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GLOBAL ESTIMATES FOR SINGULAR INTEGRALS OF THE COMPOSITE OPERATOR

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ABSTRACT. We establish the Poincaré-type inequalities for the composition of the homotopy operator and the projection operator applied to the nonhomogeneous A-harmonic equation in John domains. We also obtain some estimates for the integral of the composite operator with a singular density.

1. Introduction

In our recent paper [5], we investigated singular integrals of composite operators and established some inequalities for composite operators with singular factors. In this paper, we keep working on the same topic and derive global estimates for the singular integrals of composite operators in δ-John domains. The purpose of this paper is to estimate the Poincaré-type inequalities for the composition of the homotopy operator T and the projection operator H over the δ-John domain. Differential forms and these two key operators are widely used not only in analysis and partial differential equations [1], [3], [14], [18], but also in physics [2], [7], [9], [15]. We all know that any differential form u can be decomposed as $u = d(Tu) + T(du)$, where d is the differential operator and T is the homotopy operator. We also need to estimate the composition of the homotopy operator T and the projection operator H in many situations. For example, when we consider the decomposition of $H(u)$ in the case of the Poisson's equation, we have to study the composition $T \circ H$ of the homotopy operator T and the projection operator H. The reason that we establish inequalities with the singular weights was motivated from physics. In real applications, we often need to estimate the integrals with singular factors. For example, let us assume that the object P with mass m_1 is located at the origin and the object Q with mass m_2 is located at (x, y, z) in $\mathbb{R}^3$.

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Then, Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects P and Q is $|\mathbf{F}| = m_1 m_2 G/d^2(P,Q)$, where $d(P,Q) = \sqrt{x^2+y^2+z^2}$ is the distance between P and Q, and G is the gravitational constant. Thus, we have to evaluate a singular integral whenever the integrand contains $|\mathbf{F}|$ as a factor and the integral domain includes the origin. Also, when calculating an electric field, we will deal with the integral $E(r) = \frac{1}{4\pi\epsilon_0}\int_D \rho(x)\frac{r-x}{\|r-x\|^3}\,dx$, where $\rho(x)$ is a charge density and x is the integral variable. The integral is singular if $r \in D$. When we consider the integral of the vector field $\mathbf{F} = \nabla f$, we have to deal with the singular integral if the potential function f contains a singular factor, such as the potential energy in physics. It is clear that the singular integrals are more interesting to us because of their wide applications in different fields of mathematics and physics.

We assume that Ω is a bounded, convex domain and B is a ball in $\mathbb{R}^n$, $n \geq 2$, throughout this paper. We use σB to denote the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$, $\sigma > 0$. We do not distinguish the balls from cubes in this paper. We use $|E|$ to denote the Lebesgue measure of the set E. We say w is a weight if $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $w > 0$ a.e. Let M be a domain in an oriented, compact, C^∞ smooth Riemannian manifold of dimension $n \geq 2$. Differential forms are extensions of functions in $\mathbb{R}^n$. For example, the function $u(x_1, x_2, \ldots, x_n)$ is called a 0-form. Moreover, if $u(x_1, x_2, \ldots, x_n)$ is differentiable, then it is called a differential 0-form. The 1-form $u(x)$ in $\mathbb{R}^n$ can be written as $u(x) = \sum_{i=1}^n u_i(x_1, x_2, \ldots, x_n)\,dx_i$. If the coefficient functions $u_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, are differentiable, then $u(x)$ is called a differential 1-form. Similarly, a differential k-form $u(x)$ is generated by $\{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}\}$, $k = 1, 2, \ldots, n$, that is, $u(x) = \sum_I u_I(x)\,dx_I = \sum u_{i_1 i_2 \cdots i_k}(x)\,dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, where $I = (i_1, i_2, \ldots, i_k)$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$, $D'(M, \wedge^l)$ be the space of all differential l-forms on M and $L^p(M, \wedge^l)$ be the l-forms $u(x) = \sum_I u_I(x)\,dx_I$ on M satisfying $\int_M |u_I|^p < \infty$ for all ordered l-tuples I, $l = 1, 2, \ldots, n$. We denote the exterior derivative by $d : D'(M, \wedge^l) \to D'(M, \wedge^{l+1})$ for $l = 0, 1, \ldots, n-1$, and define the Hodge star operator $\star : \wedge^k \to \wedge^{n-k}$ as follows. If $u = u_{i_1 i_2 \cdots i_k}(x_1, x_2, \ldots, x_n)\,dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} = u_I\,dx_I, i_1 < i_2 < \cdots < i_k$, is a differential k-form, then $\star u = \star(u_{i_1 i_2 \cdots i_k}\,dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = (-1)^{\Sigma(I)} u_I\,dx_J$, where $I = (i_1, i_2, \ldots, i_k)$, $J = \{1, 2, \ldots, n\} - I$, and $\sum(I) = \frac{k(k+1)}{2} + \sum_{j=1}^k i_j$. The Hodge codifferential operator $d^\star : D'(M, \wedge^{l+1}) \to D'(M, \wedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(M, \wedge^{l+1})$, $l = 0, 1, \ldots, n-1$. We write $\|u\|_{s,M} = (\int_M |u|^s)^{1/s}$ and $\|u\|_{s,M,w} = (\int_M |u|^s w(x)\,dx)^{1/s}$, where $w(x)$ is a weight. The differential forms can be used to describe various systems of PDEs and to express different geometric structures on manifolds. For instance, some kinds of differential forms are often utilized in studying deformations of elastic bodies, the related extrema for variational integrals, and certain geometric invariance. Differential forms have become invaluable tools for many fields of sciences and engineering, see [12] and [15].

We are particularly interested in a class of differential forms satisfying the well known nonhomogeneous A-harmonic equation

$$d^\star A(x, du) = B(x, du), \tag{1.1}$$

where $A : M \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and $B : M \times \wedge^l(\mathbb{R}^n) \to \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions:

$$|A(x,\xi)| \le a|\xi|^{p-1}, \qquad A(x,\xi)\cdot\xi \ge |\xi|^p, \qquad |B(x,\xi)| \le b|\xi|^{p-1} \tag{1.2}$$

for almost every $x \in M$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1.1). If the operator $B = 0$, equation (1.1) becomes $d^\star A(x, du) = 0$, which is called the (homogeneous) A-harmonic equation. A solution to (1.1) is an element of the Sobolev space $W^{1,p}_{\text{loc}}(M, \wedge^{l-1})$ such that $\int_M A(x, du)\cdot d\varphi + B(x, du)\cdot\varphi = 0$ for all $\varphi \in W^{1,p}_{\text{loc}}(M, \wedge^{l-1})$ with compact support. Let $A : M \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ be defined by $A(x,\xi) = \xi|\xi|^{p-2}$ with $p > 1$. Then, A satisfies the required conditions and $d^\star A(x, du) = 0$ becomes the p-harmonic equation

$$d^\star(du\,|du|^{p-2}) = 0 \tag{1.3}$$

for differential forms. If u is a function (0-form), the equation (1.3) reduces to the usual p-harmonic equation $\operatorname{div}(\nabla u|\nabla u|^{p-2}) = 0$ for functions. A remarkable progress has been made recently in the study of different versions of the harmonic equations, see [1], [8], [13], [16], [17]. Let $\wedge^l M$ be the lth exterior power of the cotangent bundle, $C^\infty(\wedge^l M)$ be the space of smooth l-forms on M and $\mathcal{W}(\wedge^l M) = \{u \in L^1_{\text{loc}}(\wedge^l M) : u$ has generalized gradient$\}$. The harmonic l-fields are defined by $\mathcal{H}(\wedge^l M) = \{u \in \mathcal{W}(\wedge^l M) : du = d^\star u = 0, u \in L^p$ for some $1 < p < \infty\}$. The orthogonal complement of $\mathcal{H}$ in L^1 is defined by $\mathcal{H}^\perp = \{u \in L^1 : \langle u, h\rangle = 0$ for all $h \in \mathcal{H}\}$. Then, the Green's operator G is defined as $G : C^\infty(\wedge^l M) \to \mathcal{H}^\perp \cap C^\infty(\wedge^l M)$ by assigning $G(u)$ be the unique element of $\mathcal{H}^\perp \cap C^\infty(\wedge^l M)$ satisfying Poisson's equation $\Delta G(u) = u - H(u)$, where H is the harmonic projection operator that maps $C^\infty(\wedge^l M)$ onto $\mathcal{H}$ so that $H(u)$ is the harmonic part of u. See [14] for more properties of these operators.

The operator K_y with the case $y = 0$ was first introduced by Cartan in [3]. Then, it was extended to the following version in [6]. To each $y \in \Omega$, there corresponds a linear operator $K_y : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ defined by $(K_y u)(x; \xi_1, \ldots, \xi_{l-1}) = \int_0^1 t^{l-1} u(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1})\,dt$ and the decomposition $u = d(K_y u) + K_y(du)$. A homotopy operator $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ is defined by averaging K_y over all points $y \in \Omega$:

$$Tu = \int_\Omega \phi(y) K_y u\,dy,$$

where $\phi \in C_0^\infty(\Omega)$ is normalized so that $\int_\Omega \phi(y)\,dy = 1$.

2. Main results

We first introduce the following definition and lemmas that will be used in this paper.

DEFINITION 1. A proper subdomain $\Omega \subset \mathbb{R}^n$ is called a δ-John domain, $\delta > 0$, if there exists a point $x_0 \in \Omega$ which can be joined with any other point $x \in \Omega$ by a continuous curve $\gamma \subset \Omega$ so that

$$d(\xi, \partial\Omega) \geq \delta |x - \xi|$$

for each $\xi \in \gamma$. Here, $d(\xi, \partial\Omega)$ is the Euclidean distance between ξ and $\partial\Omega$.

LEMMA 1 ([4]). *Let ϕ be a strictly increasing convex function on $[0, \infty)$ with $\phi(0) = 0$, and D be a domain in $\mathbb{R}^n$. Assume that u is a function in D such that $\phi(|u|) \in L^1(D, \mu)$ and $\mu(\{x \in D : |u - c| > 0\}) > 0$ for any constant c, where μ is a Radon measure defined by $d\mu(x) = w(x)\,dx$ for a weight $w(x)$. Then, we have*

$$\int_D \phi\left(\frac{a}{2}|u - u_{D,\mu}|\right) d\mu \leq \int_D \phi(a|u|)\,d\mu$$

for any positive constant a, where $u_{D,\mu} = \frac{1}{\mu(D)} \int_D u\,d\mu$.

LEMMA 2 ([11]). *Let $u \in C^\infty(\wedge^l M)$ and $l = 1, 2, \ldots, n$, $1 < s < \infty$. Then, there exists a positive constant $C = C(s)$, independent of u, such that*

$$\begin{aligned} &\|dd^*G(u)\|_{s,M} + \|d^*dG(u)\|_{s,M} + \|dG(u)\|_{s,M} \\ &\quad + \|d^*G(u)\|_{s,M} + \|G(u)\|_{s,M} \leq C(s)\|u\|_{s,M}. \end{aligned}$$

We will need the following Covering lemma appearing in [10].

LEMMA 3. *Each Ω has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that $\bigcup_i Q_i = \Omega$, $\sum_{Q_i \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q} \leq N\chi_\Omega$ and some $N > 1$, and if $Q_i \cap Q_j \neq \emptyset$, then there exists a cube R (this cube need not be a member of $\mathcal{V}$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if Ω is δ-John, then there is a distinguished cube $Q_0 \in \mathcal{V}$ which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1, \ldots, Q_k = Q$ from $\mathcal{V}$ and such that $Q \subset \rho Q_i$, $i = 0, 1, 2, \ldots, k$, for some $\rho = \rho(n, \delta)$.*

LEMMA 4. *Let $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, $H : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^l)$ be the projection operator and $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator. Then, there exists a constant $C = C(n, s, \Omega)$, independent of u, such that*

$$\|T(H(u))\|_{s,B} \leq C(n, s, \Omega)|B| \operatorname{diam}(B)\|u\|_{s,B} \tag{2.1}$$

for all balls $B \subset \Omega$.

Proof. Let T be the homotopy operator and u be locally L^s integrable l form. From [6], there exists a constant $C_1(n,s,\Omega)$, independent of u, such that

$$\|Tu\|_{s,B} \le C_1(n,s,\Omega)|B| \operatorname{diam}(B)\|u\|_{s,B}. \tag{2.2}$$

By using Lemma 2, we have

$$\begin{aligned} \|\Delta G(u)\|_{s,B} & \\ &= \|(dd^* + d^*d)G(u)\|_{s,B} \le \|dd^*G(u)\|_{s,B} + \|d^*dG(u)\|_{s,B} \\ &\le C_2(s)\|u\|_{s,B}. \end{aligned} \tag{2.3}$$

Thus, by (2.2) and (2.3), we have

$$\begin{aligned} &\|TH(u)\|_{s,B} \\ &\quad\le C_1(n,s,\Omega)|B| \operatorname{diam}(B)\|H(u)\|_{s,B} \\ &\quad= C_1(n,s,\Omega)|B| \operatorname{diam}(B)\|u - \Delta G(u)\|_{s,B} \\ &\quad\le C_1(n,s,\Omega)|B| \operatorname{diam}(B)\big(\|u\|_{s,B} + \|\Delta G(u)\|_{s,B}\big) \\ &\quad\le C_1(n,s,\Omega)|B| \operatorname{diam}(B)\big(\|u\|_{s,B} + C_2(s)\|u\|_{s,B}\big) \\ &\quad\le C_3(n,s,\Omega)|B| \operatorname{diam}(B)\|u\|_{s,B} \end{aligned}$$

which ends the proof of Lemma 4. □

We considered a singular integral in the paper [5] since the type of singular integral was commonly seen when solving problems in physics and engineering fields. The integral that was considered in paper [5] was over any open ball contained in a bounded region Ω. Now we consider the integral globally on δ-John domain Ω. We state the following version of Theorem 3 in [5] as a lemma and omit the proof since the proof would be the same as the proof of Theorem 3 in [5] by using Lemma 4 and noticing that $\frac{1}{d(x,\partial\Omega)} \le \frac{1}{r_B - |x|}$ for any $x \in B$, where r_B is the radius of ball B.

LEMMA 5. *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1,2,\ldots,n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic equation in a bounded and convex domain Ω, H be the projection operator and T be the homotopy operator. Then, there exists a constant $C(n,s,\alpha,\lambda,\Omega)$, independent of u, such that*

$$\begin{aligned} &\left(\int_B |T(H(u))|^s \frac{1}{d^\alpha(x,\partial\Omega)}\,dx\right)^{1/s} \\ &\quad\le C(n,s,\alpha,\lambda,\Omega)|B|^\gamma \left(\int_{\rho B} |u|^s \frac{1}{|x - x_B|^\lambda}\,dx\right)^{1/s} \end{aligned}$$

for all balls B with $\rho B \subset \Omega$, $\rho > 1$, and any real number α and λ with $\alpha > \lambda \ge 0$ and $\gamma = 1 + \frac{1}{n} - \frac{\alpha - \lambda}{ns}$. Here x_B is the center of the ball.

THEOREM 1. *Let $u \in D'(\Omega, \wedge^1)$ be a solution of the A-harmonic equation (1.1), H be the projection operator and T be the homotopy operator. Assume that s is a fixed exponent associated with the nonhomogeneous A-harmonic equation. Then, there exists a constant $C(n, N, s, \alpha, \lambda, Q_0, \Omega)$, independent of u, such that*

$$\left(\int_\Omega |T(H(u)) - (T(H(u)))_{Q_0}|^s \frac{1}{d^\alpha(x, \partial\Omega)}\, dx\right)^{1/s} \leq C(n, N, s, \alpha, \lambda, Q_0, \Omega)\left(\int_\Omega |u|^s g(x)\, dx\right)^{1/s} \tag{2.4}$$

for any bounded and convex δ-John domain $\Omega \subset \mathbb{R}^n$, where $g(x) = \sum_i \chi_{Q_i} \frac{1}{|x - x_{Q_i}|^\lambda}$. Here α and λ are constants with $0 \leq \lambda < \alpha < \min\{n, s + \lambda + n(s-1)\}$, and the fixed cube $Q_0 \subset \Omega$, the cubes $Q_i \subset \Omega$ and the constant $N > 1$ appeared in Lemma 3, x_{Q_i} is the center of Q_i.

Proof. We use the notation appearing in Lemma 3. There is a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ for Ω such that $\Omega = \bigcup Q_i$, and $\sum_{Q_i \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q_i} \leq N\chi_\Omega$ for some $N > 1$. Since $\Omega = \bigcup Q_i$, for any $x \in \Omega$, it follows that $x \in Q_i$ for some i. Applying Lemma 5 to Q_i, we have

$$\left(\int_{Q_i} |TH(u)|^s \frac{1}{d^\alpha(x, \partial\Omega)}\, dx\right)^{1/s} \leq C_1(n, s, \alpha, \lambda, \Omega)|Q_i|^\gamma \left(\int_{\sigma Q_i} |u|^s \frac{1}{d^\lambda(x, x_{Q_i})} dx\right)^{1/s}, \tag{2.5}$$

where $\sigma > 1$ is a constant. Let $\mu(x)$ and $\mu_1(x)$ be the Radon measures defined by $d\mu = \frac{1}{d^\alpha(x, \partial\Omega)}\, dx$ and $d\mu_1(x) = g(x)\, dx$, respectively. Then,

$$\begin{aligned} \mu(Q) &= \int_Q \frac{1}{d^\alpha(x, \partial\Omega)}\, dx \\ &\geq \int_Q \frac{1}{(\mathrm{diam}(\Omega))^\alpha}\, dx = M(n, \alpha, \Omega)|Q|, \end{aligned} \tag{2.6}$$

where $M(n, \alpha, \Omega)$ is a positive constant. Then, by the elementary inequality $(a + b)^s \leq 2^s(|a|^s + |b|^s)$, $s \geq 0$, we have

$$\begin{aligned} &\left(\int_\Omega |T(H(u)) - (T(H(u)))_{Q_0}|^s \frac{1}{d^\alpha(x, \partial\Omega)}\, dx\right)^{1/s} \\ &= \left(\int_{\bigcup Q} |T(H(u)) - (T(H(u)))_{Q_0}|^s\, d\mu\right)^{1/s} \\ &\leq \Bigg(\sum_{Q \in \mathcal{V}} \Bigg(2^s \int_Q |T(H(u)) - (T(H(u)))_Q|^s\, d\mu \end{aligned} \tag{2.7}$$

$$+ 2^s \int_Q |(T(H(u)))_Q - (T(H(u)))_{Q_0}|^s \, d\mu \Big) \Big)^{1/s}$$

$$\le C_1(s) \bigg(\Big(\sum_{Q\in\mathcal{V}} \int_Q |T(H(u)) - (T(H(u)))_Q|^s \, d\mu \Big)^{1/s} + \Big(\sum_{Q\in\mathcal{V}} \int_Q |(T(H(u)))_Q - (T(H(u)))_{Q_0}|^s \, d\mu \Big)^{1/s} \bigg)$$

for a fixed $Q_0 \subset \Omega$. The first sum in (2.7) can be estimated by using Lemma 1 with $\varphi = t^s$, $a = 2$, and Lemma 5,

$$\begin{aligned} &\sum_{Q\in\mathcal{V}} \int_Q |T(H(u)) - (T(H(u)))_Q|^s \, d\mu \\ &\quad \le \sum_{Q\in\mathcal{V}} \int_Q 2^s |T(H(u))|^s \, d\mu \\ &\quad \le C_2(n,s,\alpha,\lambda,\Omega) \sum_{Q\in\mathcal{V}} |Q|^{\gamma s} \int_{\rho Q} |u|^s \, d\mu_1 \\ &\quad \le C_3(n,s,\alpha,\lambda,\Omega) |\Omega|^{\gamma s} \sum_{Q\in\mathcal{V}} \int_\Omega (|u|^s \, d\mu_1) \chi_{\rho Q} \\ &\quad \le C_4(n,N,s,\alpha,\lambda,\Omega) |\Omega|^{\gamma s} \int_\Omega |u|^s \, d\mu_1 \\ &\quad \le C_5(n,N,s,\alpha,\lambda,\Omega) \int_\Omega |u|^s g(x) \, dx. \end{aligned} \tag{2.8}$$

To estimate the second sum in (2.7), we need to use the property of δ-John domain. Fix a cube $Q \in \mathcal{V}$ and let $Q_0, Q_1, \dots, Q_k = Q$ be the chain in Lemma 3.

$$\begin{aligned} &|(T(H(u)))_Q - (T(H(u)))_{Q_0}| \\ &\quad \le \sum_{i=0}^{k-1} |(T(H(u)))_{Q_i} - (T(H(u)))_{Q_{i+1}}|. \end{aligned} \tag{2.9}$$

The chain $\{Q_i\}$ also has property that, for each i, $i = 0, 1, \dots, k-1$, with $Q_i \cap Q_{i+1} \neq \emptyset$, there exists a cube D_i such that $D_i \subset Q_i \cap Q_{i+1}$ and $Q_i \cup Q_{i+1} \subset N D_i$, $N > 1$.

$$\frac{\max\{|Q_i|, |Q_{i+1}|\}}{|Q_i \cap Q_{i+1}|} \le \frac{\max\{|Q_i|, |Q_{i+1}|\}}{|D_i|} \le C_6(N).$$

For such D_j, $j = 0, 1, \dots, k-1$, Let $|D^\star| = \min\{|D_0|, |D_1|, \dots, |D_{k-1}|\}$, then

$$\frac{\max\{|Q_i|, |Q_{i+1}|\}}{|Q_i \cap Q_{i+1}|} \le \frac{\max\{|Q_i|, |Q_{i+1}|\}}{|D^\star|} \le C_7(N). \tag{2.10}$$

By (2.6), (2.10) and Lemma 5, we have

$$
\begin{aligned}
(2.11)\quad & |(T(H(u)))_{Q_i} - (T(H(u)))_{Q_{i+1}}|^s \\
&= \frac{1}{\mu(Q_i \cap Q_{i+1})} \\
&\quad \times \int_{Q_i \cap Q_{i+1}} |(T(H(u)))_{Q_i} - (T(H(u)))_{Q_{i+1}}|^s \frac{dx}{d^\alpha(x, \partial\Omega)} \\
&\le \frac{C_8(n,\alpha,\Omega)}{|Q_i \cap Q_{i+1}|} \\
&\quad \times \int_{Q_i \cap Q_{i+1}} |(T(H(u)))_{Q_i} - (T(H(u)))_{Q_{i+1}}|^s \frac{dx}{d^\alpha(x, \partial\Omega)} \\
&\le \frac{C_8(n,\alpha,\Omega) C_7(N)}{\max\{|Q_i|, |Q_{i+1}|\}} \int_{Q_i \cap Q_{i+1}} |(T(H(u)))_{Q_i} - (T(H(u)))_{Q_{i+1}}|^s \, d\mu \\
&\le C_9(n,N,s,\alpha,\Omega) \sum_{j=i}^{i+1} \frac{1}{|Q_j|} \int_{Q_j} |T(H(u)) - (T(H(u)))_{Q_j}|^s \, d\mu \\
&\le C_{10}(n,N,s,\alpha,\lambda,\Omega) \sum_{j=i}^{i+1} \frac{|Q_j|^{\gamma s}}{|Q_j|} \int_{\rho Q_j} |u|^s \, d\mu_1 \\
&= C_{10}(n,N,s,\alpha,\lambda,\Omega) \sum_{j=i}^{i+1} |Q_j|^{\gamma s - 1} \int_{\rho Q_j} |u|^s \, d\mu_1.
\end{aligned}
$$

Since $Q \subset NQ_j$ for $j = i$, $i+1$, $0 \le i \le k-1$, from (2.11)

$$
\begin{aligned}
(2.12)\quad & |(T(H(u)))_{Q_i} - (T(H(u)))_{Q_{i+1}}|^s \chi_Q(x) \\
&\le C_{11}(n,N,s,\alpha,\lambda,\Omega) \sum_{j=i}^{i+1} \chi_{NQ_j}(x) |Q_j|^{\gamma s - 1} \int_{\rho Q_j} |u|^s \, d\mu_1 \\
&\le C_{12}(n,N,s,\alpha,\lambda,\Omega) \sum_{j=i}^{i+1} \chi_{NQ_j}(x) |\Omega|^{\gamma s - 1} \int_{\rho Q_j} |u|^s \, d\mu_1.
\end{aligned}
$$

We know that $|\Omega|^{\gamma - 1/s} < \infty$ since Ω is bounded and $\gamma - \frac{1}{s} = 1 + \frac{1}{n} + \frac{\lambda}{ns} - \frac{1}{s} - \frac{\alpha}{ns} > 0$ when $\alpha < s + \lambda + n(s-1)$. Thus, from $(a+b)^{1/s} \le 2^{1/s}(|a|^{1/s} + |b|^{1/s})$, (2.9) and (2.12),

$$
\begin{aligned}
& |(T(H(u)))_Q - (T(H(u)))_{Q_0}| \chi_Q(x) \\
&\quad \le C_{13}(n,N,s,\alpha,\lambda,\Omega) \sum_{D \in \mathcal{V}} \left(\int_{\rho D} |u|^s \, d\mu_1 \right)^{1/s} \chi_{ND}(x)
\end{aligned}
$$

for every $x \in \mathbb{R}^n$. Then

$$\sum_{Q\in\mathcal{V}} \int_Q |(T(H(u)))_Q - (T(H(u)))_{Q_0}|^s \, d\mu \\ \le C_{13}(n,N,s,\alpha,\lambda,\Omega) \int_{\mathbb{R}^n} \left| \sum_{D\in\mathcal{V}} \left(\int_{\rho D} |u|^s \, d\mu_1 \right)^{1/s} \chi_{ND}(x) \right|^s d\mu.$$

Notice that

$$\sum_{D\in\mathcal{V}} \chi_{ND}(x) \le \sum_{D\in\mathcal{V}} \chi_{\rho ND}(x) \le N\chi_\Omega(x).$$

Using elementary inequality $|\sum_{i=1}^M t_i|^s \le M^{s-1} \sum_{i=1}^M |t_i|^s$, we finally have

$$\begin{aligned} &\sum_{Q\in\mathcal{V}} \int_Q |(T(H(u)))_Q - (T(H(u)))_{Q_0}|^s \, d\mu \\ &\quad \le C_{14}(n,N,s,\alpha,\lambda,\Omega) \int_{\mathbb{R}^n} \left(\sum_{D\in\mathcal{V}} \left(\int_{\rho D} |u|^s \, d\mu_1 \right) \chi_D(x) \right) d\mu \\ &\quad = C_{14}(n,N,s,\alpha,\lambda,\Omega) \sum_{D\in\mathcal{V}} \left(\int_{\rho D} |u|^s \, d\mu_1 \right) \\ &\quad \le C_{15}(n,N,s,\alpha,\lambda,\Omega) \int_\Omega |u|^s g(x) \, dx. \end{aligned} \tag{2.13}$$

Substituting (2.8) and (2.13) in (2.7), we have completed the proof of Theorem 1. □

We know from [6] that there is a constant $C(n,s,\Omega)$, independent of u, such that $\|\nabla T(u)\|_{s,B} \le C(n,s,\Omega)|B|\|u\|_{s,B}$ for any $B \subset \Omega$ and all differential forms u. Hence, using Lemma 2, we obtain

$$\begin{aligned} &\|\nabla TH(u)\|_{s,B} \\ &\quad \le C_1(n,s,\Omega)|B|\|H(u)\|_{s,B} \\ &\quad = C_1(n,s,\Omega)|B|\|u - \Delta G(u)\|_{s,B} \\ &\quad \le C_2(n,s,\Omega)|B|\big(\|u\|_{s,B} + \|(dd^* + d^*d)G(u)\|_{s,B}\big) \\ &\quad \le C_2(n,s,\Omega)|B|\big(\|u\|_{s,B} + C_3(s)\|u\|_{s,B}\big) \\ &\quad \le C_4(n,s,\Omega)\|u\|_{s,B}. \end{aligned} \tag{2.14}$$

Using (2.14), we have the following Lemma 6 whose proof is similar to the proof of Lemma 5.

LEMMA 6. *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic equation in a bounded and convex domain Ω, H be the projection operator and T be the homotopy operator. Then, there exists* *a constant* $C(n,s,\alpha,\lambda,\Omega)$, *independent of* u, *such that*

$$\left(\int_B |\nabla T(H(u))|^s \frac{1}{d(x,\partial\Omega)^\alpha}\,dx\right)^{1/s} \le C(n,s,\alpha,\lambda,\Omega)|B|^\gamma \left(\int_{\rho B} |u|^s \frac{1}{|x-x_B|^\lambda}\,dx\right)^{1/s} \tag{2.15}$$

for all balls B *with* $\rho B\subset\Omega$ *and any real number* α *and* λ *with* $\alpha>\lambda\ge 0$ *and* $\gamma=1+\frac{1}{n}-\frac{\alpha-\lambda}{ns}$. *Here* x_B *is the center of the ball.*

Notice that (2.15) can also be written as

$$\|\nabla TH(u)\|_{s,B,w_1} \le C(n,s,\alpha,\lambda,\Omega)|B|^\gamma \|u\|_{s,\rho B,w_2}. \tag{2.15$'$}$$

Next, we prove the imbedding inequality with a singular factor in the John domain.

THEOREM 2. *Let* $u\in D'(\Omega,\wedge^1)$ *be a solution of the* A*-harmonic equation* (1.1), H *be the projection operator and* T *be the homotopy operator. Assume that* s *is a fixed exponent associated with the nonhomogeneous* A*-harmonic equation. Then, there exists a constant* $C(n,s,\alpha,\lambda,\Omega)$, *independent of* u, *such that*

$$\|\nabla(T(H(u)))\|_{s,\Omega,w_1} \le C(n,s,\alpha,\lambda,\Omega)\|u\|_{s,\Omega,w_2}, \tag{2.16}$$

$$\|T(H(u))\|_{W^{1,s}(\Omega),w_1} \le C(n,s,\alpha,\lambda,\Omega)\|u\|_{s,\Omega,w_2} \tag{2.17}$$

for any bounded and convex δ*-John domain* $\Omega\subset\mathbb{R}^n$. *Here the weights are defined by* $w_1(x)=\frac{1}{d^\alpha(x,\partial\Omega)}$ *and* $w_2(x)=\sum_i \chi_{Q_i}\frac{1}{|x-x_{Q_i}|^\lambda}$, *respectively.* α *and* λ *are constants with* $0\le\lambda<\alpha<\lambda+(n+1)s$.

Proof. Applying the Covering lemma and Lemma 6, we have (2.16) immediately. For inequality (2.17), using Lemma 5 and the Covering lemma, and noticing that $1+\frac{1}{n}-\frac{\alpha-\lambda}{ns}>0$ when $\alpha<\lambda+(n+1)s$, we have

$$\|T(H(u))\|_{s,\Omega,w_1} \le C_1(n,s,\alpha,\lambda,\Omega)|\Omega|^{1+1/n-(\alpha-\lambda)/(ns)}\|u\|_{s,\Omega,w_2}. \tag{2.18}$$

By the definition of the $\|\cdot\|_{W^{1,s}(\Omega),w_1}$ norm, we know that

$$\|T(H(u))\|_{W^{1,s}(\Omega),w_1} = \operatorname{diam}(\Omega)^{-1}\|T(H(u))\|_{s,\Omega,w_1} + \|\nabla(T(H(u)))\|_{s,\Omega,w_1}. \tag{2.19}$$

Substituting (2.16) and (2.18) into (2.19) yields

$$\|T(H(u))\|_{W^{1,s}(\Omega),w_1} \le C_2(n,s,\alpha,\lambda,\Omega)\|u\|_{s,\Omega,w_2}.$$

We have completed the proof of the Theorem 2. ☐

THEOREM 3. *Let* $u\in D^{'}(\Omega,\wedge^1)$ *be a solution of the* A*-harmonic equation* (1.1), H *be the projection operator and* T *be the homotopy operator. Assume that* s *is a fixed exponent associated with the nonhomogeneous*

A-harmonic equation. Then there exists a constant $C(n,N,s,\alpha,\lambda,Q_0,\Omega)$, independent of u, such that

$$\begin{aligned}\|T(H(u))-(T(H(u)))_{Q_0}\|_{W^{1,s}(\Omega),w_1}\\ \le C(n,N,s,\alpha,\lambda,Q_0,\Omega)\|u\|_{s,\Omega,w_2}\end{aligned} \tag{2.20}$$

for any bounded, convex δ-John domain $\Omega\subset\mathbb{R}^n$. Here the weights are defined by $w_1(x)=\frac{1}{d^\alpha(x,\partial\Omega)}$ and $w_2(x)=\sum_i\chi_{Q_i}\frac{1}{|x-x_{Q_i}|^\lambda}$, α and λ are constants with $0\le\lambda<\alpha<\min\{n,\lambda+n(s-1)\}$, and the fixed cube $Q_0\subset\Omega$ and the constant $N>1$ appeared in Lemma 3.

Proof. Since $(T(H(u)))_{Q_0}$ is a closed form, $\nabla((T(H(u)))_{Q_0})=d((T(H(u)))_{Q_0})=0$. Thus, by using Theorem 1 and (2.16), we have

$$\begin{aligned}&\|T(H(u))-(T(H(u)))_{Q_0}\|_{W^{1,s}(\Omega),w_1}\\ &\quad=\operatorname{diam}(\Omega)^{-1}\|T(H(u))-(T(H(u)))_{Q_0}\|_{s,\Omega,w_1}\\ &\qquad+\|\nabla(T(H(u))-(T(H(u)))_{Q_0})\|_{s,\Omega,w_1}\\ &\quad=\operatorname{diam}(\Omega)^{-1}\|T(H(u))-(T(H(u)))_{Q_0}\|_{s,\Omega,w_1}+\|\nabla(T(H(u)))\|_{s,\Omega,w_1}\\ &\quad\le C_1(n,N,s,\alpha,\lambda,Q_0,\Omega)\|u\|_{s,\Omega,w_2}+C_2(n,s,\alpha,\lambda,\Omega)\|u\|_{s,\Omega,w_2}\\ &\quad\le C_3(n,N,s,\alpha,\lambda,Q_0,\Omega)\|u\|_{s,\Omega,w_2}.\end{aligned}$$

Thus, (2.20) holds. The proof of Theorem 3 has been completed. □

As applications of our main results, we consider the following examples.

EXAMPLE 1. Let $B=0$, $A(x,\xi)=\xi|\xi|^{p-2}$, $p>1$, and u be a function (0-form) in (1.1). Then, the operator A satisfies the required conditions and the nonhomogeneous A-harmonic equation (1.1) reduces to the usual p-harmonic equation

$$\operatorname{div}(\nabla u|\nabla u|^{p-2})=0 \tag{2.21}$$

which is equivalent to

$$(p-2)\sum_{k=1}^{n}\sum_{i=1}^{n}u_{x_k}u_{x_i}u_{x_kx_i}+|\nabla u|^2\Delta u=0. \tag{2.22}$$

If we choose $p=2$ in (2.21), we have the Laplace equation $\Delta u=0$ for functions. Hence, the equation (2.21), (2.22) and the $\Delta u=0$ are the special cases of the nonhomogeneous A-harmonic equation (1.1). Therefore, all results proved in Theorems 1, 2 and 3 are still true for u that satisfies one of the above three equations.

EXAMPLE 2. Let $f:\Omega\to\mathbb{R}^n$, $f=(f^1,\dots,f^n)$, be a mapping of the Sobolev class $W^{1,p}_{\text{loc}}(\Omega,\mathbb{R}^n)$, $1\le p<\infty$, whose distributional differential $Df=[\partial f^i/\partial x_j]:\Omega\to GL(n)$ is a locally integrable function on Ω with values in the space $GL(n)$ of all $n\times n$-matrices, $i,j=1,2,\dots,n$. A homeomorphism $f:\Omega\to\mathbb{R}^n$ of Sobolev class $W^{1,n}_{\text{loc}}(\Omega,\mathbb{R}^n)$ is said to be K-quasiconformal, $1\le K<\infty$, if its differential matrix $Df(x)$ and the Jacobian determinant $J = J(x,f) = \det Df(x)$ satisfy

$$|Df(x)|^n \le KJ(x,f), \tag{2.23}$$

where $|Df(x)| = \max\{|Df(x)h| : |h| = 1\}$ denotes the norm of the Jacobi matrix $Df(x)$. It is well known that if the differential matrix $Df(x) = [\partial f^i/\partial x_j]$, $i,j = 1,2,\dots,n$, of a homeomorphism $f(x) = (f^1, f^2, \dots, f^n) : \Omega \to \mathbb{R}^n$ satisfies (2.23), then, each of the functions

$$u = f^i(x), \quad i = 1,2,\dots,n, \quad \text{or} \quad u = \log|f(x)|, \tag{2.24}$$

is a generalized solution of the quasilinear elliptic equation

$$\operatorname{div} A(x, \nabla u) = 0, \tag{2.25}$$

in $\Omega - f^{-1}(0)$, where

$$A = (A_1, A_2, \dots, A_n), \quad A_i(x,\xi) = \frac{\partial}{\partial \xi_i}\left(\sum_{i,j=1}^{n} \theta_{i,j}(x)\xi_i\xi_j\right)^{n/2}$$

and $\theta_{i,j}$ are some functions, which can be expressed in terms of the differential matrix $Df(x)$ and satisfy

$$C_1(K)|\xi|^2 \le \sum_{i,j}^{n} \theta_{i,j}\xi_i\xi_j \le C_2(K)|\xi|^2 \tag{2.26}$$

for some constants $C_1(K), C_2(K) > 0$. All results proved in Theorems 1, 2 and 3 are still true if u is defined in (2.24).

EXAMPLE 3. The Jacobian determinant $J(x,f)$ of a mapping f has been well studied and widely used in many areas of mathematics and physics. We know that the Jacobian $J(x,f)$ of a mapping $f : \Omega \to \mathbb{R}^n$, $f = (f^1, \dots, f^n)$ is an n-form, specifically, $J(x,f)\,dx = df^1 \wedge \cdots \wedge df^n$, where $dx = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$. Hence, Lemma 4 proved in this paper can be used to estimate the Jacobian $J(x,f)$ of a mapping f.

Our results can be applied to all differential forms or functions satisfying some version of the A-harmonic equation, the usual p-harmonic equation or the Laplace equation. The projection operator has found many applications in physics and computer sciences. For the purpose of evaluation or estimation functions related to partial differential equations, the projection operator method often plays the key role, see [7] and [9] for example. The homotopy operator is also commonly used in computer science and computer engineering [2]. Considering the length of the paper, we only list the above three examples here. For more application, see [1], [2], [12], [15].

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