence $H_2(F) = 0$ because it is free abelian. By Hurewicz, we have $\pi_2(F) = 0$ and thus, by the Poincaré Conjecture, F is diffeomorphic to the 3-disk, as claimed.

Otherwise, $\pi_3(M \setminus V) = H_3(M \setminus V) = 0$ and the Wang sequence above gives us $H_2(F) = \mathbb{Z}$ and $H_4(M \setminus V) = 0$. By the Poincaré Conjecture a simple connected 3-manifold is a holed disk and thus $F = S^2 \times [0, 1]$. Moreover, from Hurewicz we obtain $\pi_4(M \setminus V) = 0$. Furthermore, from the long exact sequence in homotopy of the fibration $f : M \setminus V \to D^4 \setminus \{0\}$ we derive:

$$0 = \pi_4(M \setminus V) \to \pi_4(D^4 \setminus \{0\}) \to \pi_3(F) = \pi_3(S^2) \to \pi_3(M \setminus V) = 0$$

But $\pi_4(S^3) = \mathbb{Z}/2\mathbb{Z}$ while $\pi_3(S^2) = \mathbb{Z}$, which contradicts the exacteness of the sequence above. This proves the claim. □

Proposition 2.8. *If the isolated critical point x of $f : M^{n+q} \to N^n$ is cone-like then there exists a compact manifold with boundary $K^{n+q} \subset M^{n+q}$ containing x and a disk neighborhood D^n containing the critical value $f(x)$ such that*:

(1) *the restriction $f : K^{n+q} \to D^n$ is proper and has only one critical point x*;
(2) *K^{n+q} is contractible.*

Proof. This is basically contained in [36]. Here is a simple proof. We consider a small enough manifold neighborhood K^{n+q} of x intersecting transversely $f^{-1}(D^n)$ and containing only one critical point. Let F^q be the fiber. Then K^{n+q} deformation retracts onto $V \cap K^{n+q}$ which is a cone $C(L)$ and thus contractible. □

Remark 2.9. Actually if $n + q \neq 4, 5$ we can assume that K^{n+q} is homeomorphic to a ball and its boundary to a sphere (see [36]). This implies that the fiber F^3 is simply connected also when $n = 3$.

Proposition 2.10. *If $m - n = 3$ and $x \in M^m$ is a cone-like isolated singularity then f is locally topologically equivalent to one of the following local models*:

(1) *a projection*;
(2) *the Kuiper map $\tau_{4,1} : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \times \mathbb{R}$, when $m = 8, n = 5$ (see Remark 2.2)*;

Proof. If x is an isolated point of V then, by [58], f is locally topologically equivalent to the cone of a Hopf fibration and thus to the Kuiper map $\tau_{4,1}$. Henceforth, we assume that z is not an isolated point in $f^{-1}(f(z))$, for any $z \in M$. Then f has a cone-like critical point x implies that $x \notin S(f)$ and hence the result follows from above. □

2.3. King's classification revisited

We consider germs of isolated singularities up to right composition by a homeomorphism, as described by King in [36] for continuous functions and in [37] for smooth functions.

Let $f : N_1^{n+q} \to N_2^n$ be a smooth map with finitely many critical points. Fix a critical point p. A smooth codimension zero compact manifold with boundary $M^{n+q} \subset N_1^{n+q}$ containing x in its interior such that the restriction $f : M^{n+q} \to D^n$ is a proper smooth map having only one critical point onto the disk $D^n \subset N_2^n$ is called an *adapted* neighborhood of the critical point. According to the construction of [36] (see below) there exist arbitrarily small adapted neighborhoods of p.

Let $f^{-1}(S^{n-1}) = E^{n+q-1} \subset \partial M^{n+q}$, $V = f^{-1}(f(p)) \subset M^{n+q}$ and F^q denote the generic fiber, which is a manifold with non-empty boundary.

The restriction $f : E^{n+q-1} \to S^{n-1}$ is a fibration with fiber F^q. Moreover, the restriction to the boundary is a trivial fibration $\partial E \to S^{n-1}$. Further $\partial V \cong \partial F$ is a closed submanifold in ∂M having trivial normal bundle and which is endowed with a natural trivialization $t : \partial F \times D^n \to \partial M$, extending the trivial fibration $\partial E \to S^{n-1}$. Specifically we obtain t by using the identification of $f : \partial M \setminus \mathrm{int}(E) \to D^n$ with the trivial fibration over D^n.

There is a retraction $r : E \to V$ obtained by contracting a vanishing compact $\mathcal{A} \subset E$ as follows:

(1) $E \setminus \mathcal{A}$ is a product, *i.e.* it is homeomorphic to $W \times S^{n-1}$ such that the restriction of $f : E \to S^{n-1}$ to $E \setminus \mathcal{A}$, namely $f|_{E\setminus\mathcal{A}} : E \setminus \mathcal{A} \cong W \times S^{n-1} \to S^{n-1}$ is the projection on the second factor. In particular, the boundary fibration $f|_{\partial E} : \partial E \to S^{n-1}$ is trivial;
(2) $W = F \setminus A$, where $A \subset F$ is some compact and $\mathcal{A} \cap F = A$. Here F is an arbitrary fiber in E and a priori A might depend on the particular fiber;
(3) $V = F/A$ and the class of A in F/A is the singular point p of the fiber V. Thus $V \setminus \{p\}$ is homeomorphic to $W = F \setminus A$;
(4) the retraction $r : E \to V$ is obtained as follows: $r|_{E\setminus\mathcal{A}} : E\setminus\mathcal{A} \cong W \times S^{n-1} \to W \cong V - \{p\}$ is given by the projection on the first factor of the product and $r(x) = p$, if $x \in \mathcal{A}$.

Let $\mathfrak{E}_{n,q}$ be the set of data $(E \to S^{n-1}, \mathcal{A} \subset E)$ as above up to the following equivalence relation. We set $(E_1 \to S^{n-1}, \mathcal{A}_1) \sim (E_2 \to S^{n-1}, \mathcal{A}_2)$ if there exist sub-fibrations $E'_j \subset E_j$ such that $\mathcal{A}_j \subset E'_j$, which are homeomorphic by a fibered homeomorphism $\varphi : E'_1 \to E'_2$ (over S^{n-1}) and there exists a fibered isotopy $h_t : E'_1 \setminus \mathcal{A}_1 \to E'_2$ such that

(1) h_1 is the restriction of $\varphi|_{E'_1\setminus\mathcal{A}_1}$;
(2) $h_0(E'_1 \setminus \mathcal{A}_1) = E'_2 \setminus \mathcal{A}_2$ and the fibered homeomorphism $h_0 : E'_1 \setminus \mathcal{A}_1 \to E'_2 \setminus \mathcal{A}_2$ comes from a homeomorphism between the compactified fibers *i.e.* the fiber restriction $h_0 : F_1 - A_1 \to F_2 - A_2$ is induced from a homeomorphism $F_1/A_1 \to F_2/A_2$.

Proposition 2.11 (King). *The class $\delta(f) = (E \to S^{n-1}, \mathcal{A} \subset E)$ in $\mathfrak{E}_{n,q}$ is a complete invariant of the germ of f, up to right equivalence i.e. up to right composition by a germ of homeomorphism of M at p.*

Proof. This is the main result of [36] in a slightly changed setting. □

Definition 2.12. The compact $A \subset F$ is *tame* if it has a mapping cylinder neighborhood *i.e.* a neighborhood N which is a codimension zero compact submanifold of F which deformation retracts onto A such that $N \setminus A$ is homeomorphic to $\partial N \times (0, 1]$.

Remark 2.13. If A is a polyhedron in F then it has a regular neighborhood and thus it is tame. If A is an Artin-Fox wild arc (see [6]) in D^3 then it is not tame.

Proposition 2.14. *Let $\mathfrak{E}^{PL}_{n,q} \subset \mathfrak{E}_{n,q}$ be the subset of equivalence classes of pairs $(E \to S^{n-1}, \mathcal{A} \subset E)$ for which $\mathcal{A}$ is tame. Then $\delta(f) \in \mathfrak{E}^{PL}_{n,q}$ classifies isolated cone-like singularities of germs up to right equivalence.*

Proof. If $\mathcal{A} \subset E$ is tame then A is tame and thus it has a mapping cylinder neighborhood $N(A) \subset F$. We could further assume that $F = N(A)$ by restricting to a smaller representative in $\mathfrak{E}^{PL}_{n,q}$. In fact, if E' denotes $E \setminus (F \setminus N(A)) \times S^{n-1}) \subset E$, then $(E' \to S^{n-1}, \mathcal{A}) = (E \to S^{n-1}, \mathcal{A}) \in \mathfrak{E}_{n,q}$.

Further, we have $V = N(A)/A$. As $N(A) \setminus A \cong \partial N(A) \times (0, 1]$, it follows that $N(A)/A$ is homeomorphic to the one point compactification of $\partial N(A) \times (0, 1]$ and thus to the cone $C(\partial N(A))$. In particular V is cone-like at p.

Conversely, assume that p is cone-like and let $N = C(L)$ be a conical neighborhood in V. Let then $E' = E \setminus (F \setminus N) \times S^{n-1}) \subset E$, so that $(E' \to S^{n-1}, \mathcal{A}) = (E \to S^{n-1}, \mathcal{A}) \in \mathfrak{E}_{n,q}$. This way we replaced V in the new representative by a conical neighborhood N and thus we can assume that $V = C(\partial V)$. Moreover, $\partial V = \partial F$ and thus F/A is homeomorphic to $C(\partial F)$. We can therefore consider that A is a spine of $F \setminus N(\partial F)$, where $N(\partial F)$ is a collar of ∂F into F. In particular we can take A to be tame, for any fiber F. This proves that there is a representative of f for which $\mathcal{A}$ is tame. □

The cone-like singularities were completely described in the second part of [36], where it is shown that any class in $\mathfrak{E}^{PL}_{n,q}$ is realized as $\delta(f)$ for some f with a cone-like isolated singularity. The missing part for completing the general classification (of not necessarily cone-like singularities) was the characterization of the set of data in $\mathfrak{E}_{n,q}$ which could be realized as obstructions $\delta(f)$ for some f. We have an immediate but implicit description from above:

Proposition 2.15. *A class in $\mathfrak{E}_{n,q}$ can be realized by a map $f : M^{n+q} \to D^n$ with an isolated critical point if and only if the mapping cylinder $M(r)$ is a manifold (at p).*

Proof. Given the data $(E \to S^{n-1}, \mathcal{A} \subset E)$ we can recover both M^{n+q} and the map f as follows. The manifold M^{n+q} (which is supposed to be an adapted neighborhood of p) is the mapping cylinder $M(r)$ of the retraction r *i.e.* $M(r) = E \times [0, 1] \cup_{(x,1)\sim r(x)} V$. Moreover, the map $f : M^{n+q} \to D^n$ is given by the formula $f(x, t) = (f(x), t)$, where D^n is identified with the mapping cylinder of the trivial map $S^{n-1} \to \{*\}$ to a point. This proves the claim. □

2.4. Contractibility of adapted neighborhoods and tameness

The main result of this section gives a topological characterization of cone-like singularities in high dimensions:

Proposition 2.16. *If the codimension is $q \leq 3$ then there exists a contractible adapted neighborhood M^{n+q} ($n \geq 4$ if $q = 3$) iff the singularity is cone-like.*

Proof. One direction is clear. If the singularity is cone-like the intersection $V \cap M$ with a small adapted neighborhood M is $C(\partial V)$ and thus contractible. But M deformation retracts onto $V \cap M$.

In order to prove the reverse implication we need the following technical lemma:

Lemma 2.17. *There exist open subsets $V_s \subset V$, $M_s \subset M$ and the sequence of concentric disks $D_s \subset D$ fulfilling the following conditions*:

(1) $\overline{V_s} \setminus \{x\}$ *is a manifold with compact boundary* $\partial\overline{V_s}$.
(2) $V_s \subset V \setminus \partial V$.
(3) *Each component of* $V \setminus \overline{V_s}$ *meets* ∂V.
(4) *Each component of* V_s *meets* x.
(5) $V_s = V \cap M_s$ *and the restriction* $f : M_s \to D_s$ *is proper with one critical point.*
(6) $\overline{M_s}$ *is a compact manifold having the same components as* M_s.
(7) *the diameter of* M_s *tends to zero, when* s *goes to infinity.*
(8) *We have* $\overline{M_s} \subset M_{s+1}$.
(9) *The restriction* $f : \overline{M}_s \cap f^{-1}(D_{s+1}) \setminus \overline{M}_{s+1} \to D_{s+1}$ *is a fibration over the disk* D_{s+1} *of radius.*
(10) *We have* $\overline{D_m} \subset D_{m+1}$ *and* $\cap_{m=0}^{\infty} D_m = \{0\}$.

Proof. This is part of the content of ([13, Lemma 3.3, page 951]), where the lemma is stated for codimension 2. However, the same proof applies word-by-word without any codimension restriction for the statements above. An alternative approach is given in the [36, proof of Proposition 1], as an application of Siebenmann's Union Lemma from [53, Lemma 6.9]. □

Let F_s denote the generic fiber of $f : M_s \to D_s$ and F be the generic fiber associated to $f : M \to D$.

The converse is given by:

Lemma 2.18. *If $\overline{M}$ is contractible and $n \geq 4$ then the singular fiber $V \cap \overline{M}$ is homeomorphic to the* 3*-disk.*

Proof. Consider the sequences V_s, M_s, D_s, F_s furnished by Lemma 2.17 where M_s are contractible. Lemma 2.7 says that F_s is a 3-disk, for all s.

By hypothesis we have $\overline{M}_s \cap f^{-1}(D_{s+1}) \setminus \overline{M}_{s+1} \to D_{s+1}$ is a fibration and thus $(\overline{F}_s \setminus F_{s+1}, \partial F_{s+1})$ is diffeomorphic to $(\overline{V}_s \setminus V_{s+1}, \partial\overline{V}_s)$. Therefore ∂V_s is a 2-sphere and $\overline{V}_s \setminus V_{s+1}$ is a cylinder.

The fiber $V \cap M \setminus \{x\}$ is the union $\cup_s(\overline{V_s} \setminus V_{s+1})$ *i.e.* the ascending union of cylinders. Since it has an exhaustion by compact sub-manifolds whose boundary are spheres it follows that $V \cap M \setminus \{x\}$ is simply connected at infinity and thus it is diffeomorphic to $D^3 \setminus \{0\}$. This implies that $V \cap M$ is homeomorphic to the 3-disk. □

Proposition 2.16 follows. □

2.5. Isolated singularities with trivial singular fibers after Hamstrom

The main result of this section concerns the singularities whose singular fiber is a disk.

Proposition 2.19. *If the codimension is* 3 *and the singular fiber is homeomorphic to a* 3*-disk then* f *is a topological fibration.*

Proof. Recall, after Hamstrom ([27]) the following:

Definition 2.20. An open proper mapping $f : X \to Y$ between metric spaces X and Y is homotopy r-regular if for each x in X and $\varepsilon > 0$ there is a $\delta > 0$ so that every mapping of a s-sphere S^s, with $s = 0, 1, \ldots, r$, into $D(x, \delta) \cap f^{-1}(y)$, for y in Y, is homotopic to 0 in $D(x, \varepsilon) \cap f^{-1}(y)$. Here $D(x, \varepsilon)$ denotes the metric disk centered at x, consisting of points whose distances to x are less than ε.

Assume now that we have an isolated singularity of codimension 3, as in the previous section. We know that $\partial \overline{V}_s$ is a 2-sphere embedded in the disk V and thus $\underline{V_s}$ is homeomorphic to a 2-disk, by the Schoenflies theorem. Since the diameter of $\overline{V}_s$ goes to zero it follows that the singular fiber $V \cap M$ satisfies the homotopy 2-regularity condition at 0. Thus f is homotopy 2-regular.

It is proved in ([27, Theorem 6.1]) that a homotopy 2-regular map f of a complete metric space X onto a finite dimensional space Y such that each fiber $f^{-1}(y)$ (for $y \in Y$) is homeomorphic to a given compact 3-manifold Q with boundary without fake 3-disks (*i.e.* having the property that each homotopy 3-cell in M is homeomorphic to the 3-cell) is actually a locally trivial fiber map. This proves the claim. □

Remark 2.21. The problem we faced above is the fact that the singularity might not be cone-like at x. If the singularity were tame in the sense developed by Kauffman and Neumann in [35] (for instance f is locally topologically equivalent to a real polynomial mapping) then we could take for M^{n+q} a ball and the arguments in [44] would immediately show that the local Milnor fiber F^3 is contractible (when dimension is $n \geq 4$) or simply connected if $n = 3$. Thus F^3 is a disk and thus the singularity is trivial, as proved in [11]. It is actually proved in ([36, page 396]) that cone-like singularities have adapted neighborhoods which are homeomorphic to balls at least if the dimension $n + q$ is not 4 or 5.

Now, if the singularity is not cone-like, we can always assume that the singular fiber V is transverse to all small enough spheres S_ε centered at x, but for an infinite discrete set of radii ε_n accumulating to 0. However, we have no local Milnor fibration and no local cone structure around a singularity. We have instead approximations of this fibration, as provided by Lemma 2.17.

2.6. Removable singularities

Definition 2.22. A smooth map $f : M \to D$ with a single critical point has a *removable* isolated singularity at p if there exists a smooth manifold X, a smooth map $g : X \to D$ without critical points and a homeomorphism $h : M \to X$ which is a diffeomorphism in a neighborhood of the boundary such that outside of an open neighborhood of the critical point we have $f = g \circ h$. The singularity is called *strongly removable* if one can obtain X by excising a smooth disk out of M and gluing it back.

A typical example of a map with a removable isolated singularity is that obtained from a map g without critical points by composing with a *near-homeomorphism*. Recall that near-homeomorphisms are limits of homeomorphisms in the compact-open topology and they were characterized by Siebenmann ([52]) as being the cell-like maps between manifolds.

Proposition 2.23. *If the codimension is at most* 2 *then isolated singularities are removable.*

Proof. Assume that we have an orientable fibration $E \to S^{n-1}$ having the fiber F^2 of dimension at most 2.

If the codimension is 0 then either $n = 2$ and E is a finite covering and thus $A \subset F$ is obviously tame since it is a finite set of points or else $n \geq 3$ and thus F is a point and A is empty.

If the codimension is 1 then the fiber is a bunch of intervals. As the boundary fibration must be trivial it follows that E is a product.

If the codimension is 2 then F is an orientable surface with boundary. A homeomorphism which is identity on the boundary should preserve the orientation. Further, a celebrated result due to Earle, Eells and Schatz tells us that all connected components of $\mathrm{Homeo}^+(F, \partial F)$ are contractible (see [19, 20]), if F is neither a disk nor an annulus. In particular, such a surface fibration over S^{n-1} (with $n \geq 2$) must be trivial. This implies that either E is trivial or the fiber is a disk or an annulus.

Moreover, if the surface is either a disk or an annulus then $\mathrm{Homeo}^+(F, \partial F)$ has the homotopy type of a circle. In particular a disk or annulus fibration over S^{n-1} is trivial whenever $n \geq 3$.

If the fibration E is trivial then the mapping cylinder $M(r)$ of any map $r : E \to F/A$ can be globally described as a quotient. In fact, $M(r)$ is given by the quotient $F \times D^n/A \times \{0\}$. This is the simplest example of a decomposition space obtained from a upper semi-continuous decomposition, which in our case has only one non-degenerate element, namely $A \times \{0\}$.

The theory of decompositions of manifolds is largely described in the Daverman monograph [16]. According to ([16, Exercise 7, page 41]) $F \times D^n/A \times \{0\}$ is a manifold iff $A \times \{0\} \subset F \times D^n$ is cellular. Recall that a subset A of an n-manifold is called *cellular* if there exists a nested sequence of n-cells B_i in that manifold such that $B_j \subset \mathrm{int}(B_{j-1})$ for any j and $A = \cap_{j=1}^{\infty} B_j$.

Remark 2.24. Cellular sets are compact and connected but not necessarily locally connected, as the graph of $\sin\left(\frac{1}{x}\right)$ is such a cellular subset of $\mathbb{R}^2$.

Moreover, it is known that the decomposition consisting of a single cellular set is shrinkable (see [16, Corollary 2.2A, page 36]) and thus the quotient of a manifold by a cellular subset is again a manifold (see [16, Theorem 2.2, page 23]) homeomorphic to the former manifold.

Alternatively, a cellular subset A in a manifold is *cell-like* (see [16, Corollary 3.2.B, page 120]) namely, for each neighborhood U of A in that manifold one can contract A to a point in U. Compact ANR are cell-like if and only if they are contractible. Cell-like maps are those continuous maps whose point inverses are cell-like sets. Siebenmann characterized in [52] the cell-like maps between manifolds of the same dimension as being precisely those maps which are near-homeomorphisms. Therefore the projection map $F \times D^k \to F \times D^k/A$, which is a cell-like map, is a near-homeomorphism.

In particular we can replace the map $M \to D^n$ by composing with the near-homeomorphism $F \times D^n \to M$ by the projection $F \times D^n \to D^n$ which has no critical points and has the same boundary restrictions. In other words the singularity of the map f is removable.

Otherwise the manifold F is a 2-disk or an annulus. Recall that we actually have a nested sequence of fibers $F_s \subset F_{s-1} \subset F$, whose intersection is the vanishing compactum $A \subset F$ in the fiber.

If all but finitely many F_s are disks then A is cellular in F. From [16] A is shrinkable and thus $V = F/A$ is homeomorphic to a disk. We can apply Hamstrom's theorem from above to obtain that f has no topological critical points.

If all but finitely many F_s are cylinders and $n = 2$ then A is a circle in F. In fact the boundary of the cylinders ∂F_s should be parallel and thus they consist in arbitrarily thin adapted neighborhoods of the intersection circle $A = \cap_{s=0}^{\infty} F_s$. Thus A is tame and hence the singularity is cone-like.

If F is non-orientable then the group of homeomorphisms is $SO(3)$ (for S^2, $\mathbb{RP}^2$) or else a circle (for the Klein bottle, disk, annulus or the Mobius band). The argument above for the annulus extends for the Mobius band as well. □

Remark 2.25. We can actually show that all codimension 2 singularities are cone-like, as claimed in Proposition 2.4. We give here an alternative proof below.

Proof. It suffices to show that $A \subset F$ is tame by constructing a regular neighborhood of it. Recall that $A = \cap_{s=0}^{\infty} F_s$, where $F_{s+1} \subset \mathrm{int}(F_s)$, for all s. Moreover, we can show that the number of components of ∂F_s is uniformly bounded. It is clear that genus of F_s is decreasing. Thus a descending sequence of surfaces like above has the homeomorphism type eventually constant. Further by finiteness and the Gruschko theorem it follows that the surfaces have eventually parallel boundaries. This implies that we have a mapping cylinder neighborhood for A. □

Proposition 2.26.

(1) *In codimension* 3 *there exist uncountably many non-equivalent germs of continuous maps with an isolated singularity.*
(2) *Any codimension* 3 *smooth map with an isolated singularity* $M \to D^n$ *for* $n \geq 4$ *is either strongly removable or else an isolated point in the fiber.*

Proof. The first point is to observe that there exist $A \subset F^3$ which are wild. For instance we can take any Artin-Fox wild arc (see [6]) in D^3 or a Whitehead continuum such that its complement in S^3 is a Whitehead manifold. It is known (see [8]) that there exist uncountably many (one-holed) Whitehead manifolds W^3 which are pairwise non-homeomorphic. According to the King classification of germs it follows that maps associated to the data ($E = D^3 \times S^{n-1}$, $\mathcal{A} = (D^3 - W^3) \times S^{n-1}$) are not right equivalent if the respective W^3 are not homeomorphic. On the other hand the mapping cylinder is $M = D^3 \times D^n / A \times \{0\}$.

Let A be an arbitrary contractible closed subset of D^3, for instance a wild arc. It follows that the decomposition given by the singleton $A \times \{0\}$ of $D^3 \times D^n$ is shrinkable, as soon as $n \geq 2$ (see [16, Corollary 8A, page 196]). Therefore M is homeomorphic to $D^3 \times D^n$ and thus it is a manifold. In particular, we obtain uncountably many germs of continuous maps $R^{n+3} \to \mathbb{R}^n$ which are pairwise non-equivalent by right composition with a homeomorphism.

Let us prove now the second claim, namely that all codimension 3 singularities are arising as claimed. We suppose henceforth that the singular fiber is not reduced to one point.

If the fiber F^3 is irreducible and the boundary is non-empty then either F^3 is a disk or else $\mathrm{Homeo}^+(F, \partial F)$ has contractible components, according to Hatcher (see [28–30]). Therefore, either F is a 3-disk or else F^3-fibrations over S^{n-1} are trivial, as soon as $n \geq 3$.

Let us analyze the case where F is not irreducible. Recall that we have a sequence of nested manifold neighborhoods $F_{s+1} \subset F_s \subset F$. If there exists at least one F_s which is irreducible then we can apply the argument from above.

Lemma 2.27. *If* $n \geq 4$ *then there exists some representative of the germ* f *of isolated singularity for which some associated fiber* F^3 *is irreducible.*

Proof. Suppose that all F_s are reducible. According to the Sphere theorem (see [31, 43]) we can detect the essential 2-spheres as follows: F_s^3 is reducible if and only if $\pi_2(F_s) \neq 0$, modulo the Poincaré conjecture.

Therefore, by the Haken-Kneser finiteness theorem there exist only finitely many embedded essential and pairwise non-parallel 2-spheres $S_{s,t} \subset F_s$. For fixed s the spheres $S_{s,t}$ are disjoint. Further, if for some pair $s_1 < s_2$ we have $S_{s_1,t} \cap S_{s_2,v} \neq \emptyset$, then we can replace $S_{s_2,v}$ by a number of spheres obtained by pushing them along the intersections. In fact the intersection of two spheres S_1 and S_2 can be made transverse by a small isotopy and $S_1 \setminus S_1 \cap S_2$ is a genus zero surface with boundary a number of circles. Each such circle, when viewed in S_2, bounds a disk in S_2. We start from a circle bounding an innermost such disk δ and then consider two parallel copies of δ corresponding to the parallel boundary circles of a cylinder neighborhood U of the circle in S_1. We remove then the cylinder U from the sphere S_1 and glue back the two copies of δ. This replaces S_1 by two spheres whose product as class in $\pi_2(M)$ is the class of S_1. Moreover the intersection between S_1 and S_2 countains one circle less than before. We can continue this procedure until S_2 gets disjoint from the spheres we create.

There are finitely many such spheres, and among them there exists at least one which is non-trivial in $\pi_2(F_{s_2})$ since the product of all the spheres obtained this way is the class of the original sphere $S_{s_2,v}$. By using a double recurrence on s_2 and s_1 we obtain that the spheres $S_{s_1,t}$ are pairwise disjoint for all s_1 and t.

Recall now that we have a fibration of $M_s \setminus V_s \to D^n \setminus \{0\}$ with fiber F_s. Its homotopy exact sequence reads:

$$\pi_{i+1}(S^{n-1}) \to \pi_i(F_s) \to \pi_i(M_s \setminus V_s) \to \pi_i(S^{n-1}) \to .$$

From Lemma 2.6 and the proof of Lemma 2.7 the inclusion $F_s \hookrightarrow M_s$ induces $\pi_i(F_s) \cong \pi_i(M_s)$, for $i \leq 2$, as soon as $n \geq 4$. Since M_s deformation retracts onto V_s the composition $F_s \hookrightarrow M_s \to V_s$ induces an isomorphism:

$$\pi_i(F_s) \cong \pi_i(F_s/A), \quad \text{for } i \leq 2.$$

We will show below that this is contradicted for $i = 2$ when all F_s are reducible.

Using again the Haken-Kneser finiteness theorem there exists some s such that one of the following holds:

(1) either there exists some infinite sequence of spheres S_{k_i,v_i} which are parallel to some $S_{s,t}$ within F_s;
Let now U_i be the annuli bounding $S_{k_i,v_i} \cup S_{k_{i+1},v_{i+1}}$ in F_s and O_i be their image in F_s/A, where S_{k_0,v_0} denotes $S_{s,t}$. Set $O_\infty = \cup_{i=0}^{\infty} O_i \cup A \subset F_s/A$ be the union of the annuli and the singular point A. As $\cup_{i=0}^{\infty} U_i$ is a punctured 3-disk and S_{k_i,v_i} accumulate on A it follows that O_∞ is the continuous image of the compactified 3-disk into F_s/A. This implies that the image of the class of $S_{s,t}$ under the map $\pi_2(F_s) \to \pi_2(F_s/A)$ vanishes. But the class of $S_{s,t}$ in F_s is non-zero since this is an essential sphere. This contradicts the isomorphism above;

(2) or else the 2-spheres $S_{k,v}$, for $k \geq s + p$ are null-homotopic in F_s, for some p. Then $S_{k,v}$ is essential in $F_k \subset F_s$ but null-homotopic in F_s. By the Sphere theorem, possibly changing the collection of essential spheres, we can assume that each $S_{k,v}$ bounds an embedded 3-ball $B_{k,v}$ in F_s. Let M_k, $E_k \to S^{n-1}$ be the adapted neighborhood and respectively fibration of the germ f, corresponding to the fiber F_k. Then $\overline{M_k} = M_k \cup_v B_{k,v} \times D^n \subset M_s$, because $M_s \setminus M_k$ is a trivial fibration over the disk D^n. Let $k \geq s + p$ and consider the

restriction of the germ $f : M_s \to D^n$ to $\overline{M_k}$. The restriction of the fibration E_s to $E_s \setminus E_k$ is a trivial fibration over S^{n-1} with fiber $F_s \setminus F_k$, because it extends over D^n. Thus $\overline{E_k} = E_s \cap \overline{M_k}$ is a sub-fibration of E_s over S^{n-1} whose fiber is $\overline{F_k} = F_k \cup_v B_{k,v}$. The fibration $\overline{E_k} \to S^{n-1}$ (with the same vanishing compactum) defines therefore the same germ f.

Now the 2-spheres $S_{k,v}$ are pairwise disjoint and $\partial F_k \subset \cup_v B_{k,v}$. Therefore every essential 2-sphere S contained in the interior of $F_k \setminus \cup_v B_{k,v}$ is parallel to one of the boundary spheres $S_{k,v}$. In fact such a 2-sphere should be homotopically trivial in F_s and thus, by the Sphere theorem we can suppose that S bounds a ball B in F_s. Thus either B is disjoint from the balls $B_{k,v}$ or there exists some v so that $B_{k,v} \subset \mathrm{int}(B)$. In the first case S is trivial in $F_k \setminus \cup_v B_{k,v}$ and in the second case we could use Brown's solution to the Schoenflies Conjecture to obtain that $S \cup S_{k,v}$ bounds an annulus in F_k and hence in $F_k \setminus \cup_v B_{k,v}$. Eventually $\overline{F_k}$ is irreducible since all essential spheres of $F_k \setminus \cup_v B_{k,v}$ were capped off by 3-balls. Thus, in this case there exists a representative of the germ f whose fiber $\overline{F}_k$ is irreducible. □

Now we can conclude as above: either F_s are disks for all large enough s or else the F_s-fibrations are trivial. If all F_s are 3-disks then A is cellular in F and thus the map $F \to F/A = V$ is a near-homeomorphism and thus the singular fiber is homeomorphic to a disk. Then Hamstrom's theorem from the previous section implies that the singularity is removable.

If some F_s fibration is trivial it follows that $M(r)$ is $F_s \times D^n / A \times \{0\}$. From [16] $M(r)$ is a manifold only if $A \times \{0\}$ is cellular in $F_s \times D^n$ and in this case the projection $F_s \times D^n \to F_s \times D^n / A \times \{0\}$ is a near-homeomorphism.

Moreover, in both cases we are able to be more precise. As $A \times \{0\}$ is cellular it admits a sequence of arbitrarily small disk neighborhoods in $F \times D^n$. Furthermore one can choose then a PL and thus a smooth neighborhood B of it. Then $B/A \times \{0\}$ is also homeomorphic to a disk since it is a quotient by a cellular subset and the restriction of the map $B \to B/A \times \{0\}$ to a collar of the boundary is the identity. Thus the boundary of $B/A \times \{0\}$ is a smooth sphere embedded in the smooth manifold M and hence $B/A \times \{0\}$ is a smooth manifold homeomorphic to a disk. As $n \geq 2$ Smale's theorem implies that $B/A \times \{0\}$ is diffeomorphic to a disk.

Let $q : B \to B/A \times \{0\}$ be a homeomorphism (recall that the projection is a near-homeomorphism), which can be assumed to be smooth on the boundary. It follows that we can obtain the manifold $F \times D^n$ from M by excising a $(n+3)$-disk corresponding to $B/A \times \{0\}$ and gluing it back by twisting the gluing map by the homeomorphism q; this is the same as gluing back B.

Therefore, with the notations from Definition 2.22, one puts $X = F \times D^n$ to find that the singularity is strongly removable, as claimed. □

Remark 2.28. It seems that isolated singularities in dimensions (6, 3) are either cone-like or else removable. From above it suffices to analyze the case when F_s are reducible. If we knew that there are adapted neighborhoods which are contractible, then the claim would follow. In fact Remark 2.9 shows that F_s should be disks-with-holes (using the Poincaré conjecture). Then one could prove that the number of components of ∂F_s is uniformly bounded, as shown in ([13, Lemma 3.4, page 952]) for codimension 2 singularities. Therefore ∂F_s should be parallel, for large enough s. Since we have embeddings $F_s \subset \text{int}(F_{s-1})$ for each s, there is no knotting phenomena and thus $A = \cap_s F_s$ is a tame subset of the fiber F. Thus the singularity is cone-like.

Remark 2.29. Examples in the next section will show that the statement of Lemma 2.27 fails when $n = 3$, since there exist fibered links whose fibers are not irreducible. Moreover, our proof of Proposition 2.26 could not work for $n = 3$ since the fibered links fibrations might also be non-trivial.

2.7. Cone-like isolated singularities in dimensions (6,3)

There exists a general recipe for constructing fibered links $\sqcup_d S^{n-1}$ into S^{2n-1}. We will restrict to the high dimensional case $n \geq 3$ below. According to Haefliger (see [24, 25]) embeddings of S^{n-1} into S^{2n-1} are trivial up to isotopy, when $n \geq 3$. Thus to each $S_1^{n-1} \subset S^{2n-1}$ one can associate a *Hopf dual* S_2^{n-1} which links S_1^{n-1} once. In fact S_2^{n-1} is the core of the complement of a regular neighborhood of S_1^{n-1}.

Consider then $S_2^{n-1}, \dots S_d^{n-1}$ be a set of pairwise disjoint Hopf duals to S_1^{n-1}. Then, by Haefliger classification theorem (see [24, 25]) the link $L = \sqcup_{j=1}^d S_j^{n-1}$ is uniquely determined, up to isotopy, by its linking matrix A_L. Since the first column and row are made of ± 1, the most important information is the linking matrix A_{L^*} of the sub-link $L^* = \sqcup_{j=2}^d S_j^{n-1}$. Observe also that links were oriented for the purpose of computing linking numbers, although the fibering property concerns unoriented links.

By convenience the diagonal of the matrix A_L will have trivial entries 0 as the links are not framed. We also set L_A for the link L with $A_{L^*} = A$. The linking numbers $lk(a, b)$ of the $(n-1)$-spheres a, b in S^{2n-1} satisfy $lk(a, b) = (-1)^n lk(b, a)$.

Proposition 2.30.

(1) *For every choice of an integral* $(-1)^n$*-symmetric* $(d-1) \times (d-1)$ *matrix* A *with trivial diagonal the link* L_A *is fibered.*
(2) *If* $n = 3$ *then every fibered link* (*with simple connected fiber*) *is isotopic to some* L_A.

Proof. The complement of a regular neighborhood of an S^{n-1} in S^{2n-1} is $S^{n-1} \times D^n$, which fibers trivially over S^{n-1}.

Let us consider now sub-fibrations of $S^{n-1} \times D^n$ for which the first projection restricts to a locally trivial fibration over S^{n-1} with fiber $D^n \setminus \sqcup_{i=1}^{d-1} D^n$. Such locally trivial fibrations η are determined up to isotopy among fibrations by the homotopy class of the classifying map $c_\eta : S^{n-1} \to \mathbb{F}_{d-1}(D^n)$, where $\mathbb{F}_{d-1}(D^n)$ denotes the configuration space of $(d-1)$ disjoint disks of equal (very small) radii in D^n. The value $c_\eta(x)$ is the class in $\mathbb{F}_{d-1}(D^n)$ of the fiber $\eta(x) \subset D^n$. Moreover $\mathbb{F}_{d-1}(D^n)$ has the homotopy type of the configuration space $\mathbb{F}_d(\mathbb{R}^n)$ of d points on $\mathbb{R}^n$.

The map $\mathbb{F}_d(\mathbb{R}^n) \to \mathbb{F}_{d-1}(\mathbb{R}^n)$ which forgets the first point of the d-tuple is a fibration whose fiber $\mathbb{F}_{1,d-1}(\mathbb{R}^n)$ has the homotopy type of a wedge $\vee_{d-1} S^{n-1}$ of $(d-1)$ spheres of dimension $(n-1)$ (see [21]). In particular, $\mathbb{F}_{d-1}(\mathbb{R}^n)$ is $(n-2)$-connected.

To every map $\mathbf{f} : S^{n-1} \to \mathbb{F}_d(\mathbb{R}^n)$ one associates d disjoint embeddings $f_j : S^{n-1} \to S^{n-1} \times D^n$ by the formula $f_j(x) = (x, \mathbf{f}(x)_j)$, where $\mathbf{f}(x)_j$ is the j-th component of the d-tuple $\mathbf{f}(x) \in \mathbb{F}_d(\mathbb{R}^n)$, and one identifies $\mathrm{int}(D^n)$ to $\mathbb{R}^n$.

Lemma 2.31. *We have an isomorphism $\pi_{n-1}(\mathbb{F}_{d-1}(\mathbb{R}^n)) \to \mathbb{Z}^{(d-2)(d-1)/2}$ which associates to the class of the map $\mathbf{f} : S^{n-1} \to \mathbb{F}_{d-1}(\mathbb{R}^n)$ the $(-1)^n$-symmetric matrix whose upper triangular entries are given by $(lk(f_i(S^{n-1}), f_j(S^{n-1})))_{1\leq i<j\leq d-1}$.*

Proof. It is known that $\pi_{n-1}(\mathbb{F}_{d-1}(\mathbb{R}^n))$ is the direct sum $\pi_{n-1}(\mathbb{F}_{d-1}(\mathbb{R}^n))) \oplus \pi_{n-1}(\mathbb{F}_{d-2}(\mathbb{R}^n))$. By functoriality it suffices then to consider only the case when $d = 2$, where the proof is elementary. In fact, let π denote the projection $\mathbb{F}_2(\mathbb{R}^n) \to S^{n-1}$ given by $\pi(x, y) = \frac{x-y}{|x-y|}$. Then $lk(f_1(S^{n-1}), f_2(S^{n-1}))$ equals the degree of $\pi(\mathbf{f})$, as an element of $\pi_{n-1}(S^{n-1})$. □

It follows that for every $(-1)^n$-symmetric matrix A with trivial diagonal there exists some sub-fibration η with fiber $D^n \setminus \sqcup_{i=1}^{d-1} D^n$ whose associated link is L_A. Thus L_A are fibered links.

Consider next $(m, n) = (6, 3)$. Then the fiber F^3 of a fibered link is simply connected and hence a disk-with-holes $D^3 \setminus \sqcup_{i=1}^{d-1} D^3$. The associated fibration is then a sub-fibration of a D^3-fibration over S^2, which is trivial (by the Hatcher solution to the Smale Conjecture in the smooth category and by the Alexander trick in the PL and topological categories). Therefore the fibered link arises as in the previous construction. Observe that the singular fiber is the cone over the union of d disjoint 2-spheres, namely a wedge of d disks of dimension 3. □

Remark 2.32. The proof from above shows that the isolated singularity so obtained is locally topologically a fibration, if $d = 1$ and $n = 3$.

Once we know all fibered links in dimension $(6, 3)$ we can construct examples of pairs of manifolds with finite φ by using a Lego construction. Let Γ be a (decorated) graph whose vertices have valence at least 3. Each vertex v of Γ is decorated by some symmetric integral $(d-1) \times (d-1)$ matrix $A(v)$. To every vertex v there is associated a fibration $S^5 - N(L_{A(v)}) \to S^2$ which extends to a smooth map with one critical point $f_v : D^6_v \to D^3$. We glue together the disks D_v using the pattern of the graph Γ by identifying one component of $\partial N(L_{A(v)})$ to one component of $\partial N(L_{A(w)})$ if v and w are adjacent in Γ. The identification has to respect the trivializations $\partial N(L_{A(v)}) \to D^3$ and hence one can take them to be the same as in the double construction. We obtain then a manifold with boundary $X(\Gamma, A(v)_{v\in\Gamma})$ endowed with a proper smooth map $f_\Gamma : X(\Gamma, A(v)_{v\in\Gamma}) \to D^3$ with n singular points (n being the number of vertices of Γ) inside one fiber. The generic fiber is $\#_g S^1 \times S^2$, where g is the rank of $H^1(\Gamma)$. If orientation reversing gluing homeomorphisms are allowed then one could also obtain non-orientable factors homeomorphic to the twisted S^2-fibration over the circle. The restriction of f_Γ to the boundary is a $\#_g S^1 \times S^2$-fibration over S^2. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ be a set of graphs associated to a family of cobounding fibrations, namely such that there exists a fibration over $D^3 \setminus \sqcup_{i=1}^{p-1} D^3$ extending the boundary fibrations restrictions of f_{Γ_i}, $1 \leq i \leq p$. Then we can glue together f_{Γ_j} to obtain some smooth map with finitely many critical points into S^3. In particular, we can realize the double of f_Γ by gluing together f_Γ and its mirror image.

Remark 2.33. All 6-manifolds M^6 admitting a smooth map $M^6 \to S^3$ with finitely many cone-like singularities should arise by this construction. However, at present we don't know whether there are any non-trivial examples, in particular we are unable to find whether $\varphi(M^6, S^3)$ equals the total number of vertices of the graphs Γ_j. Notice that a similar construction produces the examples in dimensions (4, 3) and (8, 5), but in the later case the value of $\varphi(M, N)$ can be expressed in terms of algebraic topology invariants of the manifolds M, N.

2.8. Essential singularities in codimension 3 and proof of Theorem 1.1

Let $M^{n+3} \to N^n$ be a smooth map with finitely many critical points. We have to prove that M^{n+3} is diffeomorphic to $\Sigma^{n+3} \# Q^{n+3}$ where:

(1) Σ^{n+3} is a homotopy sphere;
(2) there is a smooth (induced) map $Q^{n+3} \to N^n$ having only finitely many critical points and fulfilling:
    (a) if $n \geq 6$ or $n = 4$ then there are no critical points;
    (b) if $n = 5$ then around each critical point the map is smoothly equivalent to the Kuiper map $k_{4,1}$ *i.e.* the cone over the corresponding Hopf fibration.

Around each critical point of $f : M \to N$ we can find adapted neighborhoods $M_i \subset M$ and disks $D_i \subset N$ centered at the critical points so that $f : M_i \to D_i$ is a proper smooth map with precisely one critical point, for $i \in \{1, 2, \ldots, r\}$.

A theorem of Timourian (see [58]) states that a critical point which is an isolated point of its fiber is such that locally the map is topologically equivalent to the cone over a Hopf fibration (and thus $n = 5$). Thus the M_i, $1 \leq i \leq s$, corresponding to such critical points are homeomorphic to a disk D^{n+3} and thus diffeomorphic to D^{n+3}, by Smale's theorem, for $n \geq 2$.

From the strong removability of the remaining critical points we can find maps $f_i : M_i' \to D_i$, where M_i and M_i' are homeomorphic, for all i with $s + 1 \leq i \leq r$. Moreover $M_i \setminus U_i$ is identified with $M_i' \setminus U_i'$ for some open disks U_i and U_i' and then f_i and $f|_{M_i}$ coincide when restricted to the subset $M_i \setminus U_i$. Therefore we can glue together the various mappings f_i to obtain a smooth map $f' : M' \to N$, where $M' = \cup_{i \geq s+1} U_i' \cup \left(M \setminus \cup_{i \geq s+1} U_i\right)$. This amounts to excise a number of disjoint disks and glue them back differently. The result is then the connected sum of M with a homotopy sphere.

We excise now each $M_i \subset M'$ for all $i \leq s$ and glue it back by using the restriction of the homeomorphism which identifies $f|_{M_i}$ with the cone over the

Hopf fibration. This amounts to change M by making one more connected sum with a homotopy sphere, since these M_i (with $i \leq s$) are disks. Let then M'' be the resulting manifold. Then the induced map $M'' \to N$ is smooth and has no other critical points but the s points where it is smoothly equivalent to the Kuiper map. Moreover, M'' is homeomorphic to M and obtained from M by a connected sum with a homotopy sphere.

3. Proof of the structure Theorem 1.4

3.1. Spherical blocks and Montgomery-Samelson fibrations

Here and henceforth the dimensions m, n will be of the form $m = 2k, n = k + 1$, where $k \in \{2, 4\}$, unless the opposite is explicitly stated. The subject of this section is the description of the structure of spherical blocks, namely of manifolds M^{2k} admitting a smooth map with finitely many critical points into S^{k+1}. Specifically, we have first:

Proposition 3.1. *Assume that M^{2k} is a compact orientable manifold.*

(1) *If $(2k, k+1) = (4, 3)$ and $\varphi(M^4, S^3)$ is finite non-zero then M^4 is either homeomorphic to $\#_r S^2 \times S^2$, for some $r \geq 0$ or else homeomorphic to a fibration over S^3.*
(2) *If $(2k, k+1) = (8, 5)$ and $\varphi(M^8, S^5)$ is finite non-zero then M^8 is diffeomorphic either to*
   (a) *$\Sigma^8 \#_r S^4 \times S^4$ for some $r \geq 0$, with Σ^8 a homotopy sphere, or to*
   (b) *$\Sigma^8 \# N^8$, where $N \to S^5$ is a fibration, Σ^8 is an exotic sphere (actually the generator of the group of homotopy 8-spheres $\Gamma_8 = \mathbb{Z}/2\mathbb{Z}$), while $\Sigma^8 \# N^8$ is not a fibration over S^5.*

Recall now the following definition from [45]:

Definition 3.2. A Montgomery-Samelson fibration $f : M^m \to N^n$ with singular set $A \subset M$ is a smooth map whose restriction to $M - A$ is a locally trivial fiber bundle while the restriction to A is a homeomorphism.

As a consequence of Proposition 2.1 and Theorem 1.1 it follows that, if $m - n \leq 3$ and $(m, n) \notin \{(4, 2), (5, 2), (6, 3)\}$ then a smooth map $f : M^m \to N^n$ with finitely many critical points is a Montgomery-Samelson fibration with finite set A. We used the result of Timourian ([58]), stating that x is not an isolated point of $f^{-1}(f(x))$ unless f is locally topologically equivalent to the suspension of a Hopf fibration.

Moreover, Montgomery-Samelson fibrations of closed orientable manifolds M^m over spheres were completely characterized topologically by Antonelli, Church, Timourian and Conner (see [4, 12, 14, 15]) as follows:

Proposition 3.3. *Let $M^m \to N^n$ be a Montgomery-Samelson fibration with finite non-empty singular set A and $n \geq 2$.*

(1) *Then* $(m, n) \in \{(2, 2), (4, 3), (8, 5), (16, 9)\}$ *and, if the codimension is positive then the fiber is a sphere. In particular* $m = 2k$, $n = k+1$, *for* $k \in \{1, 2, 4, 8\}$.
(2) *If* M^m *is simply connected then* $|A| = \chi(M) = \beta_k(M) + 2$.
(3) *When* $N = S^{k+1}$ *then* M^{2k} *is homeomorphic to* $\#_r(S^k \times S^k)$, *where* $r = \frac{1}{2}\beta_k(M)$.

Proof. See [3, 4, 58]. When $N = S^2$ it was only shown in [4] that M^4 has the oriented homotopy type of $\#_r(S^2 \times S^2)$, but the classification theorem for topological 4-manifolds due to Freedman permits to conclude. We used here the Poincaré conjecture for the 3-dimensional fiber. □

Proof of Proposition 3.1. Consider now that $(m,n) \in \{(4,3),(8,5)\}$. If $\varphi(M^{2k}, S^{k+1})$ is finite non-zero then M^{2k} is either homeomorphic to a fibration (when the map is locally topologically equivalent to a projection) or else it is a genuine Montgomery-Samelson fibration with non-empty critical locus A.

If $(m, n) = (4, 3)$ then the claim follows from Proposition 3.3. Let us analyze now the case $(m, n) = (8, 5)$. Antonelli proved in [4] that a $2k$-manifold M admitting a Montgomery-Samelson fibration over S^{k+1} is a $(k-1)$-connected π-manifold. It is known (see [39, Proposition X.3.7, page 205]) that for even $k \geq 3$ any closed $(k-1)$-connected π-manifold M^{2k} is diffeomorphic to $\Sigma^{2n}\#_r(S^k \times S^k)$.

At last, if M^8 is homeomorphic to a fibration the arguments from [1] show that $M^8 = \Sigma^8\#\widehat{N}$, where $\widehat{N}$ is a fibration over S^5 and Σ^8 is an exotic sphere, as claimed. This settles Proposition 3.1. □

Remark 3.4. Kosinski ([38]) also proved that $\Sigma_1^{2k}\#_r(S^k \times S^k)$ is not diffeomorphic to $\Sigma_2^{2k}\#_r(S^k \times S^k)$ unless Σ_1 is diffeomorphic to Σ_2 so that we obtain distinct smooth structures.

3.2. Splitting off fibrations in dimension (8,5)

Proposition 3.1 says that there is essentially a unique way to find simply connected manifolds with finite $\varphi(M^m, S^n)$, in dimensions $(4, 3)$ and $(8, 5)$. Manifolds M^m endowed with some proper smooth map $M^m \to D^n$ having only finitely many critical points will be called *disk blocks*, by analogy with the spherical blocks above. The topology of disk blocks represents the essential part of the structure of pairs with finite $\varphi(M^m, N^n)$.

Remark 3.5. Isomorphism classes of S^3-fibrations over S^4 are classified by the elements of $\pi_3(\mathrm{Homeo}^+(S^3))$. Cerf's theorem ([10]) and Hatcher's proof of Smale's Conjecture ([29]) yield $\pi_3(\mathrm{Homeo}^+(S^3)) \cong \pi_3(SO(4))$. It is standard that $SO(4)$ is diffeomorphic to $SO(3) \times S^3$ and $\pi_3(SO(4)) \cong \pi_3(SO(3) \times S^3) \cong \mathbb{Z}r \oplus \mathbb{Z}s$, where the generators r and s are the following:

$$r : S^3 \to SO(4), r(x)y = xyx^{-1}$$
$$s : S^3 \to SO(4), s(x)y = xy$$

and $S^3 = \{x \in \mathbb{H}, |x| = 1\}$ is identified with set of quaternions of unit norm. Here $\mathbb{H}$ denotes the set of quaternions. We consider here the same generators (r, s) as those used by James and Whitehead in [34], which are different from the generators (h, j) used in [41]. If we denote by $\eta[a, b]$ the fibration associated to $ar + bs$ and by $\xi_{\alpha,\beta}$ the fibration associated to $\alpha h + \beta j$ then $\eta[a, b]$ is the same as $\xi_{\alpha,\beta}$ when $\alpha + \beta = b$ and $\beta = -a$. Moreover, as it is well-known, these two invariants are the same as the classical invariants of a rank 4 vector bundle on S^4, namely the Euler class e and the Pontryaguin class p_1. Specifically, we have

$$p_1(\eta[a, b]) = (2b + 4a)[S], e(\eta[a, b]) = b[S]$$

where $[S]$ is the generator of $H^4(S^4)$. The Hopf fibration is $\eta[0, 1] = \xi_{1,0}$ and has non-zero Euler class.

Let us start now from a smooth map $f : M^m \to N^n$ between compact manifolds, having only finitely many critical points. We notice first that:

Lemma 3.6. *The map f is a generalized fiber sum $g \oplus_\alpha h$ of a disk block and a fibration over N^n.*

Proof. Let $D^n \subset N^n$ be a disk containing the critical values in its interior and $X^m = f^{-1}(D^n)$. Then the restriction $f|_{M^m \setminus X^m} : M^m \setminus X^m \to N^n \setminus D^n$ is a fibration and f splits as the generalized fiber sum of $f|_{M^m \setminus X^m}$ and $f|_{X^m}$. □

Consider now that $(m, n) = (8, 5)$. If the disk block is empty or topologically trivial it follows that the manifold M^8 is a fibration up to the connect sum with a homotopy sphere. Assume from now on that the disk block h is non-trivial. Therefore, by the local structure of singularities (see Theorem 1.1) h is a Montgomery-Samelson fibration with at least one critical point. Moreover, all its critical points are locally modelled by the cone over the Hopf fibration.

Recall that the double of a manifold X with boundary is the union of two copies of X with their boundaries identified.

Proposition 3.7. *Let $h : X^8 \to D^5$ be a smooth proper map such that $h|_{\partial X}$ is a fibration over S^4. We denote by $h \oplus h : X^8 \cup_{\partial X} X^8 \to S^5$ the double of h, namely the map induced from the double of X^8 to the sphere. Let $X^8_1, X^8_2, \ldots, X^8_d, \ldots X^8_{d+f}$ be the connected components of X. Assume that each X^8_i for $1 \le i \le d$ contains at least one critical point, while those for $d + 1 \le i \le d + f$ contain none.*

Then, for each connected component X^8_i with $1 \le i \le d$ the boundary fibration $f|_{\partial X_i} : \partial X^8_i \to S^4$ is a fiber sum of Hopf fibrations (possibly trivial) and thus is isomorphic to $\eta[0, r]$, for some $r \in \mathbb{Z}$.

Proof. According to the local structure the map $f : X^8_i \to D^5$ has s critical points p_j, $j = 1, \ldots, s$ whose topological local model is the suspension of the Hopf fibration. Thus there exist local charts U^8_j around p_j and V^5_j around $q_j = H(p_j)$ such that $f : U^8_j \to V^5_j$ is the cone over the Hopf fibration $h : \partial U_j \to \partial V_j$. Thus

U_j^8 (and V_j^5) are 8-dimensional (respectively 5-dimensional) disks. Moreover, the map $f : X_i^8 \setminus \cup_{j=1}^s U_j^8 \to D^5 \setminus \cup_{j=1}^s V_j^5$ is a topological fibration.

As in [1], one of the disks U_j^8 could be removed and then glued back in order to obtain the homotopy sphere summand Σ^8 and that the restriction of f to $X_i^8 \setminus \cup_{j=1}^s U_j^8$ is a smooth fibration.

The fibration on each boundary component ∂U_j has fiber S^3, and thus the fiber of the restriction of f should be a disjoint union of S^3's. However, the base $D^5 \setminus \cup_{j=1}^s V_j^5$ is simply connected and thus $X_i^8 \setminus \cup_{j=1}^s U_j^8$ has as many connected components as the fiber. This implies that the fiber should be S^3.

Lemma 3.8. *Let $\alpha_1, \alpha_2, \ldots, \alpha_{s+1}$ be isomorphism classes of S^3-fibrations over S^4. Then there exists a S^3-fibration over $S^5 \setminus \cup_{j=1}^{s+1} D_j^5$, whose restriction to the boundary ∂D_j^5 of each 5-disk is the class α_j if and only if*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{s+1} = 0 \in \pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Proof. The holed sphere $S^5 \setminus \cup_{j=1}^{s+1} D_j^5$ retracts onto the wedge of s spheres $\vee_{\lambda=1}^s S_j^4$, which is embedded in $S^5 \setminus \cup_{j=1}^{s+1} D_j^5$. Let α be the class of a S^3-fibration over the holed 5-sphere. We associate to α the collection $(\eta_\lambda)_{1 \le \lambda \le s}$ of restrictions to each factor S_λ^4 of the wedge of spheres. This yields an injective map from the set of isomorphism classes α into the group $\oplus_{\lambda=1}^s \pi_3(SO(4))$. This map is also surjective. Given an element γ of $\oplus_{\lambda=1}^s \pi_3(SO(4))$ we consider trivial S^3-fibrations on the 4-disks $D_{\lambda,+}^4$, $\lambda = 1, \ldots, s$ and, respectively, on the wedge of disks $\vee_{\lambda=1}^s D_{j,-}^4$. We glue together these fibrations along their boundaries, namely for each λ, $\partial D_{\lambda,+}^4$ is identified to $\partial D_{\lambda,-}^4$ by using the twist $\gamma_\lambda \in \pi_3(SO(4))$. This yields a fibration over $\vee_{\lambda=1}^s S_j^4$ which extends to $S^5 \setminus \cup_{j=1}^{s+1} D_j^5$.

We have to relate now the classes η_λ and the classes α_j of the restrictions of α to the boundary components ∂D_j^5. We choose a particular embedding of $\vee_{\lambda=1}^s S_j^4$ into $S^5 \setminus \cup_{j=1}^{s+1} D_j^5$ as follows:

- S_1^4 and ∂D_1^5 bound a cylinder;
- $S_\lambda^4 \vee S_{\lambda+1}^4$ and $\partial D_{\lambda+1}^5$ bound a 5-sphere deprived of three disks, among which two disks intersect in a boundary point, for $\lambda = 1, 2, \ldots, s$;
- S_s^4 and ∂D_{s+1}^5 bound a cylinder.

Taking into account the orientations on each boundary component it follows that

$$\alpha_1 = \eta_1, \alpha_j = \eta_j - \eta_{j-1}, \text{ for } 2 \le j \le s, \quad \alpha_{s+1} = \eta_s.$$

Thus the classes $(\alpha_j)_{1 \le j \le s+1}$ extend to the holed 5-sphere if and only if their sum vanishes. $\square$

We resume the proof of Proposition 3.7. We have a fibration $f : X_i \setminus \cup_{j=1}^{s} U_j \to D^5 \setminus \cup_{j=1}^{s} V_j$ and all boundary fibrations over ∂V_j are Hopf fibrations. Let us denote by α_j the class of the boundary Hopf fibration $h : \partial U_j \to \partial V_j$. We identify each factor $\pi_3(SO(4))$ with $\mathbb{Z} \oplus \mathbb{Z}$ by taking account the boundary orientation of the fibrations. Thus each α_j is either the Hopf fibration, or its negative.

For example, the suspension of the Hopf fibration $S^8 \to S^5$ has two critical points and the classes of the associated boundary fibrations are the positive and the negative Hopf fibrations.

Since the classes α_j and $f|_{\partial X_i}$ bound it follows from the Lemma 3.8 that the isomorphism class of the S^3-fibration $f|_{\partial X_i}$ is the (algebraic) sum of a number of Hopf fibrations and thus isomorphic to $\eta[0, r]$ for some integer r. It is easy to see that this is the same as r fiber sums of Hopf fibrations. □

Lemma 3.9. *The connected components X_i^8, for $d + 1 \leq i \leq d + f$ are diffeomorphic to $D^5 \times S^3$ and the fibrations $f|_{\partial X_i}$ are trivial S^3-fibrations.*

Proof. The restrictions $f : X_i^8 \to D^5$ are fibrations whenever $d + 1 \leq i \leq d + f$ and we set F_i^3 for their respective fibers. We also set $F_i^3 = S^3$ for $i \leq d$. According to our assumptions the restriction of f is also a fibration in a neighborhood of $\sqcup_{i=1}^{d+f} \partial X_i$ and thus its fiber is $\sqcup_{i=1}^{d+f} F_i$.

Recall that the restriction to the fibrewise summand $f|_{M^8 \setminus X^8} : M^8 \setminus X^8 \to N^5 \setminus D^5$ is also a fibration and that M^8 is connected. The fiber on the boundary is known so that the fiber of $f|_{M^8 \setminus X^8}$ is $\sqcup_{i=1}^{d+f} F_i^3$. If there exist a fiber F_i^3 which is not diffeomorphic to S^3 then the monodromy cannot exchange the respective connected components of the fiber and thus $M^8 \setminus X^8$ is not connected. This would imply that M^8 is not connected, since all disk blocks X_i^8 are connected. This contradiction shows that all F_i^3 are diffeomorphic to S^3 and the claim follows. □

Lemma 3.10. *Let $g : M^8 \setminus X^8 \to N^5 \setminus D^5$ be a fibration with fiber $\sqcup_{i=1}^{r} S^3$. Then the restrictions of g to each boundary component is a trivial fibration and the fibration g factors as follows: there exists a non-ramified covering $\widehat{N^5} \setminus \sqcup_{i=1}^{r} D_i^5$ of $N^5 \setminus D^5$ of degree r and an S^3-fibration $M^8 \setminus X^8 \to \widehat{N^5} \setminus \sqcup_{i=1}^{r} D_i^5$ and their composition is g.*

Proof. The main idea is that a S^3 fibration over S^4 extends over $N^5 \setminus D^5$ if and only if it is trivial. This is a simple application of the following well-known facts: these invariants of the fibration can also be interpreted as Pontryaguin and Euler numbers of the manifold and the latter are cobordism invariants.

In order to classify (orientable) fibrations with fiber $\sqcup_{i=1}^{r} S^3$ one has to consider the group of homeomorphisms of $\sqcup_{i=1}^{r} S^3$. As a consequence of the Cerf and Hatcher theorems this homeomorphisms group has the homotopy type of $SO(4)^r \ltimes S_r$, where S_r denotes the permutation groups on r elements acting on $SO(4)^r$ by permuting the factors. In particular isomorphism classes of such fibrations over N are classified by the elements of the set of homotopy classes $[N, B(SO(4)^r \ltimes S_r)]$.

The exact sequences between Lie groups induce the following exact sequence in homotopy:

$$[N, BSO(4)]^r \to [N, B(SO(4)^r \ltimes S_r)] \to [N, BS_r].$$

Then $[N, BS_r]$ is isomorphic to the set of homomorphisms $\mathrm{Hom}(\pi_1(N), S_r)$, by associating to each finite covering its monodromy homomorphism. Moreover, given a fibration with fiber $\sqcup_{i=1}^r S^3$ we can associate a monodromy covering $\widehat{N}$ of N by crushing each sphere to a point, or equivalently, by considering the monodromy action on the set of components of the fiber.

This can be interpreted as follows. Given a homomorphism $\rho : \pi_1(N) \to S_r$, the isomorphism classes of fibrations with fiber $\sqcup_{i=1}^r S^3$ and monodromy covering associated to ρ is in bijection with $[N, BSO(4)]^r$. Furthermore each factor $[N, BSO(4)]$ contains a Euler class e and a Pontryaguin class p_1, which are pull-backs of the corresponding elements in $K(\mathbb{Z}, n)$ and $K(\mathbb{Z}, 4)$.

It is known that $\pi_j(BSO(4)) = \pi_{j-1}(SO(4))$ (see *e.g.* [55, Chapter19]) and thus

$$\pi_1(BSO(4))=\pi_3(BSO(4)) = 0, \pi_2(BSO(4))=\mathbb{Z}/2\mathbb{Z}, \pi_4(BSO(4)) = \mathbb{Z} \oplus \mathbb{Z}.$$

By general facts concerning the Moore-Postnikov decomposition of $BSO(4)$ (see [54, Chapter 8]) we have a map

$$\theta : [N, BSO(4)] \to [N, K(\mathbb{Z}/2\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4)]$$

or equivalently

$$\theta : [N, BSO(4)] \to H^2(N, \mathbb{Z}/2\mathbb{Z}) \oplus H^4(N, \mathbb{Z}) \oplus H^4(N, \mathbb{Z}).$$

When N is of dimension 4 the components of $\theta(\eta)$ are the characteristic classes of the fibration η, namely the Stiefel-Whitney class $w_2 \in H^2(N, \mathbb{Z}/2\mathbb{Z})$, the Pontryaguin class $p_1 \in H^4(N, \mathbb{Z})$ and the Euler class $e \in H^4(N, \mathbb{Z})$. Let us denote by $e \in H^4(N, \mathbb{Z})$ the corresponding element $\theta(\eta)$ also when N is of arbitrary dimension, namely the image in cohomology of the pull-back of the generator of $H^4(K(\mathbb{Z}, 4), \mathbb{Z})$. Notice that e is not an Euler class if the dimension of N is not 4.

Assume now that we have a fibration α with fiber $\sqcup_{i=1}^r S^3$ over $N \setminus D$ and fixed monodromy homomorphism $\rho : \pi_1(N \setminus D) \to S_r$. We want to compute the invariants of its restriction $\alpha|_{\partial D}$. The set of isomorphisms corresponds bijectively to elements in $[N \setminus D, BSO(4)]^r$ and thus we have characteristic classes $c = (c_i)_{i=1,r} \in \oplus_1^r H^4(M)$ corresponding to each one of the r factors.

For any characteristic class $c \in H^4(M)$ we have

$$c(\alpha|_{\partial D}) = i^* c(\eta)$$

where c, $i : \partial D \to N \setminus D$ is the inclusion and $i^* : H^4(N \setminus D) \to H^4(\partial D)$ is the map induced in cohomology. However the map i^* is trivial. For instance in the Mayer-Vietoris sequence

$$0 \to H^4(N) \to H^4(N \setminus D) \oplus H^4(D^5) \to H^4(\partial D) \to$$

the map $H^4(N) \to H^4(N \setminus D)$ is an isomorphism and thus the kernel of i^* is all of $H^4(N \setminus D)$. Thus the characteristic classes of each S^3-fibration component of $\alpha|_{\partial D}$ vanish.

This result holds more generally for any boundary 4-manifold instead of the sphere ∂D, since according to Dold and Whitney (see [18,62]) the classes w_2, p_1, e classify S^3-fibrations over 4-manifolds.

By Remark 3.5 each component of $\alpha|_{\partial D}$ is a trivial fibration. □

Lemma 3.11. *Each connected component X_i^8 with $1 \le i \le d$ is diffeomorphic to some manifold $\Sigma_i^8 \#_{r_i} S^4 \times S^4 \setminus D^5 \times S^3$, so that the restriction of f to the boundary is identified with the projection $S^3 \times \partial D^5 \to \partial D^5$ on the second factor.*

Proof. The boundary fibration $f|_{\partial X_i}$ is a trivial fibration, and hence we can glue to X_i^8 the trivial fibration over D^5 in order to obtain a manifold Y_i^8 endowed with a smooth map with only finitely many critical points into S^5.

Further ∂X_i and $X_i \setminus \cup_{j=1}^s U_j$, and hence X_i, are simply connected as S^3-fibrations over simply connected bases. Thus the manifold Y_i is a simply connected spherical block. The structure of spherical blocks from Proposition 3.1 implies that Y_i^8 is diffeomorphic to $\Sigma_i^8 \#_{r_i} S^4 \times S^4$ for some $r_i \ge 0$, with Σ_i^8 a homotopy sphere. □

Remark 3.12. The isomorphism classes of S^3-fibrations over $\#_c S^1 \times S^4$ are classified by an element in $\oplus_1^c \pi_3(SO(4)) = (\mathbb{Z} \oplus \mathbb{Z})^c$, each factor $\mathbb{Z} \oplus \mathbb{Z}$ corresponding to the isomorphism class of the restriction of the S^3-fibration to a factor $\{*\} \times S^4$.

3.3. Splitting off fibrations in dimensions (4,3)

The aim of this section is to prove the corresponding splitting result in dimension (4, 3). Some extra care is needed to describe the smooth structure of the 4-manifolds.

The first step in proving the theorem is to understand the smooth structure of spherical blocks in dimension (4, 3), by a slight strengthening of Proposition 3.1:

Proposition 3.13. *If $M^4 \to S^3$ has finitely many critical points and its generic fiber is S^1, then*

(1) *either $M^4 = \Sigma^4 \# N^4$, where N^4 is a fibration over S^4;*
(2) *or else M^4 is $\Sigma^4 \#_r S^2 \times S^2$, where Σ^4 is a homotopy 4-sphere.*

Remark 3.14. If $\pi_1(M^4) = 0$ then the generic fiber should be a circle. It will be proved later (see Proposition 4.1) that a connected spherical block M^4 which is non-fibered is simply connected. On the other hand a fibered connected spherical block M^4 is a circle fibration over S^3. Thus, the generic fiber is a circle for any connected spherical block.

Proof. By the proof of Proposition 3.1 a smooth map $f : M^4 \to S^3$ with finitely many critical points is a Montgomery-Samelson fibration. Thus f is locally topologically equivalent to either the standard projection or else to the suspension H of the Hopf fibration $S^3 \to S^2$ around branch points. We will use a refinement of the argument from [1] for converting a topological fibration into a smooth fibration at the expense of adjoining some homotopy sphere.

Around each critical point $p \in M^4$ from the branch locus there exists an open neighborhood $p \in U^4 \subset M^4$ and a homeomorphism $g : V^4 \to U^4$ such that $f \circ g : V^4 \to f(U^4)$ is smoothly equivalent to the suspension H. Moreover, by choosing a smaller neighborhood V^4, we can assume that V^4 is a 4-disk and that its image U^4 is contained in a 4-disk inside M^4. This implies that U^4 is a homotopy 4-disk. If we remove the homotopy 4-disk U^4 and glue back V^4 instead then we obtain the manifold $M^4 \# \Sigma_1^4$, where Σ_1^4 is a homotopy sphere, namely the negative (in the group of homotopy 4-spheres) of the homotopy sphere obtained by capping off the boundary of U^4 by a 4-disk. A theorem of Huebsch and Morse (see [32]) says that whenever Δ^4 is a homotopy 4-disk which can be embedded in a 4-disk there exists a smooth homeomorphism $\Delta^4 \to D^4$ with only one critical point. As a consequence there exists a smooth homeomorphism $U^4 \to V^4$ having precisely one critical point.

It follows that there is an induced smooth map $M^4 \# \Sigma_1^4 \to S^3$, which coincides with f outside V^4 and it is locally smoothly equivalent around the image of p to the suspension map H. We continue in the same way for all points of the branch locus and get a manifold $M \# \Sigma^4$ endowed with a map to S^3 whose branch locus critical points are locally smoothly equivalent to H and Σ^4 is the connected sum of the homotopy 4-spheres corresponding to each critical point.

At last, consider the critical points of f which are not in the branch locus. Locally the map f is topologically equivalent to the projection. As above, by adjoining a homotopy 4-sphere we can make the map be smoothly equivalent to a projection and hence a submersion.

At the end of this process we obtained a smooth map $f : N^4 = \Sigma^4 \# M^4 \to S^3$ locally modelled on H. If the branch locus is empty then f is a fibration.

Assume that the set of branch points is non-empty from now on.

Further S^1-fibrations over $S^2$are classified by an element of $\pi_1(\mathrm{Homeo}^+(S^1)) \cong \pi_1(SO(2)) = \mathbb{Z}$, namely an integer called the Euler class of the fibration.

Lemma 3.8 has a similar statement in this dimension, as follows:

Lemma 3.15. *Let $\alpha_1, \alpha_2, \ldots, \alpha_N$ be isomorphism classes of S^1-fibrations over S^2. Then there exists a S^1-fibration over $S^3 \setminus \cup_{j=1}^N D_j^3$, whose restriction to the boundary*

∂D_j^3 *of each 5-disk is the class* α_j *if and only if*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_N = 0 \in \pi_1(SO(2)) \cong \mathbb{Z}.$$

Proof. We omit it. □

If we excise disk neighborhoods around the s critical values of the map $f : N^4 \to S^3$ we obtain a S^1-fibration over $S^3 \setminus \cup_{j=1}^s D_j^3$ extending the boundary (signed) Hopf fibrations. Since the Euler class of the Hopf fibration is the generator $1 \in \pi_1(SO(2))$, and thus it is non-zero, Lemma 3.15 implies that $s = 2k$ is even and there exist k positive and k negative Hopf fibrations.

Let consider embedded 2-spheres $S_1^2, \ldots, S_{k-1}^2$ separating boundary components into pairs of opposite signs. Using again Lemma 3.15 a number of times we obtain that the restriction of the S^1-fibration f at S_j^2 is a trivial fibration, for each j. This means that the fibration f is the fiber sum of spherical blocks with 2 critical values along trivial fibrations.

Lemma 3.16. *If* $f : N^4 \to S^3$ *is as above and has two critical values then* N^4 *is diffeomorphic to* S^4.

Proof. We have disk neighborhoods D_+^4 and D_-^4 in N^4 around the critical points and $f : N^4 \setminus (D_+^4 \cup D_-^4) \to S^2 \times [0, 1]$ is a S^1-fibration. The restrictions of f to the boundary spheres ∂D_+^4 and ∂D_-^4 are isomorphic to the Hopf fibrations over S^2. The composition of the projection $S^2 \times [0, 1] \to [0, 1]$ with the above restriction of f is still a fibration of $N^4 \setminus (D_+^4 \cup D_-^4)$ onto $[0, 1]$, and hence a trivial one. Therefore $N^4 \setminus (D_+^4 \cup D_-^4)$ is diffeomorphic to $S^3 \times [0, 1]$. The manifold N^4 is obtained by gluing D_+^4, $S^3 \times [0, 1]$ and D_-^4 along their respective boundaries. Cerf's theorem $\Gamma_4 = 0$ (see [10]) implies that N^4 is diffeomorphic to S^4, as claimed. □

Let now M_r^4 denote the total space of the fiber sum of r blocks $S^4 \to S^3$. Thus $M_1^4 = S^4$ and $M_{r+1}^4 = M_r^4 \oplus S^4$, for $r \geq 1$. We obtain M_{r+1}^4 by deleting out neighborhoods $S^1 \times \mathrm{int}(D^3)$ of generic fibers from M_r^4 and S^4 and gluing then $M_r^4 \setminus S^1 \times \mathrm{int}(D^3)$ and $S^4 \setminus S^1 \times \mathrm{int}(D^3)$ along their boundaries, by a gluing homeomorphism respecting the product structure on the boundary.

Consider first the case $r = 2$. Since the fiber $S^1 \subset S^4$ is unknotted it follows that $S^4 \setminus S^1 \times \mathrm{int}(D^3)$ is diffeomorphic to $S^2 \times D^2$. Moreover we want to glue the two copies of $S^2 \times D^2$ such that the two boundary fibrations $S^2 \times S^1 \to S^2$ glue together, *i.e.* the gluing homeomorphisms $\Phi : S^2 \times S^1 \to S^2 \times S^1$ induces a homeomorphism of the base $\varphi : S^2 \to S^2$ so that the diagram

$$\begin{array}{ccc} S^2 \times S^1 & \stackrel{\Phi}{\to} & S^2 \times S^1 \\ \downarrow & & \downarrow \\ S^2 & \stackrel{\varphi}{\to} & S^2 \end{array}$$

is commutative. This condition implies that the result of the gluing of the two spherical blocks possesses a smooth map into $D^3 \cup_\varphi D^3$, which is a sphere S^3, since diffeomorphisms reversing the orientation of S^2 are isotopic to a symmetry, by a classical result of Smale.

The isotopy class of a gluing homeomorphism respecting the product structure corresponds to a homotopy class of a map $S^2 \to \mathrm{Homeo}^+(S^1)$, namely to an element of $\pi_2(SO(2)) = 0$. Therefore the diffeomorphism type of M_2 does not depend on the choice of this isomorphism.

Therefore M_2 is the result of gluing two copies of $S^2 \times D^2$ by using the boundary identification Φ. Observe that there is also an obvious projection $S^2 \times D^2 \to S^2$ extending the boundary projection $S^2 \times S^1 \to S^2$. The above diagram shows that the two projections on the first factor of each copy $S^2 \times D^2$ glue together to get a trivial fibration map $M_2^4 \to S^2$. Moreover, the fiber of this fibration is the union $D^2 \cup D^2$, namely a 2-sphere S^2. It follows then that $M_2 = S^2 \times S^2$.

When $r > 2$, we should also analyze what happens from S^4 when we delete neighborhoods of two fibers. Since the fibers form a trivial link in S^4 and can be separated by an embedded 3-sphere it follows that $S^4 \setminus (S^1 \times \mathrm{int}(D^3) \cup S^1 \times \mathrm{int}(D^3))$ is diffeomorphic to the connected sum out of the boundary $(S^4 \setminus S^1 \times \mathrm{int}(D^3))\#(S^4 \setminus S^1 \times \mathrm{int}(D^3))$. Recall that M_r is obtained by iterating boundary gluings of $S^4 \setminus S^1 \times \mathrm{int}(D^3)$ with $(r-2)$ copies of $S^4 \setminus (S^1 \times \mathrm{int}(D^3) \cup S^1 \times \mathrm{int}(D^3))$ glued each one to the next one and finally with one more copy of $S^4 \setminus S^1 \times \mathrm{int}(D^3)$. By the previous observation this gluing is the connected sum of $(r-1)$ manifolds, where each manifold is obtained by gluing two $S^4 \setminus S^1 \times \mathrm{int}(D^3)$. It follows then than $M_r = \#_{r-1} S^2 \times S^2$. This proves Proposition 3.13. □

3.4. End of proof of Theorem 1.4

3.4.1. *Dimension* (8, 5)

The fibrewise summand $f|_{M^8 \setminus X^8}$ extends a trivial boundary fibration. We can glue then along the boundary a trivial fibration over the disk D^5 to obtain a fibration $g : W^8 \to N^5$ with fiber $\sqcup_1^{d+f} S^3$. This implies that W^8 is a S^3-fibration over the monodromy covering $\widehat{N}$, which is a degree $(d+f)$ covering of N.

Since the disk blocks over D_i^5 are connected and disjoint, the gluing of $M^8 \setminus X^8$ and X^8 to recover M^8 corresponds to taking $d+f$ fiber sums (along $d+f$ components S^3, which are fibers of W^8) with the manifolds Y_i^8. We put also $Y_j^8 = S^3 \times S^5$, when $d+1 \leq j \leq d+f$.

Consider now $\Sigma^8 = \Sigma_1^8 \# \cdots \# \Sigma_d^8$, where Σ_j^8 are the homotopy 8-spheres associated to each factor. Then we can slide the attaching maps of Σ_i^8 in order to be attached to a small disk on W^8. This is possible since the image of the monodromy of the covering is all of the permutation group S_{d+f}, as $M^8 \setminus X^8$ is connected. We obtained the structure claimed in the theorem for $D = d+f$, where all factors Y_i^8 (with $i \geq d+1$) are diffeomorphic to $\#_{-1} S^2 \times S^2$.

Let $q = 2\sum_{i=1}^{d} r_i + 2d$. Note that $\varphi(M^8, N^5) \leq q$, because we have already a smooth map with q critical points.

3.4.2. *Dimension* (4, 3)

Consider the disk block $X^4 \to D^3$ and assume that X^4 has $d + f$ connected components X_i^4 such that the first d have at least one branch point. Then X_i^4 minus a number of spherical caps is a fibration over the holed D^3. Since the holed 3-disk is simply connected it follows that the generic fiber of $f|_{X_i^4}$ is a single S^1.

Therefore $M^4 \setminus X^4$ is a manifold with $d + f$ boundary components, which are fibrations over ∂D.

Lemma 3.17. *Let $g : M^4 \setminus X^4 \to N^3 \setminus D^3$ be a fibration with fiber $\sqcup_{i=1}^r S^1$. Then the restrictions of g to each boundary component is a trivial fibration and the fibration g factors as follows: there exists a non-ramified covering $\widehat{N^3} \setminus \sqcup_{i=1}^r D_i^3$ of $N^3 \setminus D^3$ of degree r and an S^1-fibration $M^4 \setminus X^4 \to \widehat{N^3} \setminus \sqcup_{i=1}^r D_i^3$ and their composition is g.*

Proof. We can follow the same lines as in the proof of Lemma 3.10 by replacing $BSO(4)$ with $BSO(2)$. However there is a much simpler proof. In fact a circle fibration over ∂D^3 which extends to $N^3 \setminus D^3$ is trivial because 3-manifolds are parallelizable. We leave the details to the reader. □

Lemma 3.18. *Each connected component X_i^4 with $1 \leq i \leq d$ is diffeomorphic to some manifold $\Sigma_i^4 \#_{r_i} S^2 \times S^2 \setminus D^3 \times S^1$, so that the restriction of f to the boundary is the projection $S^1 \times \partial D^3 \to \partial D^3$.*

Proof. We construct the manifold Y_i^4 by adjoining to X_i^4 a trivial S^1-fibration over D^3. Although we cannot insure that Y_i^4 is simply connected we know that Y_i^4 admits a smooth map with finitely many critical points into S^3 having the generic fiber S^1. Thus we can apply Proposition 3.13 to find that Y_i^4 is diffeomorphic to $\Sigma_i^4 \#_{r_i} S^2 \times S^2$. □

The proof of the theorem follows from these Lemmas.

4. Computing φ and the proof of Theorem 1.9

Computing the precise value for φ from the structure Theorem 1.4 is easy when the fundamental group is trivial or verifies some strong assumptions forcing the finite covering to be unique. In fact the main point is to understand whether a given M^m admits several fiber sums decompositions associated to coverings $\widehat{N^n}$ of different degrees.

Henceforth we consider two manifolds M^{2k}, N^{k+1} with finite $\varphi(M^{2k}, N^{k+1})$ and $k \in \{2, 4\}$. According to Theorem 1.4 either M^{2k} is a topological fibration over

N^{k+1} or else it is the iterated fiber sum of a fibration W^{2k} over N^{k+1} with a number of spherical blocks. In the second case we say, by a language abuse, that M^{2k} is *non-fibered* (over N^{k+1}).

We have the following result (restatement of Corollary 1.8 from Introduction):

Proposition 4.1. *Let M^{2k}, N^{k+1} be closed orientable manifolds with finite $\varphi(M^{2k}, N^{k+1})$, $k \in \{2, 4\}$ and M^{2k} non-fibered.*

If $k = 4$ then $\pi_1(M^8)$ is a finite index normal subgroup of $\pi_1(N^5)$.

If $k = 2$ then $\pi_1(M^4) \cong \pi_1(N^3)$.

Proof. Let F^{k-1} denote the fiber of $W^{2k} \to N^{k+1}$.

If $k = 4$ then $\pi_1(W^8) \cong \pi_1(M^8)$ by Van Kampen. The homotopy exact sequence of the fibration $W^8 \to N^5$ yields the exact sequence:

$$0 \to \pi_1(M^8) \to \pi_1(N^5) \to \pi_0(F^3) \to 0.$$

This means that $\pi_1(M^8)$ is a normal finite index subgroup of $\pi_1(N^5)$.

If $k = 2$ the homotopy exact sequence of the fibration reads

$$0 \to \pi_2(M^4) \to \pi_2(N^3) \to \pi_1(F^1) \to \pi_1(W^4) \to \pi_1(N^3) \to \pi_0(F^1) \to 0.$$

We obtain M^4 from W^4 by capping off some of the S^1 components of the fiber. In fact, for realizing the fiber sum we remove a neighborhood of the fiber component and replace it with the complement of a neighborhood Z^4 of the generic fiber in some $\#_r S^2 \times S^2$. However, a fiber in the boundary of Z^4 is null-homotopic in $\#_r S^2 \times S^2 \setminus Z^4$: translate the fiber until it lies in the boundary of the cone of a Hopf fibration around a critical point and then collapse it onto the vertex of the cone.

Since the monodromy homomorphism of the covering $\widehat{N^3} \to N^3$ at the level of connected components of the fiber is surjective capping off one component is equivalent to capping off any other component at the level of fundamental group. Specifically, by Van Kampen

$$\pi_1(M^4) = \pi_1(W^4)/\langle i(\pi_1(F^1))\rangle$$

where $\langle i(\pi_1(F^1))\rangle$ is the normal subgroup of $\pi_1(N^3)$ generated by the image $i(\pi_1(F^1))$ by the inclusion map $i : F^1 \hookrightarrow W^4$.

The exact sequence above shows that the image $i(\pi_1(F^1))$ is a normal subgroup, namely the kernel of $\pi_1(W^4) \to \pi_1(N^3)$. Therefore there exists a natural identification of $\pi_1(M^4)$ with $\pi_1(N^3)$. □

Definition 4.2. The group G is *co-Hopfian* if any injective homomorphism $G \to G$ is an automorphism.

Proposition 4.3. *Let M^{2k}, N^{k+1} be closed orientable manifolds with finite $\varphi(M^{2k}, N^{k+1})$, $k \in \{2, 4\}$ and M^{2k} non-fibered.*

If $k = 4$, assume that $\pi_1(M^8) \cong \pi_1(N^5)$ is a co-Hopfian group or a finitely generated free non-Abelian group. Then $\varphi(M^8, N^5) = 2r + 2D$, where $r = r_1 + r_2 + \cdots + r_D$.

If $k = 2$, assume that N^3 is an irreducible closed orientable 3-manifold which is not finitely covered by a torus bundle over S^1 nor by $\Sigma \times S^1$ (for some surface Σ). Then $\varphi(M^4, N^3) = 2r + 2D$.

Proof. Let $k = 4$. The structure theorem and Van Kampen imply that $\pi_1(W^8) \cong \pi_1(M^4)$. Moreover, the homotopy exact sequence of the fibration $W^8 \to N^5$ implies that the fiber is a single sphere, because the injective homomorphism $\pi_1(M^8) \to \pi_1(N^5)$ should be an automorphism if $\pi_1(N^5)$ is co-Hopfian.

If $\pi_1(N^5)$ is free non-Abelian then an argument also used in [22] can be applied. The image of $\pi_1(M^8)$ into $\pi_1(N^5)$ should be a normal subgroup of finite index equal to the number of components of the fiber. By the Nielsen-Schreier theorem this is a free non-Abelian group of rank greater than the rank of $\pi_1(M^8)$ and thus cannot be the image of $\pi_1(M^8)$ unless the fiber is connected.

This implies that there exists only one possible value for D, namely $D = 1$.

Lemma 4.4. *If $D = 1$ and $k \in \{2, 4\}$ then r is uniquely determined by the homology of M^{2k} and N^{k+1}. Specifically, we have:*

$$2r = \begin{cases} b_k(M^{2k}) - 2b_1(N^{k+1}), & \text{if } b_k(M) \equiv 0 (\bmod 2) \\ b_k(M^{2k}) - 2b_1(N^{k+1}) + 1, & \text{if } b_k(M) \equiv 1 (\bmod 2) \end{cases}$$

where b_j states for the j-th Betti number.

Proof. It suffices to consider $r \geq 0$ since otherwise $M^{2k} = W^{2k}$ and M^{2k} fibers over N^{k+1}.

All homology groups below will be considered with rational coefficients. The Gysin sequence of the fibration $W^{2k} - S^{k-1} \times D^{k+1} \to N^{k+1} - D^{k+1}$ yields

$$\begin{aligned} 0 &\to H_1(N^{k+1} - D^{k+1}) \to H_k(W^{2k} - S^{k-1} \times D^{k+1}) \\ &\to H_k(N^{k+1} - D^{k+1}) \xrightarrow{\beta} H_0(N^{k+1} - D^{k+1}) \end{aligned}$$

and thus

$$\mathrm{rk}_{\mathbb{Q}} H_k(W^{2k} - S^{k-1} \times D^{k+1}) = 2b_1(N^{k+1}) - \mathrm{rk}_{\mathbb{Q}} \mathrm{Im}(\beta)$$

where $\mathrm{rk}_{\mathbb{Q}}$ denotes the rank over $\mathbb{Q}$. Moreover the image of β is contained in $H_0(N^3 - D^3)$ and thus $0 \leq \mathrm{rk}_{\mathbb{Q}} \mathrm{Im}(\beta) \leq 1$. Further $H_k(N^{k+1} - D^{k+1}) \cong H_k(N^{k+1}) \oplus H_k(\partial D^{k+1})$ and the map β on $H_k(N^{k+1})$ is the cap product with the

Euler class $e_W \in H^k(N^{k+1})$ of the fibration $W^{2k} \to N^{k+1}$ in $(k-1)$-spheres and respectively trivial on the factor $H_k(\partial D^{k+1})$. It follows that

$$\mathrm{rk}_{\mathbb{Q}}\mathrm{Im}(\beta) = \begin{cases} 0, & \text{if } e_W = 0 \\ 1, & \text{otherwise .} \end{cases}$$

We want now to compute the homology of M^{2k} which is obtained by gluing $W^{2k} - S^{k-1} \times D^{k+1}$ with $Z^{2k} = \#_r S^k \times S^k \setminus S^{k-1} \times D^{k+1}$ along $S^{k-1} \times S^k$. The proof of Proposition 3.13 shows us that Z^{2k} is diffeomorphic to $\#_r S^k \times S^k \# S^k \times D^k$ and thus $H_k(Z^{2k}) = \mathbb{Q}^{2r+1}$ and $H_1(Z^{2k}) = 0$, if $r \geq 0$. By Mayer-Vietoris we derive the exact sequence:

$$\begin{aligned} \mathbb{Q} = H_2(S^1 \times S^2) &\xrightarrow{\zeta} H_k(W^{2k} - S^{k-1} \times D^{k+1}) \oplus H_k(Z^{2k}) \\ &\to H_k(M^{2k}) \xrightarrow{\nu} H_{k-1}(S^{k-1} \times S^k) = \mathbb{Q} \end{aligned}$$

so that

$$\mathrm{rk}_{\mathbb{Q}} H_k(M^{2k}) = \mathrm{rk}_{\mathbb{Q}} H_k(W^{2k} - S^{k-1} \times D^{k+1}) + 2r + 1 + \mathrm{rk}_{\mathbb{Q}}\mathrm{Im}(\nu) - \mathrm{rk}_{\mathbb{Q}}\mathrm{Im}(\zeta).$$

The map ζ is injective since $H_k(Z^{2k})$ is generated by the homology classes of the obvious embedded 2-spheres. Thus $\ker \zeta = 0$ and hence $\mathrm{rk}_{\mathbb{Q}}\mathrm{Im}(\zeta) = 1$.

Note that

$$0 \leq \mathrm{rk}_{\mathbb{Q}}\mathrm{Im}(\nu) \leq 1.$$

Observe now that the map induced by inclusion $H_{k-1}(S^{k-1} \times S^k) \to H_{k-1}(Z^{2k})$ should be trivial since the fiber $S^{k-1} \times \{*\}$ is null-homologous in Z^{2k}.

We have $H_{k-1}(W^{2k} - S^{k-1} \times D^{k+1}) \cong H_{k-1}(W^{2k})$ because any $(k-1)$-cycle can be homotoped in W^{2k} outside the fiber S^{k-1}, by general position arguments. The map induced by inclusion $H_{k-1}(S^{k-1} \times S^k) \to H_{k-1}(W^{2k} - S^{k-1} \times D^{k+1}) \cong H_{k-1}(W^{2k})$ is trivial if and only if the fiber of $W^{2k} \to N^{k+1}$ is null-homologous in W^{2k}. Therefore, by Mayer-Vietoris we have $\mathrm{rk}_{\mathbb{Q}}\mathrm{Im}(\nu) = 1$ if and only if the fiber of $W^{2k} \to N^{k+1}$ is null-homologous, and thus $\mathrm{rk}_{\mathbb{Q}}\mathrm{Im}(\nu) = 0$ otherwise.

Now, if $e_W = 0$ then the fiber cannot be null-homologous since the fibration W is trivial and thus

$$\mathrm{rk}_{\mathbb{Q}} H_k(M^{2k}) = 2b_1(N^{k+1}) + 2r.$$

When $e_W \neq 0$ we have

$$\mathrm{rk}_{\mathbb{Q}} H_k(M^{2k}) = 2b_1(N^{k+1}) + 2r - 1 + \mathrm{rk}_{\mathbb{Q}}\mathrm{Im}(\nu).$$

In particular we need that $\mathrm{rk}_{\mathbb{Q}}\mathrm{Im}(\nu) \equiv b_k(M^{2k}) (\mathrm{mod}\, 2)$ and the claimed formula follows. □

Remark 4.5. Similar computations can be done for arbitrary D but instead of $b_1(N)$ we have to use $b_1(\widehat{N})$. However, we have no control on the first Betti number of finite coverings in dimension three, out of the trivial upper bound. Thus the above result does not extend when $D > 1$.

Therefore D and r are uniquely determined by the topology of M^{2k} and N^{k+1}. Consider a smooth function $M^{2k} \to N^{k+1}$ with $\varphi(M^{2k}, N^{k+1})$ critical points. The proof of Theorem 1.4 shows that there exists some decomposition of M^{2k} as a fiber sum $W^{2k} \oplus \#_r S^k \times S^k$ such that $\varphi(M^{2k}, N^{k+1}) = 2r + 2$ (recall that $D = 1$), under the assumptions of Proposition 4.3. Then Lemma 4.4 shows that any decomposition of M^{2k} as a fiber sum $W^{2k} \oplus \#_s S^k \times S^k$, where $W^{2k} \to N^{k+1}$ factors as a spherical fibration over some degree D covering of N^{k+1}, actually forces $D = 1$ and $s = r$. Thus φ is as claimed and this ends the proof of the case $k = 4$.

Let us consider now the case $k = 2$. We can assume that the closed orientable 3-manifold N^3 is geometric, according to Perelman's solution to Thurston's geometrization Conjecture. Since $\pi_1(N^3)$ is co-Hopfian N^3 is irreducible. Moreover, if N^3 is not a Seifert fibered manifold covered by a torus bundle over S^1 nor by a $\Sigma \times S^1$, then $\pi_1(N)$ is co-Hopfian (see [60, 63]).

Let us recall that $f : M^4 \to N^3$ is a Montgomery-Samelson fibration and thus it induces a fibration $M^4 \setminus V \to N^3 \setminus B$, where B and V are the finite sets of critical values and critical points, respectively. Then the homotopy exact sequence of this fibration yields

$$0 \to \pi_2(M \setminus V) \to \pi_2(N \setminus B) \to \pi_1(F) \to \pi_1(M \setminus V) \to \pi_1(N \setminus B) \to \pi_0(F) \to 0.$$

Note that $\pi_1(M) \cong \pi_1(M \setminus V)$ and $\pi_1(N) \cong \pi_1(N \setminus B)$. We observed in the proof of the Proposition 4.1 above that the inclusion $F \hookrightarrow M$ sends $\pi_1(F)$ to 0. Therefore, we derive the exact sequence:

$$0 \to \pi_1(M) \to \pi_1(N) \to \pi_0(F) \to 0.$$

But $\pi_1(M)$ and $\pi_1(N)$ are isomorphic. Thus, if $\pi_1(N)$ is co-Hopfian then $\pi_0(F)$ is 0. This implies that the fibration W has connected fibers. Therefore $D = 1$ and hence Lemma 4.4 shows that r is uniquely determined. The same argument which was used above for $k = 4$ ends the proof. □

Remark 4.6.

(1) The braid group B_3, which is the fundamental group of the trefoil knot, is a well-known non-co-Hopfian group (see [48]). For instance there exist finite index subgroups of any index $k \equiv \pm 1 (\mathrm{mod}\, 6)$ which are isomorphic to B_3. Its quotient $B_3/\mathbb{Z} = PSL(2, \mathbb{Z})$ is also non-co-Hopfian.

(2) There exist also non-co-Hopfian 3-manifold groups, for instance the following groups:
$$\langle a, b, t | ab = ba,\ tat^{-1} = a^u b^v,\ tbt^{-1} = a^w b^z \rangle$$
where $\left(\begin{smallmatrix} u & v \\ w & z \end{smallmatrix}\right)$ is a (hyperbolic) matrix with two real distinct eigenvalues. These are fundamental groups of torus bundles over the circle which are sol-groups *i.e.* lattices in the group SOL.

Proposition 4.7. *Let M^4, N^3 be closed orientable manifolds with finite $\varphi(M^4, N^3)$ and M^4 non-fibered. Suppose that the fundamental group of the closed orientable 3-manifold N^3 is not co-Hopfian. Then, for any decomposition of M^4 as a fiber sum, the value of $2r + 2D$ is independent on the choice of the covering and*

$$\varphi(M^4, N^3) = 2r + 2D = 2\left\lceil \frac{b_2(M) + 1}{2} \right\rceil - 2b_1(N) + 2.$$

Proof. Consider first N^3 be a closed irreducible 3-manifold. Assume that we have a finite non-trivial covering $\widehat{N^3}$ of N^3 with $\pi_1(\widehat{N^3}) \cong \pi_1(N^3)$. Then $\pi_1(N^3)$ is infinite and not co-Hopfian.

Lemma 4.8. *The covering $\widehat{N^3}$ is homeomorphic to N^3.*

Proof. We will use below the fact that all 3-manifolds are geometric (*i.e.* Seifert fibered or hyperbolic or Haken or a connected sum of such manifolds), as it was established by Perelman.

The geometric 3-manifolds whose groups are not co-Hopfian are either reducible or else are finitely covered by either a torus bundle or else by a product $\Sigma \times S^1$, where Σ is a closed surface (see [60, 63]).

If N^3 is covered by $\Sigma \times S^1$ then it is a Seifert fibered 3-manifold. In particular N^3 is a closed aspherical 3-manifold whose homotopy type is completely described by $\pi_1(N^3)$. The finite covering $\widehat{N^3}$ is also an irreducible Seifert fibered 3-manifold with the same fundamental group. The asphericity implies that N^3 and $\widehat{N^3}$ are homotopy equivalent. A theorem of Scott ([51]) implies that they are actually homeomorphic.

If N^3 is finitely covered by a torus bundle over S^1 then the image of the fiber is an incompressible surface in N^3, corresponding to the embedding of $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1(N^3)$. Therefore N^3 is Haken. Since N^3 and $\widehat{N^3}$ are aspherical they are homotopy equivalent and a classical theorem of Waldhausen shows that they are homeomorphic. $\square$

In particular, the fibration W^3 is diffeomorphic to a S^1-fibration over N^3. Therefore M^3 is diffeomorphic to the fiber sum along D distinct fibers of the S^1-fibration W^4 with D respective spherical blocks $\#_{r_i} S^2 \times S^2$. Then there exists a smooth function $M^4 \to N^3$ with $2r + 2D$ critical points, which has connected fibers. Take a disk in N^3 containing the critical values and its inverse image under the smooth map, which is a connected disk block. The proof of Theorem 1.4 shows that M^4 decomposes as the fiber sum of a spherical block containing the $2r + 2D$ critical points and a circle fibration $V^4 \to N^3$. But the spherical block is diffeomorphic to $\#_{(\sum_{i=1}^{D} r_i)+D-1} S^2 \times S^2$. In the new fiber sum decomposition of M^4 we have then a covering of N^3 of degree 1 and the number of factors $S^2 \times S^2$ is $(\sum_{i=1}^{D} r_i) + D - 1$. Thus any fiber sum decomposition of M^4 leads to a fiber sum whose associated covering is of degree 1. Then Lemma 4.4 implies that the value of $(\sum_{i=1}^{D} r_i) + D - 1$ is independent on the fiber sum decomposition and thus the number of critical points, namely $2r + 2D$, equals $\varphi(M, N)$, as claimed.

Remark 4.9. If $W^{2k} \to N^{k+1}$ is a spherical fibration (with connected fiber) then it seems that the fiber sum of W^{2k} with D spherical blocks $\#_{r_i} S^k \times S^k$ along D fibers is actually diffeomorphic to the fiber sum of W^{2k} with a single spherical block $\#_{(\sum_{i=1}^{D} r_i)+D-1} S^k \times S^k$. This is not anymore clear if the fiber is not connected.

Consider now the case when N^3 is reducible. Thus either $\pi_1(N^3) = \mathbb{Z}$ or else N^3 splits as a connected sum $N_1^3 \# N_2^3 \# \cdots \# N_p^3$, where N_i^3 are either irreducible or have fundamental group $\mathbb{Z}$. It is known that a closed orientable 3-manifold with fundamental group $\mathbb{Z}$ is diffeomorphic to $S^2 \times S^1$, modulo the Poincaré conjecture (see *e.g.* [31]). Let now $\widehat{N^3}$ be a finite covering of N whose fundamental group is isomorphic $\pi_1(N)$. Then $\pi_1(\widehat{N^3})$ is either $\mathbb{Z}$ or splits as $\pi_1(N_1) * \pi_1(N_2) * \cdots * \pi_1(N_p)$. In the first case the manifolds are both $S^1 \times S^2$ and thus they are diffeomorphic.

Assume that the second alternative holds. By the affirmative solution of Kneser's Conjecture (see *e.g.* [33, 61]) we have that $\widehat{N^3}$ geometrically splits as $M_1^3 \# M_2^3 \# \cdots \# M_p^3$ where M_j^3 are closed 3-manifolds with $\pi_1(M_j^3) \cong \pi_1(N_j^3)$.

Lemma 4.10. *The irreducible 3-manifolds with the same fundamental group M_j^3 and N_j^3 are diffeomorphic.*

Proof. By the Jaco-Shalen-Johansson decomposition theorem there exists a family (possibly empty) of incompressible tori in each one of the manifolds M_j and N_j such that each connected component of the complement is atoroidal.

The family of tori is non-empty if there exists a $\mathbb{Z} \oplus \mathbb{Z}$ in the fundamental group. Therefore if the family of tori associated to one manifold is non-empty the family of tori associated to the other one is also non-empty and both 3-manifolds are Haken, since they are irreducible and contain incompressible surfaces. Waldhausen's theorem claims that a closed 3-manifold which is homotopy equivalent to a Haken manifold is actually diffeomorphic to it. Since both manifolds are irreducible they are aspherical and the isomorphism of their fundamental groups implies that they are homotopy equivalent.

If the family of tori is empty the manifolds are atoroidal. In this case Thurston's geometrization Conjecture states that the manifolds are hyperbolic. Then, by Mostow's rigidity, two hyperbolic manifolds having the same fundamental group are isometric and hence diffeomorphic. □

This implies that $\widehat{N^3}$ is diffeomorphic to N^3. Now we can apply the argument used in the case of non-co-Hopfian groups of irreducible 3-manifolds. This settles Proposition 4.7. □

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