

# Global existence and decay of small solutions for quasi-linear second-order uniformly dissipative hyperbolic-hyperbolic systems

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## Abstract

This paper is concerned with quasilinear systems of partial differential equations consisting of two hyperbolic operators interacting dissipatively. Its main theorem establishes global-in-time existence and asymptotic stability of strong solutions to the Cauchy problem close to homogeneous reference states. Notably, the operators are not required to be symmetric hyperbolic, instead merely the existence of symbolic symmetrizers is assumed. The dissipation is characterized by conditions equivalent to the uniform decay of all Fourier modes at the reference state. On a technical level, the theory developed herein uses para-differential operators as its main tool. Apparently being the first to apply such operators in the context of global-in-time existence for quasi-linear hyperbolic systems, the present work contains new results in the field of para-differential calculus. In the context of theoretical physics, the theorem applies to recent formulations for the relativistic dynamics of viscous, heat-conductive fluids notably such as that of Bemfica, Disconzi and Noronha [1] (*Phys. Rev. D*, 98:104064, 2018.).

**Keywords.** hyperbolic systems, initial value problem, global existence, asymptotic stability, para-differential operators, fluid mechanics

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# 1 Introduction and main result

In this paper, we study systems of partial differential equations that are given by the superposition of two hyperbolic operators and show that homogeneous states are nonlinearly stable in the sense that small perturbations thereof lead to global-in-time decaying solutions. Concretely, we consider the Cauchy problem for quasi-linear systems of the form

$$\sum_{j=0}^d A^j(u(t, x))u_{x_j}(t, x) = \sum_{j,k=0}^d (B^{jk}(u(t, x))u_{x_j}(t, x))_{x_k}, \quad x_0 = t \geq 0, \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d, \quad (1.1)$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^d, \quad (1.2)$$

where both the operator on the right hand side and the operator on the left hand side are hyperbolic and each of them acts dissipatively on the trajectories generated by the other one. Such systems occur in theoretical physics as recent formulations for the (special-)relativistic dynamics of viscous, heat conductive fluids [15, 16, 17, 1, 12, 2]. Our results apply to these formulations. The main theorem is the following.

**1.1 Theorem.** *Consider  $d \geq 3$ ,  $s > d/2 + 1$ ,  $\bar{u} \in \mathbb{R}^n$  and let (1.1) satisfy conditions  $(H_A)$ ,  $(H_B)$  and  $(D)$  from Section 3. Then there exist constants  $\delta > 0$  and  $C = C(\delta) > 0$  such that the following holds: For all  $u_0, u_1$  with  $u_0 - \bar{u} \in H^{s+1}(\mathbb{R}^d, \mathbb{R}^n) \cap L^1(\mathbb{R}^d, \mathbb{R}^n)$ ,  $u_1 \in H^s(\mathbb{R}^d, \mathbb{R}^n) \cap L^1(\mathbb{R}^d, \mathbb{R}^n)$  as well as  $\|u_0 - \bar{u}\|_{H^{s+1}}, \|u_1\|_{H^s}, \|u_0 - \bar{u}\|_{L^1}, \|u_1\|_{L^1} < \delta$  there exists a unique global solution  $u$  of (1.1), (1.2) satisfying  $u - \bar{u} \in C([0, \infty), H^{s+1}) \cap C^1([0, \infty), H^s)$ ,  $l = 0, \dots, s + 1$  and, for all  $t \in [0, \infty)$ ,*

$$\|u(t) - \bar{u}\|_{H^s} + \|u_t(t)\|_{H^{s-1}} \leq C(1+t)^{-\frac{d}{4}}(\|u_0 - \bar{u}\|_{H^s} + \|u_1\|_{H^{s-1}} + \|u_0 - \bar{u}\|_{L^1} + \|u_1\|_{L^1}), \quad (1.3)$$

$$\begin{aligned} \|u(t) - \bar{u}\|_{H^{s+1}}^2 + \|u_t(t)\|_{H^s}^2 + \int_0^t \|u(\tau) - \bar{u}\|_{H^{s+1}}^2 + \|u_t(\tau)\|_{H^s}^2 d\tau \\ \leq C(\|u_0 - \bar{u}\|_{H^{s+1}}^2 + \|u_1\|_{H^s}^2 + \|u_0 - \bar{u}\|_{L^1}^2 + \|u_1\|_{L^1}^2) \end{aligned} \quad (1.4)$$

While conditions  $(H_A)$  and  $(H_B)$  specify the assumed hyperbolicity, condition  $(D)$ , essentially obtained in [14], characterizes the needed decay behaviour for the Fourier modes of the associated linearized system.

Based on the famous Kawashima-Shizuta condition [24, 34], analogous results are well-known for symmetric hyperbolic-parabolic systems and first-order hyperbolic systems with relaxation, cf. [9, 33, 41, 19, 26, 42, 5] among others.<sup>1</sup> Regarding dissipative second-order hyperbolic systems there are fewer results available, cf. notably [11, 27, 32] and references therein, all of those treat systems whose structure is different from the one we consider here. The

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<sup>1</sup>Note that the often available reformulations of (1.1) as first-order hyperbolic systems do typically not satisfy the assumptions of these works.

most prominent example for equations satisfying condition (D) are probably damped wave equations with a non-linear convection term, which alternatively can be viewed as conservation laws with hyperbolic artificial viscosity. In this case, (D) reduces to Whitham's famous sub-characteristic condition [40] and various in-depth results on the asymptotic behaviour of solutions have been achieved in this context, cf. [31, 25, 39, 20, 10, 23]. Closest related to the present work are [35, 36, 37], there a predecessor of Theorem 1.1 was shown for the systems proposed in [16, 17].

The theory developed in the present work requires novel techniques in the use of para-differential operators. Developed by Bony [6] and Meyer [30, 29], such operators have been used in the context of hyperbolic equations by Gérard and Rauch [18], Taylor [38] and Métivier [28]. However, quite different from these works, we will in particular need to precisely understand how the norms of para-differential operators depending on the functions inducing their symbols.

The paper is organized as follows. In the crucial Section 2 general results on para-differential operators needed for the argumentation in Section 3 and 4 will be established. The present work apparently being the first that uses such operators to treat global-in-time solutions to quasi-linear hyperbolic systems, we prove corresponding new results on that dependence. The technical highlight in this regard will be a modified version of the strong Gårding inequality. In Section 3 we construct a para-differential operator which is specifically associated with the system's dissipativity. Section 4 is dedicated to the proof of Theorem 1.1. The challenging part is the treatment of the highest derivatives. Here we have to use the sophisticated estimates of Section 2 and the construction of Section 3. Finally, Section 5 shows that models of equations of dissipative relativistic fluid dynamics satisfy the assumptions of Theorem 1.1.

## 2 Results on para-differential operators

A tour through the theory of para-differential operators from scratch to fine properties, this section relies on Appendix C of Benzoni-Gavage and Serre [4] and Section 9 of Hörmander [22], however with strong attention to symbols induced by what later will be the solution to the PDE system considered. In its initial part interpolating between brevity and legibility, the section culminates in the aforementioned novel version of the strong Gårding inequality.

### 2.1 Notation, definitions and basics

For topological vector-spaces  $V, W$  we write  $\mathcal{L}(V, W)$  for the space of continuous linear operators from  $V$  to  $W$  (or  $\mathcal{L}(V)$  if  $W = V$ ). Throughout this section consider fixed dimensions  $n, d \in \mathbb{N}$  and let  $m$  denote some real number. For  $x, \xi \in \mathbb{R}^d$  we just write  $x\xi$  for their Euclidian scalar product.

Let  $E$  be a finite-dimensional  $\mathbb{C}$ -Banach space. We denote the  $E$ -valued Schwartz space by  $\mathcal{S}(\mathbb{R}^d, E)$ , and by  $\mathcal{S}'(\mathbb{R}^d, E) := \mathcal{L}(\mathcal{S}(\mathbb{R}^d), E)$  the space of continuous linear mappings from  $\mathcal{S}(\mathbb{R}^d)$  to  $E$ , i.e. the space of  $E$ -valued temperate distributions, both equipped with the standard locally convex topologies. For  $f \in \mathcal{S}(\mathbb{R}^d, E)$  the Fourier transform is

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx$$

with inverse

$$(\mathcal{F}^{-1}\hat{f})(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix\xi} d\xi.$$

We write  $\mathcal{F}_1$  and  $\mathcal{F}_2$  for the Fourier transform with respect to the first and the second variable for functions  $f \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d, E)$ , i.e.

$$\begin{aligned} (\mathcal{F}_1 f)(\eta, y) &= \mathcal{F}(f(\cdot, y))(\eta) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x, y) e^{-ix\eta} dx, \\ (\mathcal{F}_2 f)(x, \xi) &= \mathcal{F}(f(x, \cdot))(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x, y) e^{-iy\xi} dy. \end{aligned}$$

As usual we extend  $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$  to continuous operators on  $\mathcal{S}'(\mathbb{R}^d, E)$ ,  $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d, E)$  and unitary operators on  $L^2(\mathbb{R}^d, E)$ ,  $L^2(\mathbb{R}^d \times \mathbb{R}^d, E)$  also denoted by  $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ .

We will use  $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$ ,  $\xi \in \mathbb{R}^d$ ,  $\Lambda^m := \mathcal{F}^{-1}\langle \cdot \rangle^m \mathcal{F}$ . As usual

$$H^m(\mathbb{R}^d, E) := \{u \in L^2(\mathbb{R}^d, E) : \Lambda^m u \in L^2(\mathbb{R}^d, E)\},$$

are the  $L^2$ -based  $E$ -valued Sobolev spaces with norm

$$\|u\|_{H^m(\mathbb{R}^d, E)} := \|\Lambda^m u\|_{L^2(\mathbb{R}^d, E)}.$$

If  $E$  is a Hilbert space we consider the scalar product on  $H^m(\mathbb{R}^d, E)$

$$\langle u, v \rangle_{H^m(\mathbb{R}^d, E)} := \langle \Lambda^m u, \Lambda^m v \rangle_{L^2(\mathbb{R}^d, E)}.$$

We also use  $L^\infty$ -based Sobolev spaces

$$W^{k, \infty}(\mathbb{R}^d, E) := \{u \in L^\infty(\mathbb{R}^d, E) : \partial_x^\alpha u \in L^\infty(\mathbb{R}^d, E), |\alpha| \leq k\}$$

with norm

$$\|u\|_{W^{k, \infty}(\mathbb{R}^d, E)} = \max_{|\alpha| \leq k} \|\partial_x^\alpha u\|_{L^\infty(\mathbb{R}^d, E)}.$$

We often just write  $H^m$ ,  $\|u\|_m$ ,  $\langle u, v \rangle_m$ ,  $W^{k, \infty}$  instead of  $H^m(\mathbb{R}^d, E)$ ,  $\|u\|_{H^m(\mathbb{R}^d, E)}$ ,  $\langle u, v \rangle_{H^m(\mathbb{R}^d, E)}$ ,  $W^{k, \infty}(\mathbb{R}^d, E)$  if there is no concern for confusion, and  $\|u\|$  for  $\|u\|_0$ .

For  $A \in \mathbb{C}^{n \times n}$  we denote the adjoint matrix by  $A^* = \bar{A}^t$  and for  $T \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d, \mathbb{C}^n))$  we write  $T^*$  for the adjoint operator with respect to the  $L^2(\mathbb{R}^d, \mathbb{C}^n)$  inner product. As usual we call  $T$

self-adjoint if  $T = T^*$  and positive (strictly positive) if  $\langle Tf, f \rangle_0 \geq 0$  ( $\langle Tf, f \rangle > 0$ ), in which case we also write  $T \geq 0$  ( $T > 0$ ).

Next, we turn to the basic definitions concerning pseudo-differential operators which will be used in the present paper. We consider the following symbol classes.

**2.1 Definition.** (i)  $S^m := S^m(\mathbb{R}^d, \mathbb{C}^{n \times n})$  is the set of all functions  $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^{n \times n})$  for which for any  $\alpha, \beta \in \mathbb{N}_0$  there exists  $C_{\alpha\beta} > 0$  such that

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|}. \quad (2.1)$$

With semi-norms being the optimal constants in (2.1),  $S^m$  is a Fréchet space.

(ii)  $S_{1,1}^m := S_{1,1}^m(\mathbb{R}^d, \mathbb{C}^{n \times n})$  is the set of functions  $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$  for which for any  $\alpha, \beta \in \mathbb{N}_0^d$  there exist  $C_{\alpha\beta} > 0$  such that

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|+|\beta|} \quad (2.2)$$

for all  $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ . With semi-norms being the optimal constants in (2.2),  $S_{1,1}^m$  is a Fréchet space.

(iii) For  $a \in S_{1,1}^m$  the mapping  $\text{op}[a] \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d, \mathbb{C}^n))$  defined by

$$(\text{op}[a]f)(x) := (2\pi)^{-\frac{d}{2}} \int e^{ix\xi} a(x, \xi) \mathcal{F}f(\xi) d\xi. \quad (2.3)$$

is called the pseudo-differential operator with symbol  $a$ . We also write  $a := \text{Sym}[\text{op}[a]]$ .

As first shown in [7, 8] for  $a \in S_{1,1}^m$  the operator  $\text{op}[a]$  extends to a bounded operator from  $H^{l+m}$  to  $H^l$  only if  $\text{op}[a]^*$  also has a symbol in  $S_{1,1}^m$ . But the operator norm of  $\text{op}[a]$  can in general not be controlled by semi-norms of  $a$  uniformly over this subspace. As for our applications to dissipative hyperbolic systems it is essential that the norm of  $\text{op}[a]$  is small if the semi-norms of  $a$  are small we have to make sure that the symbols occurring in the present work belong to the following smaller subspaces.

**2.2 Definition.** For  $L \in (0, 1]$ ,  $S_{1,1}^{m,L}$  is the subspace of all  $a \in S_{1,1}^m$  such that  $\mathcal{F}_1 a$  vanishes on  $N_L := \{(\eta, \xi) \in \mathbb{R}^d \times \mathbb{R}^d : |\eta + \xi| + 1 < L|\xi|\}$  in the sense of distributions, i.e.

$$a(\mathcal{F}_1 \phi) = 0 \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d) \text{ with } \text{supp } \phi \subset N_L. \quad (2.4)$$

**2.3 Proposition.** Let  $L \in (0, 1]$ . For all  $l \in \mathbb{R}$  and  $a \in S_{1,1}^{m,L}$   $\text{op}[a]$  extends to a continuous operator from  $H^{l+m}$  to  $H^l$  and  $\text{op}$  is itself continuous from  $S_{1,1}^{m,L}$  to  $\mathcal{L}(H^{l+m}, H^l)$ .

*Proof.* Cf. [22], Proposition 9.3.1. □

The symbols in Sections 2 and 3 will be induced by functions  $(x, \xi) \mapsto F(u(x), \xi)$  where  $F \in C^\infty(\mathbb{R}^n \times \mathbb{R}^d)$ ,  $u \in W^{k,\infty}(\mathbb{R}^d, \mathbb{R}^n)$  for some  $k \in \mathbb{R}$ , i.e. they belong to the following symbol class.

**2.4 Definition.** For any  $k \in \mathbb{N}_0$  the set  $\Gamma_k^m$  of symbols of order  $m$  with regularity  $k$  is the set of functions  $A : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{C}^{n \times n}$  such that,

- (i) for almost all  $x \in \mathbb{R}^d$  the mapping  $\xi \mapsto A(x, \xi)$  is in  $C^\infty(\mathbb{R}^d, \mathbb{C}^{n \times n})$
- (ii) for any  $\alpha \in \mathbb{N}_0^d$  and  $\xi \in \mathbb{R}^d$  the mapping  $x \mapsto \partial_\xi^\alpha A(x, \xi)$  belongs to  $W^{k, \infty}(\mathbb{R}^d, \mathbb{C}^{n \times n})$  and there exists  $C_\alpha > 0$  not depending on  $\xi$  such that

$$\|\partial_\xi^\alpha A(\cdot, \xi)\|_{W^{k, \infty}} \leq C_\alpha \langle \xi \rangle^{m-|\alpha|}. \quad (2.5)$$

With the semi-norms being the optimal constants in (2.5),  $\Gamma_k^m$  is a Fréchet space.

Para-differential operators associated with symbols in  $\Gamma_k^m$  are defined as follows.

**2.5 Definition.** For  $\epsilon = (\epsilon_1, \epsilon_2)$  with  $0 < \epsilon_1 < \epsilon_2 < 1$  we call a function  $\chi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$  an admissible  $\epsilon$ -cut-off if  $\chi$  is even with respect to each variable,  $\chi(\mathbb{R}^d \times \mathbb{R}^d) \subset [0, 1]$ ,

$$\chi(\eta, \xi) = \begin{cases} 1, & |\eta| \leq \epsilon_1 |\xi| \text{ and } |\xi| \geq 1 \\ 0, & |\eta| \geq \epsilon_2 \langle \xi \rangle \text{ or } |\xi| \leq \epsilon_2 \end{cases} \quad (2.6)$$

for all  $\eta, \xi \in \mathbb{R}^d$  and for all  $\alpha, \beta \in \mathbb{N}^d$  there exists  $C_{\alpha, \beta} > 0$  such that for all  $\xi, \eta \in \mathbb{R}^d$

$$|\partial_\eta^\beta \partial_\xi^\alpha \chi(\eta, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{-|\alpha| - |\beta|}.$$

**2.6 Proposition.** Let  $\chi$  be an admissible  $\epsilon$ -cut-off. Set  $K^\chi := \mathcal{F}_1^{-1}(\chi)$  and consider the function  $R^\chi : \Gamma_k^m \rightarrow C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$  given by

$$R^\chi(A) := K^\chi *_1 A, \quad A \in \Gamma_k^m.$$

Then  $R^\chi$  defines a bounded linear operator from  $\Gamma_k^m$  to  $S_{1,1}^{m, 1-\epsilon_2} \cap \Gamma_k^m$ . Here

$$(K^\chi *_1 A)(x, \xi) = \int_{\mathbb{R}^d} K^\chi(x - y, \xi) A(y, \xi) dy.$$

*Proof.* Apart from the aspect that  $a$  is not only in  $S_{1,1}^m$  but even in  $S_{1,1}^{m, 1-\epsilon_2}$  the proof can be found in [4], Proposition C.16. But that aspect follows in a straightforward manner as  $|\eta + \xi| + 1 \leq (1 - \epsilon_2)|\xi|$  implies  $|\xi| - |\eta| + 1 \leq (1 - \epsilon_2)|\xi|$  and thus  $|\eta| \geq \epsilon_2 \langle \xi \rangle$  and  $\chi$  vanishes for such  $\eta, \xi$ .  $\square$

**2.7 Definition.** Let  $\chi$  be an admissible  $\epsilon$ -cut-off. For  $A \in \Gamma_k^m$  the  $(\chi)$ -para-differential operator with symbol  $A$  is defined by

$$\text{Op}_\chi[A] := \text{op}[R^\chi(A)].$$

As  $R^\chi \in \mathcal{L}(\Gamma_k^m, S_{1,1}^{m, 1-\epsilon_2})$ ,  $\text{Op}_\chi = \text{op} \circ R^\chi$  defines a continuous linear operator from  $\Gamma_k^m$  to  $\mathcal{L}(H^{l+m}, H^l)$  ( $l \in \mathbb{R}$ ). In particular the  $\mathcal{L}(H^{l+m}, H^l)$ -norm of  $\text{Op}_\chi[A]$  can be estimated by a constant depending on  $l, \chi$  and a finite sum of  $\Gamma_m^k$ -semi-norms of  $A$ .

The following shows that, regarding its dependence on  $\chi$ ,  $\text{op}_\chi[A]$  is determined by  $A$  up to a lower order operator, if  $k \geq 1$ .

**2.8 Lemma.** *Let  $\chi$  be an admissible  $\epsilon$ -cut-off and  $k \geq 1$ . Then the following holds:*

- (i) *The mapping  $R^\chi - \text{Id}$  is a continuous operator from  $\Gamma_k^m$  to  $\Gamma_{k-1}^{m-1}$ .*
- (ii) *If  $\tilde{\chi}$  is an admissible  $\tilde{\epsilon}$ -cut-off, then  $R^\chi - R^{\tilde{\chi}}$  is a continuous operator from  $\Gamma_k^m$  to  $S_{1,1}^{m-1,1-\tau} \cap \Gamma_{k-1}^{m-1}$  with  $\tau = \max\{\epsilon_2, \tilde{\epsilon}_2\}$ .*

*Proof.* Cf. [4], Proposition C.13, Corollary C.5. □

We end this subsection by stating two additional results on para-differential operators for later usage. The proofs are contained in [4], Appendix C. To simplify notation we fix an admissible  $\epsilon$ -cut-off  $\chi$  and suppress the dependence of  $R^\chi$  and  $\text{Op}_\chi$  on  $\chi$  in the following. We call an operator  $K$  infinitely smoothing if  $K \in \mathcal{L}(H^s, H^l)$  for all  $s, l \in \mathbb{R}$ .

**2.9 Lemma.** *Let  $b \in S^m$  be constant with respect to the first variable. Then the following holds:*

- (i)  *$\text{op}(b) - \text{Op}[b]$  is infinitely smoothing.*
- (ii)  *$\text{Op}[b] = \text{Op}[b^*]$*
- (iii)  *$\text{Op}[Ab] = \text{Op}[A]\mathcal{F}^{-1}b\mathcal{F}$  for any  $A \in \Gamma_k^\mu$ .*

**2.10 Lemma.** *For each  $k > 0$  there exists  $C > 0$  such that for all  $f \in L^\infty \cap H^k, A \in W^{1,\infty} \cap H^k$*

$$\|A - \text{Op}[A]f\|_k \leq C(\|A\|_{H^k}\|f\|_{L^\infty} + \|A\|_{W^{1,\infty}}\|f\|_{H^{k-1}}).$$

## 2.2 Adjoints and products

For the argumentation in Section 3 it will be essential to control the norms of operators  $\text{Op}[A^*] - \text{Op}[A]^*$ ,  $\text{Op}[BA] - \text{Op}[B]\text{Op}[A]$ ,  $A \in \Gamma_1^m, B \in \Gamma_1^\mu, \mu \in \mathbb{R}$ , in terms of the seminorms of  $A$  and  $B$ . While for  $a \in S_{1,1}^{m,L}, b \in S_{1,1}^{m,L}$  there exist symbols  $g \in S_{1,1}^m, h \in S_{1,1}^{m+\mu}$  such that  $\text{op}[a]^* = \text{op}[g]$ ,  $\text{op}[b]\text{op}[a] = \text{op}[h]$  and that, provided  $\partial_{x_j}a \in S_{1,1}^m (j = 1, \dots, d)$ ,  $\text{op}[a^*] - \text{op}[a]^* \in \mathcal{L}(H^{l+m-1}, H^l)$ ,  $\text{op}[b]\text{op}[a] - \text{op}[ba] \in \mathcal{L}(H^{l+m+\mu-1}, H^l)$ ,  $l \in \mathbb{R}$ , it is not true in general that  $g, h$  are again in some class  $S_{1,1}^{m,L}, S_{1,1}^{m+\mu,L}$  which would allow to control their operator norms. However, for our purposes it is sufficient to consider symbols of the particular form  $a = R(A), b = R(B)$  for  $A \in \Gamma_1^m, B \in \Gamma_1^\mu$  and we will show that in this case the symbols of  $\text{op}[a]^* = \text{Op}[A]^*, \text{op}[b]\text{op}[a] = \text{Op}[B]\text{Op}[A]$  are in fact again in  $S_{1,1}^{m,L}, S_{1,1}^{m+\mu,L}$  for some  $L \in (0, 1]$ .

As a first step note that for symbols in  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^{n \times n})$  there exist neat formulas for the symbols of adjoint and product operators, which also can be found in [22].

**2.11 Lemma.** *If  $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ , then  $\text{op}[a]^* = \text{op}[g]$  with  $\mathcal{F}_1 g(\eta, \xi) = (\mathcal{F}_1 a(-\eta, \eta + \xi))^*$ .*

**2.12 Lemma.** *If  $a, b \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^{n \times n})$ , then  $\text{op}[b] \text{op}[a] = \text{op}[h]$  with*

$$\mathcal{F}_1 h(\eta, \xi) = \int_{\mathbb{R}^d} \mathcal{F}_1 b(\eta - \theta + \xi, \theta) \mathcal{F}_1 a(\theta - \xi, \xi) d\theta.$$

The significance of this result lies in the following observation.

**2.13 Lemma.** *Let  $A \in \Gamma_0^m$ . Then there exists a sequence  $(a_\nu)_{\nu \geq 1} \subset \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^{n \times n})$  such that  $\text{op}[a_\nu]u \rightarrow \text{Op}[A]u$ ,  $\nu \rightarrow \infty$  in  $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^n)$  for all  $u \in \mathcal{S}(\mathbb{R}^d, \mathbb{C}^n)$ . Furthermore for all  $\delta \in (\epsilon_2, 1)$ ,  $\epsilon_2$  being the constant of the  $\epsilon$ -cut-off, there exists  $\nu_0 > 0$  such that  $\text{supp } \mathcal{F}_1 a_\nu \subset \{(\eta, \xi) \in \mathbb{R}^d \times \mathbb{R}^d : |\eta| \leq \delta \langle \xi \rangle\}$  for all  $\nu \geq \nu_0$ .*

*Proof.* The first part of the statement is shown as in [21], proof of Theorem 18.1.8. However, we have to slightly modify the construction to also obtain the second part. Set  $a := R(A)$ . Choose  $\hat{\phi} \in \mathcal{S}(\mathbb{R}^d)$  with  $\text{supp } \hat{\phi} \subset B_0(1)$ ,  $\mathcal{F}^{-1} \hat{\phi}(0) = 1$  and define  $\phi := \mathcal{F}^{-1} \hat{\phi}$ ,

$$a_\nu(x, \xi) := \phi(x/\nu) \phi(\xi/\nu) a(x, \xi), \quad x, \xi \in \mathbb{R}^d.$$

The asserted convergence then follows as *ibid*.

It remains to show the statement concerning the supports. Set  $\psi_\nu(x, \xi) := \phi(x/\nu) \phi(\xi/\nu)$  ( $\xi, \eta \in \mathbb{R}^d, \nu \geq 1$ ). As  $\mathcal{F}_1(a_\nu) = (2\pi)^{-d/2} (\mathcal{F}_1 \psi_\nu) *_{\mathbb{1}} (\mathcal{F}_1 a)$  and  $\mathcal{F}_1 a = \chi \mathcal{F}_1 A$ , it is sufficient to show that for given  $\delta \in (\epsilon_2, 1)$  and  $\nu$  sufficiently large  $\chi(\eta - \theta, \xi) \mathcal{F}_1 \psi_\nu(\theta, \xi) = 0$  for all  $\theta, \eta, \xi \in \mathbb{R}^d$   $|\eta| \geq \delta \langle \xi \rangle$ . Clearly

$$\mathcal{F}_1 \psi_\nu(\theta, \xi) = \nu^d \hat{\phi}(\theta \nu) \phi(\xi/\nu).$$

As by construction  $\hat{\phi}(\theta \nu) = 0$  for  $|\theta| \geq \nu^{-1}$  we can assume  $|\theta| \leq \nu^{-1}$ . Then  $|\eta| \geq \delta \langle \xi \rangle$  yields

$$|\eta - \theta| \geq |\eta| - |\theta| \geq \delta \langle \xi \rangle - \nu^{-1}.$$

Hence choosing  $\nu$  so large that  $\nu^{-1} \leq \delta - \epsilon_2$  gives (note  $\langle \xi \rangle \geq 1$ )

$$|\eta - \theta| \geq \delta \langle \xi \rangle - (\delta - \epsilon_2) \langle \xi \rangle = \epsilon_2 \langle \xi \rangle.$$

But this implies  $\chi(\eta - \theta, \xi) = 0$ , which finishes the proof.  $\square$

**2.14 Proposition.** *Let  $A \in \Gamma_0^m$ . Then there exists  $b = b(A) \in S_{1,1}^{m,1-\epsilon_2}$  such that  $\text{Op}[A]^* = \text{op}[b(A)]$ . Furthermore if  $A \in \Gamma_1$  the operator*

$$T : \Gamma_1^m \rightarrow S_{1,1}^{m-1,1-\epsilon_2}, \quad a \mapsto b(A) - R(A)^*$$

*is continuous. In particular the mapping*

$$A \mapsto \text{Op}[A^*] - \text{Op}[A]^* = \text{op}[R(A)^* - b(A)]$$

*is continuous from  $\Gamma_1^m$  to  $\mathcal{L}(H^{l+m-1}, H^l)$  for any  $l \in \mathbb{R}$ .*



*Proof.* Set  $a := R(A)$ . As  $A \in S_{1,1}^{m,1-\epsilon_2}$ , the existence of  $b := b(A) \in S_{1,1}^m$  with  $\text{op}[b] = \text{Op}[A]^*$  follows by [22], Lemma 9.4.1.

Next we prove that  $\mathcal{F}_1 b$  vanishes on  $\mathcal{N}_{1-\epsilon_2}$ . If  $A \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^{n \times n})$  also  $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^{n \times n})$  and Lemma 2.11 gives

$$\mathcal{F}_1 b(\eta, \xi) = (\mathcal{F}_1 a(-\eta, \eta + \xi))^*.$$

If now  $|\eta + \xi| + 1 \leq (1 - \epsilon_2)|\xi|$  then  $\epsilon_2|\xi| \leq |\eta|$  and thus

$$\epsilon_2 \langle \eta + \xi \rangle \leq \epsilon_2(1 + |\eta + \xi|) \leq (1 - \epsilon_2)\epsilon_2|\xi| \leq (1 - \epsilon_2)|\eta| \leq |\eta|,$$

which implies  $\mathcal{F}_1 a(-\eta, \eta + \xi) = (\chi \mathcal{F}_1 A)(-\eta, \eta + \xi) = 0$ .

For general  $A$  choose a sequence  $(a_\nu)_{\nu \geq 1} \subset \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^{n \times n})$  with  $\text{op}[a_\nu]u \rightarrow \text{Op}[A]u$  in  $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^n)$  for all  $u \in \mathcal{S}(\mathbb{R}^d, \mathbb{C}^n)$ . This implies  $\text{op}[a_\nu]^* \rightarrow \text{Op}[a]^* = \text{op}[b]$  in  $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^{n \times n})$  and it is straightforward to show that this yields  $b_\nu \rightarrow b \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^{n \times n})$ , where  $\mathcal{F}_1 b_\nu(\eta, \xi) = \mathcal{F}_1 a_\nu(-\eta, \eta + \xi)$ . By Lemma 2.13  $\mathcal{F}_1 a_\nu(\eta, \xi)$  vanishes for  $|\eta| \geq \delta \langle \xi \rangle$ , if  $\delta \in (\epsilon_2, 1)$ . As seen above this yields  $b_\nu \in S_{1,1}^{m,1-\delta}$ . In conclusion  $b = \lim_{\nu \rightarrow \infty} b_\nu \in S_{1,1}^{m,1-\delta}$  for all  $\delta > \epsilon_2$ , i.e.  $b \in S_{1,1}^{m,1-\epsilon_2}$ .

Lastly  $A \in \Gamma_1^m$  directly gives  $\partial_x^\delta A \in \Gamma_0^m$  and hence  $\partial_x^\delta R(A) = R(\partial_x^\delta A) \in S_{1,1}^m$  ( $|\delta| = 1$ ). By [22], Lemma 9.6.1 (applied to  $N = 1$ ,  $m_N = m - 1$ ) we now obtain  $b - R(A) \in S_{1,1}^{m-1}$  and its  $S_{1,1}^m$ -semi-norms are bounded by a constant times a sum of finitely many  $S_{1,1}^m$ -semi-norms of  $\partial_x^\delta R(A)$  ( $|\delta| = 1$ ). As also  $b - R(A) \in S^{m-1,1-\epsilon_2}$ , the assertion follows by the continuity of  $R$  and  $\text{op}$ .  $\square$

Concerning the analysis of product operators we first consider the difference  $R^\chi(AB) - R^\chi(A)R^\chi(B)$ .

**2.15 Lemma.** *For  $A \in \Gamma_1^m$ ,  $B \in \Gamma_1^\mu$  and an  $\epsilon$ -cut-off  $\chi$  with  $\epsilon_2 < 1/2$  we have  $R^\chi(B)R^\chi(A) \in S_{1,1}^{m+\mu,1-2\epsilon_2}$ . Furthermore the bilinear operator*

$$T : \Gamma_1^m \times \Gamma_1^m \rightarrow S_{1,1}^{m+\mu-1,1-2\epsilon_2}, \quad (A, B) \mapsto R^\chi(AB) - R^\chi(A)R^\chi(B)$$

*is continuous.*

*Proof.* We suppress the superscript  $\chi$  in the following. As  $R(A) \in S_{1,1}^m$ ,  $R(B) \in S_{1,1}^\mu$ , it is clear that  $R(B)R(A) \in S_{1,1}^{m+\mu}$ . Thus regarding the first assertion we need to show that  $R(B)R(A)$  vanishes on  $\mathcal{N}_{1-2\epsilon_2}$ . Since  $\mathcal{F}_1(R(B)R(A)) = (2\pi)^{-d/2} \mathcal{F}_1 R(B) *_1 \mathcal{F}_1 R(A)$  and  $\mathcal{F}_1 R(A) = \chi \mathcal{F}_1 A$ ,  $\mathcal{F}_1 R(B) = \chi \mathcal{F}_1 B$ , it is sufficient to prove that  $\chi(\eta - \theta, \xi) \chi(\theta, \xi)$  vanish for all  $\theta, \eta, \xi \in \mathbb{R}^d$  with  $|\eta + \xi| + 1 \leq (1 - 2\epsilon_2)|\xi|$ . Take such  $\theta, \eta, \xi$ . If  $\chi(\theta, \xi) \neq 0$  then  $|\theta| \leq \epsilon_2 \langle \xi \rangle$  and  $|\eta + \xi| + 1 \leq (1 - 2\epsilon_2)|\xi|$  implies  $|\eta| \geq 2\epsilon_2|\xi| + 1$ . Together this yields

$$|\eta - \theta| \geq |\eta| - |\theta| \geq 2\epsilon_2|\xi| + 1 - \epsilon_2 \langle \xi \rangle \geq \epsilon_2 \langle \xi \rangle.$$

Now  $\chi(\eta - \theta, \xi)$  vanishes for such  $\eta, \theta, \xi$ , which completes the argument.

In regard to the continuity we write

$$R(BA) - R(B)R(A) = R(BA) - BA + B(A - R(A)) - (R(B) - B)R(A).$$

Hence it follows from Lemma 2.8 (i) and the continuity of  $R$  that  $T$  is continuous as an operator to  $\Gamma_0^{m+\mu-1}$ . Thus the proof is finished if we show that each  $S_{1,1}^{m+\mu-1}$ -semi-norm can be bounded by a constant times a finite sum of  $\Gamma_0^{m+\mu-1}$ -semi-norms of  $T(A, B)$ . We show that even the following holds: For all  $\alpha, \beta \in \mathbb{N}_0$  there exists  $C_\beta > 0$  such that

$$|\partial_x^\beta \partial_\xi^\alpha T(A, B)(x, \xi)| \leq C_{\alpha\beta} |\partial_\xi^\alpha T(A, B)(x, \xi)| \langle \xi \rangle^{|\beta|}, \quad x, \xi \in \mathbb{R}^d. \quad (2.7)$$

By Bernstein's Lemma applied to  $\partial_\xi^\alpha T(a, b)(\cdot, \xi)$  (cf. e.g. [4], Lemma C.3) this can be deduced from the fact that for all  $\xi \in \mathbb{R}^d$

$$\text{supp}((\mathcal{F}T(A, B))(\cdot, \xi)) \subset B(0, 2\epsilon_2 \langle \xi \rangle).$$

In fact,

$$\text{supp}(\mathcal{F}(R(ba))(\cdot, \xi)) \subset \text{supp}(\chi(\cdot, \xi)) \subset B(0, 2\epsilon_2 \langle \xi \rangle)$$

holds by definition of  $\chi$  and that  $\mathcal{F}_1(R(b)R(a))$  vanishes for all  $\eta, \xi$  with  $|\eta| > 2\epsilon_2 \langle \xi \rangle$  follows by the same argumentation as in the first part of the proof.  $\square$

We can now prove our main proposition concerning products of para-differential operators.

**2.16 Proposition.** *Let  $A \in \Gamma_0^m$ ,  $B \in \Gamma_0^\mu$ . Then for  $L := (1 - \epsilon_2)^2$  there exists  $h(B, A) \in S_{1,1}^{\mu+m,L}$  such that  $\text{Op}[B] \text{Op}[A] = \text{op}[h(B, A)]$ . Furthermore the operator*

$$\Gamma_1^m \times \Gamma_1^\mu \rightarrow \mathcal{L}(H^{l+\mu+m-1}, H^l), \quad (B, A) \mapsto \text{Op}_\chi[B] \text{Op}[A] - \text{Op}[BA]$$

is continuous for all  $l \in \mathbb{R}$ .

*Proof.* The existence of a  $h = h(B, A) \in S_{1,1}^{m+\mu}$  such that

$$\text{op}[h(B, A)] = \text{op}[R(B)] \text{op}[R(A)] = \text{Op}[B] \text{Op}[A]$$

follows directly from [22], Lemma 9.5.1 as  $R(A) \in S_{1,1}^{m,1-\epsilon_2}$ . We now prove that  $h$  satisfies (2.4) for  $L = (1 - \epsilon_2)^2$ . First assume  $a := R(A), b := R(B) \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ . By Lemma 2.12

$$\mathcal{F}_1 h(\eta, \xi) = \int_{\mathbb{R}^d} \mathcal{F}_1 b(\eta - \theta + \xi, \theta) \mathcal{F}_1 a(\theta - \xi, \xi) d\theta. \quad (2.8)$$

Let  $\eta, \xi \in \mathbb{R}^d$  with  $|\eta + \xi| + 1 \leq (1 - \epsilon_2)^2 |\xi|$ . If  $\mathcal{F}_1 a(\theta - \xi, \xi) = \mathcal{F}_1 R(A)(\theta - \xi, \xi) \neq 0$  we have  $|\theta - \xi| \leq \epsilon_2 \langle \xi \rangle \leq \epsilon_2 + \epsilon_2 |\xi|$ , which gives  $(1 - \epsilon_2) |\xi| \leq |\theta| + \epsilon_2$ . We arrive at

$$\begin{aligned} |\eta + \xi - \theta| &\geq |\theta| - |\eta + \xi| \geq |\theta| - (1 - \epsilon_2)^2 |\xi| + 1 \geq |\theta| - (1 - \epsilon_2) |\theta| - (1 - \epsilon_2) \epsilon_2 + 1 \\ &= \epsilon_2 \theta + \epsilon_2 + (1 - \epsilon_2)^2 \geq \epsilon_2 \langle \theta \rangle. \end{aligned}$$

But this implies  $\mathcal{F}_1 b(\eta + \xi - \theta, \theta) = \mathcal{F}_1 R(B)(\eta + \xi - \theta, \theta) = 0$ , which finishes the argument.

For general  $A, B$  choose sequences  $(a_\nu)_{\nu \geq 1}, (b_\nu)_{\nu \geq 1} \subset \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$  with  $\text{op}[a_\nu]u \rightarrow \text{Op}[A]u$ ,  $\text{op}[b_\nu]u \rightarrow \text{Op}[B]u$  in  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^n)$  for all  $u \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^n)$  as constructed in Lemma 2.13. Then clearly

$$\text{op}[h_\nu]u = \text{op}[b_\nu] \text{op}[a_\nu]u \rightarrow \text{Op}[B] \text{Op}[A]u = \text{op}[h]u$$

in  $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^n)$ , where  $h_\nu$  is defined by (2.8) with  $a, b$  replaced by  $a_\nu, b_\nu$ . This implies  $h_\nu \rightarrow h$  in  $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C}^{n \times n})$ . As for all  $1 > \delta > \epsilon_2$   $\text{supp } \mathcal{F}_1 a_\nu, \text{supp } \mathcal{F}_1 b_\nu \subset \{(\eta, \xi) \in \mathbb{R}^d \times \mathbb{R}^d : |\eta| \leq \delta \langle \xi \rangle\}$  for  $\nu$  sufficiently large we get by the same reasoning as above that for all  $1 > \delta > \epsilon_2$   $h_\nu$  vanishes on  $\mathcal{N}_{(1-\delta)^2}$  for  $\nu$  sufficiently large, which proves that  $h$  vanishes on  $\mathcal{N}_{(1-\epsilon_2)^2}$ .

To prove the second assertion note that by Lemma 2.8 (ii), the mapping  $G \mapsto \text{Op}_\chi[G] - \text{Op}_{\tilde{\chi}}[G]$  is continuous from  $\Gamma_1^k$  to  $\mathcal{L}(H^{l+k-1}, H^l)$ ,  $k, l \in \mathbb{R}$ , for any admissible cut-offs  $\chi, \tilde{\chi}$ . Hence we can assume w.l.o.g  $\epsilon_2 < \frac{1}{2}$ . By Lemma 2.15 and the continuity of  $\text{op}$

$$(B, A) \mapsto \text{Op}[BA] - \text{op}[R(B)R(A)] = \text{op}[R(BA) - R(B)R(A)]$$

is also continuous as mapping from  $\Gamma_1^m \times \Gamma_1^m$  to  $\mathcal{L}(H^{l+\mu+m-1}, H^l)$ . What is left to show is the continuity of

$$(B, A) \mapsto \text{Op}[B] \text{Op}[A] - \text{op}[R(B)R(A)] = \text{op}[h(B, A) - R(B)R(A)].$$

As  $R(A) \in S_{1,1}^{m, 1-\epsilon_2}$  and

$$\partial_{x_j} R^{\tilde{\chi}}(A) = R(\partial_{x_j} A) \in S_{1,1}^m, \quad \partial_{x_j} R^{\tilde{\chi}}(B) = R(\partial_{x_j} B) \in S_{1,1}^m, \quad j = 1, \dots, d.$$

all semi-norms of  $h(B, A) - R(B)R(A)$  can be estimated by a constant times a finite sum of products of semi norms of  $\partial_{x_j} R(A), \partial_{x_k} R(B)$ . Thus as  $h(B, A) - R(B)R(A) \in S_{1,1}^{m-1, L}$  for  $l = \min\{1 - 2\epsilon_2, (1 - \epsilon_2)^2\}$  the assertion follows from the continuity of  $\text{op}$  and  $R$ .  $\square$

## 2.3 Estimates for operators with symbols induced by Sobolev functions

In Section 3 the results of Sections 2.1, 2.2 are applied to symbols of the form  $(x, \xi) \mapsto F(u(x), \xi)$ , where  $F \in C^\infty(\mathcal{U} \times \mathbb{R}^d, \mathbb{C}^{n \times n})$  ( $\mathcal{U} \subset \mathbb{R}^n$  some 0-neighbourhood) and  $u \in H^s(\mathbb{R}^d, \mathbb{R}^n)$  for  $s$  sufficiently large. For this purpose we prove the results below.

In the following let  $\mathcal{U} \subset \mathbb{R}^N$  be a 0-neighbourhood.

**2.17 Definition.** We denote by  $S^m(\mathcal{U}) := S^m(\mathcal{U}, \mathbb{C}^{n \times n})$  the set of all functions  $F \in C^\infty(\mathcal{U} \times \mathbb{R}^d, \mathbb{C}^{n \times n})$  for which for any  $\alpha, \beta \in \mathbb{N}_0^d$  there exists  $C_{\alpha\beta} > 0$  such that for all  $(u, \xi) \in \mathcal{U} \times \mathbb{R}^d$

$$|\partial_x^\beta \partial_\xi^\alpha F(u, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|}. \quad (2.9)$$

For functions  $F : \mathcal{U} \times \mathbb{R}^d \rightarrow \mathbb{C}^{n \times n}$  and  $u : \mathbb{R}^d \rightarrow \mathcal{U}$  we consider the composition

$$F_u : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}^{n \times n}, (x, \xi) \mapsto F(u(x), \xi).$$

**2.18 Lemma.** *Let  $F \in S^m(\mathcal{U})$  and  $u \in H^s$  with  $s > d/2$ . Then  $F_u \in \Gamma_k^m$  for  $k = [s - d/2]$  and for all  $\alpha \in \mathbb{N}_0^d$  and each  $\Gamma_k^m$ -semi-norm  $p_\alpha(F_u)$  it holds*

$$p_\alpha(F_u) \leq C_\alpha(\|u\|_s, F),$$

and if additionally  $F(0, \xi) = 0$ , then

$$p_\alpha(F_u) \leq \tilde{C}_\alpha(\|u\|_s, F)\|u\|_s,$$

where  $C_\alpha, \tilde{C}_\alpha$  depend on  $\alpha, F$  and continuously on  $\|u\|_s$ .

*Proof.* By Sobolev embedding  $H^s \hookrightarrow W^{k, \infty}$ . Thus we have  $F_u(\cdot, \xi) \in W^{k, \infty}$  and

$$\|\partial_\xi^\alpha F_u(\cdot, \xi)\|_{W^{k, \infty}} \leq C(\|u\|_{W^{k, \infty}})\|\partial_\xi^\alpha F(\cdot, \xi)\|_{W^{k, \infty}(\mathcal{U})} \leq C(\|u\|_s)C_\alpha(F)\langle \xi \rangle^{m-|\alpha|}.$$

all  $\xi \in \mathbb{R}^d$ . If  $F(0, \xi) = 0$ , we even get, for all  $\xi \in \mathbb{R}^d$ ,

$$\|\partial_\xi^\alpha F_u(\cdot, \xi)\|_{W^{k, \infty}} \leq C(\|u\|_{W^{k, \infty}})\|u\|_{W^{k, \infty}}\|\partial_\xi^\alpha F_u(\cdot, \xi)\|_{W^{k, \infty}(\mathcal{U})} \leq C(\|u\|_s, F)\|u\|_s\langle \xi \rangle^{m-|\alpha|}.$$

□

The following proposition will be central for the energy estimates in Section 3. It follows directly by the continuity of  $\text{Op} : \Gamma_k^m \rightarrow \mathcal{L}(H^{l+m}, H^l)$  and Lemma 2.18 as well as Propositions 2.14, 2.16 and the facts that  $\text{Op}[F_0]^* = \text{op}[F_0^*]$  and  $\text{op}[G_0 F_0] - \text{op}[G_0]\text{op}[F_0]$  is infinitely smoothing by Lemma 2.9.

**2.19 Proposition.** *Let  $F \in S^m(\mathcal{U})$ ,  $l \in \mathbb{R}$ . Then for all  $u \in H^s$  with  $s > d/2$  there exists  $C_l = C_l(F, \|u\|) > 0$  depending on  $l, F$  and monotonically increasing on  $\|u\|_s$  such that:*

$$(i) \quad \|\text{Op}[F_u]\|_{\mathcal{L}(H^{l+m}, H^l)} \leq C_l(\|u\|_s) \text{ and for } F(0, \cdot) = 0 \quad \|\text{Op}[F_u]\|_{\mathcal{L}(H^{l+m}, H^l)} \leq C_l\|u\|_s,$$

$$(ii) \quad \text{for } s > d/2 + 1, \quad \text{Op}[F_u]^* - \text{Op}[F_u^*] \in \mathcal{L}(H^{l-1+m}, H^m) \text{ and}$$

$$\|\text{Op}[F_u]^* - \text{Op}[F_u^*]\|_{\mathcal{L}(H^{l-1+m}, H^m)} \leq C_l\|u\|_s$$

(iii) *for  $G \in S^\mu(\mathcal{U})$  and  $s > d/2 + 1$  there exist  $C_{l,2} = C_{l,2}(G, \|u\|_s)$  depending on  $G$  and monotonically increasing on  $\|u\|_s$  such that*

$$\|\text{Op}[G_u]\text{Op}[F_u] - \text{Op}[G_u F_u]\|_{\mathcal{L}(H^{l+\mu-1+m}, H^m)} \leq C_{l,2}C_l\|u\|_s$$

*up to an infinitely smoothing operator, which is determined by  $F(0, \cdot), G(0, \cdot)$ .*

**2.20 Proposition.** Let  $F \in S^m(\mathcal{U})$  and  $u \in C^1([0, T], H^s)$  ( $T > 0$ ) for  $s > d/2$ . Then for each  $l \in \mathbb{R}$  the mapping

$$[0, T] \rightarrow \mathcal{L}(H^{l+m}, H^l), \quad t \mapsto \text{Op}[F_{u(t)}]$$

is continuously differentiable and there exists  $C_l$  depending on  $l$  and  $F$  but not on  $u$  such that for all  $t \in [0, T]$

$$\left\| \frac{d}{dt} \text{Op}[F_{u(t)}] \right\|_{\mathcal{L}(H^{l+m}, H^l)} \leq C_l \|\partial_t u(t)\|_{s_0} \quad (2.10)$$

*Proof.* If  $\frac{d}{dt} F_{u(t)} \in \Gamma_0^m$  we get by continuity and linearity of Op

$$\frac{d}{dt} \text{Op}[F_{u(t)}] = \text{Op}[\partial_t F_{u(t)}]$$

To prove this and (2.10) it is sufficient to show that for any  $\alpha \in \mathbb{N}_0^d$  there exists  $C_\alpha = C_\alpha(F)$  such that for all  $\xi \in \mathbb{R}^d$

$$\|\partial_\xi^\alpha \partial_t F_{u(t)}(\cdot, \xi)\|_{L^\infty} \leq C_\alpha \|\partial_t u(t)\|_s \langle \xi \rangle^{m-|\alpha|}.$$

Let  $\alpha \in \mathbb{N}_0^d$  and set  $F_{u(t)}^\alpha := \partial_\xi^\alpha F_{u(t)}$ . We have for all  $x, \xi \in \mathbb{R}^d$

$$\partial_t F_{u(t)}^\alpha(x, \xi) = \sum_{j=1}^n \partial_t u^j \partial_{u^j} F^\alpha(u(t, x), \xi)$$

Due to  $F \in S^m(\mathcal{U})$  this yields

$$\|\partial_t F_{u(t)}^\alpha(x, \xi)\|_{L^\infty} \leq \|\partial_t u(t)\|_{L^\infty} \sum_{|\beta|=1} \|\partial_u^\beta F^\alpha(\cdot, \xi)\|_{L^\infty} \leq C_\alpha(F) \|\partial_t u\|_s \langle \xi \rangle^{m-|\alpha|}.$$

□

Lastly we prove a version of the strict Gårding inequality for  $F \in S^m(\mathcal{U})$ . First consider the following lemma which is a modification of a construction in [21], proof of Thm. 18.1.6.

**2.21 Lemma.** *There exists an even function  $\psi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$  with unit integral,  $\text{Op}[\psi] = \text{Op}[\psi]^*$ ,  $\langle \text{op}[\psi]v, v \rangle \geq 0$  ( $v \in \mathcal{S}(\mathbb{R}^d)$ ) and  $\mathcal{F}_1 \psi$  compactly supported.*

*Proof.* Choose an even function  $\hat{\phi} \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$  with  $L^2$ -norm one and set  $\phi = \mathcal{F}_1^{-1} \hat{\phi}$ . By definition  $\mathcal{F}_1 \phi$  is compactly supported and clearly  $\phi$  is even and has  $L^2$ -norm one. Next, let  $\psi \in \mathcal{S}(\mathbb{R}^d)$  be the symbol of  $\text{op}[\psi]^* \text{op}[\psi]$ . As ibid. it follows that  $\psi$  is even and has unit integral.  $\text{op}[\psi] = \text{op}[\psi]^*$ ,  $\langle \text{op}[\psi]v, v \rangle_{L^2} \geq 0$  ( $u \in \mathcal{S}(\mathbb{R}^d)$ ) holds by definition. Now, let  $\rho$  be the symbol of  $\text{op}[\phi]^*$ . By Lemma 2.11 we get

$$\mathcal{F}_1 \rho(\eta, \xi) = (\mathcal{F}_1 \phi)^*(-\eta, \eta + \xi), \quad \eta, \xi \in \mathbb{R}^d$$

and thus by Lemma 2.12

$$\mathcal{F}_1\psi(\eta, \xi) = \int_{\mathbb{R}^d} \mathcal{F}_1\rho(\eta - \theta, \theta + \xi)\mathcal{F}_1\phi(\theta, \xi)d\theta = \int_{\mathbb{R}^d} \mathcal{F}_1\phi(\theta - \eta, \eta + \xi)\mathcal{F}_1\phi(\theta, \xi)d\theta.$$

As  $\mathcal{F}_1\phi$  is compactly supported, we can choose  $C > 0$  such that  $\mathcal{F}_1\phi(\theta, \xi) = 0$  if  $|\theta| \geq C$  or  $|\xi| \geq C$ . Then by definition  $\mathcal{F}_1\psi(\eta, \xi) = 0$  if  $|\xi| \geq C$ . Given  $|\eta| \geq 2C$  and  $|\theta| \leq C$  we conclude  $|\theta - \eta| \geq |\eta| - |\theta| \geq C$ , i.e.  $\mathcal{F}_1\phi(\theta - \eta, \eta + \xi) = 0$ . In conclusion we have proven that  $\mathcal{F}_1\psi$  is in fact compactly supported. In particular  $\psi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ .  $\square$

Next we introduce a method to decompose symbols in  $S_{1,1}^m$  into an infinite sum of infinitely smoothing symbols; cf. [22].

First, choose a function  $\rho \in \mathcal{D}(\mathbb{R}^d)$  even and monotonically decaying along rays such that  $\rho(\mathbb{R}^d) \subset [0, 1]$  and

$$\rho(\xi) = \begin{cases} 1, & |\xi| \leq \frac{1}{2} \\ 0, & |\xi| \geq 1 \end{cases}.$$

For  $\nu \in \mathbb{N}_0$  define  $\rho_\nu, \zeta_\nu \in \mathcal{D}(\mathbb{R}^d)$  by

$$\rho_\nu(\xi) := \rho(\xi/2^\nu), \quad \zeta_\nu(\xi) = \rho_{\nu+1}(\xi) - \rho_\nu(\xi), \quad \xi \in \mathbb{R}^d$$

Additionally set  $\zeta_{-1} := \rho$ .

**2.22 Definition.** For a function  $a : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}^{n \times n}$  and  $\nu \geq -1$  define

$$a_\nu(x, \xi) := a(x, \xi)\zeta_\nu(\xi).$$

Note that  $a = \sum_{\nu \geq -1} a_\nu$ .

It is straightforward to show the following.

**2.23 Lemma.** Let  $a \in S_{1,1}^m$ . Then  $a_\nu \in S^{-r}$  for all  $r \in \mathbb{R}$  and for any  $\alpha, \beta \in \mathbb{N}_0$   $x, \xi \in \mathbb{R}^d$

$$|\partial_x^\beta \partial_\xi^\alpha a_\nu(x, \xi)| \langle \xi \rangle^r \leq C 2^{\nu(r+m-|\alpha|+|\beta|)} \sum_{\gamma \leq \alpha} C_{\gamma\beta}(a),$$

where  $C_{\gamma\beta}(a)$  are semi-norms of  $a$ .

**2.24 Proposition.** Let  $s > d/2$ ,  $u \in H^{s+2}$  and  $F \in S^m(\mathcal{U})$  such that there exists an  $R > 0$  with  $F(y, \xi) + F(y, \xi)^* \geq 0$  for all  $y \in \mathcal{U}$  and  $\xi \in \mathbb{R}^d$  with  $|\xi| > R$ . Then there exists  $C = C(\|u\|_{s+2}, F) > 0$  and for all  $q \in \mathbb{R}$  there exists  $c = c(\|u\|_{s+2}, F, q) > 0$ , both increasing functions of  $\|u\|_{s+2}$ , such that for all  $v \in \mathcal{S}(\mathbb{R}^d, \mathbb{C}^n)$

$$\langle (\text{Op}[F_u] + \text{Op}[F_u]^*)v, v \rangle_{L^2} \geq -C\|u\|_{s+2}^{\frac{1}{2}}\|v\|_{(m-1)/2}^2 - c\|v\|_{-q}^2.$$

*Proof.* In the following it is straightforward to see that all constants can be chosen to be increasing functions of  $\|u\|_{s+2}$ . First note that by Proposition 2.19 for all  $l \in \mathbb{R}$

$$\|\text{Op}[F_u] + \text{Op}[F_u]^* - \text{Op}[F_u + F_u^*]\|_{\mathcal{L}(H^{l+m-1}, H^l)} \leq C_l \|u\|_{s+1}.$$

Thus

$$\langle (\text{Op}[F_u] + \text{Op}[F_u]^*)v, v \rangle_{L^2} \geq \langle \text{Op}[F_u + F_u^*]v, v \rangle_{L^2} - C \|u\|_{s+1} \|v\|_{(m-1)/2}^2, \quad v \in \mathcal{S}(\mathbb{R}^d).$$

Hence it is sufficient to prove the result for  $\text{Op}[F_u] + \text{Op}[F_u]^*$  replaced by  $\text{Op}[F_u + F_u^*]$ , i.e. we can assume w.l.o.g  $F(u, \xi) = F(u, \xi)^* \geq 0$ .

It holds  $R(F_u) = R(F_u^*) = R(F_u)^*$ . By assumption this gives pointwise in  $\mathbb{R}^d \times \{|\xi| \geq R\}$  for all  $v \in \mathbb{C}^n$

$$\begin{aligned} \langle (R(F_u))v, v \rangle_{\mathbb{C}^n} &\geq \langle (R(F_u) - F_u)v, v \rangle_{\mathbb{C}^n} \geq -|R(F_u) - F_u| |v|^2 \\ &\geq -(|R(F_u - F_0) - (F_u - F_0)| + |R(F_0) - F_0|) |v|^2. \end{aligned}$$

By Lemma 2.18  $F_u - F_0 \in \Gamma_2^m$  with all semi-norms bounded by a positive constant depending on  $F$  times  $\|u\|_{s+2}$ . By Lemma 2.8 (i) this yields  $R(F_u - F_0) - (F_u - F_0) \in \Gamma_1^{m-1}$  with semi-norms bounded in the same way. Thus

$$|R(F_u - F_0) - (F_u - F_0)| \leq C_0 \|u\|_{s+2} \langle \xi \rangle^{m-1}.$$

Using also that  $R(F_0) - F_0$  has compact support we conclude that for all  $q \in \mathbb{R}$

$$|R(F_u - F_0) - (F_u - F_0)| + |R(F_0) - F_0| \leq C_0 \|u\|_{s+2} \langle \xi \rangle^{m-1} + c_q^0 \langle \xi \rangle^{-q}.$$

Therefore on  $\mathbb{R}^d \times \{|\xi| \geq R\}$

$$a := R(F_u) + C_0 \|u\|_{s+2} \langle \xi \rangle^{m-1} + c_0 \langle \xi \rangle^{-r} \geq 0$$

and  $a = a^*$ ,  $a \in S_{1,1}^{m,1-\epsilon_2}$ . As

$$\text{Op}[F_u] = \text{op}[R(F_u)] = \text{op}[a] - C_0 \|u\|_{s+2} \text{op}[\langle \xi \rangle^{m-1}] - c_0 \text{op}[\langle \xi \rangle^{-r}]$$

it is now sufficient to show

$$\langle \text{op}[a]v, v \rangle_{L^2} \geq -C \|u\|_{s+2}^{1/2} \|v\|_{(m-1)/2}^2 - c \|v\|_{-q}$$

for all  $q \in \mathbb{R}$ .

To this end we proceed similarly as in the proof of Theorem 9.7.1 in [22] but with a crucial modification. First, decompose  $a = \sum_{\nu \geq -1} a_\nu$  according to Definition 2.22. As for all  $\nu_0$   $\bar{a}_{\nu_0} := \sum_{\nu=-1}^{\nu_0} a_\nu \in S^{-q}$  for any  $q \in \mathbb{R}$  with norm depending on  $\mu, \nu_0$  according to Lemma 2.23, i.e.  $\|\text{op}[\bar{a}_{\nu_0}]v\| \leq c_{\nu_0, \mu} \|v\|_{-r}$ , we only need to consider  $\sum_{\nu \geq \nu_0} a_\nu$  for some  $\nu_0 \in \mathbb{N}$ . Naturally, in a first step we choose  $\nu_0$  large enough to obtain  $2^{\nu_0-2} > R$  and thus by assumption

$a_\nu(x, \xi) \geq 0$  for all  $x, \xi \in \mathbb{R}^d$ ,  $\nu \geq \nu_0$ . But we will later see that we may have to choose  $\nu_0$  even larger.

W.l.o.g. assume  $u \neq 0$ . Otherwise the result readily follows as  $F_0 \geq 0$  is constant with respect to  $x$  and  $\text{Op}[F_0] - \text{op}[F_0]$  is infinitely smoothing.

Choose an even function  $\psi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$  with unit integral such that  $\text{op}[\psi] = \text{op}[\psi]^*$ ,  $\langle \text{op}[\psi]v, v \rangle \geq 0$  ( $v \in \mathcal{S}(\mathbb{R}^d)$ ) and  $\mathcal{F}_1\psi$  compactly supported as constructed in Lemma 2.21. For  $\nu \in \mathbb{N}_0$  set  $q_\nu := 2^{\nu/2}$  and write  $a_\nu = b_\nu + h_\nu$  with

$$b_\nu(x, \xi) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi((x-y)q_\nu\mu, (\xi-\theta)/(q_\nu\mu)) a_\nu(y, \theta) dy d\theta \quad (2.11)$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(y, \theta) a_\nu(x-y/(q_\nu\mu), \xi-\theta q_\nu\mu) dy d\theta, \quad (2.12)$$

where  $\mu := \|u\|_{s+2}$ . As  $a_\nu \geq 0$  and  $\text{op}[\psi]$  is a positive operator it is straightforward to obtain the positivity of  $b_\nu$ . Hence the theorem is proven provided

$$\langle \text{op}[h]v, v \rangle_{L^2} \geq -C_\mu \|u\|_{k+1}^{\frac{1}{2}} \|v\|_{(m-1)/2}, \quad v \in \mathcal{S}(\mathbb{R}^d). \quad (2.13)$$

To this end we show  $h \in S_{1,1}^{m-1,L}$  for some  $L \in (0, 1)$  and that all semi-norms of  $h$  are bounded by a constant times  $\|u\|_{s+2}^{\frac{1}{2}}$ . Then (2.13) follows from Proposition 2.3.

First we verify  $h \in S_{1,1}^{m-1}$  and the estimate on the semi-norms, i.e.

$$\left| \sum_{\nu \geq \nu_0} \partial_x^\beta \partial_\xi^\alpha h_\nu(x, \xi) \right| \leq C_{\alpha\beta} \|u\|_{s+2}^{\frac{1}{2}} \langle \xi \rangle^{m-1-|\alpha|+|\beta|}. \quad (2.14)$$

Let  $\alpha = \beta = 0$ . Fix  $\xi \in \mathbb{R}^d$  and consider  $\nu \in \mathbb{N}_0$  with  $|\xi| < 2^{\nu-2}$  or  $|\xi| > 2^{\nu+2}$ . As  $a_\nu(y, \theta) = 0$  for  $2^{\nu-1} \leq |\theta| \leq 2^{\nu+1}$  we then have  $h_\nu(x, \xi) = -b_\nu(x, \xi)$  and it follows by basic estimates (cf. [22]) that in the support of the first integrand in (2.11)

$$|\xi - \theta| \geq \frac{1}{5}(2^\nu + |\xi|)$$

and thus

$$|\xi - \theta|/q_\nu = 2^{-\nu/2} |\xi - \theta| \geq \frac{1}{5}(2^\nu + |\xi|)^{\frac{1}{2}}. \quad (2.15)$$

As  $a \in S_{1,1}^m$  and  $\text{supp } a_\nu \subset \{(x, \theta) \in \mathbb{R}^d \times \mathbb{R}^d : 2^{\nu-1} \leq |\theta| \leq 2^\nu\}$

$$|a_\nu(y, \theta)| \leq C \langle \theta \rangle^m \leq C(1 + 2^\nu)^m.$$

Hence  $\psi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$  and (2.15) yield

$$\begin{aligned} |h_\nu| &\leq C_m (1 + 2^\nu)^m \\ &\int \int (|\xi - \theta|/(q_\nu\mu))^{-2(|m|+1)} (1 + |\xi - \theta|/(q_\nu\mu))^{-n-1} (1 + |(x-y)q_\nu\mu|)^{-n-1} dy d\theta \\ &\leq C_{m,n} \mu^{2(|m|+1)} (1 + 2^\nu)^m (2^\nu + |\xi|)^{-2|m|-2} \\ &\leq C_{m,n} \mu^{2(|m|+1)} (1 + |\xi|)^{m-1} 2^{-\nu}. \end{aligned} \quad (2.16)$$



Thus

$$\sum_{\{\nu:|\xi|<2^{\nu-2} \text{ or } |\xi|>2^{\nu+2}\}} |h_\nu| \leq C \|u\|_{s+2}^{\frac{1}{2}} \langle \xi \rangle^{m-1}. \quad (2.17)$$

Now consider  $\nu \in \mathbb{N}_0$  with  $2^{\nu-2} \leq |\xi| \leq 2^{\nu+2}$ . As  $\psi$  is an even function with unit integral we get from (2.12)

$$\begin{aligned} h_\nu &= a_\nu - b_\nu = \int \int \psi(y, \theta) (a_\nu(x, \xi) - a_\nu(x - y/(q_\nu \mu), \xi - \theta q_\nu \mu)) dy d\theta \\ &= \int \int \psi(y, \theta) \left( \sum_{|\alpha+\beta|<2} \partial_x^\beta \partial_\xi^\alpha a_\nu(x, \xi) (-y)^\beta (-\theta)^\alpha - a_\nu(x - y/(q_\nu \mu), \xi - \theta q_\nu \mu) \right) dy d\theta. \end{aligned} \quad (2.18)$$

By Taylor's fomula we can estimate (w.l.o.g. assume  $|\theta| \leq |\xi|$ )

$$\begin{aligned} & \left| \sum_{|\alpha+\beta|<2} \partial_x^\beta \partial_\xi^\alpha a_\nu(x, \xi) (-y)^\beta (-\theta)^\alpha - a_\nu(x - y/(q_\nu \mu), \xi - \theta q_\nu \mu) \right| \\ & \leq C \sum_{|\alpha+\beta|=2} \sup_{x, \xi \in \mathbb{R}^d} |\partial_x^\beta \partial_\xi^\alpha a_\nu(x, \xi)| |y^\beta \theta^\alpha| (q_\nu \mu)^{|\alpha|-|\beta|}. \end{aligned} \quad (2.19)$$

Note that  $a = R(F_u)$ . By Lemma 2.18  $F_u \in \Gamma_2^m$  and thus  $\partial_x^\beta F_u \in \Gamma_{2-|\beta|}^m$  for  $|\beta| \leq 2$ . Hence for each  $\gamma \in \mathbb{N}_0^d$ ,  $\xi \in \mathbb{R}^d$

$$\|\partial_\xi^\gamma \partial_x^\beta F_u(\cdot, \xi)\|_{W^{2-|\beta|, \infty}} \leq C_\gamma \langle \xi \rangle^{m-|\gamma|}$$

and for  $|\beta| \geq 1$  we also have  $\partial_x^\beta F_u|_{u=0} = 0$ . Thus again by Lemma 2.18

$$\|\partial_\xi^\gamma \partial_x^\beta F_u(\cdot, \xi)\|_{W^{2-|\beta|, \infty}} \leq C_\gamma \|u\|_{s+2} \langle \xi \rangle^{m-|\gamma|}.$$

Clearly

$$\partial_x^\beta a = \partial_x^\beta R(F_u) = R(\partial_x^\beta F_u).$$

and we conclude from Proposition 2.6 that  $\partial_x^\beta a \in S_{1,1}^m$  and for all  $x, \xi \in \mathbb{R}^d$

$$|\partial_x^\beta \partial_\xi^\gamma a(x, \xi)| \leq C_\gamma \langle \xi \rangle^{m-|\gamma|} \begin{cases} 1, & |\beta| = 0, \\ \|u\|_{s+2}, & 1 \leq |\beta| \leq 2. \end{cases}$$

Then by Lemma 2.23

$$\sup_{x, \xi \in \mathbb{R}^d} |\partial_x^\beta \partial_\xi^\gamma a_\nu| \leq C_\gamma 2^{\nu(m-|\alpha|)} \leq C_\gamma \begin{cases} 1, & |\beta| = 0, \\ \|u\|_{s+2}, & 1 \leq |\beta| \leq 2 \end{cases} \quad (2.20)$$

From (2.18), (2.19), (2.20) and  $\mu = \|u\|_{s+2}^{\frac{1}{4}}$ ,  $q_\nu = 2^{\nu/2}$  we now get for  $2^{\nu-2} \leq |\xi| \leq 2^{\nu+2}$

$$\begin{aligned} |h_\nu| &\leq C \int \psi(y, \theta) (|\theta|^2 + |\theta||y| + |y|^2) dy d\theta \\ &\quad (2^{\nu(m-2)} 2^\nu \|u\|_{s+2}^{\frac{1}{2}} + 2^{\nu(m-1)} \|u\|_{s+2} + 2^{\nu m} \|u\|_{s+2} 2^{-\nu} \|u\|_{s+2}^{-\frac{1}{2}}) \\ &\leq C_\mu 2^{\nu(m-1)} \|u\|_{s+2}^{\frac{1}{2}} \leq C \|u\|_{s+2}^{\frac{1}{2}} \langle \xi \rangle^{m-1}, \end{aligned}$$

where we used  $\psi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$  and  $2^{\nu-2} \leq |\xi| \leq 2^{\nu+2}$  in the last line. Together with (2.17) this shows (2.14) for  $\alpha = \beta = 0$ .

Now note that  $\partial_x^\beta \partial_\xi^\alpha b_\nu$  is given by (2.12) with  $a_\nu$  replaced by  $\partial_x^\beta \partial_\xi^\alpha a_\nu$ . Hence we obtain (2.14) for  $\alpha, \beta \neq 0$  by applying the argumentation above with  $a_\nu$  replaced by  $\partial_x^\beta \partial_\xi^\alpha a_\nu$  and  $m$  replaced by  $m - |\alpha| + |\beta|$ .

To finish the proof we show that  $\mathcal{F}_1 h$  vanishes on  $\mathcal{N}_L = \{(\eta, \xi) \in \mathbb{R}^d \times \mathbb{R}^d : |\eta + \xi| < L|\xi|\}$  with  $L := \min\{1 - \epsilon_2, \frac{1}{2}\}$ . Then the estimate on the operator norm follows by the continuity of op. As  $a = R(F_u) \in S_{1,1}^{m,1-\epsilon_2}$ , it suffices to prove that  $\mathcal{F}_1 b$  vanishes on  $\mathcal{N}_{\frac{1}{2}}$ .

By standard arguments on convolution and Fourier transform we have for all  $g \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$

$$b_\nu(\mathcal{F}_1 g) = (\mu q_\nu)^{-d/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a_\nu(y, \theta) \mathcal{F}_1 f(y, \theta) d\theta dy,$$

where

$$f(\eta, \theta) = \int_{\mathbb{R}^d} \mathcal{F}_1 \psi(\eta/(q_\nu \mu), (\xi - \theta)/q_\nu \mu) g(\eta, \xi) d\eta. \quad (2.21)$$

Let  $\text{supp } g \subset \mathcal{N}_{1/2}$ . By construction we have  $\text{supp } \mathcal{F}_1 \psi \subset \{(\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d : |\eta|, |\xi| \leq D, \}$  for some  $D > 0$ . Next choose  $\nu_0 \in \mathbb{N}$  so large that  $3D\mu \leq q_{\nu_0}/2$ . Then for  $\nu \geq \nu_0$  on the support of the integrand of (2.21) we have  $|\eta|, |\xi - \theta| \leq Dq_\nu \mu$  and  $|\xi + \eta| + 1 < \frac{1}{2}|\xi|$ . The first and third inequality yield

$$|\xi| < 2|\eta| \leq 2Dq_\nu \mu$$

and thus the second one gives

$$|\theta| \leq Dq_\nu \mu + |\xi| < 3Dq_\nu \mu \leq q_\nu q_{\nu_0}/2 \leq 2^{\nu/2} 2^{\nu_0/2-1} \leq 2^{\nu-1}.$$

But this implies  $b_\nu(y, \theta) = 0$  for all  $y \in \mathbb{R}^d$ . Therefore we have proven  $b_\nu(\mathcal{F}_1 g) = 0$  for all  $\nu \geq \nu_0$  and  $\text{supp } g \subset \{(\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d : |\xi + \eta| < \frac{1}{2}|\xi|\}$ . Hence this also holds for  $b = \sum_{\nu \geq \nu_0} b_\nu$ .

□

### 3 Dissipativity

Throughout this section we consider (1.1), (1.2) with smooth matrix families  $A^j, B^{jk} : \mathcal{U} \rightarrow \mathbb{R}^{n \times n}$ ,  $u_0, u_1 : \mathbb{R}^d \rightarrow \mathcal{U}$  and  $u : [0, T] \times \mathbb{R}^d \rightarrow \mathcal{U}$  for some domain  $\mathcal{U} \subset \mathbb{R}^n$ . Carrying out the differentiation with respect to  $x_k$  on the right-hand side and distinguishing between space and time derivatives we write (1.1) as

$$-B^{00}(u)u_{tt} = \sum_{j,k=1}^d B^{jk}(u)u_{x_j x_k} + \sum_{j=1}^d (B^{0j}(u) + B^{j0}(u)u_t)_{x_j} - A^0(u)u_t - \sum_{j=1}^d A^j(u)u_{x_j} + Q(u, D_{t,x}u),$$

where  $Q$  is of the form

$$Q(u, D_{t,x}u) = \sum_{l=1}^n \sum_{j,k=0}^d Q^{ljk}(u) u_{x_k}^l u_{x_j}.$$

We will see in the proofs that the specific form of the matrices  $Q^{ljk}(u)$  does not play any role. Hence multiplying (1.1) by  $(-B^{00})^{-1}$ , we can assume  $-B^{00} = I_n$  without loss of generality, which we will always do in the following.

Next, denote by

$$\begin{aligned} B(u, \boldsymbol{\xi}) &:= \sum_{j,k=1}^d B^{jk}(u) \xi_j \xi_k, & C(u, \boldsymbol{\xi}) &:= \sum_{j=1}^d (B^{0j}(u) + B^{j0}(u)) \xi_j, \\ A(u, \boldsymbol{\xi}) &:= \sum_{j=1}^d A^j(u) \xi_j, & \boldsymbol{\xi} &= (\xi_1, \dots, \xi_n) \in \mathbb{R}^d. \end{aligned}$$

the symbols of the second and first order parts, respectively. Then the hyperbolicity of both sides of (1.1) is expressed by the following conditions:

(H<sub>A</sub>) (a) there exists a smooth bounded family of hermitian uniformly positive definite matrices  $\Sigma : \mathcal{U} \rightarrow \mathbb{R}^{n \times n}$  such that  $\Sigma(u)A^0(u)$  is symmetric and uniformly positive on  $\mathcal{U}$ ,  
(b) the matrix family  $A_0(u)^{-1}A(u, \boldsymbol{\xi})$  permits a symbolic symmetrizer  $H(u, \boldsymbol{\xi})$ ,

(H<sub>B</sub>) with

$$\mathcal{B}(u, \boldsymbol{\xi}) = \begin{pmatrix} 0 & |\boldsymbol{\xi}| I_n \\ -|\boldsymbol{\xi}|^{-1} B(u, \boldsymbol{\xi}) & iC(u, \boldsymbol{\xi}) \end{pmatrix}, \quad \boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d,$$

the matrix family  $i\mathcal{B}(u, \boldsymbol{\xi})$  permits a symbolic symmetrizer  $\mathcal{H}(u, \boldsymbol{\xi})$ .

Above we use the following notion of a symbolic symmetrizer (cf. e.g. [38]).

**3.1 Definition.** Let  $K \in C^\infty(\mathcal{U} \times \mathbb{R}^d \setminus \{0\}, \mathbb{C}^{n \times n})$ . A symbolic symmetrizer for  $K$  is a smooth mapping  $S \in C^\infty(\mathcal{U} \times \mathbb{R}^d \setminus \{0\}, \mathbb{C}^{n \times n})$  positive homogeneous of degree 0 with respect to the second argument, bounded as well as all its derivatives on  $\mathcal{U} \times \mathbb{S}^{d-1}$  such that for some  $c > 0$  and all  $(u, \boldsymbol{\xi}) \in \mathcal{U} \times \mathbb{R}^d \setminus \{0\}$

$$S(u, \boldsymbol{\xi}) = S(u, \boldsymbol{\xi})^* \geq cI_n,$$

and  $S(u, \boldsymbol{\xi})K(u, \boldsymbol{\xi}) = (S(u, \boldsymbol{\xi})K(u, \boldsymbol{\xi}))^*$ .

**3.2 Remark.**  $K$  admits a symbolic symmetrizer if  $K$  is positive homogeneous of degree 1, for all  $(u, \boldsymbol{\omega}) \in \mathcal{U} \times \mathbb{S}^{d-1}$  all eigenvalues of  $K(u, \boldsymbol{\xi})$  are real, semi-simple (i.e. their geometric and algebraic multiplicities coincide) and their multiplicities do not depend on  $(u, \boldsymbol{\omega})$  (cf. [38], Proposition 5.2 C). If this holds for  $A^0(u)^{-1}A(u, \boldsymbol{\xi})$  or  $\mathcal{B}(u, \boldsymbol{\xi})$  the respective operator is often called constantly hyperbolic.

We now fix a homogeneous state  $\bar{u} \in \mathcal{U}$  and assume the following dissipativity conditions on the coefficient matrices.

**Condition (D).** Matrices  $A^j(\bar{u}), B^{jk}(\bar{u})$  have three properties:

(D1) For every  $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$ , all restrictions, as a quadratic form, of

$$W_1 = H(\bar{u}, \boldsymbol{\omega})(A^0(\bar{u}))^{-1} \left( -B(\bar{u}, \boldsymbol{\omega}) + (A^0(\bar{u}))^{-1}(A(\bar{u}, \boldsymbol{\omega}))(A^0(\bar{u}))^{-1}A(\bar{u}, \boldsymbol{\omega}) \right. \\ \left. + C(\bar{u}, \boldsymbol{\omega})(A^0(\bar{u}))^{-1}A(\bar{u}, \boldsymbol{\omega}) \right),$$

on the eigenspaces  $E = J_E^{-1}(\mathbb{C}^n)$  of

$$W_0 = (A^0(\bar{u}))^{-1}A(\bar{u}, \boldsymbol{\omega})$$

are uniformly negative in the sense that

$$J_E^*(W_1 + W_1^*)J_E \leq -\bar{c} J_E^*J_E \quad \text{with one } \bar{c} > 0.$$

(D2) For every  $\boldsymbol{\omega} \in S^{d-1}$ , all restrictions, as a quadratic form, of

$$\mathcal{W}_1 = \mathcal{H}(\bar{u}, \boldsymbol{\omega})\mathcal{A}(\bar{u}, \boldsymbol{\omega}), \quad \mathcal{A}(\bar{u}, \boldsymbol{\omega}) = \begin{pmatrix} 0 & 0 \\ -iA(\bar{u}, \boldsymbol{\omega}) & -A^0(\bar{u}) \end{pmatrix} \quad (3.1)$$

on the eigenspaces  $\mathcal{E} = \mathcal{J}_{\mathcal{E}}^{-1}(\mathbb{C}^{2n})$  of

$$\mathcal{W}_0 = \mathcal{B}(\bar{u}, \boldsymbol{\omega}) \quad (3.2)$$

are uniformly negative in the sense that

$$\mathcal{J}_{\mathcal{E}}^*(\mathcal{W}_1 + \mathcal{W}_1^*)\mathcal{J}_{\mathcal{E}} \leq -\bar{c} I_{\mathcal{E}} \quad \text{with one } \bar{c} > 0..$$

(D3) All solutions  $(\lambda, \boldsymbol{\xi}) \in \mathbb{C} \times (\mathbb{R}^d \setminus \{0\})$  of the dispersion relation of (1.1) at  $\bar{u} = 0$  have  $\text{Re}(\lambda) < 0$ .

**3.3 Remark.** Note that as (D) is an open condition there exists a neighbourhood of  $\bar{u}$  such that  $B^{jk}(u), A^j(u)$  satisfy (D) with  $\bar{u}$  replaced by  $u$  for all  $u \in \mathcal{U}_0$  with  $\bar{c}$  independent of  $u$ .

The following remark is useful in the proofs below.

**3.4 Remark.** It is straightforward to show that (D1) and (D2) are equivalent to the same conditions with  $W_0, W_1$  replaced by

$$\bar{W}_0 := H(\bar{u}, \boldsymbol{\omega})^{\frac{1}{2}}A(\bar{u})^{-1}A(0, \boldsymbol{\omega})H(\bar{u}, \boldsymbol{\omega})^{-\frac{1}{2}}, \quad \bar{W}_1 := H(\bar{u}, \boldsymbol{\omega})^{-\frac{1}{2}}W_1H(\bar{u}, \boldsymbol{\omega})^{-\frac{1}{2}}$$

and  $\mathcal{W}_0, \mathcal{W}_1$  replaced by

$$\bar{\mathcal{W}}_0 := \mathcal{H}(\bar{u}, \boldsymbol{\omega})^{\frac{1}{2}}\mathcal{B}(\bar{u}, \boldsymbol{\omega})\mathcal{H}(\bar{u}, \boldsymbol{\omega})^{-\frac{1}{2}}, \quad \bar{\mathcal{W}}_1 := \mathcal{H}(\bar{u}, \boldsymbol{\omega})^{\frac{1}{2}}\mathcal{A}(\bar{u}, \boldsymbol{\omega})\mathcal{H}(\bar{u}, \boldsymbol{\omega})^{-\frac{1}{2}}.$$

From now on we always assume  $(H_A)$ ,  $(H_B)$  and  $(D)$ . As we could also consider (1.1), (1.2) in the variable  $u - \bar{u}$ , we can w.l.o.g. restrict our argumentation to the case  $\bar{u} = 0$ .

We write (1.1) as the first-order in time system

$$\begin{aligned} u_t &= v \\ v_t &= \sum_{j=1}^d (B^{j0} + B^{0j})(u)v_{x_j} + \sum_{j,k=1}^d B^{jk}(u)u_{x_j x_k} - A^0(u)v - \sum_{j=1}^d A^j(u)u_{x_j} + Q(u, D_{t,x}u) \end{aligned} \quad (3.3)$$

and denote by

$$\bar{\mathcal{M}}(u, \boldsymbol{\xi}) := \begin{pmatrix} 0 & I_n \\ M(u, \boldsymbol{\xi}) & N(u, \boldsymbol{\xi}) \end{pmatrix}, \quad (3.4)$$

with

$$M(u, \boldsymbol{\xi}) = -iA(u, \boldsymbol{\xi}) - B(u, \boldsymbol{\xi}), \quad N(u, \boldsymbol{\xi}) = iC(u, \boldsymbol{\xi}) - A^0(u),$$

the Fourier symbol of (3.3). We also define

$$\begin{aligned} \mathcal{M}(u, \boldsymbol{\xi}) &:= \mathcal{Z}(\boldsymbol{\xi})\bar{\mathcal{M}}(u, \boldsymbol{\xi})\mathcal{Z}(\boldsymbol{\xi})^{-1} \\ \mathcal{Z}(\boldsymbol{\xi}) &= \begin{pmatrix} \langle \boldsymbol{\xi} \rangle I_n & 0 \\ 0 & I_n \end{pmatrix}. \end{aligned}$$

First we treat the linearization of (1.1) at the reference state  $u = 0$ , i.e.

$$\sum_{j=0}^d A_j(0)u_{x_j} = \sum_{j,k=0}^d B^{ij}(0)u_{x_i x_j}. \quad (3.5)$$

Such linear systems were studied in [14], however under the stronger assumptions, that the coefficient matrices are symmetric and  $A^0$  is positive definite. Then  $(H_A)$  is clearly satisfied with  $F_A = I_n$  and  $H = A^0$ . Also, condition  $(H_B)$  (b) *ibid.* requires the existence of a matrix family  $\mathcal{S} : \mathbb{S}^{d-1} \rightarrow \mathbb{C}^{n \times n}$  such that  $i\mathcal{S}(\boldsymbol{\omega})\mathcal{B}(0, \boldsymbol{\omega})\mathcal{S}(\boldsymbol{\omega})^{-1}$  is real symmetric. But one can easily check that this can be relaxed to the assumption that  $i\mathcal{S}(\boldsymbol{\omega})\mathcal{B}(0, \boldsymbol{\omega})\mathcal{S}(\boldsymbol{\omega})^{-1}$  is hermitian, which is satisfied in the present context for  $\mathcal{S}(\boldsymbol{\omega}) := \mathcal{H}(0, \boldsymbol{\omega})^{\frac{1}{2}}$ . Lastly, we want to point out that  $(D1)$ ,  $(D2)$  *ibid.* were stated in the equivalent form mentioned in Remark 3.4.

We will make plausible below that the weaker conditions in the present work are still sufficient to retrieve the main result of [14], namely:

**3.5 Proposition.** *There exist a  $c > 0$  and a family  $\boldsymbol{\xi} \mapsto \mathcal{T}(\boldsymbol{\xi}), \mathbb{R}^d \rightarrow \mathbb{C}^{2n \times 2n}$  of linear transformations of  $\mathbb{C}^{2n}$  which, together with their inverses  $\mathcal{T}(\boldsymbol{\xi})^{-1}$ , are uniformly bounded, such that*

$$\mathcal{T}(\boldsymbol{\xi})\mathcal{M}(0, \boldsymbol{\xi})\mathcal{T}^{-1}(\boldsymbol{\xi}) + (\mathcal{T}(\boldsymbol{\xi})\mathcal{M}(0, \boldsymbol{\xi})\mathcal{T}^{-1}(\boldsymbol{\xi}))^* \leq -c\rho(\boldsymbol{\xi})I_{2n}, \quad \boldsymbol{\xi} \in \mathbb{R}^d, \quad (3.6)$$

where  $\rho(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2/(1 + |\boldsymbol{\xi}|^2)$ .

As outlined in [14] this brings about the pointwise decay of solutions in Fourier space and thus the following decay estimate for the inhomogeneous linear Cauchy problem.

**3.6 Corollary.** *For any  $s \in \mathbb{N}_0$  there exists  $C > 0$  such that the following holds: For all  $u_0 \in H^{s+1} \cap L^1$ ,  $u_1 \in H^s \cap L^1$  and  $f \in C([0, T], H^s \cap L^1)$  the solution  $u$  of*

$$f + \sum_{j=0}^d A_j(0)u_{x_j} = \sum_{j,k=0}^d B^{kj}(0)u_{x_j x_k}$$

with  $u(0) = u_0$ ,  $u_t(0) = u_1$  satisfies

$$\begin{aligned} \|u(t)\|_{s+1} + \|u_t(t)\|_s &\leq C(1+t)^{-\frac{d}{4}}(\|u_0\|_{s+1} + \|u_0\|_{L^1} + \|u_1\|_s + \|u_1\|_{L^1}) \\ &\quad C \int_0^t (1+t-\tau)^{-\frac{d}{4}}(\|f(\tau)\|_s + \|f(\tau)\|_{L^1}) d\tau \end{aligned}$$

for all  $t \in [0, T]$ .

*Proof of Proposition 3.5.* As stated above the proof can be found essentially in [14]. We just illustrate at which points it has to be slightly modified.

The existence of a bounded family  $\mathcal{T}(\boldsymbol{\xi}) \subset \text{Gl}_{2n}^2$  satisfying (3.6) is proven separately for the three different regimes  $|\boldsymbol{\xi}| \leq r_0$ ,  $r_0 \leq |\boldsymbol{\xi}| \leq r_\infty$  and  $|\boldsymbol{\xi}| \geq r_\infty$  for suitable  $r_0, r_\infty > 0$ . In the latter two cases only  $((H)_B)$  and conditions (D2), (D3) are used. The symmetry of the matrices plays no role whatsoever.

For small values of  $|\boldsymbol{\xi}|$  writing  $\boldsymbol{\xi} = \xi \boldsymbol{\omega}$  for  $\xi > 0$ ,  $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$  one finds a bounded family of invertible  $\mathcal{R}(\xi, \boldsymbol{\omega})$  with  $\mathcal{R}(\xi, \boldsymbol{\omega})^{-1}$  also bounded and (suppressing the argument  $u = 0$ )

$$\mathcal{R}(\xi, \boldsymbol{\omega}) \bar{\mathcal{M}}(\xi \boldsymbol{\omega}) \mathcal{R}(\xi, \boldsymbol{\omega})^{-1} = \begin{pmatrix} X(\xi, \boldsymbol{\omega}) & 0 \\ 0 & Y(\xi, \boldsymbol{\omega}) \end{pmatrix},$$

where

$$\begin{aligned} X(\xi, \boldsymbol{\omega}) &= i\xi(A^0)^{-1}A(\boldsymbol{\omega}) \\ &\quad + \xi^2(A^0)^{-1}(-B(\boldsymbol{\omega}) + (A^0)^{-1}(A(\boldsymbol{\omega}))(A^0)^{-1}A(\boldsymbol{\omega}) + C(\boldsymbol{\omega})(A^0)^{-1}A(\boldsymbol{\omega})) + \mathcal{O}(\xi^3) \\ Y(\xi, \boldsymbol{\omega}) &= -A^0 + \mathcal{O}(\xi^3). \end{aligned}$$

This is due to the fact that  $A^0(0)$  is invertible and again makes no use of the symmetry. Hence for

$$\check{\mathcal{R}}(\xi, \boldsymbol{\omega}) = \begin{pmatrix} H(\boldsymbol{\omega})^{\frac{1}{2}} & 0 \\ 0 & F_A^{\frac{1}{2}} \end{pmatrix} \check{\mathcal{R}}(\xi, \boldsymbol{\omega})$$

we get

$$\check{\mathcal{R}}(\xi, \boldsymbol{\omega}) \mathcal{M}(\xi \boldsymbol{\omega}) \check{\mathcal{R}}(\xi, \boldsymbol{\omega})^{-1} = \begin{pmatrix} i\xi \bar{W}_0 + \xi^2 \bar{W}_1 + \mathcal{O}(\xi^3) & \\ 0 & -F_A^{\frac{1}{2}} A^0 F_A^{-\frac{1}{2}} \end{pmatrix}.$$

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<sup>2</sup>For  $m \in \mathbb{N}$   $\text{Gl}_m$  denotes the space of invertible  $m \times m$ -matrices.

with  $\bar{W}_0, \bar{W}_1$  as in Remark 3.4. Since  $F_A^{\frac{1}{2}}A^0F_A^{-\frac{1}{2}}$  is positive definite the existence of the family  $\mathcal{T}(\xi)$  now follows for sufficiently small  $\xi$  by condition (D1) and [14], Lemma 5.<sup>3</sup>  $\square$

In Section 4 we will see that, given  $d \geq 3, s > d/2 + 1$ , Corollary 3.6 directly implies the decay of a solution to the quasi-linear problem (1.1) in  $H^{s-1}$  but only provided that its  $H^s$ -norm is a-priori known to be small. To close this gap we need to show that the  $H^s$ -norm of a small solution can be bounded by the initial conditions and  $L^2$ -norms of lower order derivatives. The rest of this section is devoted to a construction preparing such a result.

In the following for  $\xi \in \mathbb{R}^d$  we write  $\xi = \xi\omega$  with  $\xi = |\xi| \in [0, \infty), \omega = \xi/|\xi| \in \mathbb{S}^{d-1}$ . For  $r > 0, u \in \mathbb{R}^n, \xi \in \mathbb{R}^d$  and  $\omega \in \mathbb{S}^{d-1}$  by  $B^n(u, r), B^d(\xi, r), B^{\mathbb{S}}(\omega, r)$  we denote the balls with radius  $r$  and center  $u, \xi, \omega$  with respect to the metrics on  $\mathbb{R}^n, \mathbb{R}^d, \mathbb{S}^{d-1}$ . For some  $\omega^* \in \mathbb{S}^{d-1}$  and  $\delta > 0$  we use

$$P(\omega^*, \delta) = B^n(0, \delta) \times [0, \delta) \times B^{\mathbb{S}}(\omega^*, \delta)$$

**3.7 Proposition.** *There exist  $r > 0, c_\infty > 0$  and a mapping  $\mathcal{D}_\infty \in C^\infty(\Omega_\infty, \mathbb{C}^{2n \times 2n}), \Omega_\infty := \bar{\mathcal{U}}_0 \times \{\xi \in \mathbb{R}^d : |\xi| \geq r^{-1}\}, \bar{\mathcal{U}}_0 := \overline{B^n(0, r)} \subset \mathcal{U}$ , such that:*

(i) For all  $(u, \xi) \in \Omega_\infty$

$$\mathcal{D}_\infty(u, \xi) = \mathcal{D}_\infty(u, \xi)^* \geq c_\infty I_n,$$

and

$$\mathcal{D}_\infty(u, \xi)\mathcal{M}(u, \xi) + (\mathcal{D}_\infty(u, \xi)\mathcal{M}(u, \xi))^* \leq -c_\infty I_{2n}.$$

(ii) For any  $\alpha, \beta \in \mathbb{N}_0^d$  there exist  $C_{\alpha\beta} > 0$  with

$$|\partial_u^\beta \partial_\xi^\alpha \mathcal{D}_\infty(u, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{-|\alpha|}, \quad (u, \xi) \in \Omega_\infty. \quad (3.7)$$

*Proof.* Consider the mapping  $\mathcal{K} : \mathcal{U} \times (0, \infty) \times \mathbb{S}^{d-1} \rightarrow \mathbb{C}^{2n \times 2n}$  defined by

$$\mathcal{K}(u, \eta, \omega) = \begin{pmatrix} 0 & I_n \\ -i\eta A(u, \omega) - B(u, \omega) & -iC(u, \omega) - \eta A^0(u) \end{pmatrix}, \quad \omega \in \mathbb{S}^{d-1}. \quad (3.8)$$

and  $\mathcal{H}(u, \omega)$  denote the symmetrizer of  $\mathcal{B}(u, \omega)$  as in condition (H<sub>B</sub>) (b). Set

$$\mathcal{W}(u, \eta, \omega) := \mathcal{H}(u, \omega)^{\frac{1}{2}} \mathcal{K}(u, \eta, \omega) \mathcal{H}(u, \omega)^{-\frac{1}{2}}.$$

Since

$$\mathcal{K}(0, 0, \omega) = \begin{pmatrix} 0 & I_n \\ -B(0, \omega) & iC(0, \omega) \end{pmatrix} = \mathcal{B}(0, \omega)$$

---

<sup>3</sup>Note that in said Lemma it is sufficient to assume that  $i\mathcal{M}(0, \omega)$  is selfadjoint instead of requiring  $i\mathcal{M}(0, \omega)$  to be real symmetric.

and

$$\frac{\partial \mathcal{K}}{\partial \eta}(0, 0, \boldsymbol{\omega}) = \begin{pmatrix} 0 & 0 \\ -iA(0, \boldsymbol{\omega}) & -A^0(0) \end{pmatrix} = \mathcal{A}(0, \boldsymbol{\omega})$$

$\mathcal{W}$  satisfies

$$\mathcal{W}(0, 0, \boldsymbol{\omega}) = \bar{\mathcal{W}}_0, \quad \frac{\partial \mathcal{W}(0, 0, \boldsymbol{\omega})}{\partial \eta} = \bar{\mathcal{W}}_1,$$

with  $\bar{\mathcal{W}}_0, \bar{\mathcal{W}}_1$  as in Remark 3.4. Now fix  $\boldsymbol{\omega}_0 \in \mathbb{S}^{d-1}$ . By virtue of condition (D2) it follows from Lemma 5 in [14] that there exists  $\delta_0 > 0, c_0 > 0$  and  $\mathcal{T}_0 \in C^\infty(P(\boldsymbol{\omega}^*, \delta_0), \text{Gl}_{2n})$  with  $\mathcal{T}_0^{-1}$  also bounded such that pointwise on  $P(\boldsymbol{\omega}_0, \delta_0)$

$$\mathcal{T}_0 \mathcal{W} \mathcal{T}_0^{-1} + (\mathcal{T}_0 \mathcal{W} \mathcal{T}_0^{-1})^* \leq -\tilde{c} \eta I_{2n}$$

for some  $\tilde{c} > 0$ . Hence  $\tilde{\mathcal{D}}_0 := \mathcal{H}^{\frac{1}{2}} \mathcal{T}_0^* \mathcal{T}_0 \mathcal{H}^{\frac{1}{2}} \in C^\infty(P(\delta_0, \boldsymbol{\omega}_0), \mathbb{C}^{2n \times 2n})$  satisfies

$$\tilde{\mathcal{D}}_0(u, \xi, \boldsymbol{\omega}) = \tilde{\mathcal{D}}_0(u, \xi, \boldsymbol{\omega})^* \geq c I_{2n}, \quad (u, \xi, \boldsymbol{\omega}) \in P(\delta_0, \boldsymbol{\omega}_0)$$

for some  $c > 0$  and thus

$$\tilde{\mathcal{D}}_0 \mathcal{K} + (\tilde{\mathcal{D}}_0 \mathcal{K})^* \leq -c \tilde{c} \eta I.$$

In conclusion we have shown the following: For each  $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$  there exist  $\delta_\omega > 0, c_\omega > 0$  and  $\mathcal{D}_\omega \in C^\infty(P(\boldsymbol{\omega}, \delta_\omega), \mathbb{C}^{2n \times 2n})$  such that for all  $(u, \xi, \bar{\boldsymbol{\omega}}) \in P(\boldsymbol{\omega}, \delta_\omega)$

$$\begin{aligned} \mathcal{D}_\omega(u, \eta, \bar{\boldsymbol{\omega}}) &= \mathcal{D}_\omega(u, \eta, \bar{\boldsymbol{\omega}})^* \geq c_\omega I \\ \mathcal{D}_\omega(u, \eta, \bar{\boldsymbol{\omega}}) \mathcal{K}(u, \eta, \bar{\boldsymbol{\omega}}) + (\mathcal{D}_\omega(u, \eta, \bar{\boldsymbol{\omega}}) \mathcal{K}(u, \eta, \bar{\boldsymbol{\omega}}))^* &\leq -c_\omega \xi^2 I. \end{aligned} \tag{3.9}$$

As  $\mathbb{S}^{d-1}$  is compact we may choose  $\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_r$  such that

$$\bigcup_{l=1}^{\bar{l}} B^{\mathbb{S}}(\boldsymbol{\omega}_l, \delta_l/2) = \mathbb{S}^{d-1} \quad (\delta_l := \delta_{\boldsymbol{\omega}_l}).$$

Set  $r_0 = \min\{\delta_1, \dots, \delta_r\}$ ,  $c_0 = \min\{c_{\boldsymbol{\omega}_1}, \dots, c_{\boldsymbol{\omega}_r}\}$ . Then for  $l = 1, \dots, \bar{l}$  and  $P_l := B^n(0, r_0) \times [0, r_0) \times B^{\mathbb{S}}(\boldsymbol{\omega}_l, \delta_l)$  choose functions  $\phi_l \in C^\infty(\mathbb{S}^{d-1}, [0, 1])$  with  $\text{supp } \phi_l \subset B^{\mathbb{S}}(\boldsymbol{\omega}_j, \delta_l)$ ,  $\phi_l = 1$  on  $B^{\mathbb{S}}(\boldsymbol{\omega}_j, \delta_j/2)$  and extend  $\mathcal{D}_l := \mathcal{D}_{\boldsymbol{\omega}_l}$  trivially by 0 to a function defined on  $B^n(0, r_0) \times [0, r_0) \times \mathbb{S}^{d-1} =: \Omega_0$ . Define

$$\mathcal{D}_0 : \Omega_0 \rightarrow \mathbb{C}^{2n \times 2n} : (u, \eta, \boldsymbol{\omega}) \mapsto \sum_{l=1}^{\bar{l}} \phi_l(\boldsymbol{\omega}) \mathcal{D}_l(u, \eta, \boldsymbol{\omega}).$$

Then  $\mathcal{D}_0 \in C^\infty(\Omega_0, \mathbb{C}^{2n \times 2n})$ , and  $\mathcal{D}_0(u, \eta, \boldsymbol{\omega})$  is hermitian for all  $(u, \eta, \boldsymbol{\omega}) \in \Omega_0$ . Furthermore for  $(u, \eta, \boldsymbol{\omega}) \in \Omega_0$  we have  $\boldsymbol{\omega} \in B^{\mathbb{S}}(\boldsymbol{\omega}_k, \delta/2)$  for some  $k \in \{1, \dots, \bar{l}\}$  and thus as  $\mathcal{D}_l(u, \eta, \boldsymbol{\omega}) \geq 0$

$$\mathcal{D}_0(u, \eta, \boldsymbol{\omega}) = \sum_{l=1}^{\bar{l}} \phi_l(\boldsymbol{\omega}) \mathcal{D}_l(u, \eta, \boldsymbol{\omega}) \geq \mathcal{D}_k(u, \eta, \boldsymbol{\omega}) \geq c_0 I.$$

with the same reasoning we see

$$\mathcal{D}_0(u, \eta, \boldsymbol{\omega}) \mathcal{K}(u, \eta, \boldsymbol{\omega}) + (\mathcal{D}_0(u, \eta, \boldsymbol{\omega}) \mathcal{K}(u, \eta, \boldsymbol{\omega}))^* \leq -c_0 \eta I_{2n}, \quad (u, \eta, \boldsymbol{\omega}) \in \Omega_0.$$



Now note that for all  $u, \xi, \omega$

$$\begin{aligned}\xi\mathcal{K}(u, 1/\xi, \omega) &= \begin{pmatrix} 0 & \xi I_n \\ -iA(u, \omega) - \xi B(u, \omega) & -i\xi C(u, \omega) - A^0(U) \end{pmatrix} \\ &= \tilde{\mathcal{Z}}(\xi)\mathcal{M}(u, \xi\omega)\tilde{\mathcal{Z}}(\xi)^{-1},\end{aligned}$$

where

$$\tilde{\mathcal{Z}}(\xi) = \begin{pmatrix} \frac{\langle \xi \rangle}{\xi} I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

As clearly  $\tilde{\mathcal{Z}}, \tilde{\mathcal{Z}}^{-1} \in C^\infty((r_0^{-1}, \infty), \mathbb{C}^{2n \times 2n})$  are symmetric and positive definite on  $(r_0^{-1}, \infty)$ , for  $r := r_0/2$ ,  $\Omega_\infty := \overline{B^n(0, r)} \times \{\xi \in \mathbb{R}^d : |\xi| \geq r^{-1}\}$  the mapping

$$\mathcal{D}_\infty : \Omega_\infty \rightarrow \mathbb{C}^{2n \times 2n}, (u, \xi) \mapsto \tilde{\mathcal{Z}}(|\xi|)\mathcal{D}_0(u, 1/|\xi|, \xi/|\xi|)\tilde{\mathcal{Z}}(|\xi|)$$

is in  $C^\infty(\Omega_\infty, \mathbb{C}^{2n \times 2n})$  and for all  $(u, \xi) \in \Omega_\infty$

$$\mathcal{D}_\infty(u, \xi) = \mathcal{D}_\infty(u, \xi)^* \geq c_\infty I_{2n}$$

for some  $c_\infty > 0$ . Since for  $\xi = \xi\omega \in \mathcal{U}_0$

$$\mathcal{D}_\infty(u, \xi)\mathcal{M}(u, \xi) = \xi\tilde{\mathcal{Z}}(\xi)\mathcal{D}_0(u, 1/\xi, \omega)\tilde{\mathcal{Z}}(\xi)\mathcal{K}(u, 1/\xi, \omega) = \xi\tilde{\mathcal{Z}}(\xi)\tilde{\mathcal{D}}_0(u, 1/\xi, \omega)\mathcal{K}(u, 1/\xi)\tilde{\mathcal{Z}}(\xi),$$

we also have

$$\mathcal{D}_\infty(u, \xi)\mathcal{M}(u, \xi) + (\mathcal{D}_\infty(u, \xi)\mathcal{M}(u, \xi))^* \leq -c_\infty I_{2n}$$

for some  $c_\infty > 0$ .

It remains to verify (3.7). First note that the functions  $\xi \mapsto \langle |\xi| \rangle / |\xi|$ ,  $\xi \mapsto \xi_k / |\xi|$ ,  $k = 1, \dots, d$  and  $\xi \mapsto 1/|\xi|$  are positive homogeneous of degree 0 and  $-1$ , respectively. Thus for any  $\alpha \in \mathbb{N}_0^d$  there exists  $C_\alpha > 0$  such that for all  $\xi \in \mathbb{R}^d$  with  $|\xi| > 2r_0^{-1}$

$$|D^\alpha \mathcal{Z}(\xi)| + |D^\alpha(\xi_k/|\xi|)| + |D^\alpha(1/|\xi|)| \leq C_\alpha \langle \xi \rangle^{-|\alpha|}.$$

Since  $\mathcal{D}_0$  as well as all of its derivatives are bounded on  $\overline{B^n(0, r_0/2)} \times [0, r_0/2] \times \mathbb{S}^{d-1}$  the estimate (3.7) follows by product and chain rule.  $\square$

## 4 Proof of Theorem 1.1

To begin with, we remark that local well-posedness of (1.1), (1.2) follows from the existing theory for hyperbolic systems of any order [38].<sup>4</sup> Our task thus consists in showing that under an a priori smallness assumption the solution satisfies the decay and energy estimates (1.3) and (1.4), for, w.l.o.g.,  $\bar{u} = 0$ . Then we can extend them globally by standard methods (cf. e.g. [24], proof of Theorem 3.6). We show the following.

**4.1 Proposition.** *Consider  $d \geq 3$ ,  $s > d/2 + 1$  and assume  $(H_B)$ ,  $(H_A)$  and  $(D)$ . Then there exist constants  $\mu > 0$ ,  $\delta = \delta(\mu) > 0$ , and  $C = C(\mu, \delta) > 0$  (all independent of  $T$ ) such that the following holds: For all  $u_0 \in H^{s+1}$ ,  $u_1 \in H^s$  with  $\|u_0\|_{s+1} + \|u_1\|_s < \delta$  and all  $u \in C^0([0, T], H^{s+1}) \cap C^1([0, T], H^s)$  satisfying (1.1), (1.2) and*

$$\sup_{t \in [0, T]} \|u(t)\|_{s+1}^2 + \|u_t(t)\|_s^2 + \int_0^t \|u(\tau)\|_{s+1}^2 + \|u_t(\tau)\|_s^2 d\tau \leq \mu$$

we have for all  $t \in [0, T]$

$$\|u(t)\|_s + \|u_t(t)\|_{s-1} \leq C(1+t)^{-\frac{d}{4}}(\|u_0\|_s + \|u_0\|_{L^1} + \|u_1\|_{s-1} + \|u_1\|_{L^1}), \quad (4.1)$$

$$\|u(t)\|_{s+1}^2 + \|u_t(t)\|_s^2 + \int_0^t \|u(\tau)\|_{s+1}^2 + \|u_t(\tau)\|_s^2 \leq C(\|u_0\|_{s+1}^2 + \|u_0\|_{L^1}^2 + \|u_1\|_{s+1}^2 + \|u_1\|_{L^1}^2) \quad (4.2)$$

We split the proof into two parts corresponding to the following two assertions.

**4.2 Proposition.** *In the situation of Proposition 4.1 there exist  $\mu > 0$ ,  $\delta > 0$ , and  $C > 0$  such that the following holds: For all  $u_0 \in H^{s+1} \cap L^1$ ,  $u_1 \in H^s \cap L^1$  with  $\|u_0\|_{s+1} + \|u_1\|_s, \|u_0\|_{L^1} + \|u_1\|_{L^1} < \delta$  and all  $u \in C^0([0, T], H^{s+1}) \cap C^1([0, T], H^s)$  satisfying (1.1), (1.2) and*

$$\sup_{t \in [0, T]} \|u(t)\|_{s+1}^2 + \|u_t(t)\|_{s+1}^2 + \int_0^t \|u(\tau)\|_{s+1}^2 + \|u_t(\tau)\|_s^2 d\tau \leq \mu$$

(1.3) holds for all  $t \in [0, T]$ .

**4.3 Proposition.** *In the situation of Proposition 4.1 there exist  $\mu > 0$ , and  $C > 0$  such that the following holds: For all  $u_0 \in H^{s+1}$ ,  $u_1 \in H^s$  and all  $u \in C^0([0, T], H^{s+1}) \cap C^1([0, T], H^s)$  satisfying (1.1), (1.2) and*

$$\sup_{t \in [0, T]} \|u(t)\|_{s+1}^2 + \|u_t(t)\|_{s+1}^2 + \int_0^t \|u(\tau)\|_{s+1}^2 + \|u_t(\tau)\|_s^2 d\tau \leq \mu$$

we have for all  $t \in [0, T]$

$$\begin{aligned} & \|u(t)\|_{s+1}^2 + \|u_t(t)\|_s^2 + \int_0^t \|u(\tau)\|_s^2 + \|u_t(\tau)\|_{s-1}^2 d\tau \\ & \leq C(\|u_0\|_{s+1}^2 + \|u_1\|_s^2) + C \int_0^t \|u(\tau)\|_s^2 + \|u_t(\tau)\|_{s-1}^2 d\tau. \end{aligned} \quad (4.3)$$

---

<sup>4</sup>For example, the recent result in [3], which applies to the class we study in Section 5, is of this type.

From there Proposition 4.1 clearly follows by multiplying (4.3) with a sufficiently small factor integrating, (1.3) with respect to  $t$ , and adding the resulting inequalities.

For notational reasons we write the first order representation (3.3) of (1.1) in the compact form

$$U_t = L(u)U + (0, Q(u, D_{x,t}u))^t \quad (4.4)$$

with  $U = (u, u_t)$ ,

$$L(u) = \begin{pmatrix} 0 & I_n \\ \sum_{j,k=1}^d B^{jk}(u) \partial_{x_j} \partial_{x_k} - \sum_{j=1}^d A^j(u) \partial_{x_j} & \sum_{j=1}^d (\bar{B}^{j0} + \bar{B}^{0j})(u) \partial_{x_j} - A^0 \end{pmatrix}$$

*Proof of Proposition 4.2.* As  $s > d/2 + 1$  we find by Moser type inequalities (cf. [4] Appendix C and the references therein)

$$\|(L_2(u) - L_2(0))U\|_{s-1} + \|(L_2(u) - L_2(0))U\|_{L^1} \leq C_\mu \|u\|_{s-1} (\|u\|_{s+1} + \|u_t\|_s),$$

where  $L(u)U = (U_2, L_2(u)U)$ . Furthermore

$$\|Q(u, D_{x,t}u)\|_{s-1} + \|Q(u, D_{x,t}u)\|_{L^1} \leq C_\mu \|u\|_s \|u_t\|_{s-1}.$$

Now writing system (4.4) as  $L(0)U = (0, L_2(0) - L_2(u) + Q(u, D_{x,t}u))$  and applying Corollary 3.6 to  $f = (L_2(0) - L_2(u)) + Q(u, D_{x,t}u)$  with  $s$  replaced by  $s - 1$  yields

$$\begin{aligned} \|u(t)\|_s + \|u_t(t)\|_{s-1} &\leq C(1+t)^{-\frac{d}{4}} (\|u_0\|_s + \|u_0\|_{L^1} + \|u_1\|_{s-1} + \|u_1\|_{L^1}) \\ &+ C_\mu \sup_{\tau \in [0,t]} (\|u(\tau)\|_{s+1} + \|u_t(\tau)\|_s) \int_0^t (1+t-\tau)^{-\frac{d}{4}} (\|u(\tau)\|_s + \|u_t(\tau)\|_{s-1}) d\tau. \end{aligned} \quad (4.5)$$

As  $t \rightarrow (1+t)^{-\frac{d}{4}}$  is square-integrable over  $[0, \infty)$  for  $d \geq 3$  this gives (1.3) as in e.g. [24], proof of Proposition 3.3.  $\square$

*Proof of Proposition 4.3.* From now  $C_\mu$  always denotes some constant depending monotonically increasing on  $\mu$ , whose concrete value may change at every instance.

For  $0 < \epsilon < 1$  let  $J_\epsilon$  be the Friedrichs mollifier and set  $V = (\Lambda u, u_t)$ ,  $W := W_\epsilon := \Lambda^s J_\epsilon(\Lambda u, u_t)$  and

$$\begin{aligned} \mathcal{M}_u(x, \xi) &= \mathcal{M}_{u(t)}(x, \xi) = \mathcal{M}(u(t, x), \xi) \\ &= \begin{pmatrix} 0 & \langle \xi \rangle I_n \\ (-B(u, \xi) - A(u, \xi)) \langle \xi \rangle^{-1} & C(u, \xi) - A^0(u) \end{pmatrix}. \end{aligned}$$

We start with the following observation.

**4.4 Lemma.** *W satisfies the differential equation*

$$W_t = \text{Op}[\mathcal{M}_u]W + R_1, \quad (4.6)$$

for some  $R_1 \in L^2$  satisfying

$$\|R_1\| \leq C_\mu \|V\|_s^2 + C \|V\|_{s-1} \quad (4.7)$$

*Proof.* Set

$$\tilde{L}(u) := \begin{pmatrix} \Lambda I_n & 0 \\ 0 & I_n \end{pmatrix} L(u) \begin{pmatrix} \Lambda^{-1} I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

Then

$$V_t = \text{Op}[\mathcal{M}_u]V + \tilde{R}_1 \quad (4.8)$$

where

$$\tilde{R}_1 = (\tilde{L}(u) - \text{Op}[\mathcal{M}_u])V + (0, Q(u, D_{x,t}u)).$$

As we have already seen in the proof of Proposition 4.2 (now with  $s - 1$  replaced by  $s$ )

$$\|Q(u, D_{x,t}u)\|_s \leq C_\mu \|V\|_s^2.$$

By Lemma 2.9

$$\|(\tilde{L}(0) - \text{Op}[\mathcal{M}_0])V\|_s \leq C \|V\|_{s-1}$$

and due to Lemma 2.9 (iii) all terms appearing in

$$(\tilde{L}(u) - \tilde{L}(0) - \text{Op}[\mathcal{M}_u - \mathcal{M}_0])V$$

are of the form  $(a(u) - \text{Op}[a_u])f$ , where  $a$  is a smooth function with  $a(0) = 0$  and  $f \in \{\partial_t^l \partial_x^\beta u \mid l \leq 1, l + |\beta| \leq 2\} \subset H^{s-1} \hookrightarrow L^\infty$ . Hence Lemma 2.10 yields

$$\|(\tilde{L}(u) - \tilde{L}(0) - \text{Op}[\mathcal{M}_u - \mathcal{M}_0])V\|_s \leq C_\mu (\|u\|_s \|V\|_s + \|V\|_{s-1}).$$

In conclusion we have shown

$$\|\tilde{R}_1\|_s \leq C_\mu (\|V\|_s^2 + \|V\|_{s-1}). \quad (4.9)$$

Now apply  $\Lambda^s J_\epsilon$  to (4.8) and obtain

$$W_t = \text{Op}[\mathcal{M}_u]W + R_1, \quad (4.10)$$

where

$$R_1 = [\Lambda^s J_\epsilon, \text{Op}[\mathcal{M}_u]]V + \Lambda^s J_\epsilon \tilde{R}_1$$

Note that  $(J_\epsilon)_{\epsilon \in (0,1)}$  is a family of pseudo-differential operators, constant with respect to  $x$ , with symbols uniformly bounded in  $S^0$ . Thus we get from (4.9)

$$\|\Lambda^s J_\epsilon R_1\| \leq C_\mu \|V\|_s^2 + C \|V\|_{s-1}$$

and from Proposition 2.19 (iii)

$$\|[\Lambda^s J_\epsilon, \text{Op}[\mathcal{M}_u]]V\| \leq C_\mu \|u\|_s \|V\|_s + C \|V\|_{s-1},$$

which proves the assertion.  $\square$

Next, let  $\mathcal{D}_\infty \in C^\infty(\overline{B_r^n(0)} \times \{\xi \in \mathbb{R}^d : |\xi| \geq r\}, \mathbb{C}^{2n \times 2n})$  be the mapping constructed in Proposition 3.7 and extend it trivially by zero to a function defined on  $\overline{B_r^n(0)} \times \mathbb{R}^d := \mathcal{U}_0 \times \mathbb{R}^d$ . Choose a function  $\phi \in C^\infty(\mathbb{R}^d)$ , with  $0 \leq \phi \leq 1$ ,  $\phi(\xi) = 0$  for  $|\xi| \leq 2r$  and  $\phi(\xi) = 1$  for  $|\xi| \geq 3r$ . Set

$$\mathcal{D}(v, \xi) := \phi(\xi)\mathcal{D}_\infty(v, \xi), \quad (v, \xi) \in \mathcal{U}_0 \times \mathbb{R}^d$$

Let  $\mu$  be sufficiently small such that  $u(t, x) \in \overline{B_r^n(0)}$  for all  $(t, x) \in [0, T] \times \mathbb{R}^d$  and define

$$\mathcal{D}_u(x, \xi) := \mathcal{D}_{u(t)}(x, \xi) = \mathcal{D}(u(t, x), \xi), \quad (t, x, \xi) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

Choose another function  $\psi \in C^\infty(\mathbb{R}^d)$ , with  $0 \leq \psi \leq 1$ ,  $\psi(\xi) = 0$  for  $|\xi| \geq 5r$ ,  $\psi(\xi) = 1$  for  $|\xi| \leq 4r$  and define

$$\tilde{\mathcal{D}}_u(x, \xi) = \mathcal{D}_u(x, \xi) + \psi(\xi)I_{2n}.$$

**4.5 Lemma.** *The family of operators  $(\mathcal{G}_{u(t)})_{t \in [0, T]}$  defined by*

$$\mathcal{G}_{u(t)} := \frac{1}{2}(\text{Op}[\tilde{\mathcal{D}}_{u(t)}] + \text{Op}[\tilde{\mathcal{D}}_{u(t)}]^*) + \text{op}[\tilde{\mathcal{D}}_0] - \text{Op}[\tilde{\mathcal{D}}_0]$$

*is self-adjoint and uniformly positive definite in  $\mathcal{L}(L^2)$  for  $\mu$  sufficiently small. Furthermore*

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{G}_u W, W \rangle = \text{Re} \langle \mathcal{G}_u \text{Op}[\mathcal{M}_u] W, W \rangle + R_2,$$

*for some  $R_2 \in \mathbb{R}$  with*

$$\|R_2\| \leq C_\mu \|W\| (\|V\|_s^2 + \|V\|_s \|W\| + \|V\|_{s-1}).$$

*Proof.* By Proposition 3.7  $\tilde{\mathcal{D}}, \mathcal{D} \in S^0(\mathcal{U})$  and  $\tilde{\mathcal{D}}_u = \tilde{\mathcal{D}}_u^*$  is uniformly positive definite. In particular,  $\text{op}[\tilde{\mathcal{D}}_0] = \text{op}[\tilde{\mathcal{D}}_0]^*$  is a self-adjoint and uniformly positive definite operator on  $\mathcal{L}(L^2)$  (cf. Lemma 2.9). Due to *ibid.* also  $\text{Op}[\tilde{\mathcal{D}}_0]^* = \text{Op}[\tilde{\mathcal{D}}_0]$ , i.e.

$$\mathcal{G}_u = \text{op}[\tilde{\mathcal{D}}_0] + \frac{1}{2}(\text{Op}[\tilde{\mathcal{D}}_u - \tilde{\mathcal{D}}_0] + \text{Op}[\tilde{\mathcal{D}}_u - \tilde{\mathcal{D}}_0]^*).$$

Proposition 2.19 (i) gives

$$\|\text{Op}[\tilde{\mathcal{D}}_u - \tilde{\mathcal{D}}_0]\|_{\mathcal{L}(L^2)} \leq C_\mu \|u\|_s,$$

which yields the first assertion.

Now apply  $\mathcal{G}_u$  to (4.6), take the  $L^2$  scalar product with  $W$  and consider the real part to find

$$\text{Re} \langle \mathcal{G}_u W_t, W \rangle = \text{Re} \langle \mathcal{G}_u \text{Op}[\mathcal{M}_u] W, W \rangle + \text{Re} \langle \mathcal{G}_u R_1, W \rangle := \text{Re} \langle \mathcal{G}_u \text{Op}[\mathcal{M}_u] W, W \rangle + R_{21}. \quad (4.11)$$

Due to (4.7) and  $\|\mathcal{G}_u\|_{\mathcal{L}(L^2)} \leq C_\mu$ ,

$$\|R_{21}\| \leq C_\mu \|W\| (\|V\|_s^2 + \|V\|_{s-1}). \quad (4.12)$$

As  $\mathcal{G}_u$  is self-adjoint we get

$$\operatorname{Re}\langle \mathcal{G}_u W_t, W \rangle = \frac{1}{2} \frac{d}{dt} \langle \mathcal{G}_u W, W \rangle - \operatorname{Re} \left\langle \left( \frac{d}{dt} \mathcal{G}_u \right) W, W \right\rangle \quad (4.13)$$

and 2.20 (iv) yields

$$2 \left\| \frac{d}{dt} \mathcal{G}_u \right\|_{\mathcal{L}(L^2)} \leq \left\| \frac{d}{dt} \operatorname{Op}[\tilde{\mathcal{D}}_u] \right\|_{\mathcal{L}(L^2)} \leq C_\mu \|u_t\|_s. \quad (4.14)$$

The second statement then clearly follows from (4.11)-(4.14).  $\square$

The last step consists in showing the following.

**4.6 Lemma.** *It holds*

$$\operatorname{Re}\langle \mathcal{G}_u \operatorname{Op}[\mathcal{M}_u] W, W \rangle \leq -c \|W\|^2 + C_\mu \|W\|^2 (\|u\|_{s+1}^{\frac{1}{2}} + \|u\|_s) + C_\mu \|W\|_{-1}^2.$$

From Lemmas 4.5, 4.6 we obtain

$$\frac{1}{2} \frac{d}{dt} \langle \mathcal{G}_u W, W \rangle + c \|W\|^2 \leq C_\mu \|W\| (\|V\|_s^2 + \|V\|_s \|W\| + \|V\|_s^{\frac{1}{2}}) + C_\mu (\|V\|_{s-1}^2 + \|W\|_{-1}^2). \quad (4.15)$$

As  $\Lambda^{-k} W = \Lambda^{-k} W_\epsilon \rightarrow V$  as  $\epsilon \rightarrow 0$  uniformly with respect to  $t$  for  $0 \leq k \leq s$  and  $\mathcal{G}_u$  is uniformly positive definite, we find by integrating (4.15)

$$\|V(t)\|_s^2 + \int_0^t \|V\|_s^2 d\tau \leq C_\mu (\|V(0)\| + \int_0^t \|V(\tau)\|_s^3 + \|V(\tau)\|_s^{\frac{5}{2}} + \|V(\tau)\|_{s-1}) d\tau, \quad t \in [0, T],$$

which yields the assertion since  $\|V\|_s^2 = \|u\|_{s+1}^2 + \|u_t\|^2$ .  $\square$

*Proof of Lemma 4.6.* Set  $\kappa := \operatorname{op}[\tilde{\mathcal{D}}_0] - \operatorname{Op}[\tilde{\mathcal{D}}_0]$ , which is infinitely smoothing. Then

$$\mathcal{G}_u = \frac{1}{2} (\operatorname{Op}[\tilde{\mathcal{D}}_u] + \operatorname{Op}[\tilde{\mathcal{D}}_u]^*) + \kappa.$$

As  $\|\mathcal{M}_u\|_{\mathcal{L}(H^l, H^{l-1})} \leq C_\mu$ ,  $l \in \mathbb{R}$ , due to Proposition 2.19 (i) we find

$$\operatorname{Re}\langle \kappa \operatorname{Op}[\mathcal{M}_u] W, W \rangle \leq C_\mu \|W\|_{-1}^2$$

By construction  $\tilde{\mathcal{D}}_u = \tilde{\mathcal{D}}_u^*$  and thus 2.19 (ii) yields

$$\operatorname{Re} \left\langle \frac{1}{2} (\operatorname{Op}[\tilde{\mathcal{D}}_u]^* - \operatorname{Op}[\tilde{\mathcal{D}}_u]) \operatorname{Op}[\mathcal{M}_u] W, W \right\rangle \leq C_\mu \|u\|_s \|W\|^2.$$

Next note that  $\tilde{\mathcal{D}}_u(x, \cdot) - \mathcal{D}_u(x, \cdot)$  is compactly supported with support not depending on  $t$ . Therefore  $\operatorname{Op}[\tilde{\mathcal{D}}_u - \mathcal{D}_u]$  is infinitely smoothing and

$$\operatorname{Re}\langle \operatorname{Op}[\tilde{\mathcal{D}}_u - \mathcal{D}_u] \operatorname{Op}[\mathcal{M}_u] W, W \rangle \leq C_\mu \|W\|_{-1}^2.$$

In conclusion

$$\begin{aligned}
\operatorname{Re}\langle \mathcal{G}_u \operatorname{Op}[\mathcal{M}_u]W, W \rangle &= \operatorname{Re}\langle \operatorname{Op}[\mathcal{D}_u] \operatorname{Op}[\mathcal{M}_u]W, W \rangle + \operatorname{Re}\langle \kappa \operatorname{Op}[\mathcal{M}_u]W, W \rangle \\
&+ \frac{1}{2} \operatorname{Re}\langle (\operatorname{Op}[\tilde{\mathcal{D}}_u]^* - \operatorname{Op}[\tilde{\mathcal{D}}_u]) \operatorname{Op}[\mathcal{M}_u]W, W \rangle + \operatorname{Re}\langle \operatorname{Op}[\tilde{\mathcal{D}}_u - \mathcal{D}_u]\mathcal{M}_uW, W \rangle \\
&\leq \operatorname{Re}\langle \operatorname{Op}[\mathcal{D}_u] \operatorname{Op}[\mathcal{M}_u]W, W \rangle + C_\mu(\|u\|_s \|W\|^2 + \|W\|_{-1}^2)
\end{aligned} \tag{4.16}$$

By Proposition 2.19 (iii)

$$\|(\operatorname{Op}[\mathcal{D}_u] \operatorname{Op}[\mathcal{M}_u] - \operatorname{Op}[\mathcal{D}_u \mathcal{M}_u])W\| \leq C_\mu \|u\|_s \|W\| + C \|W\|_{-1}.$$

Hence

$$\operatorname{Re}\langle \operatorname{Op}[\mathcal{D}_u] \operatorname{Op}[\mathcal{M}_u]W, W \rangle \leq \operatorname{Re}\langle \operatorname{Op}[\mathcal{D}_u \mathcal{M}_u]W, W \rangle + C_\mu \|u\|_s \|W\|^2 + C \|W\|_{-1}^2. \tag{4.17}$$

Set  $\mathcal{X}_u := \mathcal{D}_u \mathcal{M}_u + c_\infty / 2I_{2n}$  with  $c_\infty$  as in Lemma 3.7. Note that  $c_\infty$  does not depend on  $\mu$ . Since  $\operatorname{Op}[I_{2n}] - \operatorname{Id}_{\mathcal{L}^2}$  is infinitely smoothing we conclude

$$\operatorname{Re}\langle \operatorname{Op}[\mathcal{D}_u \mathcal{M}_u]W, W \rangle \leq \operatorname{Re}\langle \operatorname{Op}[\mathcal{X}_u]W, W \rangle - \frac{c_\infty}{2} \|W\|^2 + C \|W\|_{-1}^2. \tag{4.18}$$

By Proposition 3.7

$$\mathcal{X}_u(x, \xi) + \mathcal{X}_u^*(x, \xi) = \mathcal{D}_u \mathcal{M}_u(x, \xi) + (\mathcal{D}_u \mathcal{M}_u)(x, \xi)^* + c_\infty \leq 0,$$

for  $x \in \mathbb{R}^d$  und  $\xi \in \mathbb{R}^d$  with  $|\xi| \geq 3r$ . Since  $u \in H^{s+1}$  and  $s+1 \geq d/2 + 2$ , Proposition 2.24 applied to  $-\mathcal{X}_u$  gives

$$\operatorname{Re}\langle \operatorname{Op}[\mathcal{X}_u]W, W \rangle \leq C_\mu (\|u\|_{s+1}^{\frac{1}{2}} \|W\|^2 + \|W\|_{-1}). \tag{4.19}$$

(4.18) and (4.19) lead to

$$\operatorname{Re}\langle \operatorname{Op}[\mathcal{D}_u \mathcal{M}_u]W, W \rangle \leq -c \|W\|^2 + C_\mu (\|u\|_{s+1}^{\frac{1}{2}} \|W\|^2 + \|W\|_{-1}^2) \tag{4.20}$$

for  $c$  independent of  $u$ . Clearly the assertion follows from (4.16), (4.17), (4.18) and (4.20).  $\square$

## 5 A class of examples from dissipative relativistic fluid dynamics

We consider the Euler-augmented Navier-Stokes formulation of dissipative relativistic fluid dynamics on flat Minkowski space-time derived in [12] as a generalization of a model proposed in [1]. For barotropic fluids it consists of a system of four equations which, using Einstein's summation convention, read

$$A^{\alpha\beta\gamma}(\psi^\epsilon) \frac{\partial \psi_\gamma}{\partial x^\delta} = \frac{\partial}{\partial x^\beta} (B^{\alpha\beta\gamma\delta}(\psi^\epsilon) \frac{\partial \psi_\gamma}{\partial x^\delta}), \quad \alpha = 0, 1, 2, 3, \tag{5.1}$$

where all Greek indices run from 0 to 3,  $A^{\alpha\beta\gamma}, B^{\alpha\beta\gamma\delta}$  are contravariant tensors and the unknown function  $\psi^\epsilon = (\psi^0, \psi^1, \psi^2, \psi^3)^t$  determining the state of the fluid is a 4-vector with respect to the Minkowski-metric of flat space-time. More specifically  $\psi^\epsilon = u^\epsilon/\theta$  with  $u^\epsilon$  being the 4-velocity,  $\theta$  the temperature of the fluid. We show that the results of the present work imply non-linear stability of the homogeneous reference state  $\bar{\psi} = \bar{u}^\epsilon/\bar{\theta}$ , where  $\bar{u}^\epsilon = (1, 0, 0, 0)$  represents the fluid's rest frame and  $\bar{\theta} > 0$  is a constant temperature.

For a fluid with equation of state  $p = \rho/r, 1 \leq r < \infty$ ,  $p$  being the pressure,  $\rho$  the specific internal energy, the coefficient matrices evaluated at  $\bar{\psi}$  are given by [12] (w.l.o.g. assume  $\bar{\theta} = 1$ )<sup>5</sup>

$$\begin{aligned} A^0(\bar{\psi}) &= \begin{pmatrix} r & 0 \\ 0 & I_3 \end{pmatrix}, \quad A^j(\bar{\psi}) = \begin{pmatrix} 0 & (e^j)^t \\ e^j & 0 \end{pmatrix}, \quad B^{00}(\bar{\psi}) = \begin{pmatrix} -r^2\mu & 0 \\ 0 & -\nu I_3 \end{pmatrix}, \\ B^{0j}(\bar{\psi}) &= B^{j0}(\bar{\psi}) = \frac{1}{2} \begin{pmatrix} 0 & -(\mu r + \nu)(e^j)^t \\ -(\mu r + \nu)e^j & 0 \end{pmatrix}, \\ B^{ij}(\bar{\psi}) &= \begin{pmatrix} -\nu\delta^{ij} & 0 \\ 0 & \eta\delta^{ij} + \frac{1}{2}(-\mu + \frac{1}{3}\eta + \zeta)(e^i \otimes e^j + e^j \otimes e^i) \end{pmatrix}, \quad i, j = 1, 2, 3, \end{aligned}$$

where  $\eta, \zeta > 0$  quantify the fluid's viscosity,  $\nu, \mu > 0$  with  $\mu > \tilde{\eta} := \frac{4}{3}\eta + \zeta$  reflect a frame change and  $A^\beta(\psi^\epsilon) := (A^{\alpha\beta\gamma}(\psi^\epsilon))_{0 \leq \alpha, \gamma \leq 3}, B^{\beta\gamma}(\psi^\epsilon) := (B^{\alpha\beta\gamma\delta}(\psi^\epsilon))_{0 \leq \alpha, \gamma \leq 3}, \beta, \delta = 0, \dots, 3$ .

We do not give the detailed non-linear formulation at this point and just refer to [12]. The only information we need for the argumentation below is the fact that for all  $\beta, \delta = 0, \dots, 3$  and all states  $\psi^\epsilon$  the coefficient matrices  $A^\beta(\psi^\epsilon), B^{\beta\delta}(\psi^\epsilon), \beta, \delta = 0, \dots, 3$  are symmetric (cf. ibid.).

We show  $(H_A), (H_B), (D)$  for the matrices  $(-B^{00})^{-1}B^{\beta\delta}, (-B^{00})^{-1}A^\beta$ .

$(H_A)$  is straightforward: As  $-B^{00}(\psi^\epsilon), A^0(\psi^\epsilon)$  are positive definite at  $\psi^\epsilon = \bar{\psi}$  and symmetric for all states they are symmetric positive definite also in a neighbourhood of  $\bar{\psi}$ . Thus  $(H_A)$  (a) is satisfied with  $F_A(u) = -B^{00}(u)$  and  $(H_A)$  (b) with  $H(u) = A^0(u)$ .

Regarding  $(H_B)$  Freistühler proved ibid. that at the reference state  $\psi^\epsilon = \bar{\psi}$  for each  $\omega \in \mathbb{S}^2$  the matrix

$$\tilde{\mathcal{B}}(\psi^\epsilon, \omega) = \begin{pmatrix} 0 & I_4 \\ -(-B^{00})^{-\frac{1}{2}}B(\psi^\epsilon, \omega)(-B^{00})^{-\frac{1}{2}} & i(-B^{00})^{-\frac{1}{2}}C(\psi^\epsilon, \omega)(-B^{00})^{-\frac{1}{2}} \end{pmatrix},$$

where

$$B(\psi^\epsilon, \omega) = \sum_{ij=0}^d B^{ij}(\psi^\epsilon)\omega_i\omega_j, \quad C(\psi^\epsilon, \omega) = 2 \sum_{j=0}^d B^{0j}(\psi^\epsilon)\omega_j, \quad \omega = (\omega_1, \dots, \omega_2) \in \mathbb{S}^2,$$

---

<sup>5</sup>Here  $e^1, e^2, e^3$  and  $\delta^{ij}$  denote the cononical basis of  $\mathbb{R}^3$  and the Kronecker symbol, respectively.



has four simple and two semi-simple purely imaginary eigenvalues. This is then also true for

$$\mathcal{B}(\psi^\epsilon, \boldsymbol{\omega}) := \begin{pmatrix} 0 & I_4 \\ (-B^{00})^{-1}B(\psi^\epsilon, \boldsymbol{\omega}) & i(-B^{00})^{-1}C(\psi^\epsilon, \boldsymbol{\omega}) \end{pmatrix} = \mathcal{T}^{-1}\tilde{\mathcal{B}}(\psi^\epsilon, \boldsymbol{\omega})\mathcal{T}$$

with  $\mathcal{T} = \text{diag}((-B^{00})^{\frac{1}{2}}, (-B^{00})^{\frac{1}{2}})$ . Now in the present context the geometric multiplicities of purely imaginary eigenvalues of  $\mathcal{B}(\psi^\epsilon, \boldsymbol{\omega})$  are state invariant properties. Therefore there exists a symbolic symmetrizer of  $\mathcal{B}$  due to Remark 3.2.

To see this invariance note that (even in the general setting in Section 3) the eigenvectors  $v = v(u, \boldsymbol{\omega}) \in \mathbb{C}^{2n} \setminus \{0\}$  to an eigenvalue  $\lambda = \lambda(u, \boldsymbol{\omega}) \in \mathbb{C}$  of  $\mathcal{B}(u, \boldsymbol{\omega})$  are exactly of the form  $v = (v_1, \lambda v_1)$  with  $v_1 \in \mathbb{C}^n$  such that  $e^{\lambda t + i\xi} v_1$  is a plane wave solution to the linearization of (1.1) at  $u$ . As (5.1) is a covariant expression,  $e^{\lambda t + i\xi} v$  being a plane wave solution with  $\lambda \in i\mathbb{R}$  is also a covariant property (cf. e.g. [13]).

It remains to show (D1), (D2), (D3). In the following we only consider matrices evaluated at  $\bar{\psi}$ . The Fourier-symbols corresponding to the differential operators in (5.1) are given by

$$A(\boldsymbol{\omega}) = \sum_{j=1}^d A^j \omega_j = \begin{pmatrix} 0 & \boldsymbol{\omega}^t \\ \boldsymbol{\omega} & 0 \end{pmatrix}, \quad B(\boldsymbol{\omega}) = \sum_{j,k=1}^d B^{jk} \omega_j \omega_k = \begin{pmatrix} -r^2 \mu & 0 \\ 0 & \eta + (-\mu + \frac{1}{3}\eta + \zeta) \boldsymbol{\omega} \otimes \boldsymbol{\omega} \end{pmatrix},$$

$$C(\boldsymbol{\omega}) = 2 \sum_{j=1}^d B^{0j} \xi_j \begin{pmatrix} 0 & -(\mu r + \nu) \boldsymbol{\omega}^t \\ -(\mu r + \nu) \boldsymbol{\omega} & 0 \end{pmatrix}, \quad \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^{d-1}.$$

It is straightforward to see that for any  $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$  the matrices  $A^0, A^j(\boldsymbol{\omega}), B^{00}, B^{jk}(\boldsymbol{\omega}), C(\boldsymbol{\omega})$  all decompose in sense of linear operators as  $A^0 = A_l^0 \oplus A_t^0$ ,  $A(\boldsymbol{\omega}) = A_l \oplus A_t$ ,  $B^{00} = B_l^{00} \oplus B_t^{00}$ ,  $B(\boldsymbol{\omega}) = B_l \otimes B_t$ ,  $C(\boldsymbol{\omega}) = C_l \oplus C_t$  with respect to the orthogonal decomposition  $\mathbb{C}^4 = (\mathbb{C} \times \boldsymbol{\omega} \mathbb{C}) \oplus (\{0\} \times \{\boldsymbol{\omega}\}^\perp)$ . Thus we can verify the conditions for  $A_l^0, A_l, B_l^{00}, B_l, C_l$  and  $A_t^0, A_t, B_t^{00}, B_t, C_t$  separately. We have

$$A_l^0 = I_2, \quad A_t = 0, \quad B^{00} = -\nu I_2, \quad B_t = \eta I_2, \quad C_t = 0.$$

As  $\eta > 0$ , these matrices correspond to coefficients of damped wave equations and it is well-known that such equations satisfy (D). One can also check this easily by virtue of [14], Theorem 4 and Lemma 5.

Next

$$A_l^0 = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}, \quad A_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$B_l^{00} = \begin{pmatrix} -r^2 \mu & 0 \\ 0 & -\nu \end{pmatrix}, \quad B_t = \begin{pmatrix} -\nu & 0 \\ 0 & \tilde{\eta} - \mu \end{pmatrix}, \quad C_t = \begin{pmatrix} 0 & -(\mu r + \nu) \\ -(\mu r + \nu) & 0 \end{pmatrix}.$$

It was shown in [14] that

$$\tilde{A}^j = (-B_l^{00})^{-\frac{1}{2}} A_l^j (-B_l^{00})^{-\frac{1}{2}}, \quad \tilde{B}^{jk} = (-B_l^{00})^{-\frac{1}{2}} B_l^{jk} (-B_l^{00})^{-\frac{1}{2}}$$

satisfy (D). But then also  $\check{A}^j := (-B_l^{00})^{-1} A_l^j$ ,  $\check{B}^{jk} := (-B_l^{00})^{-1} B_l^{jk}$  satisfy (D).

To see this note that  $\check{A}^j = S\tilde{A}^jS^{-1}$ ,  $\check{B}^{jk} = S\tilde{B}^{jk}S^{-1}$  with  $S = (-B^{00})^{\frac{1}{2}}$ . Hence we have  $\check{W}_0 = S\tilde{W}_0S^{-1}$  for  $\check{W}_0$  and  $\tilde{W}_0$  as in (D1) for the matrices  $\check{A}^j, \check{B}^{jk}$  and  $\tilde{A}^j, \tilde{B}^{jk}$ , respectively. Further the symbolic symmetrizer  $\check{H} = \check{A}^0$  of  $(\check{A}^0)^{-1}\check{A}(\omega)$  the matrix  $\check{H} = S^{-1}\tilde{A}^0S^{-1}$  is a symbolic symmetrizer for  $(\tilde{A}^0)^{-1}\tilde{A}(\omega)$ . This yields  $\check{W}_1 = S^{-1}\tilde{W}_1S^{-1}$  with  $\check{W}_1, \tilde{W}_1$  as in (D1) for the respective matrices. If now  $v$  is an eigenvector of  $\check{W}_0$ ,  $S^{-1}v$  is an eigenvector of  $\tilde{W}_0$  and as  $\tilde{A}^j, \tilde{B}^{jk}$  satisfy (D1) we get

$$\langle (\check{W}_1 + \check{W}_1^*)S^{-1}v, S^{-1}v \rangle \leq -c|S^{-1}v|^2 \leq -\check{c}|v|^2,$$

i.e.  $\langle (\check{W}_1 + \check{W}_1^*)v, v \rangle \leq -\check{c}|v|^2$ , which proves (D1) for  $\check{A}^j, \check{B}^{jk}$ .

(D2) follows analogously since with  $\mathcal{S} = \text{diag}((-B^{00})^{\frac{1}{2}}, (-B^{00})^{\frac{1}{2}})$  the matrix  $\mathcal{S}^{-1}\check{\mathcal{H}}(\omega)\mathcal{S}^{-1}$  is a symbolic symmetrizer for  $\check{\mathcal{B}}(\omega)$  if  $\tilde{\mathcal{H}}(\omega)$  is a symbolic symmetrizer for  $\tilde{\mathcal{B}}(\omega)$ .

Lastly, (D3) is satisfied trivially, as the matrices introduce equivalent systems of PDEs and thus solutions to the dispersion relation are identical for the two systems.

## Statements and declarations

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## References

- [1] F. S. Bemfica, M. M. Disconzi, and J. Noronha. Causality and existence of solutions of relativistic viscous fluid dynamics with gravity. *Phys. Rev. D*, 98(10):104064, 2018.
- [2] F. S. Bemfica, M. M. Disconzi, and J. Noronha. First-order general relativistic viscous fluid dynamics. *Phys. Rev. X*, 12(2):021044, 2022.
- [3] F. S. Bemfica, M. M. Disconzi, C. Rodriguez, and Y. Shao. Local existence and uniqueness in Sobolev spaces for first-order conformal causal relativistic viscous hydrodynamics. *Commun. Pure Appl. Anal.*, 20(6):2279–2290, 2021.
- [4] S. Benzoni-Gavage and D. Serre. *Multidimensional Hyperbolic Partial Differential Equations: First-order Systems and Applications*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2006.

- [5] S. Bianchini, B. Hanouzet, and R. Natalini. Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. *Comm. Pure Appl. Math.*, 60(11):1559–1622, 2007.
- [6] J. M. Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sc. Ec. Norm. Sup. (4)*, 14(2):209–246, 1981.
- [7] G. Bourdaud. *Sur les opérateurs pseudo-différentiels à coefficients peu réguliers*. PhD thesis, Univ. de Paris-Sud, 1983.
- [8] G. Bourdaud. Une algèbre maximale d’opérateurs pseudo-différentiels. *Comm. Partial Diff. Eq.*, 13(9):1059–1083, 1988.
- [9] G.-Q. Chen, C. D. Levermore, and T.-P. Liu. Hyperbolic conservation laws with stiff relaxation terms and entropy. *Comm. Pure Appl. Math.*, 47(6):787–830, 1994.
- [10] J. Chen and W. Wang. The point-wise estimates for the solution of damped wave equation with nonlinear convection in multi-dimensional space. *Commun. Pure Appl. Anal.*, 13(1):307–330, 2014.
- [11] P. M. N. Dharmawardane, T. Nakamura, and S. Kawashima. Global solutions to quasi-linear hyperbolic systems of viscoelasticity. *Kyoto J. Math.*, 51(2):467–483, 2010.
- [12] H. Freistühler. A class of Hadamard well-posed five-field theories of dissipative relativistic fluid dynamics. *J. Math. Phys.*, 61(3):033101, 2020.
- [13] H. Freistühler, M. Reintjes, and B. Temple. Decay and subluminality of modes of all wave numbers in the relativistic dynamics of viscous and heat conductive fluids. *J. Math. Phys.*, 62(5):053101, 2021.
- [14] H. Freistühler and M. Sroczinski. A class of uniformly dissipative symmetric hyperbolic-hyperbolic systems. *J. Differ. Equ.*, 288:40–61, 2021.
- [15] H. Freistühler and B. Temple. Causal dissipation and shock profiles in the relativistic fluid dynamics of pure radiation. *Proc. R. Soc. A*, 470(2166):20140055, 2014.
- [16] H. Freistühler and B. Temple. Causal dissipation for the relativistic dynamics of ideal gases. *Proc. R. Soc. A*, 473(2201):20160729, 2017.
- [17] H. Freistühler and B. Temple. Causal dissipation in the relativistic dynamics of barotropic fluids. *J. Math. Phys.*, 59(6):063101, 2018.
- [18] P. Gérard and J. Rauch. Propagation de la régularité locale de solutions d’équations hyperboliques non linéaires. *Ann. Inst. Fourier (Grenoble)*, 37(3):65–84, 1987.
- [19] B. Hanouzet and R. Natalini. Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. *Arch. Rational Mech. Anal.*, 169(2):89–117, 2003.

- [20] I. Hashimoto and Y. Ueda. Asymptotic behaviour of solutions for damped wave equations with non-convex convection term on the half line. *Osaka J. Math.*, 49(1):37–52, 2012.
- [21] L. Hörmander. *The analysis of linear partial differential operators III: Pseudo-differential operators*, volume 274 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, 1985.
- [22] L. Hörmander. *Lectures on Nonlinear Hyperbolic Differential Equations*, volume 26 of *Mathematiques et Applications*. Springer, Berlin, Heidelberg, New York, 1997.
- [23] M. Kato and Y. Ueda. Asymptotic profile of solutions for the damped wave equation with a nonlinear convection term. *Math. Methods Appl. Sci.*, 40(18):7760–7779, 2017.
- [24] S. Kawashima. *Systems of a Hyperbolic-Parabolic Composite Type, with Applications to the Equations of Magnetohydrodynamics*. PhD thesis, Kyoto University, 1983.
- [25] S. Kawashima and Y. Ueda. Large time behavior of solutions to a semilinear hyperbolic system with relaxation. *J. Hyperbolic Differ. Equ.*, 4(1):147–179, 2007.
- [26] S. Kawashima and W.-A. Yong. Dissipative structure and entropy for hyperbolic systems of balance laws. *Arch. Rational Mech. Anal.*, 174(3):345–364, 2004.
- [27] Y. Liu and S. Kawashima. Global existence and asymptotic decay of solutions to the nonlinear Timoshenko system with memory. *Nonlinear Anal.*, 84:1–17, 2013.
- [28] G. Métivier. Stability of multidimensional shocks. In *Advances in the theory of shock waves*, volume 47 of *Progr. Nonlinear Differential Equations Appl.*, pages 25–103. Birkhäuser, Boston, MA, 2001.
- [29] Y. Meyer. Régularité des solutions des équations aux dérivées partielles non linéaires (d’après J.-M. Bony). In *Bourbaki Seminar, Vol. 1979/80*, volume 842 of *Lecture Notes in Math.*, pages 293–302. Springer, Berlin, New York, 1981.
- [30] Y. Meyer. Remarques sur un théorème de J.-M. Bony. In *Proceedings of the Seminar on Harmonic Analysis (Pisa, 1980)*, number suppl.1 in *Rend. Circ. Mat. Palermo (2)*, pages 1–20. 1981.
- [31] R. Orive and E. Zuazua. Long-time behavior of solutions to a nonlinear hyperbolic relaxation system. *J. Differ. Equ.*, 288(1):17–38, 2006.
- [32] R. Racke, W. Wang, and R. Xue. Optimal decay rates and global existence for a semilinear Timoshenko system with two damping effects. *Math. Methods Appl. Sci.*, 40(1):210–222, 2017.
- [33] T. Ruggeri and D. Serre. Stability of constant equilibrium state for dissipative balance laws system with a convex entropy. *Quart. Appl. Math.*, 62(1):163–179, 2004.

- [34] Y. Shizuta and S. Kawashima. Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. *Hokkaido Math. J.*, 14(2):249–275, 1985.
- [35] M. Sroczinski. Asymptotic stability of homogeneous states in the relativistic dynamics of viscous, heat-conductive fluids. *Arch. Rational Mech. Anal.*, 231(1):91–113, 2019.
- [36] M. Sroczinski. *Global existence and asymptotic decay for quasilinear second-order symmetric hyperbolic systems of partial differential equations occurring in the relativistic dynamics of dissipative fluids*. PhD thesis, University of Konstanz, 2019.
- [37] M. Sroczinski. Asymptotic stability in a second-order symmetric hyperbolic system modeling the relativistic dynamics of viscous heat-conductive fluids with diffusion. *J. Differ. Equ.*, 268(2):825–851, 2020.
- [38] M. E. Taylor. *Pseudodifferential operators and nonlinear PDE*, volume 100 of *Progress in mathematics*. Birkhäuser Boston Inc., Boston, MA, 1991.
- [39] Y. Ueda, T. Nakamura, and S. Kawashima. Energy method in partial Fourier space and application to stability problems in the half space. *J. Differ. Equ.*, 250(2):1169–1199, 2015.
- [40] G. B. Whitham. *Linear and nonlinear waves*. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974.
- [41] W.-A. Yong. Basic aspects of hyperbolic relaxation systems. In *Advances in the theory of shock waves*, volume 47 of *Progr. Nonlinear Differential Equations Appl.*, pages 259–305. Birkhäuser Boston, Boston, MA, 2001.
- [42] W.-A. Yong. Entropy and global existence for hyperbolic balance laws. *Arch. Rational Mech. Anal.*, 172(2):247–266, 2004.