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GLOBAL EXISTENCE AND STABILITY FOR EULER-BERNOULLI BEAM EQUATION WITH MEMORY CONDITION AT THE BOUNDARY

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ABSTRACT. In this article we prove the existence of the solution to the mixed problem for Euler-Bernoulli beam equation with memory condition at the boundary and we study the asymptotic behavior of the corresponding solutions. We proved that the energy decay with the same rate of decay of the relaxation function, that is, the energy decays exponentially when the relaxation function decay exponentially and polynomially when the relaxation function decay polynomially.

1. Introduction

The main purpose of this work is to study the asymptotic behavior of the solutions of Euler-Bernoulli Beam Equation with boundary condition of memory type. For this, we consider the following initial boundary-value problem:

$$u_{tt} + u_{xxxx} + f(u_t) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$u(0,t) = u_x(0,t) = u_{xx}(L,t) = 0, \quad \forall t > 0, \tag{1.2}$$

$$-u(L,t) + \int_0^t g(t-\tau)u_{xxx}(L,\tau)d\tau = 0, \quad \forall t > 0, \tag{1.3}$$

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) \quad \text{in} \quad \Omega, \tag{1.4}$$

where $\Omega = [0,L]$, $\|\cdot\|$ is the norm of $L^2(\Omega)$.

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The integral equation (1.3) describe the memory effect which can be caused, for example, by the interaction with another viscoelastic element. Frictional dissipative boundary condition for the wave equation was studied by several authors, see for example([1, 3, 4, 5, 6, 8, 12, 13, 14])among others. In these works existence of solutions and exponential stabilization were proved for linear and for nonlinear equations. In contrast with the large literature for frictional dissipative, for boundary condition with memory, we have only a few works as for example([2, 3, 9, 10, 11]). The main result this paper is to show that the solutions of system (1.1)–(1.4) decays uniformly in time with the same rate of decay of the relaxation function. More precisely, denoting by κ the resolvent kernel of $g'/g(0)$, we show that the solution decays exponentially to zero provided κ decays exponentially to zero. When the resolvent kernel κ decays polynomially, we show that the corresponding solution also decays polynomially to zero.

The method used here is based on the construction of a suitable Lyapunov functional $\mathcal{L}$ satisfying

$$\frac{d}{dt}\mathcal{L}(t) \le -c_1\mathcal{L}(t) + c_2 e^{-\gamma t} \quad \text{or} \quad \frac{d}{dt}\mathcal{L}(t) \le -c_1\mathcal{L}(t)^{1+1/\alpha} + \frac{c_2}{(1+t)^{\alpha+1}}$$

for some positive constants c_1, c_2, γ, and α.

Note that, because of condition (1.2) the solution of system (1.1)–(1.4) must belong to the following space:

$$V = \{u \in H^2(0,L) \mid u(0) = u_x(0) = 0\}$$

and

$$W = \{u \in V \cap H^4(0,L) \mid u_{xx}(L) = 0\}.$$

The notation used in this paper is standard and can be found in Lions' book[7]. In the sequel by c (sometime $c_1, c_2, \cdots$) we denote various positive constants independent of t and on the initial data. The organization of this paper is as follow. In Section 2, we establish the existence of strong solutions for system (1.1)–(1.4). In Section 3, we prove the uniform rate of exponential decay. In Section 4, we prove the uniform rate of polynomial decay.

2. Existence and regularity

In this section, we shall study the existence and regularity of solutions for system (1.1)–(1.4). To facilitate our analysis, we introduce the following binary operators:

$$(f\Box\phi)(t) = \int_0^t f(t-s)|\phi(t)-\phi(s)|^2 ds,$$
$$(f*\phi)(t) = \int_0^t f(t-s)\phi(s)ds,$$

where * is the convolution product.

Differentiating (1.3), we arrive to the Volterra integral equations:

$$u_{xxx}(L,t) + \frac{1}{g(0)} g' * u_{xxx}(L,t) = \frac{1}{g(0)} u_t(L,t).$$

Applying the Volterra's inverse operator, we get

$$u_{xxx}(L,t) = \frac{1}{g(0)}\{u_t(L,t) + \kappa * u_t(L,t)\},$$

where the resolvent kernel satisfy

$$\kappa + \frac{1}{g(0)} g' * \kappa = -\frac{1}{g(0)} g'.$$

Denoting by $\tau = \frac{1}{g(0)}$, we obtain

$$u_{xxx}(L,t) = \tau\{u_t(L,t) + \kappa(0)u(L,t) - \kappa(t)u(L,0) + \kappa' * u(L,t)\}. \tag{2.1}$$

Since we are interested in relaxation function of exponential or polynomial type and identity (2.1) involve the resolvent kernel κ, we want to know if κ has the same properties. The following lemma answers this question. Let h be a relaxation function and κ its resolvent kernel, that is

$$\kappa(t) - \kappa * h(t) = h(t). \tag{2.2}$$

LEMMA 2.1. *If h is a positive continuous function, then κ also is a positive continuous function. Moreover,*

(1) *If there exist positive constants c_0 and γ with $c_0 < \gamma$ such that*

$$h(t) \le c_0 e^{-\gamma t},$$

then, the function κ satisfies

$$\kappa(t) \le \frac{c_0(\gamma-\epsilon)}{\gamma-\epsilon-c_0} e^{-\epsilon t},$$

for all $0 < \epsilon < \gamma - c_0$.

(2) *Given* $p > 1$, *let us denote by*

$$c_p = \sup_{t \in \mathbb{R}^+} \int_0^t (1+t)^p (1+t-s)^{-p} (1+s)^{-p} ds.$$

If there exists a positive constant c_0 *with* $c_0 c_p < 1$ *such that*

$$h(t) \leq c_0 (1+t)^{-p},$$

then the function κ *satisfies*

$$\kappa(t) \leq \frac{c_0}{1 - c_0 c_p} (1+t)^{-p}.$$

Proof. Note that $\kappa(0) = h(0) > 0$. Now, we take $t_0 = \inf\{t \in \mathbb{R}^+ : \kappa(t) = 0\}$, so $\kappa(t) > 0$ for all $t \in [0, t_0)$. If $t_0 \in \mathbb{R}^+$, from (2.2) we get that $-\kappa * h(t_0) = h(t_0)$ but this is contradictory. Therefore $\kappa(t) > 0$ for all $t \in \mathbb{R}_0^+$. Now, let us fix ϵ, such that $0 < \epsilon < \gamma - c_0$ and denote by

$$\kappa_\epsilon(t) = e^{\epsilon t} \kappa(t), \quad h_\epsilon(t) = e^{\epsilon t} h(t).$$

Multiplying (2.2) by $e^{\epsilon t}$, we get $\kappa_\epsilon(t) = h_\epsilon(t) + \kappa_\epsilon * h_\epsilon(t)$, hence

$$\begin{aligned} \sup_{s \in [0,t]} \kappa_\epsilon(s) \leq & \sup_{s \in [0,t]} h_\epsilon(s) \\ & + \int_0^\infty c_0 e^{(\epsilon - \gamma) s} ds \times \sup_{s \in [0,t]} \kappa_\epsilon(s) \\ \leq & \, c_0 + \frac{c_0}{\gamma - \epsilon} \sup_{s \in [0,t]} \kappa_\epsilon(s). \end{aligned}$$

Therefore

$$\kappa_\epsilon(t) \leq \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0},$$

which implies our first assertion. To show the second part let us consider the following notations

$$\kappa_p(t) = (1+t)^p \kappa(t), \quad h_p(t) = (1+t)^p h(t).$$

Multiplying (2.2) by $(1+t)^p$, we get $\kappa_p(t) = h_p(t) + \int_0^t \kappa_p(t-s)(1+t-s)^{-p}(1+t)^p h(s) ds$, hence

$$\begin{aligned} \sup_{s \in [0,t]} \kappa_p(s) \leq & \sup_{s \in [0,t]} h_p(s) + c_0 c_p \sup_{s \in [0,t]} \kappa_p(s) \\ \leq & \, c_0 + c_0 c_p \sup_{s \in [0,t]} \kappa_p(s). \end{aligned}$$

Therefore

$$\kappa_p(t) \leq \frac{c_0}{1 - c_0 c_p},$$

which proves our second assertion. $\square$

Due to this Lemma, in the remainder of this paper, we shall use (2.1) instead of (1.3). The following lemma state an important property of the convolution operator.

LEMMA 2.2. *For $f, \phi \in \mathbb{C}^1([0,\infty);\mathbb{R})$, we have*

$$\int_0^t f(t-s)\phi(s)ds \cdot \phi_t$$
$$= -\frac{1}{2}f(t)|\phi(t)|^2 + \frac{1}{2}f'\Box\phi - \frac{1}{2}\frac{d}{dt}\Big[f\Box\phi - \Big(\int_0^t f(s)ds\Big)|\phi|^2\Big].$$

The proof of this lemma follows by differentiating the term $f\Box\phi$. The first-order energy of system (1.1)–(1.4) is given by

$$E(t,u) = \frac{1}{2}\Big(\|u_t(t)\|^2 + \|u_{xx}(t)\|^2 - \tau\kappa'\Box u(L,t) + \tau\kappa(t)|u(L,t)|^2\Big).$$

We summarize the well-posedness of (1.1)–(1.4) in the following theorem.

THEOREM 2.1. *Let $\kappa \in \mathbb{C}^2(\mathbb{R}^+)$ be such that $\kappa, -\kappa', \kappa'' \geq 0$ and $f;\mathbb{R} \to \mathbb{R}$ is continuously differentiable function there exist ρ a positive constant such that*

$$f(0) = 0 \quad and \quad (f(r) - f(s), r - s) \geq \rho|r - s|^2, \quad \forall r, s \in \mathbb{R}.$$

If $u_0 \in W$, $u_1 \in L^2(\Omega)$ satisfying the compatibility condition $u_{xxx}(L,0) = \tau u_t(L,0)$, then there is only one solution u of system (1.1)–(1.4) satisfying

$$u \in L^\infty(0,\infty;V),\ u' \in L^\infty(0,\infty;V),\ u'' \in L^\infty(0,\infty;L^2(\Omega)). \tag{2.3}$$

Proof. The main idea is to use the Galerkin method. To do this let us take a basis $\{w_j\}_{j\in\mathbb{N}}$ to V which is orthonormal in $L^2(\Omega)$ and we represent by V_m the subspace of V generated by the first m vector. Standard results on ordinary differential equations guarautee that there exists only one local solution $u^m(t) = \Sigma_{j=1}^m g_{j,m}(t)w_j$, of the approximate system,

$$\begin{aligned}&(u^m_{tt}, w) + (u^m_{xx}, w_{xx}) + (f(u^m_t), w)\\ &= -\Big(\tau u^m_t(L,t) + \kappa(0)u^m(L,t) - \kappa(t)u^m(L,0)\\ &\quad + \kappa' * u^m(L,t), w(L,t)\Big),\end{aligned} \tag{2.4}$$

for all $w \in V_m$ with the initial data

$$(u^m(0), u^m_t(0)) = (u^0, u^1).$$

The extension of these solutions to the whole interval $[0,T], 0 < T < \infty$, is a consequence of the first estimate which we are going to prove below.

A Priori Estimate I.

Replacing w by $u_t^m(t)$ in (2.4), using Lemma 2.2 and from hypothesis of f, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u_t^m(t)\|^2 + \|u_{xx}^m(t)\|^2 - \tau\kappa'\Box u^m(L,t) + \tau\kappa(t)|u^m(L,t)|^2\Big) \\
&+ \rho\|u_t^m(t)\|^2 \\
\le & -\tau|u_t^m(L,t)|^2 + \tau\kappa(t)(u^m(L,0), u_t^m(L,t)) \\
&+ \frac{\tau}{2}\kappa'(t)|u^m(L,t)|^2 - \frac{\tau}{2}\kappa''\Box u^m(L,t) \\
\le & -\frac{\tau}{2}|u_t^m(L,t)|^2 + \frac{\tau}{2}\kappa^2(t)|u^m(L,0)|^2 + \frac{\tau}{2}\kappa'(t)|u^m(L,t)|^2 \\
&- \frac{\tau}{2}\kappa''\Box u^m(L,t).
\end{aligned}$$

Using $\kappa, -\kappa', \kappa'' \ge 0$, we get

$$\frac{d}{dt}E(t,u^m) + \rho\|u_t^m(t)\|^2 \le cE(0,u^m).$$

Integrating it over $[0,t]$ and taking into account the definition of the initial data of u^m, we conclude that

$$\begin{aligned}
\|u_t^m(t)\|^2 + \|u_{xx}^m(t)\|^2 + \rho\int_0^t \|u_t^m(s)\|^2 ds \\
\le c, \ \forall t \in [0,T], \quad \forall m \in \mathbb{N}.
\end{aligned} \tag{2.5}$$

A Priori Estimate II.

First of all, we are estimating $u_{tt}^m(0)$ in the L^2-norm. Considering $t = 0$ and $w = u_{tt}^m(0)$ in (2.4) and using the compatibility condition we obtain

$$\|u_{tt}^m(0)\|^2 + (u_{xxxx}^m(0), u_{tt}^m(0)) + (f(u_t^m(0)), u_{tt}^m(0)) = 0.$$

Since $u_0 \in W, u_1 \in L^2(\Omega)$, the growth hypothesis for the function f imply that $f(u_1) \in L^2(\Omega)$. Hence $\|u_{tt}^m(0)\|^2 \le C, \ \ \forall m \in \mathbb{N}$

Finally, differentiating (2.4) and multiplying the both sides of equation by $u_{tt}^m(t)$, then we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u_{tt}^m(t)\|^2 + \|u_{xxt}^m(t)\|^2\Big) + (f'(u_t^m)u_{tt}^m, u_{tt}^m) \\
= &- \tau|u_{tt}^m(L,t)|^2 - \tau\kappa(0)(u_t^m(L,t), u_t^m(L,t)) \\
&+ \tau\kappa'(t)\Big(u^m(L,0), u_{tt}^m(L,t)\Big) - \tau\Big((\kappa' * u^m(L,t))_t, u_{tt}^m(L,t)\Big).
\end{aligned}
$$

Noting that

$$
(\kappa' * u^m)_t = \kappa'(t)u_0^m + \int_0^t \kappa'(t-s)u_t^m(s)ds
$$

and using Lemma 2.2, we obtain

(2.6)
$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u_{tt}^m(t)\|^2 + \|u_{xxt}^m(t)\|^2 - \tau\kappa'\Box u_t^m(L,t) + \tau\kappa(t)|u_t^m(L,t)|^2\Big) \\
&+ (f'(u_t^m)u_{tt}^m, u_{tt}^m) \\
= &- \tau|u_{tt}^m(L,t)|^2 + \tau\kappa'(t)(u^m(L,0), u_{tt}^m(L,t)) \\
&+ \frac{\tau}{2}\kappa'(t)|u_t^m(L,t)|^2 - \frac{\tau}{2}\kappa''\Box u_t^m(L,t).
\end{aligned}
$$

We also note that from assumption on the function f, we get

$$
\Big|\Big(f'(u_t^m)u_{tt}^m, u_{tt}^m\Big)\Big| \le c\|u_{tt}^m(t)\|^2. \tag{2.7}
$$

Substitution of inequality (2.7) into (2.6), we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u_{tt}^m(t)\|^2 + \|u_{xxt}^m(t)\|^2 - \tau\kappa'\Box u_t^m(L,t) + \tau\kappa(t)|u_t^m(L,t)|^2\Big) \\
&\le c\|u_{tt}^m(t)\|^2 + \frac{\tau}{2}(\kappa'(t))^2|u^m(L,0)|^2.
\end{aligned}
$$

Integrating with respect to the time and applying Gronwall's inequality we conclude that

$$
\|u_{tt}^m(t)\|^2 + \|u_{xxt}^m(t)\|^2 \le c, \quad \forall m \in \mathbb{N}, \quad \forall t \in [0,T]. \tag{2.8}
$$

By estimates (2.5) and (2.8), we obtain

$$
\left\{
\begin{array}{lll}
(u^m) & \text{is bounded in} & L^\infty(0,T;V), \\
(u_t^m) & \text{is bounded in} & L^\infty(0,T;V), \\
(u_{tt}^m) & \text{is bounded in} & L^\infty(0,T;L^2(\Omega)).
\end{array}
\right.
$$

Therefore, we can get subsequences, if necessary, denoted by (u^m), such that

$$\begin{cases} u^m & \rightarrow & u \text{ weakly star in } & L^\infty(0,T;V), \\ u_t^m & \rightarrow & u_t \text{ weakly star in } & L^\infty(0,T;V), \\ u_{tt}^m & \rightarrow & u_{tt} \text{ weakly star in } & L^\infty(0,T;L^2(\Omega)). \end{cases}$$

We can use Lions-Aubin Lemma to get the necessary compactness in order to pass (2.4) to the limit. Then it is matter of routine to conclude the existence of global solutions in [0, T]. The uniqueness is straightforward by standard methods and Gronwall's inequality. □

3. Exponential decay

In this section, we shall study the asymptotic behavior of the solutions of system (1.1)–(1.4) when the resolvent kernel κ is exponentially decreasing, that is, there exist positive constants b_1, b_2 such that

$$\kappa(0) > 0, \quad \kappa'(t) \leq -b_1\kappa(t), \quad \kappa''(t) \geq -b_2\kappa'(t). \tag{3.1}$$

Note that this conditions implies that $\kappa(t) \leq \kappa(0)e^{-b_1 t}$.

Our point of departure will be to establish some ineqalities for the strong solution of system (1.1)–(1.4).

LEMMA 3.1. *Any strong solution u of system* (1.1)–(1.4) *satisfy*

$$\begin{aligned}\frac{d}{dt}E(t) \leq &-\frac{\tau}{2}|u_t(L,t)|^2 + \frac{\tau}{2}\kappa^2(t)|u(L,t)|^2 \\ &+\frac{\tau}{2}\kappa'(t)|u(L,t)|^2 - \frac{\tau}{2}\kappa''\Box u(L,t) - \rho\|u_t(t)\|^2.\end{aligned}$$

Proof. Multiplying (1.1) by u_t and integrating by parts over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\Big(\|u_t(t)\|^2 + \|u_{xx}(t)\|^2\Big) + \Big(f(u_t(t)), u_t(t)\Big) = -\Big(u_{xxx}(L,t), u_t(L,t)\Big).$$

Substituting the boundary term, using Lemma 2.1 and assumption of f, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u_t(t) + \|u_{xx}(t)\|^2 - \tau\kappa'\Box u(L,t) + \tau\kappa(t)|u(L,t)|^2\Big) \\
&+ \Big(f(u_t(t)), u_t(t)\Big) \\
= &-\tau|u_t(L,t)|^2 + \frac{\tau}{2}\kappa'(t)|u(L,t)|^2 - \frac{\tau}{2}\kappa''\Box u(L,t) \\
&+ \tau\kappa(t)\Big(u(L,0), u_t(L,t)\Big) \\
\leq &-\frac{\tau}{2}|u_t(L,t)|^2 + \frac{\tau}{2}\kappa'(t)|u(L,t)|^2 - \frac{\tau}{2}\kappa''\Box u(L,t) \\
&+ \frac{\tau}{2}\kappa^2(t)|u(L,t)|^2.
\end{aligned}$$

Using assumption of f, we obtain

$$\begin{aligned}
\frac{d}{dt}E(t) \leq &-\frac{\tau}{2}|u_t(L,t)|^2 + \frac{\tau}{2}\kappa^2(t)|u(L,t)|^2 + \frac{\tau}{2}\kappa'(t)|u(L,t)|^2 \\
&- \frac{\tau}{2}\kappa''\Box u(L,t) - \rho\|u_t(t)\|^2.
\end{aligned}$$

□

Let us consider the following binary operator:

$$(\kappa \diamond \varphi)(t) = \int_0^t \kappa(t-s)(\varphi(t) - \varphi(s))ds.$$

Then applying the Hölder's inequality for $0 \leq \mu \leq 1$, we have

$$|(\kappa \diamond \varphi)(t)|^2 \leq \Big[\int_0^t |\kappa(s)|^{2(1-\mu)}ds\Big](|\kappa|^{2\mu}\Box\varphi)(t). \tag{3.2}$$

Let us introduce the following functional:

$$\Psi(t) = \delta(u(t), u_t(t)) \ \text{ with } \ 0 < \delta < \frac{1}{2}.$$

The following lemma plays an important role for the construction of the Lyapunov functional.

LEMMA 3.2. *For any strong solution of system* (1.1)–(1.4), *we get*

$$\begin{aligned}
\frac{d}{dt}\Psi(t) \leq &-\delta\|u_{xx}(t)\|^2 + c\|u_t(t)\|^2 + c\|u(t)\|^2 + c|u_t(L,t)|^2 + c|u(L,t)|^2 \\
&+ c\kappa(t)|u(L,0)|^2 + c\kappa(0)|\kappa'|\Box u(L,t) + c\kappa(t)|u(L,t)|^2.
\end{aligned}$$

Proof. From (1.1), it follows that

$$\begin{aligned}\frac{d}{dt}\Psi(t) &= \delta(u(t),u_{tt}(t)) + \delta\|u_t(t)\|^2 \\ &= \delta\|u_t(t)\|^2 - \delta\|u_{xx}(t)\|^2 - \delta\Big(f(u_t(t)),u(t)\Big) \\ &\quad - \delta\tau(u_t(L,t),u(L,t)) + \delta\tau\kappa(t)(u(L,0),u(L,t)) \\ &\quad - \delta\tau\Big(\kappa(0)u(L,t) + \kappa' * u(L,t), u(L,t)\Big).\end{aligned} \tag{3.3}$$

Note that

$$\begin{aligned}&-\kappa(0)u(L,t) - \kappa' * u(L,t) \\ &\qquad = -\int_0^t \kappa'(t-s)\Big[u(L,s)-u(L,t)\Big]ds - \kappa(t)u(L,t) \\ &\qquad \le \Big(\int_0^t |\kappa'(s)|ds\Big)^{\frac{1}{2}}\Big[|\kappa'|\Box u(L,t)\Big]^{\frac{1}{2}} + \kappa(t)|u(L,t)| \\ &\qquad \le |\kappa(t)-\kappa(0)|^{\frac{1}{2}}\Big[|\kappa'|\Box u(L,t)\Big]^{\frac{1}{2}} + \kappa(t)|u(L,t)|.\end{aligned} \tag{3.4}$$

Using (3.3), (3.4), and Young's inequality, it follows that

$$\begin{aligned}\frac{d}{dt}\Psi(t) &\le \delta\|u_t(t)\|^2 - \delta\|u_{xx}(t)\|^2 - \delta\Big(f(u_t(t)),u(t)\Big) \\ &\quad - \delta\tau(u_t(L,t),u(L,t)) + \delta\tau\kappa(t)(u(L,0),u(L,t)) \\ &\quad + \delta\tau\Big(|\kappa(t)-\kappa(0)|^{\frac{1}{2}}\Big[|\kappa'|\Box u(L,t)\Big]^{\frac{1}{2}}, u(L,t)\Big) + \delta\tau\kappa(t)|u(L,t)|^2 \\ &\le -\delta\|u_{xx}(t)\|^2 + c\|u_t(t)\|^2 + c\|u(t)\|^2 + c|u_t(L,t)|^2 + c|u(L,t)|^2 \\ &\quad + c\kappa(t)|u(L,0)|^2 + c\kappa(0)|\kappa'|\Box u(L,t) + c\kappa(t)|u(L,t)|^2.\end{aligned}$$

□

Let us introduce the Lyapunov functional

$$\mathcal{L}(t) = NE(t) + \Psi(t), \quad \text{with} \quad N > 0. \tag{3.5}$$

Using Young's inequality and taking N large enough we find that

$$q_0E(t) \le \mathcal{L}(t) \le q_1E(t), \quad \text{for some positive constants } q_0 \text{ and } q_1. \tag{3.6}$$

We will show later that the functional $\mathcal{L}$ satisfies the inequality of the following Lemma.

LEMMA 3.3. *Let f be a real positive function of class $\mathbb{C}^1$. If there exists positive constants γ_0, γ_1 and c_0 such that $f'(t) \le -\gamma_0 f(t) + c_0e^{-\gamma_1 t}$, then there exist positive constants γ and c such that $f(t) \le (f(0) + c)e^{-\gamma t}$.*

Proof. First, let us suppose that $\gamma_0 < \gamma_1$. Define $F(t)$ by

$$F(t) = f(t) + \frac{c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t}.$$

Then

$$F'(t) = f'(t) - \frac{\gamma_1 c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t} \le -\gamma_0 F(t).$$

Integrating from 0 to t, we arrive to

$$F(t) \le F(0) e^{-\gamma_0 t} \Rightarrow f(t) \le \Big(f(0) + \frac{c_0}{\gamma_1 - \gamma_0} \Big) e^{-\gamma_0 t}.$$

Now, we shall assume that $\gamma_0 \ge \gamma_1$. In this conditions, we get

$$f'(t) \le -\gamma_1 f(t) + c_0 e^{-\gamma_1 t} \Rightarrow (e^{\gamma_1 t} f(t))' \le c_0.$$

Integrating from 0 and t, we obtain

$$f(t) \le (f(0) + c_0 t) e^{-\gamma_1 t}.$$

Since $t \le (\gamma_1 - \epsilon) e^{(\gamma_1 - \epsilon) t}$ for any $0 < \epsilon < \gamma_1$, we conclude that

$$f(t) \le (f(0) + c_0(\gamma_1 - \epsilon)) e^{-\epsilon t}.$$

This completes the proof. □

Finally, we shall show the main result of this section.

THEOREM 3.1. *Let us suppose that the initial data $(u_0, u_1) \in W \times L^2(\Omega)$ and that the resolvent κ satisfies the conditions* (3.1). *Then there exist positive constants α_1 and γ_1 such that*

$$E(t) \le \alpha_1 e^{-\gamma_1 t} E(0) \quad \textit{for all} \quad t \ge 0.$$

Proof. We will suppose that $(u_0, u_1) \in (H^4(\Omega) \cap W) \times W$ and satisfies the compatibility conditions $u_{xxx}(L, 0) = \tau u_t(L, 0)$; our conclusion will follow by standard density arguments. Using Lemmas 3.1 and Lemma 3.2, we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \le & - \Big(\frac{\tau}{2} N - c\Big) |u_t(L,t)|^2 + \frac{\tau}{2} N \kappa^2(t) |u(L,t)^2| + \frac{\tau}{2} N \kappa'(t) |u(L,t)|^2 \\
& - \frac{\tau}{2} N \kappa'' \Box u(L,t) - (N\rho - \delta) \|u_t(t)\|^2 - \delta \|u_{xx}(t)\|^2 \\
& + c\|u(t)\|^2 + c|u(L,t)|^2 + c\kappa(t)|u(L,0)|^2 \\
& + c\kappa(0)|\kappa'|\Box u(L,t) + c\kappa(t)|u(L,t)|^2.
\end{aligned}$$

Then, choosing N large enough and $N\rho > \delta$, $\frac{\tau}{2} N > c$, we obtain

$$\frac{d}{dt}\mathcal{L}(t) \le -q_2 E(t) + c\kappa^2(t) E(0), \quad \text{where} \quad q_2 > 0 \quad \text{is a small constant.}$$

Here we use (3.1) to conclude the following estimates for the corresponding two terms appearing in Lemma 3.1.

$$-\frac{\tau}{2}\kappa''\Box u(L,t) \leq c_1\kappa'\Box u(L,t),$$
$$\frac{\tau}{2}\kappa'|u(L,t)|^2 \leq -c_1\kappa|u(L,t)|^2.$$

Finally, from (3.1) and (3.6), we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{q_2}{q_1}\mathcal{L}(t) + cE(0)\exp(-2b_1t).$$

Using the exponential decay of the resolvent kernel κ and Lemma 3.3, we conclude

$$\mathcal{L}(t) \leq \{\mathcal{L}(0) + c\}e^{-\gamma_1 t} \quad \text{for all} \quad t \geq 0.$$

Use of (3.6) now completes the proof. □

4. Polynomial rate of decay

The proof of the existence of global solutions for (1.1)–(1.4) with resolvent kernel κ decaying polynomially is essentially the same as in Section 3. Here our attention will be focused on the uniform rate of decay when the resolvent κ decays polynomially such as $(1+t)^{-p}$. In this case, we will show that the solution also decays polynomially with the same rate. We shall use the following hypotheses:

$$\begin{aligned} 0 < \kappa(t) &\leq b_0(1+t)^{-p}, \\ -b_1\kappa^{\frac{p+1}{p}} \leq \kappa'(t) &\leq -b_2\kappa^{\frac{p+1}{p}}, \\ b_3(-\kappa')^{\frac{p+2}{p+1}} \leq \kappa''(t) &\leq b_4(-\kappa')^{\frac{p+2}{p+1}}, \end{aligned} \tag{4.1}$$

where $p > 1$ and b_i, $i = 0, 1, \cdots, 4$, are positive constants.

Also we assume that

$$\int_0^\infty |\kappa'(t)|^r dt < \infty \quad \text{if} \quad r > \frac{1}{p+1}. \tag{4.2}$$

The following lemmas will play an important role in the sequel.

LEMMA 4.1. *Let m and h be integrable functions, $0 \leq r < 1$ and $q > 0$. Then, for $t \geq 0$:*

$$\int_0^t |m(t-s)h(s)|ds$$
$$\leq \left(\int_0^t |m(t-s)|^{1+\frac{1-r}{q}}|h(s)|ds\right)^{\frac{q}{q+1}} \left(\int_0^t |m(t-s)|^r|h(s)|ds\right)^{\frac{1}{q+1}}.$$

Proof. Let

$$v(s) = |m(t-s)|^{1-\frac{r}{q+1}}|h(s)|^{\frac{q}{q+1}}, w(s) = |m(t-s)|^{\frac{r}{q+1}}|h(s)|^{\frac{1}{q+1}}.$$

Then using Hölder's inequality with $\delta = \frac{q}{q+1}$ for v and $\delta^* = q+1$ for w, we arrive to the conclusion. □

LEMMA 4.2. *Let $p > 1$, $0 \leq r < 1$ and $r \geq 0$. Then for $r > 0$,*

$$\left(|\kappa'|\Box u(L,t)\right)^{\frac{1+(1-r)(p+1)}{(1-r)(p+1)}}$$
$$\leq 2\Big(\int_0^t |\kappa'(s)|^r ds \|u\|^2_{L^\infty(0,T;L^2(0,L))}\Big)^{\frac{1}{(1-r)(p+1)}} \left(|\kappa'|^{1+\frac{1}{p+1}}\Box u(L,t)\right),$$

and for $r = 0$,

$$\left(|\kappa'|\Box u(L,t)\right)^{\frac{p+2}{p+1}}$$
$$\leq 2\left(\int_0^t \|u(s)\|^2_{L^2(0,L)}ds + t\|u(s)\|^2_{L^2(0,L)}\right)^{\frac{1}{p+1}} \left(|\kappa'|^{1+\frac{1}{p+1}}\Box u(L,t)\right).$$

Proof. The above inequality is a immediate consequence of Lemma 4.1 with $m(s) = |\kappa'(s)|$, $h(s) = |u(x,t) - u(x,s)|^2$, $q = (1-r)(p+1)$, and t fixed. □

LEMMA 4.3. *Let $\alpha > 0$, $\beta \geq \alpha + 1$, and $f \geq 0$ be differentiable function satisfying $f'(t) \leq \frac{-\bar{c_1}}{f(0)^{\frac{1}{\alpha}}} f(t)^{1+\frac{1}{\alpha}} + \frac{\bar{c_2}}{(1+t)^\beta} f(0)$ for $t \geq 0$ and some positive constants $\bar{c_1}$, $\bar{c_2}$. Then there exists a constant $\bar{c_3} > 0$ such that for $t \geq 0$, $f(t) \leq \frac{\bar{c_3}}{(1+t)^\alpha} f(0)$.*

Proof. Let $t \geq 0$ and

$$F(t) = f(t) + \frac{2\bar{c_2}}{\alpha}(1+t)^{-\alpha}f(0).$$

Then

$$\begin{aligned} F' &= f' - 2\bar{c_2}(1+t)^{-(\alpha+1)}f(0) \\ &\leq \frac{-\bar{c_1}}{f(0)^{\frac{1}{\alpha}}} f^{1+\frac{1}{\alpha}} - \bar{c_2}(1+t)^{-(\alpha+1)}f(0), \end{aligned}$$

where we used $\beta \geq \alpha + 1$. Hence

$$F' \leq \frac{-c}{f(0)^{\frac{1}{\alpha}}} \left(f^{1+\frac{1}{\alpha}} + (1+t)^{-(\alpha+1)} f(0)^{1+\frac{1}{\alpha}} \right) \leq \frac{-c}{F(0)^{\frac{1}{\alpha}}} F^{1+\frac{1}{\alpha}}.$$

Integration yields $F(t) \leq \frac{F(0)}{(1+ct)^{\alpha}} \leq \frac{c}{(1+t)^{\alpha}} f(0)$, hence $f(t) \leq \frac{\bar{c_3}}{(1+t)^{\alpha}} f(0)$ for some $\bar{c_3}$, which proves Lemma 4.3. □

THEOREM 4.1. *Assume that* $(u_0, u_1) \in W \times L^2(0,L)$ *and that the resolvent* κ *satisfies* (4.1). *Then there exists a positive constant* c *for which*

$$E(t) \leq \frac{c}{(1+t)^{p+1}} E(0).$$

Proof. We will suppose that $(u_0, u_1) \in H^2(0,L) \cap W \times W$ and satisfies the compatibility condition; our conclusion will follows by standard density arguments. We define the functional $\mathcal{L}$ as in (3.5) and we have the equivalence to the energy term E as given in (3.6) again.

From the Lemmas 3.1 and 3.2, we conclude that

$$\begin{aligned} \frac{d}{dt}\mathcal{L}(t) \leq & -c_1\Big(|u_t(L,t)|^2 + \kappa''\Box u(L,t) \\ & + \|u(t)\|^2 + \|u_{xx}(t)\|^2\Big) + c_2\kappa^2(t)E(0). \end{aligned} \tag{4.3}$$

From hypothesis (4.1), we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{L}(t) \leq & -c_1\Big(|u_t(L,t)|^2 + (-\kappa')^{1+\frac{1}{p+1}}\Box u(L,t) \\ & + \|u(t)\|^2 + \|u_{xx}(t)\|^2\Big) + c_2\kappa^2(t)E(0). \end{aligned} \tag{4.4}$$

From Lemma 4.2, we get

$$\begin{aligned} &(-\kappa')^{1+\frac{1}{p+1}}\Box u(L,t) \\ &\geq \frac{c}{\left(\int_0^t |\kappa'|^r ds\right)^{\frac{1}{(1-r)(p+1)}} E(0)^{\frac{1}{(1-r)(p+1)}}} \left((-\kappa')\Box u(L,t)\right)^{1+\frac{1}{(1-r)(p+1)}}. \end{aligned} \tag{4.5}$$

On the other hand, since the energy is bounded, we have

$$\begin{aligned} &\Big(|u_t(L,t)|^2 + \|u(t)\|^2 + \|u_{xx}(t)\|^2\Big)^{1+\frac{1}{(1-r)(p+1)}} \\ &\leq cE(0)^{\frac{1}{(1-r)(p+1)}} \Big(|u_t(L,t)|^2 + \|u(t)\|^2 + \|u_{xx}(t)\|^2\Big). \end{aligned} \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4), we arrive at

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{E(0)^{\frac{1}{(1-r)(p+1)}}}\Big[\big(|u_t(L,t)|^2 + \|u(t)\|^2 + \|u_{xx}(t)\|^2\big)^{1+\frac{1}{(1-r)(p+1)}} + \big(|\kappa'|\Box u(L,t)\big)^{1+\frac{1}{(1-r)(p+1)}}\Big] + c\kappa^2(t)E(0).$$

Taking into account (3.6), we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{\frac{1}{(1-r)(p+1)}}}\mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}} + c\kappa^2(t)E(0). \tag{4.7}$$

Applying the Lemma 4.3 with $f = \mathcal{L}$ and $\beta = 2p$, we have:

$$\mathcal{L}(t) \leq \frac{c}{(1+t)^{(1-r)(p+1)}}\mathcal{L}(0). \tag{4.8}$$

Since $(1-r)(p+1) > 1$,

$$\int_0^\infty E(s)ds \leq c\int_0^\infty \mathcal{L}(s)ds \leq c\mathcal{L}(0), \tag{4.9}$$

$$t\|u(t)\|^2_{L^2(0,L)} \leq ct\mathcal{L}(t) \leq c\mathcal{L}(0), \tag{4.10}$$

$$\int_0^t \|u(s)\|^2_{L^2(0,L)}ds \leq c\int_0^\infty \mathcal{L}(t)dt \leq c\mathcal{L}(0). \tag{4.11}$$

In this conditions applying Lemma 4.2 for $r = 0$, we get

$$(-\kappa')^{1+\frac{1}{p+1}}\Box u(L,t) \geq \frac{c}{E(0)^{\frac{1}{p+1}}}\Big((-\kappa)\Box u(L,t)\Big)^{1+\frac{1}{p+1}}.$$

Using the same arguments as in the derivation of (4.7), we have

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{\frac{1}{p+1}}}\mathcal{L}(t)^{1+\frac{1}{p+1}} + c\kappa^2(t)E(0).$$

Applying the Lemma 4.3, we obtain

$$\mathcal{L}(t) \leq \frac{c}{(1+t)^{p+1}}\mathcal{L}(0).$$

Finally, from (3.6) we obtain $E(t) \leq \frac{c}{(1+t)^{p+1}}E(0)$, which complete the present proof. □

References

[1] P. L. Chow and J. L. Menaldi, *Boundary stabilization of a nonlinear string with nerodynamic force, Control of Partial Differential Equations*, Lecture notes in pure and Appl. Math. **165** (1994), 63–79.

[2] M. Ciarletta, *A differential problem for heat equation with a boundary condition with memory*, Appl. Math. Lett. **10** (1997), no. 1, 95–191.

[3] M. Fabrizio and M. Morro, *A boundary condition with memory in Electromagretism*, Arch. Ration. Mech. Anal. **136** (1996), 359–381.

[4] V. Komornik, *Rapid boundary stabilization of the wave equation*, SIAM J. Control Optim. **29** (1991), 197–208.

[5] V. Komornik and E. Zuazua, *A direct method for the boundary stabilization of the wave equation*, J. Math. Pures Appl. **69** (1990), 33–54.

[6] I. Lasiecka, *Global uniform decay rates for the solution to the wave equation with nonlinear boundary conditions*, Appl. Anal. **47** (1992), 191–212.

[7] J. L. Lions, *Quelques Methodes de resolution de problémes aux limites non lineaires*, Dunod Gauthiers Villars, Paris, 1969.

[8] M. Milla Miranda and L. A. Medeiros, *On boundary value problem for wave equations : Existence Uniqueness-Asymptotic behavior*, Rev. Math. Apl. **17** (1996), 47–73.

[9] J. E. Muñoz Rivera and D. Andrade, *Exponential decay of nonlinear wave equation with a viscoelastic boundary condition*, Math. Methods Appl. Sci. **23** (2000), 41–61.

[10] T. Qin, *Breakdown of solutions to nonlinear wave equation with a viscoelastic boundary condition*, Arab. J. Sci. Engng. **19** (1994), no. 2A, 195–201.

[11] ———, *Global solvability of nonlinear wave equation with a viscoelastic boundary condition*, Chin. Ann. Math. **14B** (1993), no. 3, 335–346.

[12] M. Tucsnak, *Boundary stabilization for stretched string equation*, Differential Integral Equation **6** (1993), no. 4, 925–935.

[13] J. Vancostenoble and P. Martinez, *Optimality of energy estimate for the wave equation with nonlinear boundary velocity feedbacks*, SIAM J. Control Optim. **39** (2000), no. 3, 776–797.

[14] E. Zuazua, *Uniform stabilization of the wave equation by nonlinear boundary feedback*, SIAM J. Control Optim. **28** (1990), 466–477.

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