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Global Existence and Blow-up for a Shallow Water Equation

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1. - Introduction

An interesting phenomenon in water channels is the appearance of waves with length much greater than the depth of the water. In 1895 D. J. Korteweg and G. de Vries started the mathematical theory of this phenomenon and derived a model describing unidirectional propagation of waves of the free surface of a shallow layer of water. This is the well-known KdV equation:

$$\begin{cases} u_t - 6uu_x + u_{xxx} = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & \qquad\quad x \in \mathbb{R}, \end{cases}$$

where u describes the free surface of the water; for a presentation of the physical derivation of the equation, see [27]. The beautiful structure behind the KdV equation initiated a lot of mathematical investigations, see [5], [20], [21], [22].

Recently, R. Camassa and D. Holm [9] proposed a new model for the same phenomenon:

$$\text{(1.1)} \qquad \begin{cases} u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & \qquad\quad x \in \mathbb{R}. \end{cases}$$

The variable $u(t,x)$ in (1.1) represents the fluid velocity at time t in the x direction in appropriate nondimensional units (or, equivalently, the height of the water's free surface above a flat bottom). Unlike KdV, which is derived by asymptotic expansions in the equation of motion, (1.1) is obtained by using an asymptotic expansion directly in the Hamiltonian for Euler's equations in the shallow water regime, cf. [9], [11]. Equation (1.1) was derived 15 years ago by Fuchssteiner and Fokas (see [16], [17]) as a bi-Hamiltonian generalization of KdV. The novelty of Camassa and Holm's work, cf. [25], was the physical derivation of (1.1) and the discovery that the equation has solitary waves that retain their individuality under interaction and eventually emerge with their original shapes and speeds. Such solitary waves are called solitons.

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After Camassa and Holm established that equation (1.1) has also physical meaning, numerous papers were devoted to its study, cf. [1], [2], [6], [7], [8], [10], [14], [17], [25] and the citations therein. Despite this abundant literature on the Camassa-Holm equation, the problem of well-posedness (and the related question of existence of global solutions) seems not yet to have been treated.

The aim of this paper is to prove local well-posedness of strong solutions to (1.1) for a large class of initial data, and to analyze global existence and blow-up phenomena. In addition, we introduce the notion of weak solutions to (1.1) suitable for soliton interaction.

Our results bring to light an interesting feature of the Camassa-Holm model. Whereas some initial data produce global solutions, others yield solutions having finite life-span. More surprisingly, it is neither the smoothness nor the size of the initial data that influence the life-span but the shape of the initial data. In particular, there are smooth initial data with arbitrary small support and arbitrary small $C^k(\mathbb{R})$-norm, $k \in \mathbb{N}$, for which the resulting solution does not exist globally. It is also worthwhile to note how these solutions exit the phase space, namely their magnitude remains bounded but the slope becomes infinite, which can be understood as a breaking of waves, cf. [27].

MAIN RESULT (a) Well-posedness. *Given $u_0 \in H^3(\mathbb{R})$, there exists a maximal $T = T(u_0) > 0$ and a unique solution*

$$u = u(\cdot, u_0) \in C\big([0, T); H^3(\mathbb{R})\big) \cap C^1\big([0, T); H^2(\mathbb{R})\big)$$

to problem (1.1). *Moreover, the solution depends continuously on the initial data, i.e., the mapping $u_0 \mapsto u(\cdot, u_0) : H^3(\mathbb{R}) \to C([0, T); H^3(\mathbb{R})) \cap C^1([0, T); H^2(\mathbb{R}))$ is continuous.*

(b) Global existence. *Assume $u_0 \in H^3(\mathbb{R})$ is such that the associated potential $y_0 = u_0 - u_0''$ belongs to $L_1(\mathbb{R})$ and does not change sign. Then the solution $u(\cdot, u_0)$ exists globally.*

(c) Blow-up. *Assume $u_0 \in H^3(\mathbb{R})$ is odd and with $u_0'(0) < 0$. Then the maximal interval of existence is finite, i.e., $T = T(u_0) < \infty$.*

To compare the hypotheses for global existence and blow-up, note that the potential $y_0 = u_0 - u_0''$ is odd, provided u_0 is odd, so that the potential y_0 changes sign if $u_0'(0) < 0$.

As an alternative model to KdV, Benjamin, Bona, and Mahoney [3] proposed the so-called BBM-equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \qquad t > 0, \ x \in \mathbb{R}.$$

Numerical work of Bona, Pritchard, and Scott [4] shows that the solitary waves of the BBM-equation are not solitons.

As noted by Whitham [27], it is intriguing to find mathematical equations including the phenomena of breaking and peaking, as well as criteria for the occurence of each. He observed that solutions of the KdV-equation do not break as physical water waves do (recently, Bourgain [5] proved that even for initial data in $L_2(\mathbb{R})$ the solutions of KdV exist globally in time). Whitham suggested to replace the KdV-model by the nonlocal equation

$$u_t + uu_x + \mathbb{K}(u) = 0, \qquad t > 0,\ \ x \in \mathbb{R},$$

for which he conjectured that breaking solutions would exist. Here $\mathbb{K}$ is a Fourier operator with symbol $k(\xi) = \sqrt{(\tanh \xi)/\xi}$. Whitham's conjecture was proved recently in [23]. The numerical calculations carried out for the Whitham equation do not support any strong claim that soliton interaction can be expected, cf. [15].

On the other hand, Camassa, Holm and Hyman [10] show that the solitary waves have a discontinuity in the first derivative at their peak and that soliton interactions occur for (1.1). In Sections 5 and 6 we introduce the notion of weak solutions to (1.1) as a suitable frame for soliton interaction.

The advantage of the new equation in comparison with the well-established models KdV, BBM and the Whitham equation is clear: The Camassa-Holm equation has peaked solitons, breaking waves, and permanent waves.

Some recent results from harmonic analysis depending heavily on the famous "T(1)" theorem [26] will enable us to apply Kato's theory for abstract quasi-linear evolution equations of hyperbolic type to prove well-posedness of (1.1) in $H^3(\mathbb{R})$.

To obtain global existence from local results is a matter of a priori estimates. Although the bi-Hamiltonian structure of (1.1) provides an infinite number of conservation laws that are functionally independent (see [9], [11]), in striking contrast to the KdV-equation where the conservation laws immediately yield a priori estimates for the solution in any $H^r(\mathbb{R})$-space with $r \geq 0$, in our case the $H^1(\mathbb{R})$-norm of a solution to (1.1) is a conservation law — but there is no way to find conservation laws controlling the $H^2(\mathbb{R})$-norm. To guarantee global existence for the class of initial data described in part b) of the Main Result, we will have to develop a more refined method than simply seeking conservation laws.

The spatial anti-symmetry of (1.1) allows us to specify a large class of smooth initial data with arbitrary support for which the corresponding solution does not exist globally.

Finally, let us mention that it is also interesting (see [9]) to look for spatially periodic solutions of (1.1). For results in that direction we refer to [12], [13].

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2. – Well-posedness

In this section we introduce the potential associated to the solution of (1.1) and we reformulate (1.1) as a quasi-linear evolution equation for this potential. The new form is suitable to be analyzed with Kato's method for abstract quasi-linear evolution equations of hyperbolic type.

For convenience we state here Kato's theorem in the form appropriate for our purposes.

Consider the abstract quasi-linear evolution equation in the Hilbert space X:

$$\frac{dv}{dt} + A(v)v = 0, \quad t \geq 0, \quad v(0) = v_0. \tag{2.1}$$

Let Y be a second Hilbert space such that Y is continuously and densely injected into X and let $Q : Y \to X$ be a topological isomorphism. Assume that

(i) $A(y) \in \mathcal{L}(Y, X)$ for $y \in Y$ with

$$\left|\big(A(y) - A(z)\big)w\right|_X \leq \mu_A |y - z|_X |w|_Y, \quad y, z, w \in Y,$$

and $A(y)$ is quasi-m-accretive, uniformly on bounded sets in Y.

(ii) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y) \in \mathcal{L}(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\left|\big(B(y) - B(z)\big)w\right|_X \leq \mu_B |y - z|_Y |w|_X, \quad y, z \in Y, \quad w \in X.$$

Here μ_A and μ_B depend only on $\max\{|y|_Y, |z|_Y\}$.

THEOREM 2.1 (Kato [19], [20]). *Assume that* (i) *and* (ii) *hold. Given $v_0 \in Y$, there is a maximal $T > 0$, depending on v_0, and a unique solution v to* (2.1) *such that*

$$v = v(\cdot, v_0) \in C\big([0, T); Y\big) \cap C^1\big([0, T); X\big).$$

Moreover, the map $v_0 \mapsto v(\cdot, v_0)$ is continuous from Y to $C([0, T); Y) \cap C^1([0, T); X)$.

We provide now the framework in which we shall reformulate problem (1.1).

All spaces of functions are over $\mathbb{R}$ and for simplicity we drop $\mathbb{R}$ in our notation of function spaces if there is no ambiguity. Additionally, we denote by $|\cdot|_r$ the norm in the Sobolev spaces H^r, $r \geq 0$.

Set $X = L_2$, $Y = H^1$, and $Q = (I - \partial_x^2)^{\frac{1}{2}}$. With $y = u - u_{xx}$ as the potential, we rewrite (1.1) in the equivalent form

$$\begin{cases} y_t + \big(Q^{-2}y\big)y_x + 2y\big(Q^{-2}y\big)_x = 0 & \text{in } L_2, \\ y(0) = y_0, \end{cases} \tag{2.2}$$

which is of type (2.1) with

$$A(y) := \big(Q^{-2}y\big)\partial_x + 2\big(Q^{-2}y\big)_x \mathrm{Id}, \quad y \in H^1,$$

where $\mathrm{dom}(A(y)) := \{v \in L_2 : (Q^{-2}y)v \in H^1\}$.

THEOREM 2.2. *Given $y_0 \in H^1$, there is a maximal $T > 0$, depending on y_0, and a unique solution y to* (2.2) *such that*

$$y = y(\cdot, y_0) \in C\big([0,T); H^1\big) \cap C^1\big([0,T); L_2\big).$$

Moreover, the map $y_0 \mapsto y(\cdot, y_0)$ is continuous from H^1 to $C([0,T); H^1) \cap C^1([0,T); L_2)$.

The remainder of this section is devoted to the proof of Theorem 2.2. In the following, K stands for a generic constant.

We first study the linear operator $A(y)$, where $y \in H^1$ is fixed. In order to do so, let $m \in H^3$ be given and define the linear operator in L_2:

$$\operatorname{dom}(D) := \big\{v \in L_2 : mv \in H^1\big\}, \quad Dv := (mv)_x - m_x v.$$

To explain the connection between the linear operators D and $A(y)$, observe that the usual pointwise multiplication $H^1 \times H^1 \to H^1$ has a unique continuous extension to a map $H^1 \times H^{-1} \to H^{-1}$, which we will not distinguish notationally. Therefore, given $v \in L_2$, we obtain by approximation the generalized Leibniz formula

$$(mv)_x = m_x v + m v_x \text{ in } H^{-1}.$$

Suppose now $v \in \operatorname{dom}(D)$. Then $(mv)_x$ and $m_x v$ belong to L_2, and therefore we find

$$Dv = (mv)_x - m_x v = m v_x \in L_2.$$

Choose now $m = Q^{-2} y \in H^3$. Then D is the principal part of $A(y)$. Since $(Q^{-2}y)_x$ clearly belongs to L_∞, we see that $A(y)$ is a well-defined operator in L_2.

We assume only mild regularity properties of m in the definition of D. Thus m may vanish on an arbitrary subset of $\mathbb{R}$. In this sense, $D : v \mapsto m v_x$ should be regarded as a (possibly) degenerate linear differential operator.

PROPOSITION 2.3. *D is quasi-m-accretive in L_2.*

PROOF. a) We first prove that $C^\infty(\mathbb{R}) \cap L_2$ is a core for D in L_2. For this, choose $\rho \in C_c^\infty(\mathbb{R})$ with $\rho \geq 0$, $\int_{\mathbb{R}} \rho = 1$, and let $\rho_n(x) = n\rho(nx)$, $n \geq 1$, be the usual mollifiers on $\mathbb{R}$. We denote by $*$ the convolution. Fix $v \in \operatorname{dom}(D)$. It suffices to show that

$$D(\rho_n * v) \to D(v) \text{ in } L_2. \tag{2.3}$$

Note that

$$D(\rho_n * v) = m(\rho_n * v)_x - \rho_n * \big[(mv)_x - m_x v\big] + \rho_n * \big[(mv)_x - m_x v\big]$$

and therefore it is enough to prove that

$$\lim_{n\to\infty}\big(m(\rho_n * v)_x - \rho_n * \big[(mv)_x - m_x v\big]\big) = 0 \text{ in } L_2. \tag{2.4}$$

We define the operators

$$P_n v = m(\rho_n * v)_x - \rho_n * \big[(mv)_x - m_x v\big], \quad v \in \mathrm{dom}(D), \quad n \geq 1,$$

and we will prove that $\{P_n\}_{n\geq 1}$ can be extended to a family of uniformly bounded linear operators on L_2.

For $x \in \mathbb{R}$ we have

$$\begin{aligned} P_n v(x) &= \int_{\mathbb{R}} (\rho_n)_x(y)\big(m(x) - m(x-y)\big)v(x-y)dy + \big(\rho_n * (m_x v)\big)(x) \\ &= n^2 \int_{\mathbb{R}} \rho_x(ny)\big(m(x) - m(x-y)\big)v(x-y)dy + \big(\rho_n * (m_x v)\big)(x). \end{aligned}$$

Thus, given $n \geq 1$, the operator P_n can be extended to the whole of L_2.

Observe that

$$\begin{aligned} &n^2 \int_{\mathbb{R}} \rho_x(ny)\big(m(x) - m(x-y)\big)v(x-y)dy \\ &\qquad = n\int_{-\lambda}^{\lambda} \rho_x(y)\left(m(x) - m\left(x - \frac{y}{n}\right)\right) v\left(x - \frac{y}{n}\right) dy \end{aligned}$$

where $\mathrm{supp}(\rho) \subset [-\lambda, \lambda]$ and therefore

$$\begin{aligned} &\left| n^2 \int_{\mathbb{R}} \rho_x(ny)\big(m(x) - m(x-y)\big)v(x-y)dy \right| \\ &\qquad \leq \sup_{x\in R} |m_x| \int_{-\lambda}^{\lambda} |\rho_x(y)||y| \left| v\left(x - \frac{y}{n}\right)\right| dy, \quad n \geq 1. \end{aligned}$$

If $C = \sup_{x\in\mathbb{R}} |m_x|^2 \int_{-\lambda}^{\lambda} |\rho_x(y)y|^2 dy$ we deduce by Schwarz's inequality and Fubini's theorem that

$$\begin{aligned} &|n^2 \int_{\mathbb{R}} \rho_x(ny)\big(m(x) - m(x-y)\big)v(x-y)dy|_0^2 \\ &\qquad \leq C\int_{\mathbb{R}}\int_{-\lambda}^{\lambda} v^2\left(x - \frac{y}{n}\right) dy\, dx = C\int_{-\lambda}^{\lambda}\int_{\mathbb{R}} v^2\left(x - \frac{y}{n}\right) dx\, dy \\ &\qquad = C\int_{-\lambda}^{\lambda} |v|_0^2 dy = 2\lambda C |v|_0^2. \end{aligned}$$

Combining this with Young's inequality

$$|\rho_n * (m_x v)|_0^2 \leq \int_{\mathbb{R}} |m_x v|^2 \leq |m_x|_{L_\infty}^2 |v|_0^2 \leq K|m|_2^2 |v|_0^2,$$

we deduce that $P_n \in \mathcal{L}(L^2)$ with

$$|P_n|^2_{\mathcal{L}(L_2)} \leq 2\lambda C + K|m|_2^2, \quad n \geq 1.$$

For $v \in C_c^\infty(\mathbb{R})$ relation (2.4) is obvious and, in view of the uniform boundedness of the operators P_n, we deduce that

$$\lim_{n\to\infty} P_n v = 0, \quad v \in L_2.$$

Thus (2.4) holds for all $v \in \mathrm{dom}(D)$ and the proof of (2.3) is complete.

b) We show now that the operator $D_0 := D + \frac{1}{2}m_x \mathrm{Id}$ with $\mathrm{dom}(D_0) = \mathrm{dom}(D)$ is skew-adjoint in L_2.

Fix $w \in \mathrm{dom}(D_0^*)$. The functional

$$F(\varphi) := (D_0\varphi, w)_0 = \int_{\mathbb{R}} \left(m\varphi_x + \frac{1}{2}m_x\varphi \right) w = (\varphi, D_0^* w)_0, \quad \varphi \in \mathcal{D}(\mathbb{R}),$$

is continuous with respect to the norm in L_2. We deduce that $mw \in H^1$ and

$$\begin{aligned} F(\varphi) = -\int_{\mathbb{R}} \left(\varphi(mw)_x - \frac{1}{2}\varphi m_x w \right) &= -\int_{\mathbb{R}} \varphi \left(m w_x + \frac{1}{2}m_x w \right) \\ &= -(\varphi, D_0 w), \quad \varphi \in \mathcal{D}(\mathbb{R}). \end{aligned}$$

Thus $w \in \mathrm{dom}(D_0)$ and $D_0 w = -D_0^* w$. This proves that $D_0^* \subset -D_0$.

Conversely, choose $v \in \mathrm{dom}(D_0)$ and let $v_n = \rho_n * v$, $n \geq 1$. Using the approximation result proved in step a), we get for any $z \in \mathrm{dom}(D_0)$ the idenity

$$\begin{aligned} (D_0 z, v)_0 = \lim_{n\to\infty} (D_0 z, v_n)_0 &= \lim_{n\to\infty} \int_{\mathbb{R}} \left(m z_x + \frac{1}{2}m_x z \right) v_n \\ &= -\lim_{n\to\infty} \int_{\mathbb{R}} \left(m(v_n)_x + \frac{1}{2}m_x v_n \right) z \\ &= -\int_{\mathbb{R}} \left(m(v)_x + \frac{1}{2}m_x v \right) z = -(z, D_0 v)_0 \end{aligned}$$

which proves that $-D_0 \subset D_0^*$.

c) Since iD_0 is self-adjoint in L_2, Stone's theorem implies that D_0 is the infinitesimal generator of a strongly continuous contraction semigroup on L_2. In addition, $D_0 - D$ is a bounded operator on L_2. Hence D generates a strongly continuous semigroup $\{U(s)\}_{s\geq 0}$ on L_2 with $|U(s)|_{\mathcal{L}(L_2)}$ bounded by $\exp(\frac{s}{2}|m_x|_{L_\infty})$ for $s \geq 0$, cf. Theorem 3.1.1 in [24]. This implies the assertion. □

One more delicate point we have to clear is the analysis of the operator $B(y) = QA(y)Q^{-1} - A(y)$ for $y \in H^1$. Using methods from harmonic analysis we will derive a representation and estimates for this operator on L_2.

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing functions and let $\mathcal{F}$ be the Fourier transform. Given $y \in H^1$, we introduce the multiplication operators $M(y)$ and $M_x(y)$ on L_2 defined by $M(y)v = (Q^{-2}y)v$ and $M_x(y)v = (Q^{-2}y)_x v$, respectively. We also denote by $[\cdot, \cdot]$ the usual commutator.

LEMMA 2.4. *Assume that* $y \in H^1$. *Then*

$$B(y) = [Q, M(y)]\partial_x Q^{-1} + 2[Q, M_x(y)]Q^{-1}$$

on functions in $\mathcal{S}(\mathbb{R})$.

PROOF. By direct computation we obtain

$$B(y) = [Q, M(y)]\partial_x Q^{-1} - M(y)[\partial_x, Q]Q^{-1} + 2[Q, M_x(y)]Q^{-1}.$$

It suffices to prove that $[\partial_x, Q] \equiv 0$. But this follows from the representations

$$\partial_x f = -\mathcal{F}^{-1}(i\xi\mathcal{F}(f)), \quad Qf = \mathcal{F}^{-1}\left(\sqrt{1+\xi^2}\mathcal{F}(f)\right). \qquad \square$$

PROPOSITION 2.5. *Given* $y \in H^1$, *the operator* $B(y)$ *can be extended to an operator in* $\mathcal{L}(L_2)$ *which is uniformly bounded for* $|y|_1$ *bounded.*

PROOF. It is obvious that $\partial_x Q^{-1}$ and $[Q, M_x(y)]Q^{-1}$ extend to bounded linear operators on L_2 which are uniformly bounded for $|y|_1$ bounded.

Clearly, $Q^{-2}y \in H^3$ and Q is a first order pseudo-differential operator. Therefore the arguments in [26], Section VII.3.5 show that $[Q, M(y)]$ extends to a bounded linear operator on L_2 whose norm in $\mathcal{L}(L_2)$ is estimated by $K|\partial_x(Q^{-2}y)|_{L_\infty}$ and hence by $K|y|_1$, where K is a universal constant. $\square$

REMARK 2.6. a) The crucial commutator property used before is well-known for multiplication operators induced by C^∞-functions, cf. Chapter VI in [26]. We actually need much weaker smoothness properties of the multiplier. The remarkable fact that the C^∞-smoothness can be brought down to $W^{1,\infty}(\mathbb{R})$ is the result of a recent theory based on the famous "T(1)" theorem (see [26], Chapter VII]). Note that the commutator results in [19] and [20] are not sharp enough for our purposes.

b) Given $y \in H^1$, the operator $B(y)$ extends also to an operator $B_1(y) \in \mathcal{L}(H^1)$, which is uniformly bounded on bounded sets in H^1. To see this, note first that $Q^{-2}y \in H^3$. Therefore $[Q, M_x(y)]Q^{-1}$ obviously extends to a bounded linear operator on H^1. To prove that this also holds true for the operator $[Q, M(y)]\partial_x Q^{-1}$, it is enough to estimate the operator $\partial_x[Q, M(y)]Q^{-1}$ in $\mathcal{L}(L_2)$. Since

$$\partial_x[Q, M(y)]Q^{-1} = QM_x(y)Q^{-1} + M_x(y) + [Q, M(y)]\partial_x Q^{-1},$$

this follows again by the arguments of Stein [26], Section VII.3.5. $\square$

PROOF OF THEOREM 2.2. Recall that K stands for a generic constant.

(i) It is clear that $A(y) \in \mathcal{L}(H^1, L_2)$. Moreover, Proposition 2.3 shows that the principal part $P(y)$ of $A(y)$ is quasi-m-accretive, uniformly on bounded sets in H^1. Since $A(y) - P(y)$ is a bounded linear operator on L^2 which is uniformly bounded on bounded sets in H^1, $A(y)$ is uniformly quasi-m-accretive.

Let $y, z, w \in H^1$ be given. We have

$$\big|(A(y) - A(z))w\big|_0^2 \le K\big|Q^{-2}(y - z)\big|_2^2 |w|_1^2 \le K\big|Q^{-2}\big|_{\mathcal{L}(L_2,H^2)} |y - z|_0^2 |w|_1^2.$$

(ii) As noted before, $B(y) = [Q, M(y)]\partial_x Q^{-1} + 2[Q, M_x(y)]Q^{-1}$ for $y \in Y$, and the first assertion was already proved in Proposition 2.5. To conclude, observe that $B(y) - B(z) = B(y - z)$.

The result now follows from Theorem 2.1. □

PROPOSITION 2.7. *Assume that* $y_0 \in H^2$. *Then the solution* y *to equation* (2.2) *posseses the additional regularity* $y \in C([0, T); H^2) \cap C^1([0, T); H^1)$.

PROOF. Given $y_0 \in H^2$, let $y \in C([0, T); H^1) \cap C^1([0, T); L_2)$ be the solution of (2.2) constructed in Theorem 2.2.

Fix $T_0 \in (0, T)$. Given $t \in [0, T_0]$, let $A_1(t)$ denote the part of $A(y(t))$ in H^1. From Propositions 2.3 and 2.5 and Lemma 5.4.4 in [6] we know that the family $\{A_1(t) : t \in [0, T_0]\}$ is stable in H^1.

Clearly $Q : H^2 \to H^1$ is a topological isomorphism and Remark 2.6 b) shows that

$$QA_1(t)Q^{-1} - A_1(t) = B_1(t), \quad t \in [0, T_0],$$

defines a norm-continuous family of operators in $\mathcal{L}(H^1)$. Additionally, it is not difficult to see that

$$\mathrm{dom}\big(A_1(t)\big) = \big\{v \in H^1 : \big(Q^{-2}y(t)\big)_x v \in H^2\big\}, \quad t \in [0, T_0].$$

Hence $H^2 \subset \mathrm{dom}(A_1(t))$ for $t \in [0, T_0]$, since H^2 is a Banach algebra.

Consequently, Corollary 5.4.7 in [6] guarantees that the linear evolution problem in H^1

$$\frac{dv}{dt} + A_1(t)v = 0, \quad v(0) = y_0,$$

has a unique solution $v \in C([0, T_0]; H^2) \cap C^1([0, T_0]; H^1)$.

Obviously, v is also a classical solution (in the sense of [6]) of the following linear evolution problem in L_2:

$$\frac{dv}{dt} + A_0(t)v = 0, \quad v(0) = y_0, \tag{2.5}$$

where $A_0(t) := A(y(t))$.

Finally, the proof of Theorem 2.2 and again Corollary 5.4.7 from [6] show that problem (2.5) is well-posed in L_2. But y is also a solution of problem (2.5) and therefore we conclude that $v = y \in C([0, T_0]; H^2) \cap C^1([0, T_0]; H^1)$, implying the assertion. □

3. – Global Existence

In this section we are going to prove that if the intial data $y_0 \in H^1$ for equation (2.2) belongs to L_1 and does not change sign, then the H^1-norm of the corresponding solution $y(t)$ cannot blow-up in finite time. In view of Theorem 2.2, this proves global existence of the solution.

We will denote by $y = u - u_{xx}$ the potential associated to (1.1). Obviously, u is a solution to (1.1) if and only if y is a solution to (2.2).

The bi-Hamiltonian structure of (1.1) yields an infinite number of conservation laws which are functionally independent, cf. [9]. Among these, we have the conserved quantity

$$\frac{1}{2}\int_{\mathbb{R}} (u^2 + u_x^2) = \frac{1}{2}\int_{\mathbb{R}} (u_0^2 + u_{0,x}^2). \tag{3.1}$$

This conservation law (the physical interpretation of which is energy) does not provide an L_2 a priori estimate for y, nor do the others available by the method described in [9]. To realize our goal, we would like to control the H^1-norm of the solution y. The situation described above is in striking contrast to the KdV-equation where the conservation laws yield immediate a priori estimates of the solution in any H^r space, $r \geq 0$. Actually, the blow-up result of the next section will show that there is no way to find conservation laws controlling the norm of u in H^2.

We do not want to make use of the theory of infinite-dimensional Hamiltonian systems to obtain the conservation law (3.1). Therefore, we provide an alternative derivation. In addition, we will find two other conserved quantities:

$$\int_{\mathbb{R}} \sqrt{y_+}, \quad \int_{\mathbb{R}} \sqrt{y_-}, \tag{3.2}$$

where y_+ and y_- stand for the positive and the negative part of y, respectively. Note that the conservation laws (3.2) are not among those found in [9].

Lemma 3.1. *If $y_0 \in H^1$ and $u = Q^{-2}y$ then, as long as the solution $y(t)$ exists, we have*

$$\int_{\mathbb{R}} (u^2 + u_x^2) = \int_{\mathbb{R}} (u_0^2 + u_{0,x}^2).$$

Proof. Integration by parts yields

$$\int_{\mathbb{R}} yu = \int_{\mathbb{R}} (u^2 + u_x^2)$$

and therefore

$$\frac{d}{dt}\int_{\mathbb{R}} (u^2 + u_x^2) = \int_{\mathbb{R}} yu_t + \int_{\mathbb{R}} y_t u.$$

On the other hand

$$\int_{\mathbb{R}} y u_t = \int_{\mathbb{R}} u_t u - \int_{\mathbb{R}} u_t u_{xx} = \int_{\mathbb{R}} u_t u - \int_{\mathbb{R}} u_{txx} u = \int_{\mathbb{R}} y_t u.$$

Hence

$$\frac{d}{dt} \int_{\mathbb{R}} (u^2 + u_x^2) = 2 \int_{\mathbb{R}} y_t u,$$

and using equation (2.2), we find

$$\frac{d}{dt} \int_{\mathbb{R}} (u^2 + u_x^2) = -4 \int_{\mathbb{R}} u u_x y - 2 \int_{\mathbb{R}} u^2 y_x = 0. \qquad \square$$

LEMMA 3.2. *If $y_0 \in H^1$ then, as long as the solution $y(t)$ exists, we have*

$$\int_{\mathbb{R}} \sqrt{y_+} = \int_{\mathbb{R}} \sqrt{(y_0)_+}, \qquad \int_{\mathbb{R}} \sqrt{y_-} = \int_{\mathbb{R}} \sqrt{(y_0)_-}\,.$$

PROOF. Let $\varepsilon > 0$. It is known, cf. [18] Section 7.4, that if Ω is a bounded domain in $\mathbb{R}^2$ and $v \in H^1(\Omega)$, then $\sqrt{\varepsilon + v_+}, \sqrt{\varepsilon + v_-} \in H^1(\Omega)$ with

$$\nabla \sqrt{\varepsilon + v_+} = \frac{\nabla v}{2\sqrt{\varepsilon + v_+}} \chi_{[v>0]}, \qquad \nabla \sqrt{\varepsilon + v_-} = \frac{\nabla v}{2\sqrt{\varepsilon + v_-}} \chi_{[v<0]},$$

where χ stands for the characteristic function.

Fix $t_0 > 0$ in the maximal interval of existence of the solution $y(t)$ of (2.2) with initial data y_0. Note that the restriction of y to $[0, t_0] \times [-n, n]$ belongs to $H^1([0, t_0] \times [-n, n])$ for every $n \geq 1$. Therefore, the above mentioned result implies

$$\begin{aligned}
\frac{d}{dt} \int_{-n}^{n} \sqrt{\varepsilon + y_+} &= \frac{1}{2} \int_{-n}^{n} \frac{y_t}{\sqrt{\varepsilon + y_+}} \chi_{[y>0]} \\
&= - \int_{-n}^{n} \frac{y u_x}{\sqrt{\varepsilon + y_+}} \chi_{[y>0]} - \frac{1}{2} \int_{-n}^{n} \frac{y_x u}{\sqrt{\varepsilon + y_+}} \chi_{[y>0]} \\
&= - \int_{-n}^{n} \sqrt{\varepsilon + y_+} u_x \chi_{[y>0]} + \varepsilon \int_{-n}^{n} \frac{u_x}{\sqrt{\varepsilon + y_+}} \chi_{[y>0]} \\
&\quad - \frac{1}{2} \int_{-n}^{n} \frac{y_x u}{\sqrt{y_+}} \chi_{[y>0]}.
\end{aligned}$$

Performing in the first integral an integration by parts, we get

$$\frac{d}{dt} \int_{-n}^{n} \sqrt{\varepsilon + y_+} = -u \sqrt{\varepsilon + y_+} \Big|_{-n}^{n} + \varepsilon \int_{-n}^{n} \frac{u_x}{\sqrt{\varepsilon + y_+}} \chi_{[y>0]} + R(n, t, \varepsilon)$$

where $R(n, t, \varepsilon)$ comes from the evaluation of $\sqrt{\varepsilon} u$ at the interior points of $[-n, n]$ delimiting intervals where $y > 0$. By the fundamental theorem of calculus we have

$$\left| \varepsilon \int_{-n}^{n} \frac{u_x}{\sqrt{\varepsilon + y_+}} \chi_{[y>0]} + R(n, t, \varepsilon) \right| \le 2\sqrt{\varepsilon} \max_{z \in [-n,n]} |u(t, z)| + 2\sqrt{\varepsilon} \int_{-n}^{n} |u_x|.$$

Using Schwarz's inequality, we get from the previous inequalities that

$$\left| \int_{-n}^{n} \sqrt{\varepsilon + y_+} - \int_{-n}^{n} \sqrt{\varepsilon + (y_0)_+} \right| \le \int_0^t C(n, s, \varepsilon) ds$$
$$+ 2t_0\sqrt{\varepsilon} \left(\max_{z \in [-n,n]} |u(t, z)| + 2n \left(\int_{-n}^{n} |u_x|^2 \right)^{1/2} \right),$$

where $C(n, s, \varepsilon) := |u(s, n)\sqrt{\varepsilon + y_+(s, n)} - u(s, -n)\sqrt{\varepsilon + y_+(s, -n)}|$, $n \ge 1$, $s \in [0, t_0]$.

Taking into account the continuous dependence of the solution with respect to time, given by Theorem 2.3, note the inequalities

$$|u(t)|_0 + |u(t)|_{L_\infty} + |y(t)|_{L_\infty} \le 2|u(t)|_1 + |y(t)|_1 \le M(t_0), \quad t \in [0, t_0]. \tag{3.3}$$

Define $C(n, s) := |u(s, n)\sqrt{y_+(s, n)} - u(s, -n)\sqrt{y_+(s, -n)}|$, $n \ge 1$, $s \in [0, t_0]$. Let $\varepsilon \to 0$ to find

$$\left| \int_{-n}^{n} \sqrt{y_+} - \int_{-n}^{n} \sqrt{(y_0)_+} \right| \le \int_0^t C(n, s) ds,$$

For fixed $t \in [0, t_0]$ we have that $u(t), y(t) \in H^1 \subset C_0(\mathbb{R})$ and therefore

$$C(n, s) \to 0 \text{ as } n \to \infty, \tag{3.4}$$

pointwise on $[0, t_0]$.

On the other hand

$$\int_{-n}^{n} \sqrt{(y_0)_+} - \int_0^{t_0} C(n, s) ds \le \int_{-n}^{n} \sqrt{y(t_0)_+} \le \int_{-n}^{n} \sqrt{(y_0)_+} + \int_0^{t_0} C(n, s) ds. \tag{3.5}$$

By (3.3) we obtain that $C(n, s) \le 2[M(t_0)]^{3/2}$ for all $s \in [0, t_0]$ and $n \ge 1$. Therefore relation (3.4) proves, in view of Lebesgue's dominated convergence theorem, that

$$\int_0^{t_0} C(n, s) ds \to 0 \text{ as } n \to \infty.$$

Letting $n \to \infty$ in (3.5) the monotone convergence theorem implies that

$$\int_{\mathbb{R}} \sqrt{y(t_0)_+} = \int_{\mathbb{R}} \sqrt{(y_0)_+}.$$

The arbitraryness of t_0 proves the assertion for y_+. There is no difference in dealing with the negative part of y. □

From Lemma 3.2 we obtain the following nice sign-preserving property of solutions to the equation (2.2):

COROLLARY 3.3. *If the initial data $y_0 \in H^1$ does not change sign, neither does the corresponding solution $y(t)$ as long as it exists.*

LEMMA 3.4. *Assume that $y_0 \in H^1$ does not change sign. Then*

$$\int_{\mathbb{R}} y = \int_{\mathbb{R}} y_0$$

as long as the solution $y(t)$ of (2.2) with initial data y_0 exists.

PROOF. Pick $t_0 > 0$ in the maximal interval of existence of the solution $y(t)$ of (2.2) with initial data y_0 and, to fix ideas, assume that $y_0 \geq 0$.

By Corollary 3.3 we know that $y(t) \geq 0$ for $t \in [0, t_0]$. Since $\chi_{[-n,n]}y \in C^1([0, t_0]; L_1(\mathbb{R}))$ we find that

$$\frac{d}{dt}\int_{-n}^{n} y = \int_{-n}^{n} y_t = -\int_{-n}^{n} (2yu_x + y_x u),$$

where we used equation (2.2).

Integration by parts yields for $t \in [0, t_0]$ and $n \geq 1$:

$$\begin{aligned}\frac{d}{dt}\int_{-n}^{n} y &= -\left(\int_{-n}^{n} yu_x\right) - (yu)|_{-n}^{n} \\ &= -\int_{-n}^{n} (uu_x - u_{xx}u_x) - (yu)|_{-n}^{n} \\ &= \left(-\frac{1}{2}u^2 + \frac{1}{2}u_x^2 - yu\right)\Big|_{-n}^{n} =: C(n, t)\,.\end{aligned}$$

Since H^1 is a Banach algebra contained in $C_0(\mathbb{R})$, we find that $C(n, t) \to 0$ as $n \to \infty$ for every $t \in [0, t_0]$. On the other hand, using the continuity of the solution with respect to time, we get

$$|C(n,t)| \leq K\left(|u(t)|_2^2 + |y(t)|_1|u(t)|_1\right) \leq KM(t_0), \quad t \in [0, t_0],$$

for a universal constant K.

These two facts, the monotone convergence theorem, and Lebesgue's dominated convergence theorem enable us to obtain the assertion of the lemma from the relation

$$\int_{-n}^{n} y = \int_{-n}^{n} y_0 + \int_0^{t_0} C(n, s)ds, \quad n \geq 1,$$

by letting $n \to \infty$. □

We shall now prove the main result of this section.

THEOREM 3.5.. *Assume that $y_0 \in H^1 \cap L_1$ does not change sign. Then the solution $y(t)$ to* (2.2) *with initial data y_0 exists globally.*

PROOF. a) We consider first the case when $y_0 \geq 0$. Corollary 3.3 then ensures that the solution $y(t)$ remains non-negative as long as it exists. By the maximum principle, this is also true for $u = Q^{-2}y$. Note in addition that by the montone convergence theorem and the fact that $u_x \in C_0(\mathbb{R})$, we get

$$\int_{\mathbb{R}} u = \lim_{n\to\infty} \int_{-n}^{n} u = \lim_{n\to\infty} \int_{-n}^{n} (u - u_{xx}) = \lim_{n\to\infty} \int_{-n}^{n} y = \int_{\mathbb{R}} y,$$

a relation which by Lemma 3.4 and the hypothesis $y_0 \in L_1$ yields

$$\int_{\mathbb{R}} u = \int_{\mathbb{R}} y = \int_{\mathbb{R}} y_0 < \infty, \tag{3.6}$$

as long as the solution exists.

We have that

$$\int_{\mathbb{R}} y \geq \int_{-\infty}^{x} y = \int_{-\infty}^{x} (u - u_{xx}) = \left(\int_{-\infty}^{x} u \right) - u_x(\cdot, x) \geq -u_x(\cdot, x)$$

since $u_x \in C_0(\mathbb{R})$. Combining this with (3.6) we get

$$-u_x(t,x) \leq |y(t)|_{L_1} = |y_0|_{L_1}, \quad x \in \mathbb{R},$$

for any t in the maximal interval of existence of the solution.

Let us now consider the case when $y_0 \leq 0$. Then, as above, $y(t)$ and $u(t)$ are both non-positive as long as they exist and

$$-\int_{\mathbb{R}} u = -\int_{\mathbb{R}} y = -\int_{\mathbb{R}} y_0 < \infty. \tag{3.7}$$

In this case we have

$$0 \geq \int_{-\infty}^{x} y = \left(\int_{-\infty}^{x} u \right) - u_x(\cdot, x),$$

and we deduce from (3.7) that

$$-u_x(t,x) \leq |u(t)|_{L_1} = |y_0|_{L_1}, \quad x \in \mathbb{R},$$

for any t in the maximal interval of existence of the solution.

Summing up, we have

$$-u_x(t,x) \leq |y_0|_{L_1}, \quad x \in \mathbb{R}, \quad t \in [0,T), \tag{3.8}$$

provided $y_0 \in L_1$ does not change sign. In (3.8) and henceforth, we denote by $[0, T)$ the maximal interval of existence of the solution of (2.2) with initial data y_0.

b) We are now going to provide an L^2 a priori bound for the solution y. Using (2.2), integration by parts yields

$$\begin{aligned}\frac{d}{dt}|y|_0^2 &= 2(y_t, y)_0 = -2(uy_x, y)_0 - 4(u_x y, y)_0 \\ &= -\int_{\mathbb{R}} u(y^2)_x - 4\int_{\mathbb{R}} u_x y^2 = -3\int_{\mathbb{R}} u_x y^2.\end{aligned}$$

Hence (3.8) implies that $\frac{d}{dt}|y|_0^2 \leq K|y|_0^2$, proving that

$$|y(t)|_0 \leq Ke^{tK}, \quad t \in [0, T). \tag{3.9}$$

c) To obtain an L^2 bound for y_x one can formally differentiate $|y_x|_0^2$ and mimic the arguments used in b). However, the justification of this formal derivation is not trivial. We therefore approximate y_0 in H^1 by functions $y_0^n \in H^2$ having the same sign as y_0. More precisely, let ρ_n, $n \geq 1$, be the mollifiers used in the proof of Proposition 2.3, and define $y_0^n := \rho_n * y_0$ for $n \geq 1$. Then $y_0^n \in H^2 \cap L_1$ and y_0^n has the same sign as y_0. In addition, we have by Young's inequality that

$$|y_0^n|_{L_1} \leq |y_0|_{L_1}, \quad |y_0^n|_1 \leq |y_0|_1, \quad n \geq 1. \tag{3.10}$$

Let us write $y^n = y^n(\cdot, y_0^n)$ for the solution of problem (2.2) with initial data y_0^n. By Proposition 2.7 we know that

$$y^n \in C\big([0, T_n); H^2\big) \cap C^1\big([0, T_n); H^1\big), \quad n \geq 1.$$

Hence $t \mapsto |y_x^n(t)|_0^2$ is continuously differentiable on $[0, T_n)$ and we obtain with $u_n := Q^{-2}y^n$ the identity

$$\frac{d}{dt}|y_x^n|_0^2 = 2\big(y_{tx}^n, y_x^n\big)_0 = -6\big((y_x^n)^2, u_x^n\big)_0 - 4\big(y^n y_x^n, u_{xx}^n\big)_0 - 2\big(y_{xx}^n y_x^n, u^n\big)_0.$$

Recalling that $u_{xx}^n = u^n - y^n$, we get

$$\big(y^n y_x^n, u_{xx}^n\big)_0 = \big(y^n y_x^n, u^n\big)_0,$$

since $\int_{\mathbb{R}} \partial_x (y^n)^3 = 0$. Consequently, we find the estimate

$$\frac{d}{dt}|y_x^n|_0^2 = -5\big((y_x^n)^2, u_x^n\big)_0 - 4\big(y^n y_x^n, u^n\big)_0 \leq K|y_x^n|_0^2 + K|y^n|_0|y_x^n|_0,$$

where we used (3.8), (3.10), and Lemma 3.1, respectively. Finally, we get

$$\frac{d}{dt}|y_x^n|_0^2 \le K\left(|y^n|_0^2 + |y_x^n|_0^2\right) \text{ on } [0, T_n).$$

Invoking the L_2-estimate (3.9) proved in part b) and Gronwall's lemma, we find that each y^n exists globally in H^1 with

$$|y^n(t)|_1 \le Ke^{tK}, \quad t \in [0, \infty), \quad n \ge 1.$$

Since $y(\cdot, y_0) \in C([0, T); H^1)$ depends continuously on $y_0 \in H^1$, we find that for every $T_0 \in (0, T)$ an $N(T_0) \in \mathbb{N}$ such that

$$|y^n(t) - y(t)|_1 \le 1, \quad n \ge N(T_0), \quad t \in [0, T_0].$$

We now obtain

$$|y(t)|_1 \le 1 + Ke^{tK}, \quad t \in [0, T),$$

and the proof is complete. □

REMARK 3.6. As an inspection of the proof of Theorem 3.5 shows, the sign-preserving property of the solution y allows us to work with initial conditions $y_0 \in H^1 \cap L_1$ without assuming faster decay at infinity.

4. – Blow-up

In this section we prove that there are smooth initial data for which the corresponding solution to (1.1) does not exist globally, as indicated in [9].

Let us first derive a useful identity satisfied by a solution to (1.1).

For this, recall that

$$\mathcal{F}^{-1}\left(\frac{1}{1+\xi^2}\right)(x) = \int_{\mathbb{R}} e^{ix\xi}\frac{d\xi}{1+\xi^2} = \frac{1}{2}e^{-|x|} =: p(x), \quad x \in \mathbb{R}.$$

Therefore the operator Q^{-2} can be represented as the following convolution operator:

$$Q^{-2}f = p * f, \quad f \in L_2.$$

Given $u_0 \in H^3$, let

$$u \in C\big([0, T); H^3\big) \cap C^1\big([0, T); H^2\big)$$

be the solution of (1.1) with initial data u_0 (as constructed in Section 2). Using equation (1.1) it is not difficult to verify that

$$Q^2(u_t + uu_x) = -2uu_x - u_x u_{xx} = -\partial_x\left(u^2 + \frac{1}{2}u_x^2\right).$$

Hence

$$u_t + uu_x = -\partial_x \left(p * \left(u^2 + \frac{1}{2}u_x^2\right)\right) \text{ in } C([0,T); H^1), \tag{4.1}$$

where $*$ stands for the convolution with respect to the spatial variable.

Differentiating (4.1) with respect to x we get

$$\begin{aligned} u_{tx} + u_x^2 + uu_{xx} &= -\partial_x^2 \left(p * \left(u^2 + \frac{1}{2}u_x^2\right)\right) \\ &= (Q^2 - \mathrm{Id})\left(p * \left(u^2 + \frac{1}{2}u_x^2\right)\right) \\ &= u^2 + \frac{1}{2}u_x^2 - p * \left(u^2 + \frac{1}{2}u_x^2\right), \end{aligned}$$

and therefore

$$u_{tx} + uu_{xx} = u^2 - \frac{1}{2}u_x^2 - p * \left(u^2 + \frac{1}{2}u_x^2\right). \tag{4.2}$$

Note that (4.2) holds true in the space $C([0,T); H^1)$.

We are now going to prove the main result of this section:

THEOREM 4.1. *Assume that $u_0 \in H^3$ is odd and $u_0'(0) < 0$. Then the corresponding solution of* (1.1) *does not exist globally. The maximal time of existence is estimated above by* $1/(2|u_0'(0)|)$.

PROOF. Let $[0,T)$ be the maximal interval of existence of the corresponding solution $u \in C([0,T); H^3) \cap C^1([0,T); H^2)$ of (1.1). Note that

$$v(t,x) := -u(t,-x), \quad t \in [0,T), \quad x \in \mathbb{R},$$

is also a solution of (1.1) in $C([0,T); H^3) \cap C^1([0,T); H^2)$ with initial data u_0. By uniqueness we conclude that $v \equiv u$ and therefore $u(t,\cdot)$ is odd for any $t \in [0,T)$. In particular, by continuity with respect to the spatial variable of u and u_{xx}, we get

$$u(t,0) = u_{xx}(t,0) = 0 \text{ for } t \in [0,T). \tag{4.3}$$

Define $g(t) := u_x(t,0)$ for $t \in [0,T)$ and note that $g \in C^1([0,T), \mathbb{R})$. From (4.2) and (4.3) we get

$$\frac{dg}{dt}(t) = -\frac{1}{2}g^2(t) - \frac{1}{2}\int_{\mathbb{R}} e^{-|y|}\left(u^2 + \frac{1}{2}u_x^2\right) dy \le -\frac{1}{2}g^2(t), \quad t \in [0,T).$$

Consequently,

$$\frac{1}{g(t)} \ge \frac{1}{g_0} + \frac{t}{2}, \quad t \in [0,T),$$

and therefore $T < -2/g_0$. In particular, the solution does not exist globally. □

REMARK 4.2. a) Due to the invariance of (1.1) with respect to the transformation $(t, x) \mapsto (t, x + \kappa)$, where $\kappa \in \mathbb{R}$, the result of Theorem 4.1 holds true for all initial data $u_0 \in H^3$ which are point-symmetric.

b) From the conservation law (3.1) we immediately see that the magnitude of the solution u remains bounded as long as the solution exists. Moreover the proof of Theorem 3.5 shows that the solution exists globally, provided we can bound u_x pointwise from below (uniformly in x and t). Therefore, in order to have blow-up, the slope of the solution must become infinite, which can be understood as a breaking of waves.

COROLLARY 4.3. *The only equilibrium point of* (1.1) *in* H^3 *is the trivial solution. It is unstable.*

PROOF. Note that (1.1) can be written in the form

$$u_t - u_{xxt} = -\left(\frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx}\right)_x .$$

An equilibrium solution $u \in H^3$ therefore satisfies

$$\frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx} = 0, \quad x \in \mathbb{R}.$$

Integration over $\mathbb{R}$ yields $u \equiv 0$.

Since in every neighborhood of zero in H^3 there are odd functions u_0 with $u_0'(0) < 0$, Theorem 4.1 implies that zero is unstable. □

5. – Weak Solutions

Observe that the class

$$C\big([0, T); H^3\big) \cap C^1\big([0, T); H^2\big) \tag{5.1}$$

is optimal in order to solve problem (1.1) in the space $C([0, T); L_2)$. In the following, we mean by a **strong solution** to the Camassa-Holm equation (1.1) a function in (5.1), satisfying (1.1) in $C([0, T); L_2)$.

A particular feature of the Camassa-Holm equation is the soliton interaction of solitary waves with corners at their peaks, discovered in [9], [10]. Clearly, such solutions do not belong to the space (5.1). To provide a mathematical framework for the study of soliton interaction we shall introduce the notion of weak solutions to problem (5.1).

To do this, let us first rewrite equation (1) as a conservation law. More precisely, let $p(x) := (1/2)\exp(-|x|)$ be the Fourier transform of the Poisson kernel on $\mathbb{R}$. As seen in the previous section, the resolvent $(1-\partial_x^2)^{-1}$ can be represented as the following convolution operator:

$$(1-\partial_x^2)^{-1} f = p * f, \qquad f \in L_2.$$

Assume now that $u_0 \in H^3$ and let

$$u \in C\big([0,T); H^3\big) \cap C^1\big([0,T); H^2\big)$$

be the corresponding strong solution of (1.1). We have

$$u_t + u u_x = -\frac{1}{2}\partial_x \left(p * \left(u^2 + \frac{1}{2} u_x^2 \right) \right) \text{ in } C\left([0,T); H^1\right),$$

see (4.1). Introducing the nonlinear operator

$$F(v) := \left(\frac{1}{2} v^2 + p * \left(v^2 + \frac{1}{2} v_x^2 \right) \right), \qquad v \in H^1(\mathbb{R}),$$

equation (1.1) can formally be rewritten as the conservation law

$$u_t + F(u)_x = 0, \qquad u(0,\cdot) = u_0. \tag{5.2}$$

Let now $u_0 \in H^1$ be given. A function $u : [0,T) \times \mathbb{R}$ is called a **weak solution** to (1.1), if u belongs to $L_{\infty,\mathrm{loc}}([0,T); H^1)$ and satisfies the identity

$$\int_0^T \int_{\mathbb{R}} \big(u\phi_t + F(u)\phi_x \big)\, dx\, dt + \int_{\mathbb{R}} u_0(x)\phi(0,x)\, dx = 0$$

for all $\phi \in C^{1,c}([0,T) \times \mathbb{R})$, where $\phi \in C^{1,c}([0,T) \times \mathbb{R})$ if it is the restriction to $[0,T) \times \mathbb{R}$ of a continuously differentiable function on $\mathbb{R}^2$ with compact support contained in $(-T,T) \times \mathbb{R}$. A weak solution is called **global**, if it is a weak solution on $[0,T)$ for every $T > 0$. Our definition is justified by

PROPOSITION 5.1. a) *Every strong solution is a weak solution.*
b) *If u is a weak solution and u belongs to the class* (5.1) *then it is a strong solution.*

PROOF. a) Let $u \in C([0,T); H^3) \cap C^1([0,T); H^2)$ be a strong solution on $[0,T)$ of equation (1.1). Then u belongs clearly to $L_{\infty,\mathrm{loc}}([0,T); H^1)$, satisfies the initial data pointwise and we have that $F(u(t)) \in H^2$ for $t \in [0,T)$. Hence the equation $u_t + F(u)_x = 0$ holds true in $C([0,T); L_2)$. Integration by parts now implies that u is a weak solution of (1.1).

b) Let u be a weak solution belonging to the class (5.1). Then $u_0 := u(0,\cdot) \in H^3$. Hence there exists a unique strong solution v with initial data u_0. By a) the function v is a weak solution to (1.1). Thus integration by parts yields

$$\int_0^T \int_{\mathbb{R}} \big(u_t + F(u)_x\big)\phi\, dx\, dt = \int_0^T \int_{\mathbb{R}} \big(v_t + F(v)_x\big)\phi\, dx\, dt, \quad \phi \in C^{1,c}\big([0,T) \times \mathbb{R}\big).$$

Since $C^{1,c}([0,T)\times\mathbb{R})$ is dense in $L_2([0,T)\times\mathbb{R})$ we find that

$$u_t + F(u)_x = v_t + F(v)_x \text{ in } L_2\big([0,T)\times\mathbb{R}\big). \tag{5.3}$$

But the regularity assumption on u implies that (5.3) actually holds true in $C([0,T);H^2)$. Applying the operator $(1-\partial_x^2)$ to this equation and using (4.1) and the fact that v is a strong solution to (1.1), we deduce that

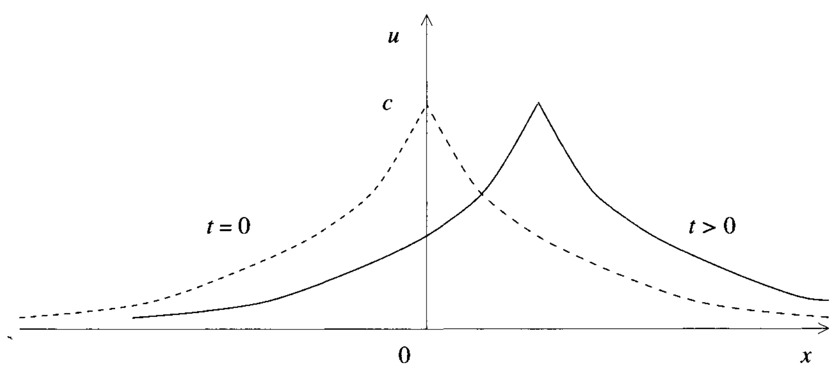
$$u_t - u_{txx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0 \text{ in } C\big([0,T);L_2\big).$$

Thus u is in fact a strong solution. □

EXAMPLE 5.2 (Solitary Wave). By computation one can check that the travelling wave

$$v_c(t,x) := ce^{-|x-ct|}, \qquad t>0,\ x\in\mathbb{R},$$

is for any $c>0$ a global weak solution to (1.1) with initial data $u_0(x) = ce^{-|x|}$, $x\in\mathbb{R}$. Note that v_c has a corner at its peak and that the speed of the wave equals its amplitude.



Suprisingly, there are no travelling waves for (1.1) which are strong solutions. To see this, pick $w_0\in H^3$ and assume that $w(t,x) := w_0(t-cx)$ is a strong solution of (1.1). Then we get

$$cw_0' - cw_0''' - 3w_0w_0' + 2w_0'w_0'' + w_0w_0''' = 0 \text{ in } L_2.$$

We find that

$$\left(cw_0 - cw_0'' - \frac{3}{2}w_0^2 + w_0w_0'' + \frac{1}{2}(w_0')^2\right)' = 0 \text{ in } L_2,$$

and therefore

$$cw_0 - cw_0'' - \frac{3}{2}w_0^2 + w_0w_0'' + \frac{1}{2}(w_0')^2 = 0 \text{ in } H^1,$$

since $w_0\in H^3\subset C_0^2(\mathbb{R})$. Multiplying this identity with $2w_0'$, a further integration shows that

$$cw_0^2 - c(w_0')^2 - w_0^3 + w_0(w_0')^2 = (w_0-c)\big[(w_0')^2 - w_0^2\big] = 0.$$

Since w_0 belongs to H^3, this is impossible. □

6. – Soliton Interaction

In this section we use the framework of weak solutions to describe the soliton interaction for (1.1): we present below with some additions the findings of Camassa and Holm [9].

Motivated by the form of the solitary waves, let us make the following Ansatz for two interacting solitary waves:

$$u(t,x) = p_1(t)e^{-|x-q_1(t)|} + p_2(t)e^{-|x-q_2(t)|}, \quad t, x \in \mathbb{R}, \tag{6.1}$$

where $p_1, p_2, q_1, q_2 \in W^1_\infty(\mathbb{R})$.

Tedious computations (fix $x \in \mathbb{R}$ and split the spatial integration over $\mathbb{R}$ in $F(u)$ according to the order of magnitude of x, $q_1(t)$ and $q_2(t)$) show that the Ansatz (6.1) is a weak solution of (1.1) if and only if the variables $p_1(t), p_2(t), q_1(t)$ and $q_2(t)$ satisfy the following system of ordinary differential equations with discontinuous right-hand side:

$$\begin{cases} q_1' = p_1 + p_2 e^{-|q_1-q_2|}, \\ q_2' = p_1 e^{-|q_1-q_2|} + p_2, \\ p_1' = p_1 p_2 \ \mathrm{sign}(q_1 - q_2) \ e^{-|q_1-q_2|}, \\ p_2' = p_1 p_2 \ \mathrm{sign}(q_2 - q_1) \ e^{-|q_1-q_2|}. \end{cases} \tag{6.2}$$

Observe that (6.2) is a Hamiltonian system with Hamiltonian

$$\frac{1}{2} \sum_{i,j=1,2} p_i(t) p_j(t) e^{-|q_i(t)-q_j(t)|}.$$

It is useful to note that the system (6.2) admits also the conserved quantity $p_1(t) + p_2(t)$.

We assume that the two solitary waves are initially well-separated, with asymptotic speeds (and amplitudes) c_1 and respectively c_2 with $c_1 > c_2 > 0$, so that a collision between them occurs if the faster wave is to the left of the slower.

For the Ansatz (6.1) we therefore assume that

$$p_1(t) \to c_1, \ p_2(t) \to c_2, \ \big(q_1(t) - c_1 t\big) \to 0, \ \big(q_2(t) - c_2 t\big) \to 0 \text{ as } t \to -\infty.$$

If we change the variables in (6.2) to the new canonical variables

$$\begin{aligned} P &= p_1 + p_2, \quad Q = q_1 + q_2, \\ p &= p_1 - p_2, \quad q = q_1 - q_2, \end{aligned}$$

we obtain the equivalent system

$$\begin{cases} P' = 0, & Q' = P(1 + e^{-|q|}) \\ p' = \frac{1}{2}(P^2 - p^2)\mathrm{sign}(q) e^{-|q|}, & q' = p(1 - e^{-|q|}) \end{cases} \tag{6.3}$$

with the Hamiltonian

$$H = \frac{1}{2}P^2\big(1 + e^{-|q|}\big) + \frac{1}{2}p^2\big(1 - e^{-|q|}\big) = c_1^2 + c_2^2.$$

Since P is constant, we find $2H = (c^2 + p^2) + (c^2 - p^2)e^{-|q|}$, where $c := c_1 + c_2$, and therefore

$$e^{-|q|} = \frac{2H - c^2 - p^2}{c^2 - p^2}.$$

Assume that at some instant t the peaks overlap, i.e. we have $q(t) = 0$. Then we would get

$$c_1^2 + c_2^2 = H = c^2 = (c_1 + c_2)^2,$$

which is impossible since $c_1 > c_2 > 0$. Hence, q does not vanish and, since the faster wave starts to the left of the slower, we have $q < 0$. In particular we see that solutions to system (6.3) are unique and smooth.

The equation for p becomes $p' = \frac{1}{2}(p^2 - (c_1 - c_2)^2)$, and by integration we obtain

$$\ln\left|\frac{p(t) - L}{p(t) + L}\right| = Lt + k, \tag{6.4}$$

where $L = c_1 - c_2$ and $k = \ln\left|\frac{p(0)-L}{p(0)+L}\right|$.

Since

$$p' = \frac{1}{2}(c^2 - p^2)\left(-\frac{d}{dq}\, e^q\right)$$

we deduce

$$dp = \frac{1}{2}(c^2 - p^2)d\big(1 - e^q\big)\frac{dt}{dq}$$

and, in view of $dq = p(1 - e^q)dt$, we find

$$\frac{2pdp}{c^2 - p^2} = d\ln(1 - e^q)$$

and we obtain $|c^2 - p^2| = \frac{e^{-k_1}}{1-e^q} > 0$ for some constant $k_1 > 0$. Since $\lim_{t\to-\infty} p(t) = c_1 - c_2$ we conclude that always $c^2 > p^2(t)$ and therefore the differential equation $p' = -\frac{1}{2}(c^2 - p^2)e^q < 0$ shows that $p(t)$ decreases from the value L as t increases.

From (6.4) we find now

$$p(t) = L\,\frac{1 - \gamma e^{Lt}}{1 + \gamma e^{Lt}}, \quad t \in \mathbb{R}, \tag{6.5}$$

since $p^2(t) < c^2$ for all time; here $\gamma = e^k$. We obtain

$$q(t) = \ln\frac{L^2\gamma e^{Lt}}{\big(c_1\gamma e^{Lt} + c_2\big)\big(c_2\gamma e^{Lt} + c_1\big)}, \quad t \in \mathbb{R}.$$

From here we deduce the asymptotic behaviour

$$\lim_{t\to\infty}\big(q(t)+Lt\big)=\ln\frac{L^2}{c_1c_2}-\ln\gamma,\quad \lim_{t\to-\infty}\big(q(t)-Lt\big)=\ln\frac{L^2}{c_1c_2}+\ln\gamma,$$

and since $\lim_{t\to-\infty}(q(t)-Lt)=0$, we find $\gamma=(c_1c_2)/L^2$. Therefore

$$\lim_{t\to\infty}\big(q(t)+Lt\big)=2\ln\frac{L^2}{c_1c_2}. \tag{6.6}$$

Integrating the differential equation $Q'=c(1+e^q)$ (we have an exact formula for q), we find

$$Q(t)=Q(0)+ct+\ln\left(\frac{(c_1\gamma e^{Lt}+c_2)(c_2\gamma+c_1)}{(c_2\gamma e^{Lt}+c_1)(c_1\gamma+c_2)}\right)$$

and therefore

$$\lim_{t\to\infty}\big(Q(t)-ct\big)=Q(0)+\ln\left(\frac{c_1c_2\gamma+c_1^2}{c_1c_2\gamma+c_2^2}\right),$$
$$\lim_{t\to-\infty}\big(Q(t)-ct\big)=Q(0)+\ln\left(\frac{c_2^2\gamma+c_1c_2}{c_1^2\gamma+c_1c_2}\right).$$

Using the relation $\lim_{t\to-\infty}\big(Q(t)-ct\big)=0$, we can compute $Q(0)=\ln(\frac{c_1^2\gamma+c_1c_2}{c_2^2\gamma+c_1c_2})$, deducing that

$$\lim_{t\to\infty}\big(Q(t)-ct\big)=2\ln\left(\frac{c_1}{c_2}\right). \tag{6.7}$$

Combining (6.5)-(6.7) with the fact that P is constant, we conclude that as $t\to\infty$,

$$p_1(t)\to c_2,\ p_2(t)\to c_1,\ \text{and}\ \big(q_1(t)-c_2t\big)\to 2\ln\left(\frac{L}{c_2}\right),$$
$$\big(q_2(t)-c_1t\big)\to 2\ln\left(\frac{c_1}{L}\right).$$

To summarize,

$$u(t,x)\approx c_1e^{-|x-c_1t|}+c_2e^{-|x-c_2t|}\ \text{as}\ t\to-\infty,$$
$$u(t,x)\approx c_1e^{-|x-c_1t-2\ln\frac{c_1}{c_1-c_2}|}+c_2e^{-|x-c_2t-2\ln\frac{c_1-c_2}{c_2}|}\ \text{as}\ t\to\infty.$$

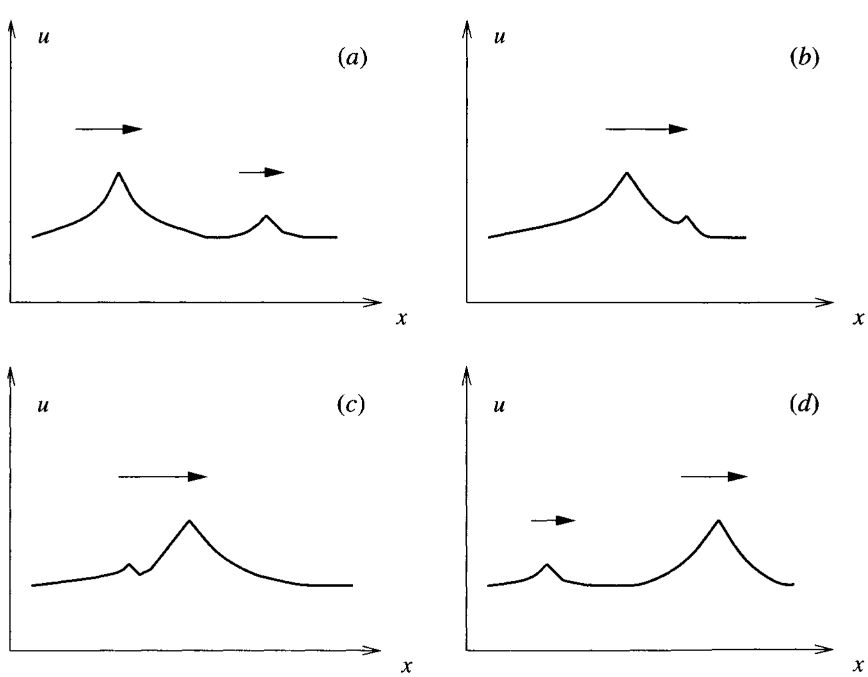
We see from these formulas that

i) if $c_1 > 2c_2$, both waves experience a forward shift;
ii) if $c_1 = 2c_2$, no shift occurs for the shorter while the taller is shifted forward;
iii) if $1 < c_1 < 2c_2$, the taller wave is shifted forward while the shorter is shifted backward.

These conclusions explain why the Ansatz (6.1) is called a **two-soliton solution** of equation (1.1). Namely, the solution is formed initially of two waves (which are almost solitary) with the taller one to the left of the shorter. The taller wave catches the shorter and they collide but no overlapping of the peaks occurs. After the collision the taller wave reappears to the right of the shorter one and moves away as time goes by. The interaction is a purely nonlinear process and it is the appearance of phase shifts which is the hallmark of this type of nonlinear interaction: the taller wave has moved forward, and the shorter backward, forward or not at all (depending on the ratio of the initial speeds), relative to the positions they would have reached if the interaction were linear.

The interaction phenomena described before is sketched below for times

$$(a)\ t = t_1;\ \ ;(b)\ t = t_2 > t_1;\ (c)\ t = t_3 > t_2;\ (d)\ t = t_4 > t_3.$$



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