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GLOBAL DIMENSION OF TILED ORDERS OVER A DISCRETE VALUATION RING

BY

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ABSTRACT. Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and the quotient field K. Let $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(K)$ be a tiled R-order, where $\lambda_{ij} \in \mathbf{Z}$ and $\lambda_{ii} = 0$ for $1 \leq i \leq n$. The following results are proved. Theorem 1. *There are, up to conjugation, only finitely many tiled R-orders in* $M_n(K)$ *of finite global dimension.* Theorem 2. *Tiled R-orders in* $M_n(K)$ *of finite global dimension satisfy DCC.* Theorem 3. *Let* $\Lambda \subseteq M_n(R)$ *and let* Γ *be obtained from* Λ *by replacing the entries above the main diagonal by arbitrary entries from R. If* Γ *is a ring and if* gl dim $\Lambda < \infty$, *then* gl dim $\Gamma < \infty$. Theorem 4. *Let* Λ *be a tiled R-order in* $M_4(K)$. *Then* gl dim $\Lambda < \infty$ *if and only if* Λ *is conjugate to a triangular tiled R-order of finite global dimension or is conjugate to the tiled R-order* $\Gamma = (\mathfrak{m}^{\gamma_{ij}}) \subseteq M_4(R)$, *where* $\gamma_{ii} = \gamma_{1i} = 0$ *for all i, and* $\gamma_{ij} = 1$ *otherwise.*

Introduction. This paper is a continuation of the author's previous paper, *Global dimension of tiled orders over commutative noetherian domains* [7]. Throughout this paper R will denote a discrete valuation ring (DVR) with maximal ideal $\mathfrak{m}$, generated by t, and the quotient field K. In this paper we will use notations and terminologies of [7]. Let Λ be a tiled R-order in $M_n(K)$, i.e., an R-order in $M_n(K)$ containing n orthogonal idempotents. If a tiled R-order Λ in $M_n(K)$ contains the usual system e_{ii}, $1 \leq i \leq n$, of n orthogonal idempotents, then $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(K)$, where $\lambda_{ii} = 0$ and $\lambda_{ij} \in \mathbf{Z}$ for all i,j [7]. Furthermore, by conjugating if necessary, we may assume that $\lambda_{ij} \geq 0$ for all i, j (cf. Lemma 1.1). One of the main results in this paper shows that if $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ is a tiled R-order of finite global dimension, then $\lambda_{ij} \leq n - 1$ for all i, j; hence it follows that there are only finitely many tiled R-orders in $M_n(R)$ of finite global dimension. Using this we show that if $S_1, S_2, \cdots, S_k$ is a finite family of

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orthogonal idempotents in $M_n(K)$, and if $\mathcal{S}$ is the collection of all tiled R-orders in $M_n(K)$ of finite global dimension containing some S_i, then $\mathcal{S}$ satisfies the descending chain condition (DCC). This shows that the conjecture of R. B. Tarsey [12] is true for a wide class of R-orders in $M_n(K)$. The complete classification given in Theorem 4.2 shows that if Λ is a tiled R-order in $M_4(K)$, and if $\operatorname{gl\,dim} \Lambda < \infty$, then $\operatorname{gl\,dim} \Lambda \leq 3$. Since there is a tiled R-order in $M_4(K)$ of global dimension 3 [5], [12], this upper bound is best possible. An intrinsic characterization of a reduced triangular tiled R-order $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$, obtained in Theorem 3.3, is of independent interest. We recall that a tiled R-order $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ is reduced if $\lambda_{ij} > 0$ or $\lambda_{ji} > 0$ whenever $i \neq j$, and that Λ is a triangular tiled R-order if $\lambda_{ij} = 0$ whenever $i \leq j$. Lastly, let $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(K)$ be a tiled R-order. Since Λ is a ring, we have

0.1. $\lambda_{ij} \leq \lambda_{ik} + \lambda_{kj}$ for $1 \leq i, j, k \leq n$.

0.2. If Λ is a triangular tiled R-order, then $\lambda_{ij} \geq \lambda_{ik}$ and $\lambda_{ki} \geq \lambda_{ji}$ whenever $j \leq k$.

We will have several occasions of using 0.1 and 0.2, and sometimes we use them without giving a reference.

The main results of this paper were announced in [6].

1. Preliminaries.

In this section we prove some preliminary results which will be needed in the sequel.

Lemma 1.1. *Let $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(K)$ be a tiled R-order. Then there exists a tiled R-order $\Gamma = (\mathfrak{m}^{\gamma_{ij}}) \subseteq M_n(R)$ such that $\gamma_{1j} = 0$ for all j and $\Gamma = y\Lambda y^{-1}$ for some unit y in $M_n(K)$. Furthermore, $ye_{ii}y^{-1} = e_{ii}$ for $1 \leq i \leq n$.*

Proof. Let y be the diagonal matrix in $M_n(K)$ with $t^{\lambda_{1i}}$ as the (i, i)th entry, where $\mathfrak{m} = tR$. Set $\Gamma = y\Lambda y^{-1}$. Then a direct computation shows that Γ and y satisfy the conditions of the lemma.

Definition 1.2. If Λ and Γ are tiled R-orders in $M_n(K)$, then Λ and Γ are *permutationally conjugate* if one is obtained from the other by permuting rows and columns, equivalently, $\Lambda = \epsilon\Gamma\epsilon^{-1}$ for some permutation matrix ϵ in $M_n(K)$.

Lemma 1.3. *Let $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ be a reduced tiled R-order, where $\lambda_{1j} = 0$ for all j. Then Λ is permutationally conjugate to a tiled R-order $\Gamma = (\mathfrak{m}^{\gamma_{ij}}) \subseteq M_n(R)$, where $\gamma_{1j} = 0$ for all j, and $\gamma_{ij} > 0$ whenever $i > j$.*

Proof. We use induction on n. If $n = 2$, then the assertion is trivial. Let $n \geq 3$. Since $\lambda_{1j} = 0$ for all j and since Λ is reduced, therefore by Lemma 1.7 of [7] we have an integer $l > 1$ such that $\lambda_{li} > 0$ whenever $i \neq l$. By interchanging the lth and the nth rows and columns, we may further assume that $l = n$. Thus,

$\lambda_{ni} > 0$ whenever $i \neq n$, and $\lambda_{1j} = 0$ for all j. Clearly, $e\Lambda e$ is a reduced tiled R-order contained in $M_{n-1}(R)$, where $e = \Sigma_{i=1}^{n-1} e_{ii}$. Hence by the induction hypothesis, $e\Lambda e$ is permutationally conjugate to a tiled R-order $\Gamma' = (\mathfrak{m}^{\gamma'_{ij}}) \subseteq M_{n-1}(R)$, where $\gamma'_{1j} = 0$, $\gamma'_{ij} > 0$ whenever $i > j$. Thus $\Gamma' = y'(e\Lambda e)y'^{-1}$ for some permutation matrix $y' = (y'_{ij})$ in $M_{n-1}(K)$. Let $y = (y_{ij})$ in $M_n(K)$ with $y_{nn} = 1$, $y_{nj} = y_{jn} = 0$ for $j \neq n$, and $y_{ij} = y'_{ij}$ otherwise. Then $\Gamma = y\Lambda y^{-1}$ fulfills the requirements of the lemma.

Let $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ be a tiled R-order. Let x be the diagonal matrix in $M_n(K)$ with t on the diagonal. Let $\overline{\Lambda} = \Lambda/\Lambda x = \Lambda/\Lambda\mathfrak{m}$. Then $\overline{\Lambda} \cong \Lambda \otimes_R R/\mathfrak{m}$ as $R/\mathfrak{m}$-algebras and thus $\overline{\Lambda}$ is a finite dimensional $R/\mathfrak{m}$-algebra. Obviously $\overline{\Lambda}$ is isomorphic to the $R/\mathfrak{m}$-algebra $(\mathfrak{m}^{\lambda_{ij}}/\mathfrak{m}^{\lambda_{ij}+1})$, where the multiplication is induced from that in Λ, i.e., if $(a_{ij} + \mathfrak{m}^{\lambda_{ij}+1})$ and $(b_{ij} + \mathfrak{m}^{\lambda_{ij}+1})$ are in $(\mathfrak{m}^{\lambda_{ij}}/\mathfrak{m}^{\lambda_{ij}+1})$, then $(a_{ij} + \mathfrak{m}^{\lambda_{ij}+1})(b_{ij} + \mathfrak{m}^{\lambda_{ij}+1}) = (\Sigma_{k=1}^{n} a_{ik}b_{kj} + \mathfrak{m}^{\lambda_{ij}+1})$. From now on we will always identify the two $R/\mathfrak{m}$-algebras $\overline{\Lambda}$ and $(\mathfrak{m}^{\lambda_{ij}}/\mathfrak{m}^{\lambda_{ij}+1})$. Let $\overline{e}_{ii} = e_{ii} + \Lambda\mathfrak{m}$, $1 \leq i \leq n$. Then $\overline{e}_{ii}$ are orthogonal indecomposable idempotents in $\overline{\Lambda}$ and $\Sigma_{i=1}^{n} \overline{e}_{ii} = 1$. Furthermore, $\overline{P}_i = \overline{e}_{ii}\overline{\Lambda}$, $1 \leq i \leq n$, are, up to isomorphism, the only principal right projectives of $\overline{\Lambda}$. Since $\mathfrak{m}^{\alpha}/\mathfrak{m}^{\alpha+1} \cong R/\mathfrak{m}$ for every nonnegative integer α, $[\overline{P}_i : R/\mathfrak{m}] = n$. Also, if Λ is reduced, then by Lemma 1.3 of [7], $J(\overline{\Lambda})$ is obtained from $\overline{\Lambda}$ by replacing the diagonal entries $R/\mathfrak{m}$ by zero. We now show that if M is a finitely generated right $\overline{\Lambda}$-module with $[M : R/\mathfrak{m}] \not\equiv 0 \bmod n$, then $\mathrm{hd}_{\overline{\Lambda}}\, M = \infty$.

Proposition 1.4. *Let E be a finite dimensional algebra over a field F. Assume that for every indecomposable idempotent e in E, $[eE : F] \equiv 0 \bmod l$, where l is independent of e. Then, for any finitely generated right E-module M with $[M : F] \not\equiv 0 \bmod l$, we have $\mathrm{hd}_E\, M = \infty$.*

Proof. Since E is a finite dimensional algebra over the field F, the algebra E is artinian. Hence, by Theorem 56.6 of [3, p. 382], if P is a finitely generated projective right E-module, then $P \cong \bigoplus_{i \in I} e_iE$, where $|I| < \infty$ and the e_i are indecomposable idempotents in E. By the hypothesis $[e_iE : F] \equiv 0 \bmod l$; therefore $[P : F] \equiv 0 \bmod l$ for any finitely generated projective right E-module. Now assume that $\mathrm{hd}_E\, M = \beta < \infty$. Then we have an exact sequence.

$$0 \longrightarrow X_\beta \xrightarrow{\delta_\beta} X_{\beta-1} \xrightarrow{\delta_{\beta-1}} \cdots \longrightarrow X_1 \xrightarrow{\delta_1} X_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

where X_i are finitely generated projective right E-modules. By Corollary 2 of [2, p. 151], we have $[M : F] = \Sigma_{i=0}^{\beta} (-1)^i [X_i : F] \equiv 0 \bmod l$. But this contradicts the hypothesis that $[M : F] \not\equiv 0 \bmod l$. Thus $\mathrm{hd}_E\, M = \infty$.

Corollary 1.5. *Let $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ be a tiled R-order. Let $\overline{\Lambda} = \Lambda/\Lambda\mathfrak{m}$. If M is a finitely generated right $\overline{\Lambda}$-module with $[M:R/\mathfrak{m}] \not\equiv 0 \bmod n$, then* $\mathrm{hd}_{\overline{\Lambda}} M = \infty$.

Let $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ be a tiled R-order. Let $A = (R, R, \cdots, R)$ be a free left R-module of rank n. Then A is a right $M_n(R)$-module naturally. This module multiplication induces a $(R - \Lambda)$ bimodule structure on A. Further, if M is a nonzero Λ-submodule of A, then, since R is a principal ideal domain, M is also a free R-module of rank n (cf. remarks at the end of §1 of [7]).

Corollary 1.6. *Let $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ be a tiled R-order. Let $\overline{\Lambda} = \Lambda/\Lambda\mathfrak{m} = \Lambda/\Lambda x$. Let A be a free left R-module of rank n treated as a right Λ-module naturally. Let M be a nonzero Λ-submodule of A. If $\overline{M} = M/Mx$ and if $\overline{M}_{\overline{\Lambda}} = B_{\overline{\Lambda}} \oplus C_{\overline{\Lambda}}$ is a nontrivial decomposition of $\overline{M}$ as a right $\overline{\Lambda}$-module, then* $\mathrm{hd}_{\overline{\Lambda}} \overline{M} = \infty$ *and* $\mathrm{hd}_{\Lambda} M = \infty$.

Proof. Clearly $\mathrm{hd}_{\overline{\Lambda}} \overline{M} = \infty$, by Corollary 1.5. Hence $\mathrm{hd}_{\Lambda} M = \infty$, by Theorem 9 of [8, p. 178].

Lemma 1.7. *Let $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ be a tiled R-order. Let $\overline{\Lambda} = \Lambda/\Lambda\mathfrak{m}$. Let $A = (R, R, \cdots, R)$ be a free left R-module of rank n. Treat A as a right Λ-module naturally. If $B = (\mathfrak{m}^{\alpha_1}, \mathfrak{m}^{\alpha_2}, \cdots, \mathfrak{m}^{\alpha_n}) \subseteq A$, where $0 \leq \alpha_i$ are integers, then*

(1) *B is a Λ-submodule of A if and only if $\lambda_{ij} \geq \alpha_j - \alpha_i$ for all i, j.*

(2) *If B is a Λ-submodule of A, then*

$$\overline{B} = B/B\mathfrak{m} = (\mathfrak{m}^{\alpha_1}/\mathfrak{m}^{\alpha_1+1}, \cdots, \mathfrak{m}^{\alpha_s}/\mathfrak{m}^{\alpha_s+1}, 0, \cdots, 0)$$
$$\oplus (0, \cdots, 0, \mathfrak{m}^{\alpha_{s+1}}/\mathfrak{m}^{\alpha_{s+1}+1}, \cdots, \mathfrak{m}^{\alpha_n}/\mathfrak{m}^{\alpha_n+1})$$

as right $\overline{\Lambda}$-modules if and only if $\lambda_{ij} \geq \alpha_j - \alpha_i$ for all i, j; $\lambda_{ij} > \alpha_j - \alpha_i$ for $1 \leq i \leq s < j \leq n$; and $\lambda_{ij} > \alpha_j - \alpha_i$ for $1 \leq j \leq s < i \leq n$. Further, if these conditions hold, then $\mathrm{hd}_{\Lambda} B = \infty$.

(3) *If B is a Λ-submodule of A, then*

$$\overline{B} = B/B\mathfrak{m} = (0, \cdots, 0, \mathfrak{m}^{\alpha_s}/\mathfrak{m}^{\alpha_s+1}, 0, \cdots, 0)$$
$$\oplus (\mathfrak{m}^{\alpha_1}/\mathfrak{m}^{\alpha_1+1}, \cdots, \mathfrak{m}^{\alpha_{s-1}}/\mathfrak{m}^{\alpha_{s-1}+1}, 0, \mathfrak{m}^{\alpha_{s+1}}/\mathfrak{m}^{\alpha_{s+1}+1}, \cdots, \mathfrak{m}^{\alpha_n}/\mathfrak{m}^{\alpha_n+1})$$

as right $\overline{\Lambda}$-modules if and only if $\lambda_{ij} \geq \alpha_j - \alpha_i$ for all i, j, $\lambda_{sj} > \alpha_j - \alpha_s$ and $\lambda_{js} > \alpha_s - \alpha_j$ whenever $j \neq s$. Further, if these conditions hold, then $\mathrm{hd}_{\Lambda} B = \infty$.

Proof. The proof is a straightforward computation and we leave it to the reader. That $\mathrm{hd}_{\Lambda} B = \infty$ in (2) and (3) follows from Corollary 1.6.

2. Tiled orders in $M_n(K)$. In this section we show that, up to conjugation, there are only finitely many tiled R-orders in $M_n(K)$ of finite global dimension (Theorem 2.3). We also show that certain large classes of tiled R-orders in $M_n(K)$ of finite global dimension satisfy DCC (Theorem 2.5).

Lemma 2.1. *If $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ is a tiled R-order with* gl dim $\Lambda < \infty$, *then for every integer k, $1 \leq k \leq n-1$, there exist integers $i \geq k+1$ and $j \leq k$ such that $\lambda_{ij} \leq 1$.*

Proof. Fix $k \geq 1$. Suppose that $\lambda_{ij} \geq 2$ whenever $i \geq k+1$ and $j \leq k$. Set $\alpha_i = 1$ for $1 \leq i \leq k$ and $\alpha_i = 0$ for $k+1 \leq i \leq n$. Then it is easy to check that the conditions of Lemma 1.7 (1) and (2) for the right Λ-module B are satisfied with $s = k$; and therefore $\operatorname{hd}_\Lambda B = \infty$. This is impossible as gl dim $\Lambda < \infty$. Thus for some integers $i \geq k+1$ and $j \leq k$ we must have $\lambda_{ij} \leq 1$.

Lemma 2.2. *Let $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ be a tiled order with* gl dim $\Lambda < \infty$. *Assume that λ_{i1} is an increasing function of i. Then,*

(1) $0 \leq \lambda_{i+1,1} - \lambda_{i1} \leq 1$ *for* $1 \leq i \leq n-1$,

(2) $\lambda_{i1} \leq i-1$ *for* $1 \leq i \leq n$,

(3) *if* $\lambda_{l1} < l-1$ *for some l, then* $\lambda_{i1} < i-1$ *whenever* $i \geq l$.

Proof. Fix an integer k between 1 and $n-1$. By Lemma 2.1 we have integer integers $s \geq k+1$ and $j \leq k$ such that $\lambda_{sj} \leq 1$. Hence by 0.1 and the monotonicity of λ_{i1} we have

$$\lambda_{k1} \leq \lambda_{k+1,1} \leq \lambda_{s1} \leq \lambda_{sj} + \lambda_{j1} \leq 1 + \lambda_{k1}.$$

Thus $\lambda_{k1} \leq \lambda_{k+1,1} \leq 1 + \lambda_{k1}$, which proves (1). For (2) we use an induction on i. Since $\lambda_{11} = 0$, the statement is true for $i = 1$. Assume that $\lambda_{i1} \leq i-1$. Then by using (1) of this lemma we have $\lambda_{i+1,1} \leq 1 + \lambda_{i1} \leq i$. This completes the induction and proves (2) The proof of (3) is similar.

In the next theorem we show that if we consider the class of all tiled R-orders of finite global dimension in $M_n(K)$ containing n orthogonal idempotents, then up to conjugation this class is finite.

Theorem 2.3. *Let R be a DVR with maximal ideal $\mathfrak{m}$ and quotient field K. Then*:

(1) *If $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ is a tiled R-order with* gl dim $\Lambda < \infty$, *then $\lambda_{ij} \leq n-1$ for $1 \leq i, j \leq n-1$.*

(2) *There are only finitely many tiled R-orders in $M_n(R)$ of finite global dimension containing a fixed set of n orthogonal idempotents.*

(3) *There are, up to conjugation, only finitely many tiled R-orders in $M_n(K)$ of finite global dimension.*

Proof. First we note that to prove (1) it is enough to show that $\lambda_{i1} \leq n-1$ for all i, since by interchanging the 1st and the jth rows and columns we can always assume that $j=1$. Furthermore, by permuting rows and columns of Λ through 2 to n we may as well assume that λ_{i1} is an increasing function of i. But then by Lemma 2.2(2) we have $\lambda_{i1} \leq i-1 \leq n-1$ for all i. Thus $\lambda_{ij} \leq n-1$ for $1 \leq i, j \leq n$.

We now prove (2). Let f_i, $1 \leq i \leq n$, be a fixed set of n orthogonal idempotents in $M_n(R)$. $M_n(R)$ contains e_{ii}, $1 \leq i \leq n$, and f_i and e_{ii} are local idempotents with $\Sigma_{i=1}^n f_i = 1 = \Sigma_{i=1}^n e_{ii}$; therefore by Proposition 3 of [9, p. 77] we have a unit u in $M_n(R)$ and a permutation π on the numbers 1 to n such that $e_{ii} = uf_{\pi(i)}u^{-1}$ for $1 \leq i \leq n$. Thus, if Λ is a tiled R-order in $M_n(R)$ containing f_i, $1 \leq i \leq n$, then $u\Lambda u^{-1}$ is a tiled R-order in $M_n(R)$ containing e_{ii}, $1 \leq i \leq n$. Hence to complete the proof we must show that there are only finitely many tiled R-orders in $M_n(R)$ of finite global dimension containing e_{ii}, $1 \leq i \leq n$. But this is obvious in view of (1).

To prove (3), let Λ be an R-order in $M_n(K)$ containing n orthogonal idempotents. Then Λ is conjugate to a tiled R-order Γ in $M_n(K)$ containing e_{ii}, $1 \leq i \leq n$, which in turn, by Lemma 1.1, is conjugate to a tiled R-order $\Delta = (\mathfrak{m}^{\delta_{ij}}) \subseteq M_n(R)$. Now the assertion follows trivially from (1) and (2).

This complete the proof of the theorem.

Proposition 2.4. *Let $f_1, f_2, \cdots, f_n$ be n orthogonal idempotents. Let $\mathcal{S}$ be the set of all tiled R-orders Λ in $M_n(K)$ such that* gl dim $\Lambda < \infty$ *and $f_i \in \Lambda$ for $1 \leq i \leq n$. Then $\mathcal{S}$ satisfies the descending chain condition.*

Proof. Let $\Lambda_1 \supseteq \Lambda_2 \supseteq \cdots \supseteq \Lambda_j \supseteq \Lambda_{j+1} \supseteq \cdots$ be a descending chain of tiled R-orders in $\mathcal{S}$. By Proposition 3 of [9, p. 77] we have a unit u in $M_n(K)$ such that, for all j, $u\Lambda_j u^{-1}$ is a tiled R-order in $M_n(K)$ containing e_{ii}, $1 \leq i \leq n$. By Lemma 1.1 we have a unit y in $M_n(K)$ such that $yu\Lambda_1 u^{-1}y^{-1} \subseteq M_n(R)$ and $ye_{ii}y^{-1} = e_{ii}$ for all i. Set $z = yu$. Then clearly

$$z\Lambda_1 z^{-1} \supseteq z\Lambda_2 z^{-1} \supseteq \cdots \supseteq z\Lambda_j z^{-1} \supseteq z\Lambda_{j+1} z^{-1} \supseteq \cdots$$

is a descending chain of tiled R-orders in $M_n(R)$. Furthermore, for all j, gl dim $z\Lambda_j z^{-1} < \infty$ and $e_{ii} \in z\Lambda_j z^{-1}$, $1 \leq i \leq n$. Hence by Theorem 2.3(2) we have an integer l such that $z\Lambda_j z^{-1} = z\Lambda_{j+1} z^{-1}$ for all $j \geq l$. Consequently $\Lambda_j = \Lambda_{j+1}$ for all $j \geq l$. This completes the proof.

Theorem 2.5. *Let R be a DVR with quotient field K. Let $S_1, S_2, \cdots, S_k$ be a finite collection of sets, where each S_j is a set of n orthogonal idempotents in $M_n(K)$. Let $\mathcal{S}$ be the collection of all tiled R-orders Λ in $M_n(K)$ such that $S_j \subset \Lambda$ for some j and* gl dim $\Lambda < \infty$. *Then $\mathcal{S}$ satisfies DCC.*

Proof. Let

$$(*) \qquad \Lambda_1 \supseteq \Lambda_2 \supseteq \cdots \supseteq \Lambda_i \supseteq \Lambda_{i+1} \supseteq \cdots$$

be a descending chain of tiled R-orders in $\mathcal{S}$. Let $\mathcal{S}_j = \{\Lambda_i: \Lambda_i \supset S_j\}$, $1 \le j \le k$. If $\mathcal{S}_j$ is nonempty, then by Proposition 2.4 we have a natural number μ_j such that $\Lambda_i = \Lambda_{\mu_j}$ for all $i \ge \mu_j$. If $\mathcal{S}_j$ is empty set $\mu_j = 0$. Let $\mu = \max_{1\le j\le k} \mu_j$. Let $i \ge \mu$. Since $\Lambda_i \in \mathcal{S}_j$ for some j, therefore $\Lambda_i = \Lambda_{\mu_j} = \Lambda_\mu$. This shows that the chain $(*)$ terminates. This completes the proof.

The above theorem shows that for a large class of R-orders in $M_n(K)$, Tarsey's conjecture [12] is true.

Theorem 2.6. *Let* $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ *be a tiled R-order with* gl dim $\Lambda < \infty$. *Let* Γ *be the set of matrices obtained from* Λ *by replacing the entries above the main diagonal by arbitrary entries from R. If* Γ *is a ring, then* gl dim $\Gamma < \infty$.

Proof. By the hypothesis $\Gamma = (\mathfrak{m}^{\gamma_{ij}}) \subseteq M_n(R)$, where $\gamma_{ij} = \lambda_{ij}$ for $i > j$ and $\gamma_{ij} = 0$ otherwise, is a ring. Hence Γ is a triangular tiled R-order. By Theorem 1 of [5], to show that gl dim $\Gamma < \infty$ it is enough to show that $\gamma_{k+1,k} \le 1$ for $1 \le k \le n-1$. Fix an integer k between 1 and $n-1$. Since gl dim $\Lambda < \infty$, therefore by Lemma 2.1 we have integers $i \ge k+1$ and $j \le k$ such that $\lambda_{ij} \le 1$. Since $i > j$, $\gamma_{ij} = \lambda_{ij} \le 1$. But then, by 0.2, we have $\gamma_{k+1,k} \le \gamma_{k+1,j} \le \gamma_{ij} \le 1$. Thus $\gamma_{k+1,k} \le 1$. This completes the proof.

Lemma 2.7. *Let* $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ *be a reduced tiled R-order with* gl dim $\Lambda < \infty$. *Then for any integer* k, $1 \le k \le n$, *there exists an integer* $\mu_k \ne k$, *depending on* k, *such that* $\lambda_{k,\mu_k} + \lambda_{\mu_k,k} = 1$.

Proof. Fix $k \le n$. Suppose that $\lambda_{jk} + \lambda_{kj} \ge 2$ for all $j \ne k$. Λ is reduced, therefore by Remark 2 at the end of §1 of [7], $J(\Lambda)$ is obtained from Λ by replacing the diagonal entries R by $\mathfrak{m}$. It is easy to see that the right Λ-module $J_k = e_{kk}\Lambda$ satisfies the conditions of Lemma 1.7(3) with $s = k$, and therefore $\mathrm{hd}_\Lambda\, J_k = \infty$. This contradicts the hypothesis that gl dim $\Lambda < \infty$. Thus for some integer $\mu_k \ne k$ we must have $\lambda_{\mu_k,k} + \lambda_{k,\mu_k} \le 1$. Since Λ is reduced and $\mu_k \ne k$, $\lambda_{\mu_k,k} + \lambda_{k,\mu_k} = 1$.

Corollary 2.8. *Let* $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ *be a tiled R-order. Assume that* $\lambda_{1j} = 0$ *for all* j *and* $\lambda_{ij} > 0$ *whenever* $i \ge 2$ *and* $i \ne j$. *Then* gl dim $\Lambda < \infty$ *if and only if* $\lambda_{ij} = 1$ *whenever* $i \ge 2$ *and* $i \ne j$.

Proof. The "if" part follows from Proposition 3.3 of [7]. We now prove the "only if" part. Clearly Λ is reduced. Hence by Lemma 2.7, for every integer

i, $1 \leq i \leq n$, we have an integer $\mu_i \neq i$ such that $\lambda_{\mu_i,i} + \lambda_{i,\mu_i} = 1$. If $\mu_i \geq 2$ and $i \geq 2$, then by the hypothesis we have $\lambda_{\mu_i,i} + \lambda_{i,\mu_i} \geq 2$. Thus, if $i \geq 2$, then we must have $\mu_i = 1$, so that $\lambda_{\mu_i,i} = \lambda_{1i} = 0$ and $\lambda_{i1} = \lambda_{i,\mu_i} = 1$. Hence $0 < \lambda_{ij} \leq \lambda_{i1} + \lambda_{1j} \leq 1$, whenever $i \geq 2$ and $i \neq j$. This completes the proof.

In [5] we have seen that the triangular tiled R-order $\Omega_n = (\mathfrak{m}^{\omega_{ij}}) \subseteq M_n(R)$, where $\omega_{ij} = i - j$ for $i > j$ and $\omega_{ij} = 0$ otherwise, plays an important role. We now show that if $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ is a tiled order of finite global dimension and if $\lambda_{ij} = n - 1$ for some $i \neq j$, then Λ is permutationally conjugate to the tiled R-order Ω_n. This in particular shows that if we disturb even slightly the "upper triangle" of Ω_n by replacing R by a proper ideal of R, then we end up with a tiled R-order of infinite global dimension. First we need a proposition.

Proposition 2.9. *Let* $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ *be a tiled R-order. Assume that*

$$\lambda_{i,i-1} = 1 \quad \text{for } 2 \leq i \leq n, \qquad \lambda_{i,i-3} = 3 \quad \text{for } 4 \leq i \leq n,$$
$$\lambda_{i,i-2} = 2 \quad \text{for } 3 \leq i \leq n, \qquad \lambda_{ij} \geq 3 \quad \text{for } i - j \geq 4.$$

Then gl dim $\Lambda < \infty$ *if and only if* Λ *is a triangular tiled R-order.*

Proof. The "if" part follows from Theorem 1 of [5]. We now prove the "only if" part. First, we observe that if $\lambda_{i,i+1} = 0$ for all i, then $\lambda_{i,i+2} = 0$ for all i, since by 0.1 we have $0 \leq \lambda_{i,i+2} \leq \lambda_{i,i+1} + \lambda_{i+1,i+2} \leq 0$. Repeating this argument one can show that $\lambda_{i,i+j} = 0$ for all $j \geq 1$, so that Λ is a triangular tiled R-order. Thus to prove the "only if" part it is enough to show that $\lambda_{i,i+1} = 0$ for all $i \geq 1$. Since $\lambda_{ij} > 0$ whenever $i > j$, Λ is reduced. By the assumption gl dim $\Lambda < \infty$, therefore by Lemma 2.7 we have natural numbers $\mu_1 \neq 1$ and $\mu_n \neq n$ such that $\lambda_{1,\mu_1} + \lambda_{\mu_1,1} = 1$ and $\lambda_{n,\mu_n} + \lambda_{\mu_n,n} = 1$. Also by the hypothesis $\lambda_{21} = 1$, $\lambda_{i1} \geq 2$ for $3 \leq i \leq n$; and $\lambda_{n,n-1} = 1$, $\lambda_{ni} \geq 2$ for $1 \leq i \leq n-2$. Hence, we must have $\mu_1 = 2$, $\mu_n = n - 1$ and $\lambda_{12} = 0 = \lambda_{n-1,n}$. If $n = 3$, then we are done. So assume that $n \geq 4$. Fix an integer k, where $2 \leq k \leq n - 2$. Set $\alpha_i = 2$ for $i \leq k - 1$, $\alpha_k = \alpha_{k+1} = 1$ and $\alpha_i = 0$ for $k + 2 \leq i \leq n$. If $\lambda_{k,k+1} > 0$, then one can easily check that the conditions of Lemma 1.7(1) and (2) for the right Λ-module B are satisfied with $s = k$, and therefore $\mathrm{hd}_\Lambda B = \infty$. This contradicts the assumption that gl dim $\Lambda < \infty$. Thus we must have $\lambda_{k,k+1} = 0$. This completes the proof of the proposition.

Corollary 2.10. *Let* $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ *be a tiled R-order. Assume that* $\lambda_{ij} = i - j$ *whenever* $i > j$. *Then* gl dim $\Lambda < \infty$ *if and only if* $\Lambda = \Omega_n$, *where* $\Omega_n = (\mathfrak{m}^{\omega_{ij}}) \subseteq M_n(R)$ *with* $\omega_{ij} = i - j$ *whenever* $i > j$ *and* $\omega_{ij} = 0$ *otherwise.*

Proof. The proof is a direct application of Proposition 2.9.

Theorem 2.11. *Let R be a DVR with maximal ideal $\mathfrak{m}$ and the quotient field K. Let $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ be a tiled R-order of finite global dimension. If $\lambda_{ij} = n-1$ for some $i \neq j$, then Λ is permutationally conjugate to Ω_n, where Ω_n is as defined in Corollary 2.10.*

Proof. If Λ is not reduced, then we have $\lambda_{kl} = \lambda_{lk} = 0$ for some $k \neq l$. Hence, Λ is Morita equivalent to the tiled R-order Γ obtained from Λ by deleting the lth row and the lth column. Since gl dim Γ = gl dim $\Lambda < \infty$, Theorem 2.3(1) yields $\lambda_{ij} \leq n-2$ for $1 \leq i, j \leq n$, $i \neq l$, $j \neq l$. By using 0.1, it is easy to see that $\lambda_{ki} = \lambda_{li}$ and $\lambda_{ik} = \lambda_{il}$ for $1 \leq i \leq n$, and therefore we must have $\lambda_{ij} \leq n-2$ for all i, j. But by the hypothesis $\lambda_{ij} = n-1$ for some $i \neq j$; hence it follows that Λ is reduced. We now observe that to prove the theorem it is enough to show that Λ is permutationally conjugate to a tiled R-order $\Gamma = (\mathfrak{m}^{\gamma_{ij}}) \subseteq M_n(R)$, where $\gamma_{ij} = i - j$ for $i > j$, since then by Corollary 2.10 we have $\Gamma = \Omega_n$. By interchanging suitable rows and columns we may assume that $\lambda_{n1} = n-1$. By Theorem 2.3(1) we have $\lambda_{i1} \leq n-1$ for all i. By permuting rows and columns of Λ through 2 to n, we may further assume that λ_{i1} is an increasing function of i. But $\lambda_{n1} = n-1$; therefore by Lemma 2.2(2) and (3) we must have $\lambda_{i1} = i-1$ for $1 \leq i \leq n-1$. Hence, by 0.1, we have $i = \lambda_{i+1,1} \leq \lambda_{i+1,i} + \lambda_{i1} = \lambda_{i+1,i} + i - 1$ for all i. This shows that $\lambda_{i+1,i} \geq 1$. By Lemma 2.1 we have integers $s \geq i+1$ and $j \leq i$ such that $\lambda_{sj} \leq 1$. By the monotonicity of λ_{i1} and 0.1 we have

$$i = \lambda_{i+1,1} \leq \lambda_{s1} \leq \lambda_{sj} + \lambda_{j1} \leq 1 + j - 1 = j \leq i.$$

Thus we have $i \leq s - 1 = \lambda_{s1} \leq j \leq i$, and therefore $i = j = s - 1$ and $\lambda_{i+1,i} = \lambda_{sj} \leq 1$. All this shows that $\lambda_{i+1,i} = 1$ for all i. We now show that $\lambda_{ij} = i - j$ whenever $i > j$. By 0.1 we have $\lambda_{i1} \leq \lambda_{ij} + \lambda_{j1}$, i.e., $i - 1 \leq \lambda_{ij} + j - 1$. Hence $\lambda_{ij} \geq i - j$. To show that $\lambda_{ij} \leq i - j$ whenever $i > j$ we use induction on i. When $i = 2$, we have $j = 1$. Since $\lambda_{21} = 1$, the statement is true when $i = 2$. Let $i \geq 3$ and let $j < i$. By 0.1 we have $\lambda_{ij} \leq \lambda_{i,i-1} + \lambda_{i-1,j}$. Hence by the induction hypothesis we have $\lambda_{ij} \leq 1 + (i-1) - j = i - j$. This completes the induction and shows that $\lambda_{ij} = i - j$ whenever $i > j$. This completes the proof.

3. Characterization of triangular tiled orders. In this section we obtain an intrinsic characterization of a triangular tiled order, i.e., we give, in terms of λ_{ij}, a necessary and sufficient condition for a tiled R-order $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ to be conjugate to a triangular tiled R-order in $M_n(R)$. If $n = 2$, Λ is always conjugate to a triangular tiled R-order by Lemma 1.1. So throughout this section we assume that $n \geq 3$.

Lemma 3.1. *Let* $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ *be a tiled R-order. Fix a natural number* $i \leq n$. *Let* $\{i_1, i_2, \cdots, i_{n-1}\}$ *be a permutation of the set* $\{1, 2, \cdots, i-1, i+1, \cdots, n\}$. *If for some fixed integer* j, *where* $1 \leq j \leq n-1$, *we have* $\lambda_{i,i_s} + \lambda_{i_s,i_{s+1}} = \lambda_{i,i_{s+1}}$ *whenever* $s \geq j$, *then*

(a) $\lambda_{i,i_l} + \lambda_{i_l,i_k} = \lambda_{i,i_k}$ *for* $k \geq l \geq j$.

(b) *Furthermore, if* Λ *is reduced, then* $\lambda_{i,i_k} + \lambda_{i_k i_l} > \lambda_{i,i_l}$ *for* $k > l \geq j$.

Proof. First, we prove (a) by using an induction on k. Fix $l \geq j$. Obviously (a) holds when $k = l$. By 0.1 we have

$$\begin{aligned}\lambda_{i,i_{k+1}} &\leq \lambda_{i,i_l} + \lambda_{i_l,i_{k+1}} \leq \lambda_{i,i_l} + \lambda_{i_l,i_k} + \lambda_{i_k,i_{k+1}} \\ &= \lambda_{i,i_k} + \lambda_{i_k,i_{k+1}}, \quad \text{by the induction hypothesis,} \\ &= \lambda_{i,i_{k+1}} \quad \text{by the hypothesis as } k \geq j.\end{aligned}$$

Thus we have proved that

$$\lambda_{i,i_{k+1}} \leq \lambda_{i,i_l} + \lambda_{i_l,i_{k+1}} \leq \lambda_{i,i_{k+1}}.$$

Consequently, $\lambda_{i,i_{k+1}} = \lambda_{i,i_l} + \lambda_{i_l,i_{k+1}}$. This completes the induction and proves (a).

We now prove (b). By (a) we have $\lambda_{i,i_k} + \lambda_{i_k,i_l} = \lambda_{i,i_l} + \lambda_{i_l,i_k} + \lambda_{i_k,i_l}$ whenever $k \geq l \geq j$. Since Λ is reduced, $\lambda_{i_l,i_k} + \lambda_{i_k,i_l} > 0$ whenever $k \neq l$. Thus we have $\lambda_{i,i_k} + \lambda_{i_k,i_l} > \lambda_{i,i_l}$ whenever $k > l \geq j$. This completes the proof of the lemma.

Proposition 3.2. *Let* $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ *be a tiled R-order. If for some fixed integer* i, *where* $1 \leq i \leq n$, *there exists a permutation* $\{i_1, i_2, \cdots, i_{n-1}\}$ *of the set* $\{1, 2, \cdots, i-1, i+1, \cdots, n\}$ *such that* $\lambda_{i,i_k} + \lambda_{i_k,i_{k+1}} = \lambda_{i,i_{k+1}}$ *for* $1 \leq k \leq n-2$, *then* Λ *is conjugate to a triangular tiled R-order.*

Proof. Set $i_0 = i$. Hence we have $\lambda_{i,i_0} + \lambda_{i_0,i_k} = \lambda_{i,i_k}$ for all $k \geq 0$. By the hypothesis $\lambda_{i,i_k} + \lambda_{i_k,i_{k+1}} = \lambda_{i,i_{k+1}}$ for $k \geq 1$, therefore using Lemma 3.1 with $j = 1$ one concludes that

$$(\#) \qquad \lambda_{i,i_l} + \lambda_{i_l,i_k} = \lambda_{i,i_k} \quad \text{whenever } k \geq l \geq 0.$$

Let $y = (y_{sj})$ and $z = (z_{sj})$ be the matrices in $M_n(K)$, where if $m = tR$, then

$$y_{k+1,i_k} = t^{\lambda_{i,i_k}}, \; z_{i_k,k+1} = t^{-\lambda_{i,i_k}} \quad \text{for } 0 \leq k \leq n-1;$$

and

$$y_{sj} = z_{sj} = 0 \quad \text{otherwise.}$$

Then, $yz = zy = 1$. Set $\Gamma = y\Lambda y^{-1}$. We show that $\Gamma = (\Gamma_{sj}) \subseteq M_n(R)$ and is a a triangular tiled R-order. To show this we must show that $\Gamma_{sj} = R$ whenever $s \leq j$, $\Gamma_{sj} \subseteq R$ whenever $s > j$. Clearly, $\Gamma = y\Lambda y^{-1} = (y_{sj})(\Lambda_{sj})(z_{sj})$; therefore using the matrix multiplication we get

$$\Gamma_{sj} = \sum_{u,v} y_{su}\Lambda_{uv}z_{vj} = y_{s,i_{s-1}}\Lambda_{i_{s-1},i_{j-1}}z_{i_{j-1},j}$$

$$= t^{\lambda_{i,i_{s-1}}} \cdot \mathfrak{m}^{\lambda_{i_{s-1},i_{j-1}}} \cdot t^{-\lambda_{i,i_{j-1}}} = \mathfrak{m}^{\lambda_{i,i_{s-1}}+\lambda_{i_{s-1},i_{j-1}}-\lambda_{i,i_{j-1}}}.$$

Now from 0.1 and (#) it follows that $\Gamma_{sj} \subseteq R$ whenever $s > j$ and $\Gamma_{sj} = R$ whenever $s \leq j$. Thus Γ is a triangular tiled R-order.

Theorem 3.3. *Let $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$, $n \geq 3$, be a reduced tiled R-order, where R is a DVR with maximal ideal $\mathfrak{m}$. Then Λ is conjugate to a triangular tiled R-order in $M_n(R)$ if and only if for some natural number $i \leq n$, there exists a permutation $\{i_1, i_2, \cdots, i_{n-1}\}$ of $\{1, 2, \cdots, i-1, i+1, \cdots, n\}$ such that*

$$\lambda_{i,i_k} + \lambda_{i_k,i_{k+1}} = \lambda_{i,i_{k+1}} \quad \textit{for } 1 \leq k \leq n-2.$$

Proof. The "if" part follows from Proposition 3.2. We now prove the "only if" part. So, assume that Λ is conjugate to a triangular tiled R-order in $M_n(R)$. By Proposition 1.9 of [7] we have a natural number $i \leq n$ such that

$$(*) \qquad \overline{P}_i \supsetneqq \overline{P}_iJ(\overline{\Lambda}) \supsetneqq \cdots \supsetneqq \overline{P}_iJ^{n-1}(\overline{\Lambda}) \supsetneqq \overline{P}_iJ^n(\overline{\Lambda}) = 0$$

is a composition series of $\overline{P}_i$ considered as a right $\overline{\Lambda}$-module, where $\overline{P}_i$ and $\overline{\Lambda}$ are as defined in §1 of this paper. Since $[\overline{P}_i\colon R/\mathfrak{m}] = n$ and since $(*)$ is a composition series, it follows that $[\overline{P}_iJ^k(\overline{\Lambda})\colon R/\mathfrak{m}] = n-k$ for $k \geq 1$. We claim that there exists a permutation $\{i_1, i_2, \cdots, i_{n-1}\}$ of the set $\{1, 2, \cdots, i-1, i+1, \cdots, n\}$ such that $\overline{P}_iJ^{s+1}(\overline{\Lambda})$ is obtained from $\overline{P}_iJ^s(\overline{\Lambda})$ by replacing the i_sth entry $\mathfrak{m}^{\lambda_{i,i_s}}/\mathfrak{m}^{\lambda_{i,i_s}+1}$ by zero. We will construct i_s inductively. Recall that, since Λ is reduced, $J(\overline{\Lambda})$ is obtained from $\overline{\Lambda}$ by replacing the diagonal entries $R/\mathfrak{m}$ by zero. Since $\overline{P}_iJ^2(\overline{\Lambda})$ is a right $\overline{\Lambda}$-module, $\overline{P}_iJ^2(\overline{\Lambda}) \supseteq \overline{P}_iJ^2(\overline{\Lambda})\overline{e}_{jj}$ for all j. Also, $\overline{P}_iJ(\overline{\Lambda}) \supsetneqq \overline{P}_iJ^2(\overline{\Lambda})$, $[\overline{P}_iJ^k(\overline{\Lambda})\colon R/\mathfrak{m}] = n-k$ for $k = 1, 2$. Therefore we obtain an integer $i_1 \neq i$ such that $\overline{P}_iJ^2(\overline{\Lambda})$ is obtained from $\overline{P}_iJ(\overline{\Lambda})$ by replacing the i_1th entry $\mathfrak{m}^{\lambda_{i,i_1}}/\mathfrak{m}^{\lambda_{i,i_1}+1}$ by zero. A similar argument and induction proves our claim. We observe that in particular $\overline{P}_iJ^{s+1}(\overline{\Lambda})$ is obtained from $\overline{P}_iJ(\overline{\Lambda})$ by replacing i_kth entry, $1 \leq k \leq s \leq n-1$, $\mathfrak{m}^{\lambda_{i,i_k}}/\mathfrak{m}^{\lambda_{i,i_k}+1}$ by zero. To complete the proof we now show that

$$\lambda_{i,i_k} + \lambda_{i_k,i_{k+1}} = \lambda_{i,i_{k+1}} \quad \text{whenever } 1 \le k \le n-2.$$

Since $\overline{P}_i J^{n-1}(\overline{\Lambda}) = \overline{P}_i J^{n-2}(\overline{\Lambda}) J(\overline{\Lambda})$, it follows that

$$(\mathfrak{m}^{\lambda_{i,i_{n-2}}}/\mathfrak{m}^{\lambda_{i,i_{n-2}}+1}) \cdot (\mathfrak{m}^{\lambda_{i_{n-2},i_{n-1}}}/\mathfrak{m}^{\lambda_{i_{n-2},i_{n-1}}+1})$$
$$= \mathfrak{m}^{\lambda_{i,i_{n-1}}} \bmod \mathfrak{m}^{\lambda_{i,i_{n-1}}+1}$$

Since the multiplication in $\overline{\Lambda}$ is induced by that in Λ, and since by 0.1 we have $\lambda_{i,i_{n-2}} + \lambda_{i_{n-2},i_{n-1}} \ge \lambda_{i,i_{n-1}}$, it follows that $\lambda_{i,i_{n-2}} + \lambda_{i_{n-2},i_{n-1}} = \lambda_{i,i_{n-1}}$. If $n = 3$, we are done. If $n \ge 4$, we use an induction on s. So, assume that $\lambda_{i,i_k} + \lambda_{i_k,i_{k+1}} = \lambda_{i,i_{k+1}}$ for $k \ge s+2$.

Then Lemma 3.1(b) yields $\lambda_{i,i_k} + \lambda_{i_k,i_{s+2}} > \lambda_{i,i_{s+2}}$ whenever $k > s+2$. Since $\overline{P}_i J^{j+1}(\overline{\Lambda})$ is obtained from $\overline{P}_i J(\overline{\Lambda})$ by replacing the i_lth entry, $1 \le l \le j \le n-1$, by zero, and since $\overline{P}_i J^{s+2}(\overline{\Lambda}) = \overline{P}_i J^{s+1}(\overline{\Lambda}) J(\overline{\Lambda})$, we must have

$$(\mathfrak{m}^{\lambda_{i,i_{s+2}}}/\mathfrak{m}^{\lambda_{i,i_{s+2}}+1}) = \sum_{l=s+1;\, l\ne s+2}^{n-1} (\mathfrak{m}^{\lambda_{i,i_l}}/\mathfrak{m}^{\lambda_{i,i_l}+1}) \cdot (\mathfrak{m}^{\lambda_{i_l,i_{s+2}}}/\mathfrak{m}^{\lambda_{i_l,i_{s+2}}+1})$$
$$\equiv \sum_{l=s+1;\, l\ne s+2}^{n-1} (\mathfrak{m}^{\lambda_{i,i_l}+\lambda_{i_l,i_{s+2}}}) \bmod \mathfrak{m}^{\lambda_{i,i_{s+2}}+1}$$

This together with the induction hypothesis yields $\lambda_{i,i_{s+1}} + \lambda_{i_{s+1},i_{s+2}} = \lambda_{i,i_{s+2}}$. This completes the induction on s and also completes the proof of the "only if" part.

4. Tiled orders in $M_n(K)$, where $2 \le n \le 4$. In this section we study tiled R-orders in $M_n(K)$ of finite global dimension with the restriction that $2 \le n \le 4$. The machinery developed in the first three sections enables us to give a complete classification of tiled R-orders in $M_4(K)$ of finite global dimension (Theorem 4.2). As another application of the developed machinery we prove Proposition 4.1, first proved by R. B Tarsey ([11], [12]). Our proof is different from that given by Tarsey and is also less computational. Throughout this section Ω_n will denote the tiled R-order $(\mathfrak{m}^{\omega_{ij}}) \subseteq M_n(R)$, where $\omega_{ij} = i - j$ for $i > j$ and $\omega_{ij} = 0$ otherwise.

Proposition 4.1. (a) *Let Λ be a tiled R-order in $M_n(K)$, where $n = 2$ or 3. Then* gl dim $\Lambda < \infty$ *if and only if Λ is conjugate to a triangular tiled R-order in $M_n(R)$ of finite global dimension.*

(b) *$M_2(R)$ and Ω_2 are, up to conjugation, the only tiled R-orders in $M_2(K)$ of finite global dimension.*

(c) *There are, up to conjugation, only four tiled R-orders in* $M_3(K)$ *of finite global dimension, and these are defined as follows:* (i) $M_3(R)$, (ii) Ω_3; (iii) $\Gamma = (\mathfrak{m}^{\gamma_{ij}}) \subseteq M_3(R)$, *where* $\gamma_{ij} = 1$ *whenever* $i > j$ *and* $\gamma_{ij} = 0$ otherwise; (iv) $\Gamma = (\mathfrak{m}^{\gamma_{ij}}) \subseteq M_3(R)$, *where* $\gamma_{31} = \gamma_{32} = 1$ *and* $\gamma_{ij} = 0$ *otherwise.*

Proof. The "if" part of (a) is trivial. We now prove (b), (c) and the "only if" part of (a) simultaneously. As seen before Λ is conjugate to a tiled R-order containing e_{ii}, $1 \le i \le n$. So we may as well assume that Λ is of the form $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(K)$. By Lemma 1.1 we may further assume that $\lambda_{ij} \ge 0$ for all i, j; and $\lambda_{1i} = 0$ for all i. Now let $n = 2$. If Λ is not reduced then we must have $\lambda_{21} = 0$, so that $\Lambda = M_2(R)$. If Λ is reduced then by Theorem 2.3(1) we have $\lambda_{21} = 1$ i.e., $\Lambda = \Omega_2$. Now let $n = 3$. By Theorem 2.3 we have $\lambda_{ij} \le 2$ for all i, j; and if $\lambda_{ij} = 2$ for some $i \ne j$, then by Theorem 2.11 we have that Λ is conjugate to the tiled R-order Ω_3. So assume that $\lambda_{ij} \le 1$. If Λ is reduced, then by Lemma 1.3 we may further assume that $\lambda_{ij} = 1$ for $i > j$ and $\lambda_{23} = 0$ or 1. If $\lambda_{23} = 0$ then Λ is the tiled R-order defined in (iii) of (c). If $\lambda_{23} = 1$, then let $y = (y_{ij})$ in $M_3(K)$, where $y_{12} = 1$, $y_{21} = y_{33} = t$ (where $tR = \mathfrak{m}$), and $y_{ij} = 0$ otherwise. A direct computation shows that $y\Lambda y^{-1} = \Omega_3$. Now assume that Λ is not reduced. Then we have $\lambda_{kl} = \lambda_{lk} = 0$ for some $k \ne l$. If $l = 1$, then by interchanging suitable rows and columns we may assume that $k = 2$, i.e., $\lambda_{21} = \lambda_{12} = 0$. Then using (0.1) one gets that $\lambda_{23} = 0$ and $\lambda_{31} = \lambda_{32} \le 1$.

So, either $\Lambda = M_3(R)$ or Λ is the order defined in (iv) of (c). Now assume that both of k and l are different from 1, so that $\lambda_{23} = \lambda_{32} = 0$. By 0.1, we have $\lambda_{21} = \lambda_{31} \le 1$. If $\lambda_{21} = \lambda_{31} = 0$, then $\Lambda = M_3(R)$. If not, set $y = (y_{ij})$ in $M_3(K)$, where $y_{12} = y_{23} = 1$, $y_{31} = t$ and $y_{ij} = 0$ otherwise. Then $y\Lambda y^{-1}$ is the tiled R-order defined in (iv) of (c).

Lastly it is easy to see that none of the tiled R-orders defined in (c) is conjugate to the other. This completes the proof of the proposition.

Theorem 4.2. *Let* R *be a discrete valuation ring with maximal ideal* $\mathfrak{m}$ *generated by* t, *and quotient field* K. *Let* Λ *be a tiled R-order in* $M_4(K)$. *Then* gl dim $\Lambda < \infty$ *if and only if* Λ *is conjugate to a triangular tiled R-order in* $M_4(R)$ *of finite global dimension or* Λ *is conjugate to the tiled R-order* $\Gamma = (\mathfrak{m}^{\gamma_{ij}}) \subseteq M_4(R)$, *where* $\gamma_{1i} = 0 = \gamma_{ii}$ *for all* i, *and* $\gamma_{ij} = 1$ *otherwise.*

Proof. The "if" part follows from Corollary 2.8. We now prove the "only if" part. As Λ is conjugate to a tiled R-order in $M_4(K)$ containing e_{ii}, $1 \le i \le 4$, we may as well assume that Λ is of the form $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_4(K)$. By Lemma 1.1, we may further assume that $\lambda_{ij} \ge 0$, $\lambda_{1i} = 0$ for all i, j. First we consider the case when Λ is reduced. Then, by Lemma 1.3, we may in addition assume that

$\lambda_{ij} > 0$ whenever $i > j$. Thus we have $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_4(R)$ with $\lambda_{ii} = \lambda_{1i} = 0$ for all i and $\lambda_{21}, \lambda_{31}, \lambda_{32}, \lambda_{41}, \lambda_{42}, \lambda_{43}$ are strictly positive. Hence we must consider various cases according as $\lambda_{23}, \lambda_{24}, \lambda_{34}$ are strictly positive or not. It is easy to see that up to conjugation we have to discuss only the following five types of tiled R-orders:

Type I. $\lambda_{23} = \lambda_{24} = \lambda_{34} = 0$.

Type II. $\lambda_{23}, \lambda_{24}, \lambda_{34} > 0$.

Type III. $\lambda_{23}, \lambda_{24} > 0$, $\lambda_{34} = 0$.

Type IV. $\lambda_{23} = 0 = \lambda_{24}$, $\lambda_{34} > 0$.

Type V. $\lambda_{23} > 0$, $\lambda_{24} = \lambda_{34} = 0$.

Since Type I is a case of triangular tiled R-order, this case is settled. In Type II, since gl dim $\Lambda < \infty$, we must have $\lambda_{ij} = 1$ whenever $i \neq j$ and $i \geq 2$, by Corollary 2.8. Thus $\Lambda = \Gamma$. Now let us discuss Type III. Clearly Λ is reduced, gl dim $\Lambda < \infty$, $\lambda_{23} + \lambda_{32} \geq 2$ and $\lambda_{24} + \lambda_{42} \geq 2$; therefore, by applying Lemma 2.7 with $k = 2$, we get that $\lambda_{21} = 1$. Since $0 < \lambda_{2i} \leq \lambda_{21} + \lambda_{1i} = 1$ for $i = 3, 4$, $\lambda_{23} = \lambda_{24} = 1$. But then $\lambda_{21} + \lambda_{13} = \lambda_{23}$ and $\lambda_{23} + \lambda_{34} = \lambda_{24}$. Therefore Proposition 3.2 applies with $i = 2$ for the permutation $\left(\begin{smallmatrix}1 & 3 & 4\\ 1 & 3 & 4\end{smallmatrix}\right)$, so that Λ is conjugate to a triangular tiled R-order. We now look at Type IV. Since $\lambda_{1j} = \lambda_{2j} = 0$ for $j = 2, 3, 4$, therefore $e_{11}\Lambda e'$ is a projective right $e'\Lambda e'$-module, where $e' = \Sigma_{i=2}^4 e_{ii}$. Hence by the analogue of Theorem 2.5 of [7], we have gl dim $e'\Lambda e' < \infty$ and $\Sigma_{i\neq 1} \mathfrak{m}^{\lambda_{1i}+\lambda_{i1}} = R$ or $\mathfrak{m}$ as $\mathfrak{m}$ is the only proper ideal I of R with gl dim $(R/I) < \infty$. Since $\lambda_{1i} = 0$ for all i, we have $\Sigma_{i\neq 1} \mathfrak{m}^{\lambda_{1i}+\lambda_{i1}} = \Sigma\, \mathfrak{m}^{\lambda_{i1}}$. Since Λ is reduced, $\lambda_{i1} > 0$ for $i \neq 1$. Hence we must have $\Sigma_{i\neq 1} \mathfrak{m}^{\lambda_{i1}} = \mathfrak{m}$. Thus $\lambda_{i1} = 1$ for some $i \geq 2$. Since $\lambda_{23} = 0 = \lambda_{24}$ and since $0 < \lambda_{21} \leq \lambda_{2i} + \lambda_{i1} = \lambda_{i1}$ for $i = 3, 4$, therefore it follows that $\lambda_{21} = 1$. Again, $\lambda_{32}, \lambda_{34}, \lambda_{42}, \lambda_{43} > 0$ and gl dim $e'\Lambda e' < \infty$, therefore by Corollary 2.8 we must have $\lambda_{32} = \lambda_{34} = \lambda_{42} = \lambda_{43} = 1$. Clearly $0 < \lambda_{i1} \leq \lambda_{i2} + \lambda_{21} = 2$ for $i = 3, 4$; therefore $\lambda_{i1} = 1$ or 2 whenever $i = 3$ or 4. If $\lambda_{31} = 1$ then $\lambda_{31} + \lambda_{12} = \lambda_{32}$, $\lambda_{32} + \lambda_{24} = \lambda_{34}$, so that Λ is conjugate to a triangular tiled R-order, by Proposition 3.2. If $\lambda_{41} = 1$, then $\lambda_{41} + \lambda_{12} = \lambda_{42}$, $\lambda_{42} + \lambda_{23} = \lambda_{43}$, so again, by Proposition 3.2, we have that Λ is conjugate to a triangular tiled R-order. If $\lambda_{31} = \lambda_{41} = 2$, then let $y = (y_{ij})$ in $M_4(K)$, where $y_{12} = 1 = y_{33} = y_{44}$, $y_{21} = t$, $y_{ij} = 0$ otherwise. Computation shows that $y\Lambda y^{-1} = \Gamma$.

Lastly we turn to Type V. Since the number of R in Λ is 9 and since the number of R in Ω_4 is 10, Λ cannot be permutationally conjugate to Ω_4. Thus, by Theorems 2.3(1) and 2.11, we must have $\lambda_{ij} \leq 2$ for all i, j. Since $\lambda_{i4} = 0$ for all i, we have, by Lemma 2.7 applied to Λ with $k = 4$, that $\lambda_{4i} = 1$ for some $i \leq 3$. If $\lambda_{41} = 1$, then, since Λ is a ring, it follows, by 0.1, that $\lambda_{21} = \lambda_{23} = 1$. Hence we have $\lambda_{24} + \lambda_{41} = \lambda_{21}$, $\lambda_{21} + \lambda_{13} = \lambda_{23}$. Thus, by Proposition 3.2, Λ

is conjugate to a triangular tiled R-order. So assume that $\lambda_{41} = 2$ and $\lambda_{4i} = 1$ for $i = 2$ or 3. By interchanging the 2nd and the 3rd rows and columns we may assume that $\lambda_{43} = 1$. Note that this permutation keeps us in Type V. Since $0 < \lambda_{23} \le \lambda_{24} + \lambda_{43} = 1$, $\lambda_{23} = 1$. Also, by Lemma 2.7, applied to Λ with $k = 1$, we have $\lambda_{21} = 1$ or $\lambda_{31} = 1$. Since all $\lambda_{ij} \le 2$, to complete the discussion of Type V, we have to discuss the following three subcases:

(a) $\lambda_{21} = 1 = \lambda_{31}$;

(b) $\lambda_{21} = 2$, $\lambda_{31} = 1$;

(c) $\lambda_{21} = 1$, $\lambda_{31} = 2$.

Case (a). Let $\lambda_{21} = \lambda_{31} = 1$. Clearly, $0 < \lambda_{32} \le \lambda_{31} + \lambda_{12} = 1$; therefore $\lambda_{32} = 1$. If $\lambda_{42} = 1$, then it is easy to check that

$$tP_1 + P_4 = J_2, \quad tP_1 \cap P_4 \cong J_1; \quad P_2 + P_3 = J_1, \quad P_2 \cap P_3 = J_2,$$

where $P_i = e_{ii}\Lambda$ and $J_i = e_{ii}J(\Lambda)$. By Theorem 1 of [10], Λ is a semiperfect ring; therefore, by using Remarks (1) and (3) at the end of §1 of [7], it is easy to see that none of J_1 and J_2 is projective as a right Λ-module. Hence by using obvious short exact sequences and Theorem 2 of [8, p. 169] it follows that $\mathrm{hd}_\Lambda J_2 = \infty$. But this contradicts the hypothesis that gl dim $\Lambda < \infty$. Thus $\lambda_{42} = 2$. But then we have $\lambda_{43} + \lambda_{31} = \lambda_{41}$, $\lambda_{41} + \lambda_{12} = \lambda_{42}$, so that Λ is conjugate to a triangular tiled R-order, by Proposition 3.2.

Case (b). Let $\lambda_{21} = 2$, $\lambda_{31} = 1$. Since $\lambda_{23} = 1$, we have $\lambda_{24} + \lambda_{43} = \lambda_{23}$, $\lambda_{23} + \lambda_{31} = \lambda_{21}$. Hence Λ is conjugate to a triangular tiled R-order, by Proposition 3.2.

Case (c). Let $\lambda_{21} = 1$, $\lambda_{31} = 2$. Recall that $\lambda_{41} = 2$. Hence by Lemma 3.1, applied with $k = 2$, we have $\lambda_{32} = 1$ or $\lambda_{42} = 1$. Since $0 < \lambda_{32} \le \lambda_{34} + \lambda_{42}$ and $\lambda_{34} = 0$, we have, in any case, $\lambda_{32} = 1$. Further if $\lambda_{42} = 1$, then we have $\lambda_{34} + \lambda_{42} = \lambda_{32}$, $\lambda_{32} + \lambda_{21} = \lambda_{31}$. Thus Proposition 3.2 guarantees that Λ is conjugate to a triangular tiled R-order. So assume that $\lambda_{42} = 2$. Then one shows, since $\mathfrak{m} = tR$, that

$$\begin{aligned} P_2 + P_3 &= J_1, & P_2 \cap P_3 &= J_3, \\ tP_2 + P_4 &= J_3, & tP_2 \cap P_4 &= tJ_2 \cong J_2, \\ tP_1 + P_4 &= J_2, & tP_1 \cap P_4 &= J_4, \\ t^2P_1 + tP_3 &= J_4, & t^2P_1 \cap tP_3 &= t^2J_1 \cong J_1. \end{aligned}$$

Then by using obvious short exact sequences and Theorem 2 of [8, p. 169] we get $\mathrm{hd}_\Lambda J_i = \infty$ for all i. Thus $\lambda_{42} = 2$ is impossible. This completes the discussion of Cases (a), (b), and (c) anf hence of Type V also. Thus the assertion of the theorem is proved when Λ is reduced.

Now assume that Λ is not reduced. Then we have $\lambda_{kl} = \lambda_{lk} = 0$ for some $k \neq l$. Recall that $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_4(R)$ and $\lambda_{ii} = \lambda_{1i} = 0$ for all i. First suppose that $\lambda_{k1} = \lambda_{1k} = 0$ for some $k \geq 2$. We may assume that $k = 2$. Then, since Λ is a ring, $\lambda_{23} = \lambda_{24} = 0$. If one of λ_{34} or λ_{43} is zero, then clearly Λ is permutationally conjugate to a triangular tiled R-order. So assume that $\lambda_{34}, \lambda_{43} > 0$. Since Λ is a ring we have $\lambda_{31} = \lambda_{32} > 0$, $\lambda_{41} = \lambda_{42} > 0$. Clearly $e_{11}\Lambda e'$ is a projective right $e'\Lambda e'$-module where $e' = e_{22} + e_{33} + e_{44}$, and gl dim $\Lambda < \infty$; therefore by the analogue of Proposition 1.10(2) of [7] we have gl dim $e'\Lambda e' < \infty$. Hence by Corollary 2.8 we have $\lambda_{31} = \lambda_{32} = \lambda_{34} = \lambda_{41} = \lambda_{42} = \lambda_{43} = 1$. Then $\lambda_{31} + \lambda_{12} = \lambda_{32}$, $\lambda_{32} + \lambda_{24} = \lambda_{34}$, so that Λ is conjugate to a triangular tiled R-order by Proposition 3.2. Now assume that $\lambda_{kl} = \lambda_{lk} = 0$ where $k \neq l$, $k, l \geq 2$. The proof of this case is similar to the above and we leave it to the reader. This completes the proof of the theorem.

Corollary 4.3. *If* $\Lambda \subseteq M_n(K)$, $2 \leq n \leq 4$, *is an arbitrary tiled R-order of finite global dimension, then* gl dim $\Lambda \leq n - 1$.

Proof. Follows from Proposition 4.1, Theorem 4.2, Theorem 1 of [5] and Proposition 3.3 of [7].

The above corollary shows that the conjecture of R. B. Tarsey [12] about the bound on the global dimension is true when $n \leq 4$.

Theorem 4.4. *Let R be a Dedekind domain with quotient field K. Let Λ be an arbitrary tiled R-order in $M_n(K)$, where $2 \leq n \leq 4$. If* gl dim $\Lambda < \infty$, *then* gl dim $\Lambda \leq n - 1$.

Proof. Follows from Corollary 4.3 and the corollary to Proposition 2.6 of [1].

K. L. Fields [4] in answer to a question of Kaplansky has constructed orders T and S in a central simple algebra Q over the quotient field of a DVR, such that $T \subset S$ gl dim $T = 2$ and gl dim $S = \infty$. We give simpler examples showing even worse behavior, viz., we construct a sequence of orders $\Lambda_1 \subsetneq \Lambda_2 \subsetneq \Lambda_3 \subsetneq \Lambda_4$ with gl dim Λ_1 = gl dim $\Lambda_3 = \infty$, gl dim $\Lambda_2 = 3$, gl dim $\Lambda_4 = 2$. Let R be a DVR with maximal ideal $\mathfrak{m}$, generated by t, and quotient field K. Define Λ_i, $1 \leq i \leq 4$, by

$$\Lambda_1 = \begin{pmatrix} R & R & R & R \\ \mathfrak{m} & R & \mathfrak{m} & R \\ \mathfrak{m}^2 & \mathfrak{m} & R & R \\ \mathfrak{m}^2 & \mathfrak{m}^2 & \mathfrak{m} & R \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} R & R & R & R \\ \mathfrak{m} & R & \mathfrak{m} & R \\ \mathfrak{m} & \mathfrak{m} & R & R \\ \mathfrak{m}^2 & \mathfrak{m}^2 & \mathfrak{m} & R \end{pmatrix},$$

$$\Lambda_3 = \begin{pmatrix} R & R & R & R \\ \mathfrak{m} & R & \mathfrak{m} & R \\ \mathfrak{m} & \mathfrak{m} & R & R \\ \mathfrak{m}^2 & \mathfrak{m} & \mathfrak{m} & R \end{pmatrix}, \qquad \Lambda_4 = \begin{pmatrix} R & R & R & R \\ \mathfrak{m} & R & R & R \\ \mathfrak{m} & \mathfrak{m} & R & R \\ \mathfrak{m}^2 & \mathfrak{m} & \mathfrak{m} & R \end{pmatrix}.$$

Clearly $\Lambda_1 \subsetneq \Lambda_2 \subsetneq \Lambda_3 \subsetneq \Lambda_4$, and one can easily verify that the Λ_i's are rings, hence tiled R-orders in $M_4(K)$. In the proof of Theorem 4.2 we have seen that $\text{gl dim } \Lambda_1 = \infty$ and $\text{gl dim } \Lambda_3 = \infty$. Since Λ_4 is not hereditary, therefore by Theorems 1 and 2 of [5] we have $\text{gl dim } \Lambda_4 = 2$. We now show that $\text{gl dim } \Lambda_2 = 3$. Let $J_i = e_{ii}J(\Lambda_2)$, $P_i = e_{ii}\Lambda_2$. It is easy to check that

$$tP_1 + P_4 = J_2, \quad tP_1 \cap P_4 = J_4 = tP_3 \cong P_3,$$
$$P_2 + P_3 = J_1, \quad P_2 \cap P_3 = J_2 \cong J_3,$$

and J_2 is not a projective right Λ_2-module. Using obvious short exact sequences and Theorem 2 of [8, p. 169] one concludes that $\text{hd } J(\Lambda_2) = 2$, so that $\text{gl dim } \Lambda_2 = 3$, by Lemma 1.2 of [7].

5. **Some remarks.** Let R be a DVR with maximal ideal $\mathfrak{m}$ and quotient field K. Let Λ be a triangular tiled R-order in $M_n(K)$, i.e., $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$, where $\lambda_{ij} = 0$ whenever $i \leq j$. Let $e = \Sigma_{i=1}^{n-1} e_{ii}$. Then, since $e\Lambda e_{nn}$ is a projective left $e\Lambda e$-module, from Theorem 2.5 of [7] it follows that $\text{gl dim } \Lambda < \infty$ if and only if $\text{gl dim } e\Lambda e < \infty$ and $J(\Lambda)e_{nn}$ is a projective left Λ-module.

It is also easy to see that if $\Gamma = (\mathfrak{m}^{\gamma_{ij}}) \subseteq M_4(R)$ is a tiled R-order where $\gamma_{1i} = 0$ for all i, and $\gamma_{ij} = 1$ for $i \geq 2$ and $i \neq j$, then $J(\Gamma)e_{44}$ is a projective left Γ-module and that, by Corollary 2.8, $\text{gl dim } e\Gamma e < \infty$, where $e = e_{11} + e_{22} + e_{33}$.

All this together with the classification given in Proposition 4.1 and Theorem 4.2 shows that, if Λ is a tiled R-order in $M_n(K)$, $2 \leq n \leq 4$, containing n orthogonal idempotents $f_1, f_2, \cdots, f_n$, then $\text{gl dim } \Lambda < \infty$ if and only if there exists a natural number $l \leq n$ such that $J(\Lambda)f_l$ is a projective left Λ-module and $\text{gl dim } g\Lambda g < \infty$, where $g = \Sigma_{i \neq l} f_i$.

We say that a tiled R-order Λ in $M_n(K)$ containing n orthogonal idempotents $f_1, f_2, \cdots, f_n$ has the property **P** if there exists a natural number $l \leq n$ such that

($\mathbf{P}_1$) $J(\Lambda)f_l$ is a projective left Λ-module or $f_lJ(\Lambda)$ is a projective right Λ-module,

($\mathbf{P}_2$) $\text{gl dim } g\Lambda g < \infty$, where $g = \Sigma_{i \neq l} f_i$.

We conjecture that if Λ is a tiled R-order in $M_n(K)$, then $\text{gl dim } \Lambda < \infty$ if and only if Λ has the property **P**. Since every tiled R-order Λ in $M_n(K)$ is conjugate to a tiled R-order in $M_n(R)$ containing e_{ii}, $1 \leq i \leq n$, it is enough to prove the conjecture for the class of tiled R-orders $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$. One can show that if $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_n(R)$ is a tiled R-order and if $J(\Lambda)e_{ll}$ (resp. $e_{ll}J(\Lambda)$) is a projective left (resp. right) Λ-module, then $f\Lambda e_{ll}$ (resp. $e_{ll}\Lambda f$) is a projective left (resp. right) $f\Lambda f$-module, where $f = \Sigma_{i \neq l} e_{ii}$; furthermore $\Sigma_{i \neq l} \mathfrak{m}^{\lambda_{li}}\mathfrak{m}^{\lambda_{il}} = R$ or $\mathfrak{m}$. Hence if our conjecture is true, then from Theorem 2.5 of [7] and induction it will follow that $\text{gl dim } \Lambda \leq n - 1$. This then would show that the conjecture of R. B.

Tarsey [12] about the bound on the global dimension of orders in $M_n(K)$ is true at least for the class of tiled R-orders. We note that the "sufficiency" of our conjecture follows from Theorem 2.5 of [7]. Lastly we construct an example of a tiled R-order $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_5(R)$ to show that the alternatives permitted in condition (P_1) are necessary.

Let $\Lambda = (\mathfrak{m}^{\lambda_{ij}}) \subseteq M_5(R)$, where $\lambda_{23} = \lambda_{45} = \lambda_{1j} = \lambda_{jj} = 0$ for all j, $\lambda_{31} = \lambda_{51} = \lambda_{52} = 2$ and $\lambda_{ij} = 1$ otherwise. One can easily check that Λ is a ring hence a tiled R-order and that $J(\Lambda)e_{55} \cong \Lambda e_{44}$. Hence $J(\Lambda)e_{55}$ is a projective left Λ-module. Computation shows that $\operatorname{hd} J_1 = 1 = \operatorname{hd} J_3$, $\operatorname{hd} J_2 = 2 = \operatorname{hd} J_5$, $\operatorname{hd} J_4 = 3$, where $J_i = e_{ii}J(\Lambda)$. Thus, by Lemma 1.2 of [7], gl dim $\Lambda = 4$. Also, if $e = \sum_{i=1}^{4} e_{ii}$, gl dim $e\Lambda e = 3$; and none of J_i is a projective right Λ-module. Hence the alternatives permitted in the condition (P_1) are necessary.

At the end we make the following remark:

Remark. One can observe that in §1 to 4, we have not made any use of commutativity of R. Hence all the proofs go through when R is a local left and right principal ideal domain.

BIBLIOGRAPHY

1. M. Auslander and O. Goldman, *Maximal orders*, Trans. Amer. Math. Soc. 97 (1960), 1–24. MR 22 #8034.

2. N. Bourbaki, *Eléments de mathématique*. Part. 1. Fasc. VI. Livre II: *Algèbre*. Chap. 2: *Algèbre linéaire*, 3ième éd., Actualités Sci. Indust., no. 1236, Hermann, Paris, 1962. MR 27 #5765.

3. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Appl. Math., vol. 11, Interscience, New York, 1962. MR 26 #2519.

4. K. L. Fields, *Examples of orders over discrete valuation rings*, Math. Z. 111 (1969), 126–130. MR 40 #182.

5. Vasanti A. Jategaonkar, *Global dimension of triangular orders over a discrete valuation ring*, Proc. Amer. Math. Soc. 38 (1973), 8–14.

6. ———, *Global dimension of tiled orders over a DVR*, Notices Amer. Math. Soc. 19 (1972), A-299. Abstract #72T-A66.

7. ———, *Global dimension of tiled orders over commutative noetherian domains*, Trans. Amer. Math. Soc. 190 (1974), 357–374.

8. Irving Kaplansky, *Fields and rings*, Univ. of Chicago Press, Chicago, Ill., 1969. MR 42 #4345.

9. J. Lambek, *Lectures on rings and modules*, Blaisdell, Waltham, Mass., 1966. MR 34 #5857.

10. Bruno J. Mueller, *On semiperfect rings*, Illinois J. Math. 14 (1970), 464–467. MR 41 #6909.

11. R. B. Tarsey, *Orders*, Thesis, The University of Chicago, Chicago, Ill. 1969.

12. ———, *Global dimension of orders*, Trans. Amer. Math. Soc. 151 (1970), 335–340. MR 42 #3125.

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