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Abstract

Far away from a fundamental magnetic monopole of the SU(5) grand unified theory, the full theory reduces to one whose gauge group is unbroken SU(3) ⊗ U(1). The algebra of generators of smooth gauge transformations in this unbroken theory, however, will not in general contain any subalgebra isomorphic to su(3). This means that global color rotations are not always defined. The resulting inability to classify semiclassical dyon states by their color climinates an apparent paradox related to the vacuum angle θ.

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GLOBAL COLOR IS NOT ALWAYS DEFINED*

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ABSTRACT

Far away from a fundamental magnetic monopole of the SU(5) grand unified theory, the full theory reduces to one whose gauge group is unbroken $SU(3) \times U(1)$. The algebra of generators of smooth gauge transformations in this unbroken theory, however, will not in general contain any subalgebra isomorphic to su(3). This means that global color rotations are not always defined. The resulting inability to classify semiclassical dyon states by their color eliminates an apparent paradox related to the vacuum angle θ .

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Recently one of us[1] has performed an analysis similiar to Witten's[2] of the low-lying states in the fundamental monopole sector of the SU(5) grand unified theory.[3] The result is that in the presence of a non-zero vacuum angle θ, both the electric charge and the color hyper-charge (to which it is proportional) take on an arbitrary value. While this result was surprising for the electric charge in Witten's original analysis, it at first appears to be a disaster for the hypercharge, which is presumably embedded in the compact color SU(3) symmetry group.

The only way out of this dilemma seemed to be the possibility that in fact no group of gauge transformations isomorphic to SU(3) exists. While this situation may seem to be no less paradoxical than the original paradox, we show in this letter that it indeed obtains. This proof holds for the full quantum theory and gives necessary and sufficient conditions for the unbroken local symmetries of a grand unified theory to be integrable into well-defined global symmetries.

We can see the essential details of the problem in an SU(3) theory spontaneously broken to SU(2) × U(1) by an adjoint Higgs field, whose expectation value in the nonsingular "radial" gauge of Ref. 3 can be taken as $\langle\phi(\hat{r})\rangle = \begin{bmatrix} 1 & \\ & -\frac{1}{2} + \frac{3}{2}\vec{\tau}\cdot\hat{r} \end{bmatrix}$. $\vec{\tau}$ are the three Pauli matrices. Clearly the generator of unbroken Abelian transformations $T_{E.M.} = \begin{bmatrix} 1 & \\ & \frac{3}{2}\vec{\tau}\cdot\hat{r} - \frac{1}{2} \end{bmatrix}$ and one of the nonabelian ones $\sigma_3 = \begin{bmatrix} 1 & \\ & -\frac{1}{2}(\vec{\tau}\cdot\hat{r} + 1) \end{bmatrix}$ are everywhere well-defined; they can be obtained by transforming the constant matrices $\begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix}$, $\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}$ respectively from the singular "unitary" gauge of Ref. 3. When we attempt the same thing with the remaining two independent matrices commuting with $\phi(\hat{z})$, however, we find that they become singular at the south pole when we transform to radial gauge:

$$\sigma_1 = \begin{bmatrix} 0 & \cos\theta/2 & \sin\theta/2\ e^{-i\phi} \\ \cos\theta/2 & 0 & 0 \\ \sin\theta/2\ e^{i\phi} & 0 & 0 \end{bmatrix}$$

$$\sigma_2 = \begin{bmatrix} 0 & -i\cos\theta/2 & -i\sin\theta/2\ e^{-i\phi} \\ i\cos\theta/2 & 0 & 0 \\ i\sin\theta/2\ e^{i\phi} & 0 & 0 \end{bmatrix}$$

It seems possible, therefore, that some global gauge trans-formations may not exist. Is there any precedent for such a situa-tion? Yes there is. Consider the problem of defining a Euclidean algebra of vector fields on the two-dimensional manifold S^2. Near every point a full algebra of small motions is defined, but it is well known that not even a single smooth nonvanishing tangent vector field exists globally. This is so in spite of the fact that S^2 and its tangent bundle can be imbedded in $\mathbb{R}^3$, which has a perfectly good algebra of global translations and rotations. Similarly in gauge theories it is possible for a twisted unbroken theory to be the re-duction of a trivial unified theory.

To proceed beyond this heuristic level it is convenient to describe the unbroken theory in Wu and Yang's two-patch formalism.[4] The theory is defined on the space $\mathbb{R} \times \{\mathbb{R}^3 - B\}$, where B is a ball occupied by the monopole core. We restrict this space further to a single sphere S^2 at some given time, and divide it into two hemi-spheres U_α $\alpha = 1,2$. Writing points of S^2 as $\hat{r}$, we have $\hat{z} \in U_1$, $U_1 \cup U_2 = S^2$, $U_1 \cap U_2 = E$, the equator. All local quantities are represented as "twisted functions", pairs f_α of functions defined on U_α respectively and related at the equator by a gauge transformation[5] : $f_2(\theta = \frac{\pi}{2},\phi) = f_1(\theta = \frac{\pi}{2}, \phi)^{g(\phi)}$ where $g: E \to G$ and G is SU(2) × U(1), realized as unimodular 3×3 matrices[6] commuting with $\phi = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix}$. Then a local gauge transformation on S^2 is represented by $h_\alpha : U_\alpha \to G$, whose matching condition we obtain as follows:

$f_1 \to (f_1)^{h_1}$. On the equator,

$(f_2)^{g^{-1}} \to (f_2{}^{g^{-1}})^{h_1} = (f_2)^{h_1 g^{-1}}$. Hence

$f_2 \to f_2{}^{h_2}$, where

$$h_2(\theta = \frac{\pi}{2}, \phi) = g(\phi)\, h_1\, (\theta = \frac{\pi}{2}, \phi)\, g^{-1}(\phi) . \qquad (1)$$

The generators of infinitesimal gauge transformations are likewise twisted functions $\psi_\alpha : U_\alpha \to \mathfrak{g}$, the Lie algebra of G, with the same condition (1).

Suppose we have a set of such generators ψ_α^i , i = 1, ⋯ 4. These act on the various physical quantities, and we can commute them to get new generators. Clearly any smooth basis of "generators of global unbroken gauge transformations" must satisfy compatibility, eq. 1, plus $[\psi_\alpha^i\,(\hat{r}), \psi_\alpha^j\,(\hat{r})] = f_{ijk}\,\psi_\alpha^k\,(\hat{r})$, where f_{ijk} are the structure constants of $\mathfrak{g}$. We can also demand that the U(1) generator be properly normalized: $Tr(\psi_\alpha^4)^2$ = const. Both these conditions on ψ are consistent with eq. 1 at the equator. They amount to the requirement that $\psi_\alpha^i\,(\hat{r}) = A_\alpha^{ij}(\hat{r})\,\psi_1^j\,(\hat{z})$ where $A_\alpha : U_\alpha \to \mathrm{Aut}\,\mathfrak{g}$, the group of automorphisms of $\mathfrak{g}$.

Theorem[7]: All automorphisms of su(n) can be written as $\psi^i \to g\psi^i g^{-1}$ or $g(\psi^{i*})g^{-1}$, $g \in U(N)$. Call the former A_g^{ij}.

Corollary: The connected component of Aut su(n) containing $\mathbb{1}$ is

isomorphic to $SU(N)/\mathbb{Z}_N$ acting by conjugation.

To apply all this, we note that the fundamental monopole has transition function[8] $g(\phi) = \exp(2Q\phi)$ where $Q = \begin{bmatrix} 0 & \\ & \frac{1}{2i}\tau_3 \end{bmatrix}$. Thus on the equator $A_2^{ij}\,(\theta = \frac{\pi}{2}, \phi) = A^{ik}_{g(\phi)}\; A_1^{kj}(\theta = \frac{\pi}{2}, \phi)$. Since each U_α is a disk, each A on the equator defines a loop in Aut $\mathfrak{g}$ which is contractible: $A_\alpha(\phi) \sim 0$, where ~ means "is homotopic to". We have on the equator $0 \sim A_2 = A_g A_1 \sim A_g$. Conversely, if $A_g \sim 0$ we can define an acceptable set of ψ_α^i by taking $A_1 = \mathbb{1}$ on U_1 and A_2 = the supposed homotopy between A_g and the null path. *Thus global color can be well-defined if and only if $A_g \sim 0$.* The derivation of this condition evidently did not depend upon the specific gauge group.

Proposition: In the presence of the fundamental monopole, no smooth basis of generators of global unbroken gauge transformations exists.

Proof: $g(\phi)$ acts trivially on the Abelian generator. On the others, the upper corner of $g(\phi) = \begin{bmatrix} 1 & \\ & e^{-i\phi} \end{bmatrix} \in U(2)$. This projects to $\begin{bmatrix} e^{+\frac{i\phi}{2}} & \\ & e^{-\frac{i\phi}{2}} \end{bmatrix} \in SU(2)/\mathbb{Z}_2$, a closed loop. When lifted to the covering group SU(2), however, it is clearly *not* closed. Hence $g(\phi)$ is not homotopically trivial.

The above result makes minimal use of the specific monopole vector potential. In fact, closer inspection shows that, for general Q satisfying the Dirac quantization condition, the issue of color definition is settled completely by the one topological invariant k of the monopole.[9] k is the number of times $g(\phi)$ winds around the U(1); color can be globally defined iff k is even. Since the full quantum theory does not mix different topological sectors, our result is valid beyond the semiclassical approximation. In particular, it is not spoiled by confinement effects. Of course it is still limited to energies low enough for the unbroken theory to be a good approximation to the full one, and to the region far from the monopole core. Similarly the addition of matter fields changes nothing; these live on bundles associated to the principal bundle and derive their definition of gauge transformations from it.

The above results can be trivially extended to the general case. In the $SU(5) \to SU(3) \times U(1)$ theory, global color is defined iff $k \equiv 0 \bmod 3$. Hence color is undefined in the presence of the fundamental monopole, resolving the original paradox. On the other hand, the purely electromagnetic monopole[8] has $k = 3 \equiv 0 \bmod 3$ and so presents no problem, as expected.

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5. In a more elevated treatment g is the transition function of a nontrivial principal fiber bundle over S^2, and we are seeking an su(2) in the infinite-dimensional algebra of sections $\Gamma(\mathcal{E})$ where $\mathcal{E}$ is the associated bundle $\mathcal{E} = P \times_{Ad} \mathfrak{g}$.
6. This is the only place where the nature of the underlying unified theory is used.
7. N. Jacobson, Lie Algebras (New York, Wiley, 1962), chap. 9.
8. S. Coleman, "The Magnetic Monopole 50 Years Later", Harvard University preprint HUTP-82/A032 (Erice lectures 1981).
9. This is geometrically obvious: the existence of sections of the associated bundle depends on the structure of the principal bundle; its connection is irrelevant.