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GLOBAL EXISTENCE AND STABILITY OF MILD SOLUTIONS TO THE BOLTZMANN SYSTEM FOR GAS MIXTURES

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Abstract. We present the global existence and stability of mild solutions to the Boltzmann system with inverse power molecular interactions for a binary gas mixture, when initial data are sufficiently small and decay exponentially in phase space. For the existence and stability of mild solutions, we employ a modified Kaniel-Shinbrot's scheme and a weighted nonlinear functional approach. Time-asymptotic convergence toward the free molecular motion is established using a weighted collision potential, and we show that the weighted L^1-distance between two mild solutions is uniformly controlled by that of initial data.

1. Introduction. This paper is devoted to the global existence and stability of mild solutions to the Boltzmann system for a binary gas mixture in a near free molecular regime, which is a perturbation of a background vacuum state. Consider the dynamics of a dilute binary gas mixture consisting of two neutral species A and B. In kinetic theory, the dynamics of dilute gases are effectively described by velocity moments of the velocity distribution function. We denote by F_A and F_B the velocity distribution functions of species A and B respectively. Then the spatial-temporal evolution of these

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velocity distribution functions is described by the Boltzmann system [5]:

$$\begin{aligned} &\partial_t F_A + \xi \cdot \nabla_x F_A = Q^{AA}(F_A, F_A) + Q^{AB}(F_A, F_B), \qquad x, v \in \mathbb{R}^3,\ t > 0, \\ &\partial_t F_B + \xi \cdot \nabla_x F_B = Q^{BB}(F_B, F_B) + Q^{BA}(F_B, F_A), \\ &(F_A, F_B)(x, \xi, 0) = (F_{A0}, F_{B0})(x, \xi), \end{aligned} \tag{1.1}$$

where $Q^{\alpha\beta}(F_\alpha, F_\beta)$ is the binary collision operator between two species α and β:

$$Q^{\alpha\beta}(F_\alpha, F_\beta)(\xi) \equiv \int_{\mathbb{R}^3 \times \mathbf{S}^2_+} q^{\alpha\beta}(\xi - \xi_*, \omega)(F_\alpha(\xi')F_\beta(\xi'_*) - F_\alpha(\xi)F_\beta(\xi_*))d\omega d\xi_*, \tag{1.2}$$

where we used standard abbreviated notation $F(\xi) = F(x, \xi, t)$, and the pre-collisional velocities (ξ, ξ_*) and post-collisional velocities (ξ', ξ'_*) satisfy the conservations of momentum and energy: $\alpha, \beta \in \{A, B\}$,

$$\begin{aligned} m_\alpha \xi' + m_\beta \xi'_* &= m_\alpha \xi + m_\beta \xi_*, \\ m_\alpha |\xi'|^2 + m_\beta |\xi'_*|^2 &= m_\alpha |\xi|^2 + m_\beta |\xi_*|^2, \end{aligned} \tag{1.3}$$

where m_α and m_β are the masses of gas particles in species A and B respectively. These conservation laws result in the collision transformation: $\omega \in \mathbf{S}^2_+$,

$$\xi' = \xi + \frac{2m_\beta}{m_\alpha + m_\beta}[(\xi_* - \xi) \cdot \omega]\omega, \qquad \xi'_* = \xi_* - \frac{2m_\alpha}{m_\alpha + m_\beta}[(\xi_* - \xi) \cdot \omega]\omega. \tag{1.4}$$

Since our framework is near a vacuum, we employ the notation in [13]:

$$F^\sharp_\alpha(x, \xi, t) \equiv F_\alpha(x + t\xi, \xi, t), \qquad Q^{\alpha\beta\sharp}(F_\alpha, F_\beta)(x, \xi, t) \equiv Q^{\alpha\beta}(F_\alpha, F_\beta)(x + t\xi, \xi, t).$$

We integrate (1.1) along the particle path to find the mild form:

$$\begin{aligned} F^\sharp_A(x, \xi, t) &= F_{A0}(x, \xi) + \int_0^t [Q^{AA\sharp}(F_A, F_A) + Q^{AB\sharp}(F_A, F_B)](x, \xi, s)ds, \\ F^\sharp_B(x, \xi, t) &= F_{B0}(x, \xi) + \int_0^t [Q^{BB\sharp}(F_B, F_B) + Q^{BA\sharp}(F_B, F_A)](x, \xi, s)ds. \end{aligned} \tag{1.5}$$

Definition 1.1. Let T be a given positive number, and let $F_A, F_B \in C([0, T)$ be nonnegative functions. Then $L^1_+(\mathbb{R}^3 \times \mathbb{R}^3)$ are the mild solutions of (1.1) with nonnegative initial data F_{A0}, F_{B0} respectively if and only if for all $t \in [0, T)$ and a.e. $(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3$, F_A, F_B satisfy the system of integral equations (1.5) in the pointwise sense.

For a single-component gas in a near free molecular regime, there are a lot of works available on the global existence of solutions [3, 4, 7, 12, 13, 14, 21, 22] and uniform stability with respect to initial data [6, 9, 10], while the corresponding issues for multi-component gases have not been much addressed in the previous literature (see [18]). The mathematical analysis of (1.1) with suitable boundary and initial conditions is a very formidable job; hence several simplified kinetic models [8, 11, 15] similar in spirit to the BGK model for the Boltzmann equation have been studied. Recently K. Aoki and S. Takata and their group have systematically studied multi-component gases on the evaporation, condensation problem, hydrodynamic limit and gas separations using asymptotic analysis and numerical experiments in a series of papers [1, 2, 16, 17, 19, 20, 23, 24].

The purpose of this paper is to establish the global existence and uniform stability of mild solutions to (1.1) in the framework of [9, 12]. First we modify Illner-Shinbrot's scheme for a single-component gas to a binary gas mixture. Secondly, we introduce nonlinear collision potentials which incorporate the multi-component nature of gases. Throughout the paper, we use the following notation: for $\alpha \in \{A, B\}$,

$$\varphi_{\alpha,r}(\xi) \equiv (1+m_\alpha|\xi|^2)^{\frac{r}{2}}, \quad r \geq 0, \qquad \|F_\alpha(t)\|_{L^1_{\alpha,r}} \equiv \|F_\alpha(\cdot,\cdot,t)\|_{L^1(\varphi_{\alpha,r}dxd\xi)},$$
$$\mathcal{M}_\alpha(x,\xi) \equiv e^{-m_\alpha(p|x|^2+q|\xi|^2)}, \qquad p,\ q>0,$$
$$\mathcal{S}_\alpha \equiv \{F \in C(\mathbb{R}_+; L^1(\mathbb{R}^3\times\mathbb{R}^3)) \mid |F^\sharp(x,\xi,t)| \leq \delta_\alpha \mathcal{M}_\alpha(x,\xi) \quad \text{for some } \delta_\alpha > 0\},$$
$$\|F\|_\alpha \equiv \sup_{0\leq t<\infty} \sup_{x,\xi\in\mathbb{R}^3} [\, e^{m_\alpha(p|x|^2+q|\xi|^2)}|F^\sharp(x,\xi,t)|\,].$$

We next list the main assumptions $\mathcal{A}$ employed in this paper.

- $\mathcal{A}1$: The collision kernel satisfies the inverse power law potential law between gas particles and the angular cut-off assumption:

$$q^{\alpha\beta}(\xi-\xi_*,\omega) = b_\gamma(\theta)|\xi-\xi_*|^\gamma, \quad \frac{b_\gamma(\theta)}{|\cos\theta|} \leq q_* < \infty, \quad \gamma \in (-2,1],$$
$$\theta \equiv \cos^{-1}\Big(\frac{(\xi-\xi_*)\cdot\omega}{|\xi-\xi_*|}\Big),$$

where q_* is a positive constant independent of α and β.
- $\mathcal{A}2$: The initial data satisfy the regularity assumption and are sufficiently small:

$$F_{\alpha 0} \in C(\mathbb{R}^3\times\mathbb{R}^3)\cap\mathcal{S}_\alpha, \qquad 0<\delta_\alpha \ll 1, \qquad \alpha\in\{A,B\}.$$

The main results of this paper are as follows.

THEOREM 1.1. Suppose the main assumptions $\mathcal{A}$ hold. Then the Boltzmann system (1.1) has nonnegative mild solutions $F_A, F_B \in C(\mathbb{R}_+; L^1(\mathbb{R}^3\times\mathbb{R}^3))$ such that

$$0 \leq F_\alpha^\sharp \leq 2\delta_\alpha\mathcal{M}_\alpha, \qquad \alpha\in\{A,B\}.$$

REMARK 1.1. For a single-component gas system, the global existences of mild and classical solutions have been established in [12, 13, 14].

THEOREM 1.2. Suppose the main assumptions $\mathcal{A}$ hold, and let F_α and $\bar{F}_\alpha$ be the mild solutions of (1.1) corresponding to the initial data $F_{\alpha 0}$ and $\bar{F}_{\alpha 0}$ respectively. Then we have

$$\sup_{t\geq 0} \sum_{\alpha\in\{A,B\}} \|F_\alpha(t)-\bar{F}_\alpha(t)\|_{L^1_{\alpha,r}} \leq G \sum_{\alpha\in\{A,B\}} \|F_{\alpha 0}-\bar{F}_{\alpha 0}\|_{L^1_{\alpha,r}},$$

where G is a positive constant independent of t.

REMARK 1.2. For $\varphi_{\alpha,0}\equiv 1$, we recover the well-known uniform L^1-stability estimate (see [6, 9, 10]).

The rest of this paper is organized as follows. In Section 2, we study several basic estimates. In Section 3, we present the global existence of mild solutions using a modified Kaniel-Shinbrot iteration scheme. In Section 4, we study a mollifying procedure and approximate Boltzmann equations. In Section 5, we study a large-time asymptotics of mild solutions using a generalized collision potential. Finally Section 6 is devoted to the uniform $L^1_{\alpha,r}$-stability of mild solutions.

2. Preliminaries. In this section, we provide basic properties of the collision operator $Q^{\alpha\beta}(F_\alpha, F_\beta)$ and several basic estimates which will be crucial to the global existence and stability analysis of mild solutions.

2.1. *Properties of collision operator.* Recall the collision operator for a binary gas mixture:

$$Q^{\alpha\beta}(F_\alpha, F_\beta)(\xi) \equiv \int_{\mathbb{R}^3\times\mathbf{S}^2_+} q^{\alpha\beta}(\xi-\xi_*,\omega)(F_\alpha(\xi')F_\beta(\xi'_*) - F_\alpha(\xi)F_\beta(\xi_*))d\omega d\xi_*.$$

Note that $Q^{\alpha\alpha}(F_\alpha, F_\alpha)$ and $Q^{\alpha\beta}(F_\alpha, F_\beta)$, $\alpha \neq \beta$ denote the self and cross collisions between component molecules in a gas mixture. The structure of the self and cross collision operators yields the following identities.

LEMMA 2.1. Let F_α and F_β be measurable functions decaying at infinity in phase space. Then for any measurable function $\psi = \psi(\xi)$ and $\alpha \neq \beta \in \{A, B\}$, we have

$$(1)\ \int_{\mathbb{R}^3} \psi(\xi) Q^{\alpha\alpha}(F_\alpha, F_\alpha)(\xi)d\xi = \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbf{S}^2_+} q^{\alpha\alpha}(\xi-\xi_*,\omega)[\psi(\xi)+\psi(\xi_*) -\psi(\xi')-\psi(\xi'_*)] \times (F_\alpha(\xi')F_\alpha(\xi'_*) - F_\alpha(\xi)F_\alpha(\xi_*))d\omega d\xi_* d\xi;$$

$$(2)\ \int_{\mathbb{R}^3} \psi(\xi) Q^{\alpha\beta}(F_\alpha, F_\beta)(\xi)d\xi = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbf{S}^2_+} q^{\alpha\beta}(\xi-\xi_*,\omega)[\psi(\xi)-\psi(\xi')] \times (F_\alpha(\xi')F_\beta(\xi'_*) - F_\alpha(\xi)F_\beta(\xi_*))d\omega d\xi_* d\xi.$$

Proof. (1) Since $q^{\alpha\alpha}(\xi-\xi_*,\omega) = |\xi-\xi_*|^\gamma b_\gamma(\theta)$, we have

$$q^{\alpha\alpha}(\xi-\xi_*,\omega) = q^{\alpha\alpha}(\xi_*-\xi,\omega) = q^{\alpha\alpha}(\xi'-\xi'_*,\omega) = q^{\alpha\alpha}(\xi'_*-\xi',\omega).$$

The change of variables

$$(\xi,\xi_*) \leftrightarrow (\xi_*,\xi), \qquad (\xi,\xi_*) \leftrightarrow (\xi',\xi'_*)$$

yields

$$\begin{aligned}
&\int_{\mathbb{R}^3} \psi(\xi) Q^{\alpha\alpha}(F_\alpha, F_\alpha)(\xi)d\xi \\
&\quad = \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbf{S}^2_+} q^{\alpha\alpha}(\xi-\xi_*,\omega)\psi(\xi)(F_\alpha(\xi')F_\alpha(\xi'_*) - F_\alpha(\xi)F_\alpha(\xi_*))d\omega d\xi_* d\xi \\
&\quad = \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbf{S}^2_+} q^{\alpha\alpha}(\xi-\xi_*,\omega)\psi(\xi_*)(F_\alpha(\xi')F_\alpha(\xi'_*) - F_\alpha(\xi)F_\alpha(\xi_*))d\omega d\xi_* d\xi \\
&\quad = \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbf{S}^2_+} q^{\alpha\alpha}(\xi-\xi_*,\omega)(-\psi(\xi'))(F_\alpha(\xi')F_\alpha(\xi'_*) - F_\alpha(\xi)F_\alpha(\xi_*))d\omega d\xi_* d\xi \\
&\quad = \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbf{S}^2_+} q^{\alpha\alpha}(\xi-\xi_*,\omega)(-\psi(\xi'_*))(F_\alpha(\xi')F_\alpha(\xi'_*) - F_\alpha(\xi)F_\alpha(\xi_*))d\omega d\xi_* d\xi.
\end{aligned}$$

We now add the above four equalities to get the desired result.

(2) We use the relation

$$q^{\alpha\beta}(\xi-\xi_*,\omega) = q^{\alpha\beta}(\xi'-\xi'_*,\omega)$$

and the change of variables $(\xi,\xi_*) \leftrightarrow (\xi',\xi'_*)$ to obtain

$$\int_{\mathbb{R}^3} \psi(\xi) Q^{\alpha\beta}(F_\alpha, F_\alpha)(\xi)d\xi$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbf{S}^2_+} q^{\alpha\beta}(\xi-\xi_*,\omega)\psi(\xi)(F_\alpha(\xi')F_\beta(\xi'_*)-F_\alpha(\xi)F_\beta(\xi_*))d\omega d\xi_* d\xi \\
&= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbf{S}^2_+} q^{\alpha\beta}(\xi-\xi_*,\omega)(-\psi(\xi'))(F_\alpha(\xi')F_\beta(\xi'_*)-F_\alpha(\xi)F_\beta(\xi_*))d\omega d\xi_* d\xi.
\end{aligned}
$$

□

The collision operators satisfy the following collision-invariant properties.

PROPOSITION 2.1. Let F_α and F_β be measurable functions decaying sufficiently fast at infinity in phase space. For $\alpha \neq \beta \in \{A, B\}$, we have

(1) $\displaystyle\int_{\mathbb{R}^3} (1, m_\alpha\xi, m_\alpha|\xi|^2)Q^{\alpha\alpha}(F_\alpha, F_\alpha)d\xi = 0,$

(2) $\displaystyle\int_{\mathbb{R}^3} Q^{\alpha\beta}(F_\alpha, F_\beta)d\xi = 0,$

(3) $\displaystyle\int_{\mathbb{R}^3} m_\alpha\xi Q^{\alpha\beta}(F_\alpha, F_\beta)d\xi + \int_{\mathbb{R}^3} m_\beta\xi Q^{\beta\alpha}(F_\beta, F_\alpha)d\xi = 0,$

(4) $\displaystyle\int_{\mathbb{R}^3} m_\alpha|\xi|^2 Q^{\alpha\beta}(F_\alpha, F_\beta)d\xi + \int_{\mathbb{R}^3} m_\beta|\xi|^2 Q^{\beta\alpha}(F_\beta, F_\alpha)d\xi = 0.$

Proof. The proof follows from Lemma 2.1 and (1.3) directly. □

We next briefly discuss the macroscopic observables, which are the moments of the velocity distribution functions F_A and F_B: for $\alpha \in \{A, B\}$,

$$
\begin{aligned}
& n_\alpha \equiv \int_{\mathbb{R}^3} F_\alpha(x,\xi)d\xi, \quad \rho_\alpha = m_\alpha n_\alpha, \quad \rho = \sum_{\alpha\in\{A,B\}} \rho_\alpha, \\
& \rho_\alpha v_\alpha \equiv \int_{\mathbb{R}^3} m_\alpha \xi F_\alpha d\xi, \quad \rho v = \sum_{\alpha\in\{A,B\}} \rho_\alpha v_\alpha, \\
& \frac{3}{2} n\kappa_B T + \frac{1}{2}\rho|v|^2 = \sum_{\alpha\in\{A,B\}} \int_{\mathbb{R}^3} \frac{m_\alpha}{2}|\xi|^2 F_\alpha d\xi,
\end{aligned}
$$

where $\rho_\alpha = m_\alpha n_\alpha$ is the mass density of the species α, $\rho = \sum_{\alpha\in\{A,B\}} \rho_\alpha$ is the total mass density, and $n = \sum_{\alpha\in\{A,B\}} n_\alpha$ is the total number density.

Note that for the special case of all species being mechanically identical, the total distribution function $F = \sum_{\alpha\in\{A,B\}} F_\alpha$ satisfies the Boltzmann equation for a single-component gas:

$$
\begin{aligned}
& \partial_t F + \xi\cdot\nabla_x F = Q(F,F), \qquad x, \xi \in \mathbb{R}^3,\ t > 0, \\
& Q(F,F)(\xi) \equiv \int_{\mathbb{R}^3\times\mathbf{S}^2_+} q(\xi-\xi_*,\omega)(F(\xi')F(\xi'_*)-F(\xi)F(\xi_*))d\omega d\xi_*,
\end{aligned}
\tag{2.1}
$$

where $q(\xi-\xi_*,\omega) = q^{\alpha\alpha}(\xi-\xi_*,\omega)$.

2.2. *Basic estimates.* In this part, we present several estimates that are needed in Section 3.

LEMMA 2.2 ([9]). Let $x \in \mathbb{R}^3$ and $V \neq 0$. Then we have

$$\int_0^\infty e^{-p|x+\tau V|^2} d\tau \leq \sqrt{\frac{\pi}{p}} \frac{1}{|V|} \quad \text{for } \ p > 0.$$

LEMMA 2.3. For $\alpha, \beta \in \{A, B\}$, we have

$$\mathcal{M}_\alpha(x + t(\xi - \xi'), \xi')\mathcal{M}_\beta(x + t(\xi - \xi'_*), \xi'_*) = \mathcal{M}_\alpha(x, \xi)\mathcal{M}_\beta(x + t(\xi - \xi_*), \xi_*).$$

Proof. We use (1.3) to obtain

$$\begin{aligned}
&m_\alpha|x + t(\xi - \xi')|^2 + m_\beta|x + t(\xi - \xi'_*)|^2 \\
&\quad = (m_\alpha + m_\beta)|x + t\xi|^2 - 2t(x + t\xi) \cdot (m_\alpha\xi' + m_\beta\xi'_*) + t^2(m_\alpha|\xi'|^2 + m_\beta|\xi'_*|^2) \\
&\quad = (m_\alpha + m_\beta)|x + t\xi|^2 - 2t(x + t\xi) \cdot (m_\alpha\xi + m_\beta\xi_*) + t^2(m_\alpha|\xi|^2 + m_\beta|\xi_*|^2) \\
&\quad = m_\alpha|(x + t\xi) - t\xi|^2 + m_\beta|(x + t\xi) - t\xi_*|^2 \\
&\quad = m_\alpha|x|^2 + m_\beta|x + t(\xi - \xi_*)|^2.
\end{aligned}$$

This yields the desired result. □

We set gain and loss parts of the collision operator:

$$\begin{aligned}
Q_+^{\alpha\beta}(F_\alpha, F_\beta)(\xi) &\equiv \int_{\mathbb{R}^3 \times \mathbf{S}^2_+} q^{\alpha\beta}(\xi - \xi_*, \omega)F_\alpha(\xi')F_\beta(\xi'_*)d\omega d\xi_*, \\
Q_-^{\alpha\beta}(F_\alpha, F_\beta)(\xi) &\equiv F_\alpha(\xi) \underbrace{\int_{\mathbb{R}^3 \times \mathbf{S}^2_+} q^{\alpha\beta}(\xi - \xi_*, \omega)F_\beta(\xi_*)d\omega d\xi_*}_{\equiv R^{\alpha\beta}(F_\beta)}.
\end{aligned}$$

LEMMA 2.4. Let $F \in \mathcal{S}_\alpha$, $G \in \mathcal{S}_\beta$. Then we have the following estimate:

$$\int_0^t |Q_+^{\alpha\beta\sharp}(F, G)(x, \xi, \tau)|d\tau \leq C_0\|F\|_\alpha\|G\|_\beta\mathcal{M}_\alpha(x, \xi),$$

where C_0 is a positive constant given by

$$C_0 \equiv \pi q_* \sqrt{\frac{\pi}{pm_\beta}} \Big[\frac{4\pi}{\gamma + 2} + \sqrt{\big(\frac{\pi}{qm_\beta}\big)^3}\Big].$$

Proof. We use Lemma 2.3 to get

$$\begin{aligned}
&\int_0^t |Q_+^{\alpha\beta\sharp}(F, G)(x, \xi, \tau)|d\tau \\
&\quad = \int_0^t \Big| \int_{\mathbb{R}^3 \times \mathbf{S}^2_+} q^{\alpha\beta}(\xi - \xi_*, \omega)F^\sharp(x + \tau(\xi - \xi'), \xi', \tau)G^\sharp(x + \tau(\xi - \xi'_*), \xi'_*, \tau)d\omega d\xi_* \Big| d\tau \\
&\quad \leq \pi q_*\|F\|_\alpha\|G\|_\beta \int_0^t \int_{\mathbb{R}^3} |\xi - \xi_*|^\gamma \mathcal{M}_\alpha(x + \tau(\xi - \xi'), \xi')\mathcal{M}_\beta(x + \tau(\xi - \xi'_*), \xi'_*)d\xi_* d\tau \\
&\quad \leq \pi q_*\|F\|_\alpha\|G\|_\beta\mathcal{M}_\alpha(x, \xi) \int_0^t \int_{\mathbb{R}^3} |\xi - \xi_*|^\gamma \mathcal{M}_\beta(x + \tau(\xi - \xi_*), \xi_*)d\xi_* d\tau.
\end{aligned}$$

We use Lemma 2.2 and Fubini's theorem to obtain

$$\int_0^t \int_{\mathbb{R}^3} |\xi - \xi_*|^\gamma \mathcal{M}_\beta(x + \tau(\xi - \xi_*), \xi_*)d\xi_* d\tau$$

$$\begin{aligned}
&\le \int_{\mathbb{R}^3} |\xi-\xi_*|^\gamma e^{-qm_\beta|\xi_*|^2}\Big[\int_0^\infty e^{pm_\beta|x+\tau(\xi-\xi_*)|^2}d\tau\Big]d\xi_* \\
&\le \sqrt{\frac{\pi}{pm_\beta}}\int_{\mathbb{R}^3}|\xi-\xi_*|^{\gamma-1}e^{-qm_\beta|\xi_*|^2}d\xi_* \\
&\le \sqrt{\frac{\pi}{pm_\beta}}\Big[\int_{|\xi-\xi_*|\le 1}|\xi-\xi_*|^{\gamma-1}d\xi_* + \int_{|\xi-\xi_*|>1}e^{-qm_\beta|\xi_*|^2}d\xi_*\Big] \\
&= \sqrt{\frac{\pi}{pm_\beta}}\quad\Big[\frac{4\pi}{\gamma+2}+\sqrt{\big(\frac{\pi}{qm_\beta}\big)^3}\Big].
\end{aligned}$$

□

3. Global existence of mild solutions. In this section, we will present a slightly modified Kaniel-Shinbrot type iteration scheme originally introduced in [13] for (2.1). The details can be found in [12, 13]. We first rewrite (1.1) as

$$\begin{aligned}
&\partial_t F_A + \xi\cdot\nabla_x F_A + F_A\Big[R^{AA}(F_A)+R^{AB}(F_B)\Big] = Q_+^{AA}(F_A,F_A)+Q_+^{AB}(F_A,F_B),\\
&\partial_t F_B + \xi\cdot\nabla_x F_B + F_B\Big[R^{BB}(F_B)+R^{BA}(F_A)\Big] = Q_+^{BB}(F_B,F_B)+Q_+^{BA}(F_B,F_A).
\end{aligned}\tag{3.1}$$

Define four sequences $\{u_A^k\}$, $\{l_A^k\}$, $\{u_B^k\}$, $\{l_B^k\}$ as the solutions of the following linear ODE system:

$$\begin{aligned}
&\frac{d}{dt}l_A^{(k+1)\sharp} + l_A^{(k+1)\sharp}\Big[R^{AA\sharp}(u_A^k)+R^{AB\sharp}(u_B^k)\Big] = Q_+^{AA\sharp}(l_A^k,l_A^k)+Q_+^{AB\sharp}(l_A^k,l_B^k),\\
&\frac{d}{dt}u_A^{(k+1)\sharp} + u_A^{(k+1)\sharp}\Big[R^{AA\sharp}(l_A^k)+R^{AB\sharp}(l_B^k)\Big] = Q_+^{AA\sharp}(u_A^k,u_A^k)+Q_+^{AB\sharp}(u_A^k,u_B^k),\\
&\frac{d}{dt}l_B^{(k+1)\sharp} + l_B^{(k+1)\sharp}\Big[R^{BB\sharp}(u_B^k)+R^{BA\sharp}(u_A^k)\Big] = Q_+^{BB\sharp}(l_B^k,l_B^k)+Q_+^{BA\sharp}(l_B^k,l_A^k),\\
&\frac{d}{dt}u_B^{(k+1)\sharp} + u_B^{(k+1)\sharp}\Big[R^{BB\sharp}(l_B^k)+R^{BA\sharp}(l_A^k)\Big] = Q_+^{BB\sharp}(u_B^k,u_B^k)+Q_+^{BA\sharp}(u_B^k,u_A^k),\\
&\qquad u_A^{k+1}(0)=F_{A0},\quad l_A^{k+1}(0)=F_{A0},\quad u_B^{k+1}(0)=F_{B0},\quad l_B^{k+1}(0)=F_{B0}.
\end{aligned}\tag{3.2}$$

DEFINITION 3.1. Let T be a positive constant. Four functions $(l_\alpha^0, l_\alpha^1, u_\alpha^0, u_\alpha^1)$ satisfy "Beginning condition (BC)" if and only if for $t\in[0,T)$

$$0\le l_\alpha^0(t)\le l_\alpha^1(t)\le u_\alpha^1(t)\le u_\alpha^0(t).$$

For a positive number T, we set

$$\mathcal{S}_{\alpha,T}\equiv\{F\in C^0(\mathbb{R}^3\times\mathbb{R}^3\times[0,T])\ |\quad \|F\|_\alpha\le\delta_\alpha,\text{for some }\delta_\alpha\}.$$

LEMMA 3.1. Let T be a given positive number and F_{A0} and F_{B0} be initial data such that

$$F_{A0}\in\mathcal{S}_A \quad\text{and}\quad F_{B0}\in\mathcal{S}_B,$$

and assume that (BC) holds:

$$\begin{aligned}
&0\le l_A^0(t)\le l_A^1(t)\le u_A^1(t)\le u_A^0(t),\\
&0\le l_B^0(t)\le l_B^1(t)\le u_B^1(t)\le u_B^0(t),\qquad 0\le t<T,
\end{aligned}$$

and that

$$u_A^0(t)\in\mathcal{S}_{A,T},\qquad u_B^0(t)\in\mathcal{S}_{B,T}.$$

Then the linear system (3.2) has the unique solutions

$$l_A^{k\sharp}, u_A^{k\sharp} \in \mathcal{S}_{A,T} \qquad \text{and} \qquad l_B^{k\sharp}, u_B^{k\sharp} \in \mathcal{S}_{B,T}, \qquad \text{for all } k \geq 1$$

satisfying

$$\begin{aligned} &0 \leq l_A^{k-1}(t) \leq u_A^k(t) \leq u_A^k(t) \leq u_A^{k-1}(t), \\ &0 \leq l_B^{k-1}(t) \leq l_B^k(t) \leq u_B^k(t) \leq u_B^{k-1}(t), \qquad 0 \leq t < T. \end{aligned}$$

Proof. We integrate (3.2) along the particle path $(x + s\xi, s)$ to get

$$\begin{aligned} l_A^{k\sharp} &= F_{A0} e^{-\int_0^t [R^{AA\sharp}(u_A^{k-1}) + R^{AB\sharp}(u_B^{k-1})] d\tau} \\ &\quad + \int_0^t e^{-\int_\tau^t [R^{AA\sharp}(u_A^{k-1}) + R^{AB\sharp}(u_B^{k-1})] ds} \Big[Q_+^{AA\sharp}(l_A^{k-1}, l_A^{k-1}) + Q_+^{AB\sharp}(l_A^{k-1}, l_B^{k-1}) \Big] d\tau, \end{aligned} \tag{3.3}$$

$$\begin{aligned} l_A^{k+1\sharp} &= F_{A0} e^{-\int_0^t [R^{AA\sharp}(u_A^{k}) + R^{AB\sharp}(u_B^{k})] d\tau} \\ &\quad + \int_0^t e^{-\int_\tau^t [R^{AA\sharp}(u_A^{k}) + R^{AB\sharp}(u_B^{k})] ds} \Big[Q_+^{AA\sharp}(l_A^{k}, l_A^{k}) + Q_+^{AB\sharp}(l_A^{k}, l_B^{k}) \Big] d\tau. \end{aligned} \tag{3.4}$$

Suppose that

$$u_A^k \leq u_A^{k-1} \qquad \text{and} \qquad l_A^{k-1} \leq l_A^k.$$

Then it follows from the monotonicity properties of $R^{\alpha\beta\sharp}$ and $Q_+^{\alpha\beta\sharp}$ that

$$\int_0^t [R^{AA\sharp}(u_A^k) + R^{AB\sharp}(u_B^k)] d\tau \;\leq\; \int_0^t [R^{AA\sharp}(u_A^{k-1}) + R^{AB\sharp}(u_B^{k-1})] d\tau, \tag{3.5}$$

$$Q_+^{AA\sharp}(l_A^k, l_A^k) + Q_+^{AB\sharp}(l_A^k, l_B^k) \;\geq\; Q_+^{AA\sharp}(l_A^{k-1}, l_A^{k-1}) + Q_+^{AB\sharp}(l_A^{k-1}, l_B^{k-1}). \tag{3.6}$$

We combine (3.5) and (3.6) to conclude

$$l_A^{k\sharp}(t) \leq l_A^{k+1\sharp}(t).$$

Other inequalities can be treated similarly. For existence, once $\{u_A^{k-1}\}$, $\{l_A^{k-1}\}$, $\{u_B^{k-1}\}$, $\{l_B^{k-1}\}$ exist for $0 \leq t < T$, then (3.3) and (3.4) imply the unique existence of $\{u_A^k\}$, $\{l_A^k\}, \{u_B^k\}$, $\{l_B^k\}$. Hence Lemma 2.4 yields

$$u_A^{k\sharp},\; l_A^{k\sharp} \in \mathcal{S}_{A,T} \qquad \text{and} \qquad u_B^{k\sharp},\; l_B^{k\sharp} \in \mathcal{S}_{B,T}.$$

□

Now if we can find suitable functions $u_A^0(t)$, $u_B^0(t)$, $l_A^0(t)$, $l_B^0(t)$ satisfying (BC), then the four sequences are monotone and will converge to $u_A(t)$, $u_B(t)$, $l_A(t)$, $l_B(t)$ respectively. Integrate system (3.2) over t and apply Lebesgue's Dominated Convergence Theorem (in short LDCT). Then we obtain the following system of separated Boltzmann equations:

$$\begin{aligned} u_\alpha^\sharp &= F_{\alpha 0} + \int_0^t \Big(Q_+^{\alpha\alpha\sharp}(u_\alpha, u_\alpha) + Q_+^{\alpha\beta\sharp}(u_\alpha, u_\beta) - u_\alpha R^{\alpha\alpha\sharp}(l_\alpha) - u_\alpha R^{\alpha\beta\sharp}(l_\beta) \Big) ds, \\ l_\alpha^\sharp &= F_{\alpha 0} + \int_0^t \Big(Q_+^{\alpha\alpha\sharp}(l_\alpha, l_\alpha) + Q_+^{\alpha\beta\sharp}(l_\alpha, l_\beta) - l_\alpha R^{\alpha\alpha\sharp}(u_\alpha) - l_\alpha R^{\alpha\beta\sharp}(u_\beta) \Big) ds. \end{aligned}$$

So if we can show furthermore that $u_A(t) = l_A(t)$, $u_B(t) = l_B(t)$ in a suitable norm, our job is done.

The proof of Theorem 1.1. We set

$$l_A^0 = 0, \quad l_B^0 = 0, \quad u_A^0 = 2\delta_A e^{-m_A(p|x|^2+q|\xi|^2)} \quad \text{and} \quad u_B^0 = 2\delta_B e^{-m_B(p|x|^2+q|\xi|^2)}.$$

In (3.2), we have

$$\begin{aligned} l_\alpha^{1\sharp} &= F_{\alpha 0} e^{-\int_0^t [R^{\alpha\alpha\sharp}(u_\alpha^0)+R^{\alpha\beta\sharp}(u_\beta^0)]ds}, \\ u_\alpha^{1\sharp} &= F_{\alpha 0} + \int_0^t \Big[Q_+^{\alpha\alpha\sharp}(u_\alpha^0, u_\alpha^0) + Q_+^{\alpha\beta\sharp}(u_\alpha^0, u_\beta^0)\Big] ds. \end{aligned}$$

These imply

$$l_\alpha^0 = 0 \le l_\alpha^1 \le u_\alpha^1.$$

On the other hand, it follows from Lemma 2.4 that

$$\begin{aligned} u_\alpha^{1\sharp} &= F_{\alpha 0} + \int_0^t \Big[Q_+^{\alpha\alpha\sharp}(u_\alpha^0, u_\alpha^0) + Q_+^{\alpha\beta\sharp}(u_\alpha^0, u_\beta^0)\Big] ds \\ &\le \delta_\alpha(1 + 4\delta_\alpha + 4\delta_\beta) e^{-m_\alpha(p|x|^2+q|\xi|^2)}, \end{aligned}$$

Since $\delta_\alpha, \delta_\beta \ll 1$, we have

$$u_\alpha^{1\sharp} \le 2\delta_\alpha e^{-m_\alpha(p|x|^2+q|\xi|^2)} = u_\alpha^{0\sharp}.$$

Hence we have shown that (BC) holds. We next show that

$$u_\alpha = l_\alpha.$$

Consider the equation for $u_\alpha - l_\alpha,\ \alpha \in \{A, B\}$:

$$\begin{aligned} u_\alpha^\sharp - l_\alpha^\sharp = \int_0^t \Big[& Q_+^{\alpha\alpha\sharp}(u_\alpha - l_\alpha, l_\alpha) + Q_+^{\alpha\alpha\sharp}(u_\alpha, u_\alpha - l_\alpha) \\ & + Q_+^{\alpha\beta\sharp}(u_\alpha, u_\beta - l_\beta) + Q_+^{\alpha\beta\sharp}(u_\alpha - l_\alpha, l_\beta) - (u_\alpha - l_\alpha) R^{\alpha\alpha\sharp}(l_\alpha) \\ & + l_\alpha R^{\alpha\alpha\sharp}(u_\alpha - l_\alpha) - (u_\alpha - l_\alpha) R^{\alpha\beta\sharp}(l_\beta) + l_\alpha R^{\alpha\beta\sharp}(u_\beta - l_\beta)\Big] ds. \end{aligned}$$

It follows from Lemma 2.3 that

$$\begin{aligned} |u_\alpha^\sharp - l_\alpha^\sharp| &\le \mathcal{O}(1)\Big[\|u_\alpha - l_\alpha\|\|u_\beta\| \\ &\qquad + \|u_\alpha\|\|u_\beta - l_\beta\| + \|u_\alpha - l_\alpha\|\|u_\alpha\| + \|l_\alpha\|\|u_\beta - l_\beta\|\Big]\mathcal{M}_\alpha \\ &\le \mathcal{O}(1)(\delta_\alpha + \delta_\beta)\Big[\|u_\alpha - l_\alpha\| + \|u_\beta - l_\beta\|\Big]\mathcal{M}_\alpha. \end{aligned}$$

Hence we have

$$\|u_\alpha - l_\alpha\| \le \mathcal{O}(1)(\delta_\alpha + \delta_\beta)\Big[\|u_\alpha - l_\alpha\| + \|u_\beta - l_\beta\|\Big].$$

This yields

$$\|u_A - l_A\| + \|u_B - l_B\| \le \mathcal{O}(1)(\delta_A + \delta_B)\Big[\|u_A - l_A\| + \|u_B - l_B\|\Big].$$

Hence for sufficiently small δ_A, δ_B, we have $u_A = l_A$, $u_B = l_B$. □

REMARK 3.1. The uniqueness of the mild solution can be shown by the uniform L^1-stability estimate in Theorem 1.2.

In the next section, we study the mollification of continuous mild solutions constructed in this section. Since the nonlinear functional approach proposed in [9] can be applied for classical solutions, we cannot employ the nonlinear functional approach directly; hence we first apply the nonlinear functional approach for mollified solutions, and then passing the mollification parameter to zero, we obtain estimates for the original continuous mild solutions.

4. Mollification procedure and approximate Boltzmann system. In this section, we present a mollification procedure and several estimates of the mild solutions following [6, 10]. The detailed explanation can be found in [6, 10], and for the simplicity of presentation, we suppress the t-dependence in F_α and $Q^{\alpha\beta}(F_\alpha, F_\beta)$:

$$F_\alpha(x,\xi) \equiv F_\alpha(x,\xi,t) \qquad \text{and} \qquad Q^{\alpha\beta}(F_\alpha,F_\beta)(x,\xi) \equiv Q^{\alpha\beta}(F_\alpha,F_\beta)(x,\xi,t).$$

Let $\phi \in C_c^\infty(\mathbb{R}^3)$ and ϕ_ε be the standard mollifier and a corresponding rescaled mollifier, respectively; i.e.,

$$0 \le \phi \le 1, \quad \mathrm{supp}(\phi) \subset B_1(0), \quad \int_{\mathbb{R}^3} \phi(x)dx = 1,$$
$$\phi_\varepsilon(x) \equiv \frac{1}{\varepsilon^3}\phi\Big(\frac{x}{\varepsilon}\Big), \quad \mathrm{supp}(\phi_\varepsilon) \subset B_\varepsilon(0), \quad x \in \mathbb{R}^3,\ \varepsilon > 0,$$

where $B_\varepsilon(0)$ is the ball with radius ε centered at the origin. We denote the mollifications of F_α and $Q^{\alpha\beta}(F_\alpha, F_\beta)$ as

$$F_{\alpha\varepsilon} \equiv F_\alpha *_x \phi_\varepsilon \qquad \text{and} \qquad Q_\varepsilon^{\alpha\beta}(F_\alpha,F_\beta) \equiv Q^{\alpha\beta}(F_\alpha,F_\beta) *_x \phi_\varepsilon,$$

where $*_x$ denotes a convolution with respect to the x-variable.

Consider the mollified Boltzmann system:

$$\begin{cases} \partial_t F_{A\varepsilon} + \xi\cdot\nabla_x F_{A\varepsilon} = Q^{AA}(F_{A\varepsilon},F_{A\varepsilon}) + Q^{AB}(F_{A\varepsilon},F_{B\varepsilon}) + P_\varepsilon^{AA} + P_\varepsilon^{AB}, \\ \partial_t F_{B\varepsilon} + \xi\cdot\nabla_x F_{B\varepsilon} = Q^{BB}(F_{B\varepsilon},F_{B\varepsilon}) + Q^{BA}(F_{B\varepsilon},F_{A\varepsilon}) + P_\varepsilon^{BB} + P_\varepsilon^{BA}, \\ F_{A\varepsilon}(x,\xi,0) = F_{A0} *_x \phi_\varepsilon, \quad F_{B\varepsilon}(x,\xi,0) = F_{B0} *_x \phi_\varepsilon. \end{cases}$$

Here $P_\varepsilon^{\alpha\beta}$ denotes the perturbation given by

$$P_\varepsilon^{\alpha\beta} \equiv Q_\varepsilon^{\alpha\beta}(F_\alpha,F_\beta) - Q^{\alpha\beta}(F_{\alpha\varepsilon},F_{\beta\varepsilon}), \quad \alpha,\beta \in \{A,B\};$$

moreover, we also define a notation:

$$P(x,\xi,\varepsilon) \equiv \sum_{\alpha,\beta\in\{A,B\}} |P_\varepsilon^{\alpha\beta}|.$$

Below we list a series of lemmata given in [10] without proof.

LEMMA 4.1 ([10]). Let (F_A, F_B) be mild solutions in Theorem 1. Then we have

$$F_{\alpha\varepsilon}^\sharp(x,\xi) \le \delta_\alpha \mathcal{M}_{\alpha\varepsilon}(x,\xi) \quad \text{and} \quad \mathcal{M}_{\alpha\varepsilon}(x,\xi) \le e^{-m_\alpha(\frac{p}{2}|x|^2 + q|\xi|^2)}, \qquad \alpha \in \{A,B\}.$$

LEMMA 4.2 ([10]). Suppose the main assumptions $\mathcal{A}$ hold. Then we have

$$\int_{\mathbb{R}^3\times\mathbf{S}_+^2} q^{\alpha\beta}(\xi-\xi_*,\omega)\mathcal{M}_{\alpha\varepsilon}(x+t(\xi-\xi_*),\xi_*)d\omega d\xi_*$$

$$\leq \mathcal{O}(1) \begin{cases} \dfrac{1}{(t+1)^{\gamma+3}}, & \text{if} \quad -2 < \gamma \leq 0, \\ \dfrac{(1+|x|^{\gamma}+|\xi|^{\gamma})}{(t+1)^{\gamma+3}}, & \text{if} \quad 0 < \gamma \leq 1, \end{cases}$$

where $\mathcal{O}(1)$ does not depend on the mollification parameter ε or on t.

LEMMA 4.3 ([10]). For $\gamma \in (-2, 1]$, we have

$$\int_{\mathbb{R}^3 \times \mathbb{R}^+} |\xi - \xi_*|^{\gamma - 1} \mathcal{M}_{\alpha\varepsilon}(x + t(\xi - \xi_*) + \tau n(\xi, \xi_*), \xi_*) d\tau d\xi_* \leq \mathcal{O}(1).$$

LEMMA 4.4. Suppose the main assumptions $\mathcal{A}$ hold, and $(F_\alpha, F_\beta) \in \mathcal{S}_\alpha \times \mathcal{S}_\beta$. Then we have

$$\Big(Q^{\alpha\beta\sharp}_{+\varepsilon}(F_\alpha, F_\beta) + Q^{\alpha\beta\sharp}_{-\varepsilon}(F_\alpha, F_\beta)\Big)(x, \xi) \leq \mathcal{O}(1)\delta_\alpha\delta_\beta \frac{e^{-m_\alpha(\frac{p}{3}|x|^2 + \frac{q}{2}|\xi|^2)}}{(t+1)^{\gamma+3}}.$$

Proof. Since the pointwise estimate for the loss operator can be treated as that for the gain operator, we only consider the gain operator. Recall that

$$F^{\sharp}_{\alpha\varepsilon}(x, \xi) \leq \delta_\alpha \mathcal{M}_{\alpha\varepsilon}(x, \xi).$$

Then we use Lemmas 2.3, 4.1, and 4.2 to find

$$\begin{aligned}
&Q^{\alpha\beta\sharp}_{+}(F_{\alpha\varepsilon}, F_{\beta\varepsilon})(x, \xi) \\
&\quad = \int_{\mathbb{R}^3 \times \mathbf{S}^2_+} q^{\alpha\beta}(\xi - \xi_*, \omega) F^{\sharp}_{\alpha\varepsilon}(x + t(\xi - \xi'), \xi') F^{\sharp}_{\beta\varepsilon}(x + t(\xi - \xi'_*), \xi'_*) d\omega d\xi_* \\
&\quad \leq \mathcal{O}(1)\delta_\alpha\delta_\beta \int_{\mathbb{R}^3 \times \mathbf{S}^2_+} q^{\alpha\beta}(\xi - \xi_*, \omega) \mathcal{M}_{\alpha\varepsilon}(x + t(\xi - \xi'), \xi') \\
&\qquad\qquad\qquad\qquad \times \mathcal{M}_{\beta\varepsilon}(x + t(\xi - \xi'_*), \xi'_*) d\omega d\xi_* \\
&\quad \leq \mathcal{O}(1)\delta_\alpha\delta_\beta \int_{\mathbb{R}^3 \times \mathbf{S}^2_+} q^{\alpha\beta}(\xi - \xi_*, \omega) \mathcal{M}_{\alpha\varepsilon}(x, \xi) \mathcal{M}_{\beta\varepsilon}(x + t(\xi - \xi_*), \xi_*) d\omega d\xi_* \\
&\quad \leq \mathcal{O}(1)\delta_\alpha\delta_\beta e^{-m_\alpha(\frac{p}{2}|x|^2 + q|\xi|^2)} \int_{\mathbb{R}^3 \times \mathbf{S}^2_+} q^{\alpha\beta}(\xi - \xi_*, \omega) e^{-m_\beta(\frac{p}{2}|x + t(\xi - \xi_*)|^2 + q|\xi_*|^2)} d\omega d\xi_* \\
&\quad = \mathcal{O}(1)\delta_\alpha\delta_\beta e^{-m_\alpha(\frac{p}{2}|x|^2 + q|\xi|^2)} \int_{\mathbb{R}^3} |\xi - \xi_*|^{\gamma} e^{-m_\beta(\frac{p}{2}|x + t(\xi - \xi_*)|^2 + q|\xi_*|^2)} d\xi_* \\
&\quad \leq \mathcal{O}(1)\delta_\alpha\delta_\beta \frac{e^{-m_\alpha(\frac{p}{3}|x|^2 + \frac{q}{2}|\xi|^2)}}{(t+1)^{\gamma+3}}.
\end{aligned}$$

□

For convenience of notation, we define

$$\begin{aligned}
P^{\alpha\beta}_{+} &\equiv Q^{\alpha\beta}_{+\varepsilon}(F_\alpha, F_\beta) - Q^{\alpha\beta}_{+}(F_{\alpha\varepsilon}, F_{\beta\varepsilon}), \\
P^{\alpha\beta}_{-} &\equiv Q^{\alpha\beta}_{-\varepsilon}(F_\alpha, F_\beta) - Q^{\alpha\beta}_{-}(F_{\alpha\varepsilon}, F_{\beta\varepsilon}).
\end{aligned}$$

Then it is easy to see that $P^{\alpha\beta}_{\varepsilon} = P^{\alpha\beta}_{+} - P^{\alpha\beta}_{-}$.

LEMMA 4.5. Suppose the main assumptions $\mathcal{A}$ in Section 1 hold. Then we have

$$\begin{aligned}
&(1)\ \lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} |P^{\alpha\beta\sharp}_{\varepsilon}| d\xi dx ds = 0, \\
&(2)\ \lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} P^{\sharp}(x, \xi, \varepsilon) d\xi dx ds = 0.
\end{aligned}$$

Proof. Since the estimate (2) follows from (1), we only prove the estimate (1). Consider

$$\int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} |P_-^{\alpha\beta\sharp}| d\xi dx ds = \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} \Big| Q_{-\varepsilon}^{\alpha\beta\sharp}(F_\alpha, F_\beta) - Q_-^{\alpha\beta\sharp}(F_{\alpha\varepsilon}, F_{\beta\varepsilon}) \Big| d\xi dx ds.$$

Below we show that the right-hand side of the above inequality goes to zero as $\varepsilon \to 0$. From the elementary property of the mollification,

$$q^{\alpha\beta}(\xi-\xi_*,\omega) F_{\alpha\varepsilon}^\sharp F_{\beta*\varepsilon}^\sharp \to q^{\alpha\beta}(\xi-\xi_*,\omega) F_\alpha^\sharp F_{\beta*}^\sharp \qquad \text{a.e. } (x,\xi) \quad \text{as} \quad \varepsilon \to 0,$$

and it follows from Lemma 4.1 that

$$q^{\alpha\beta}(\xi-\xi_*,\omega) F_{\alpha\varepsilon}^\sharp F_{\beta*\varepsilon}^\sharp \le \mathcal{O}(1)\delta_\alpha\delta_\beta e^{-\frac{m_\alpha q}{2}|\xi|^2} e^{-\frac{m_\beta q}{2}|\xi_*|^2} \in L^1(\mathbb{R}^3 \times \mathbf{S}^2).$$

Hence by LDCT,

$$Q_-^{\alpha\beta\sharp}(F_{\alpha\varepsilon}, F_{\beta\varepsilon}) \to Q_-^{\alpha\beta\sharp}(F_\alpha, F_\beta) \qquad \text{a.e. } (x,\xi) \quad \text{as} \quad \varepsilon \to 0.$$

On the other hand, it follows from the elementary property of the mollification that we also have

$$Q_{-\varepsilon}^{\alpha\beta\sharp}(F_\alpha, F_\beta) \to Q_-^{\alpha\beta\sharp}(F_\alpha, F_\beta) \qquad \text{a.e. } (x,\xi) \quad \text{as} \quad \varepsilon \to 0.$$

We combine the above estimate to yield

$$P_-^{\alpha\beta} = Q_{-\varepsilon}^{\alpha\beta}(F_\alpha, F_\beta) - Q_-^{\alpha\beta}(F_{\alpha\varepsilon}, F_{\beta\varepsilon}) \to 0 \qquad \text{a.e. } (x,\xi) \quad \text{as} \quad \varepsilon \to 0.$$

or equivalently,

$$P_-^{\alpha\beta} \to 0 \qquad \text{a.e. } (x,\xi) \quad \text{as} \quad \varepsilon \to 0.$$

Note that

$$\begin{aligned} |P_-^{\alpha\beta\sharp}| &\le |Q_{-\varepsilon}^{\alpha\beta\sharp}(F_\alpha, F_\alpha)| + |Q_-^{\alpha\beta\sharp}(F_{\alpha\varepsilon}, F_{\alpha\varepsilon})| \\ &\le \mathcal{O}(1)\Big(\frac{e^{-m_\alpha(\frac{p}{2}|x|^2+\frac{q}{2}|\xi|^2)}}{(t+1)^{\gamma+3}}\Big)_\varepsilon + \mathcal{O}(1)\frac{e^{-m_\alpha(\frac{p}{3}|x|^2+\frac{q}{2}|\xi|^2)}}{(t+1)^{\gamma+3}} \\ &\le \mathcal{O}(1)\frac{e^{-m_\alpha(\frac{p}{3}|x|^2+\frac{q}{2}|\xi|^2)}}{(t+1)^{\gamma+3}} \in L^1(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+). \end{aligned}$$

We apply LDCT again to obtain

$$\lim_{\varepsilon\to 0} \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} |P_-^{\alpha\beta\sharp}| d\xi dx ds = 0.$$

Similarly, we have

$$\lim_{\varepsilon\to 0} \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} |P_+^{\alpha\beta\sharp}| d\xi dx ds = 0.$$

Hence we obtain

$$\int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} |P^{\alpha\beta\sharp}| d\xi dx ds \le \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} |P_-^{\alpha\beta\sharp}| d\xi dx ds + \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} |P_+^{\alpha\beta\sharp}| d\xi dx ds \to 0$$

as $\varepsilon \to 0$. □

For later use, we define

$$\mathcal{E}_\varepsilon^{\alpha\beta}(x,\xi,t) \equiv \int_0^\infty \int_{\mathbb{R}^3} |\xi-\xi_*|^\gamma Q_+^{\alpha\beta\sharp}(F_{\alpha\varepsilon},F_{\beta\varepsilon})(x+(t+\tau)(\xi-\xi_*),\xi_*)d\xi_* d\tau,$$
$$\mathcal{G}_\varepsilon^{\alpha\beta}(x,\xi,t) \equiv \int_0^\infty \int_{\mathbb{R}^3} |\xi-\xi_*|^\gamma P_\varepsilon^{\alpha\beta\sharp}(x+(t+\tau)(\xi-\xi_*),\xi_*)d\xi_* d\tau,$$
$$\mathcal{E}_\varepsilon(t) \equiv \sum_{\alpha,\beta\in\{A,B\}} \mathcal{E}_\varepsilon^{\alpha\beta}(t), \qquad \mathcal{G}_\varepsilon(t) \equiv \sum_{\alpha,\beta\in\{A,B\}} \mathcal{G}_\varepsilon^{\alpha\beta}(t).$$

LEMMA 4.6. Suppose that the main assumptions $\mathcal{A}$ in Section 1 hold, and let $(F_\alpha, F_\beta) \in (\mathcal{S}_\alpha \times \mathcal{S}_\beta)$ be the mild solutions constructed in Section 3. Then we have

$$(\mathcal{E}_\varepsilon + \mathcal{G}_\varepsilon)(x,\xi,t) \le \frac{\mathcal{O}(1)}{(t+1)^{\gamma+3}}, \qquad x,\xi\in\mathbb{R}^3,\ t\ge 0,$$

where $\mathcal{O}(1)$ is a bounded function independent of x,ξ,t.

Proof. (1) Estimate of $\mathcal{E}_\varepsilon^{\alpha\beta}(t)$: For fixed $x,\xi\in\mathbb{R}^3$, we have

$$\begin{aligned} &Q_+^{\alpha\beta\sharp}(F_{\alpha\varepsilon},F_{\beta\varepsilon})(x+(t+\tau)(\xi-\xi_*),\xi_*) \\ &\qquad \le \frac{\mathcal{O}(1)\delta_\alpha\delta_\beta}{(t+1)^{\gamma+3}} e^{-m_\alpha(\frac{p}{3}|x+(t+\tau)(\xi-\xi_*)|^2+\frac{q}{2}|\xi_*|^2)}. \end{aligned}$$

This yields

$$\begin{aligned} &\mathcal{E}_\varepsilon^{\alpha\beta}(x,\xi,t) \\ &\quad \le \frac{\mathcal{O}(1)}{(t+1)^{\gamma+3}} \int_{\mathbb{R}^3} |\xi-\xi_*|^\gamma e^{-m_\alpha\frac{q}{3}|\xi_*|^2}\Big(\int_0^\infty e^{-m_\alpha\frac{p}{2}|x+(t+\tau)(\xi-\xi_*)|^2}d\tau\Big)d\xi_* \\ &\quad \le \frac{\mathcal{O}(1)}{(t+1)^{\gamma+3}} \int_{\mathbb{R}^3} |\xi-\xi_*|^{\gamma-1} e^{-m_\alpha\frac{q}{3}|\xi_*|^2} d\xi_* \\ &\quad \le \frac{\mathcal{O}(1)}{(t+1)^{\gamma+3}}, \end{aligned}$$

where we used Lemma 2.2. We add $\mathcal{E}^{\alpha\beta}$ over all α,β to get

$$\mathcal{E}_\varepsilon(x,\xi,t) \le \frac{\mathcal{O}(1)}{(t+1)^{\gamma+3}}.$$

(2) The estimate for $\mathcal{G}_\varepsilon^{\alpha\beta}(t)$: Note that

$$\begin{aligned} |P_\pm^{\alpha\beta}(x,\xi,t)| &\le |Q_{\pm\varepsilon}^{\alpha\beta\sharp}(F_\alpha,F_\beta)(x,\xi,t)| + |Q_\pm^{\alpha\beta\sharp}(F_{\alpha\varepsilon},F_{\beta\varepsilon})(x,\xi,t)| \\ &\le \mathcal{O}(1)\Big(\frac{e^{-m_\alpha(\frac{p}{2}|x|^2+\frac{q}{2}|\xi|^2)}}{(t+1)^{\gamma+3}}\Big)_\varepsilon + \mathcal{O}(1)\frac{e^{-m_\alpha(\frac{p}{3}|x|^2+\frac{q}{2}|\xi|^2)}}{(t+1)^{\gamma+3}} \\ &\le \mathcal{O}(1)\frac{e^{-m_\alpha(\frac{p}{3}|x|^2+\frac{q}{2}|\xi|^2)}}{(t+1)^{\gamma+3}}. \end{aligned}$$

Then by the same analysis as in (1), we obtain the desired result. □

5. Large-time asymptotics of mild solutions. In this section, we study the large-time behavior of mild solutions constructed in Section 3 using the collision potential approach. We slightly generalize the collision potential for a single-component Boltzmann equation in [9, 10] using the weight $\varphi_{\alpha,r}$ in the velocity space. Recall that mild solutions F_A, F_B satisfy

$$\begin{aligned}
&\partial_t F_A + \xi\cdot\nabla_x F_A = Q^{AA}(F_A,F_A) + Q^{AB}(F_A,F_B), \qquad x,v\in\mathbb{R}^3,\ t>0,\\
&\partial_t F_B + \xi\cdot\nabla_x F_B = Q^{BB}(F_B,F_B) + Q^{BA}(F_B,F_A),\\
&(F_A,F_B)(x,\xi,0) = (F_{A0},F_{B0})(x,\xi),
\end{aligned}$$

and set

$$f_A \equiv \varphi_{\alpha,r}F_A, \qquad f_B \equiv \varphi_{\alpha,r}F_B.$$

Then it is easy to see that the $L^1_{\alpha,r}$-distance of F_A, F_B is equal to the L^1-distance of f_A, f_B, and f_A, f_B satisfy

$$\begin{aligned}
&\partial_t f_A + \xi\cdot\nabla_x f_A = \widehat{Q}^{AA}(f_A,f_A) + \widehat{Q}^{AB}(f_A,f_B), \qquad x,v\in\mathbb{R}^3,\ t>0,\\
&\partial_t f_B + \xi\cdot\nabla_x f_B = \widehat{Q}^{BB}(f_B,f_B) + \widehat{Q}^{BA}(f_B,f_A),\\
&(f_A,f_B)(x,\xi,0) = (f_{A0},f_{B0})(x,\xi),
\end{aligned}\tag{5.1}$$

where $\widehat{Q}^{\alpha\beta}(f_\alpha,f_\beta)$ has the following form:

$$\begin{aligned}
\widehat{Q}^{\alpha\beta}(f_\alpha,f_\beta)(\xi) \equiv \int_{\mathbb{R}^3\times\mathbf{S}^2_+} q^{\alpha\beta}(\xi-\xi_*,\omega)\Big(&\frac{\varphi_{\alpha,r}(\xi)}{\varphi_{\alpha,r}(\xi')\varphi_{\beta,r}(\xi'_*)} f_\alpha(\xi')f_\beta(\xi'_*)\\
&- \frac{1}{\varphi_{\beta,r}(\xi_*)} f_\alpha(\xi)f_\beta(\xi_*)\Big)d\omega d\xi_*.
\end{aligned}$$

Consider the mollified system of (5.1):

$$\begin{aligned}
&\partial_t f_{A\varepsilon} + \xi\cdot\nabla_x f_{A\varepsilon} = \widehat{Q}^{AA}(f_{A\varepsilon},f_{A\varepsilon}) + \widehat{Q}^{AB}(f_{A\varepsilon},f_{B\varepsilon}) + \widehat{P}^{AA}_\varepsilon + \widehat{P}^{AB}_\varepsilon,\\
&\partial_t f_{B\varepsilon} + \xi\cdot\nabla_x f_{B\varepsilon} = \widehat{Q}^{BB}(f_{B\varepsilon},f_{B\varepsilon}) + \widehat{Q}^{BA}(f_{B\varepsilon},f_{A\varepsilon}) + \widehat{P}^{BB}_\varepsilon + \widehat{P}^{BA}_\varepsilon,\\
&f_{A\varepsilon}(x,\xi,0) = f_{A0} *_x \phi_\varepsilon, \quad f_{B\varepsilon}(x,\xi,0) = f_{B0} *_x \phi_\varepsilon.
\end{aligned}\tag{5.2}$$

Here $\widehat{P}^{\alpha\beta}_\varepsilon$ denotes the perturbation given by

$$\widehat{P}^{\alpha\beta}_\pm \equiv \widehat{Q}^{\alpha\beta}_{\pm\varepsilon}(f_\alpha,f_\beta) - \widehat{Q}^{\alpha\beta}_\pm(f_{\alpha\varepsilon},f_{\beta\varepsilon}), \quad \widehat{P}^{\alpha\beta}_\varepsilon \equiv \widehat{P}^{\alpha\beta}_+ - \widehat{P}^{\alpha\beta}_-, \quad \alpha,\beta\in\{A,B\}.$$

LEMMA 5.1. The modified collision kernels $\widehat{Q}^{\alpha\beta}_\pm, \widehat{P}^{\alpha\beta}_\pm$ are majorized by $Q^{\alpha\beta}_\pm$.

(1) $|\widehat{Q}^{\alpha\beta}_\pm(f_\alpha,f_\beta)(x,\xi)| \le |Q^{\alpha\beta}_\pm(f_\alpha,f_\beta)(\xi)|$.
(2) $|\widehat{P}^{\alpha\beta}_\pm(x,\xi)| \le |Q^{\alpha\beta}_{\pm\varepsilon}(f_\alpha,f_\beta)(x,\xi)| + |\widehat{Q}^{\alpha\beta}_\pm(f_{\alpha\varepsilon},f_{\beta\varepsilon})(x,\xi)|$.

Proof. The proof follows directly from the fact that

$$\begin{aligned}
\frac{\varphi_{\alpha,r}(\xi)}{\varphi_{\alpha,r}(\xi')\varphi_{\beta,r}(\xi'_*)} &\le \Big(\frac{1+m_\alpha|\xi|^2}{(1+m_\alpha|\xi'|^2)(1+m_\beta|\xi'_*|^2)}\Big)^{\frac{r}{2}}\\
&\le \Big(\frac{1+m_\alpha|\xi|^2+m_\beta|\xi_*|^2+m_\alpha m_\beta|\xi'|^2|\xi'_*|^2}{(1+m_\alpha|\xi'|^2)(1+m_\beta|\xi'_*|^2)}\Big)^{\frac{r}{2}} \le 1,
\end{aligned}$$

where we used the conservation of energy:

$$m_\alpha|\xi'|^2 + m_\beta|\xi'_*|^2 = m_\alpha|\xi|^2 + m_\beta|\xi_*|^2.$$

□

We now define the collision potentials for $f_\varepsilon = (f_{A\varepsilon}, f_{B\varepsilon})$ corresponding to the mollified mild solutions $F_{A\varepsilon}, F_{B\varepsilon}$:

$$\mathcal{D}^{\alpha\beta}(f_\varepsilon(t)) \equiv \int_{\mathbb{R}^3\times\mathbb{R}^3} f^\sharp_{\alpha\varepsilon}(x,\xi)\Big[\int_{\mathbb{R}^3}\int_0^\infty |\xi-\xi_*|^\gamma \times f^\sharp_{\beta\varepsilon}(x+(t+\tau)(\xi-\xi_*),\xi_*)d\tau d\xi_*\Big]d\xi dx,$$

$$\Lambda^{\alpha\beta}(f_\varepsilon(t)) \equiv \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |\xi-\xi_*|^\gamma f^\sharp_{\alpha\varepsilon}(x,\xi) f^\sharp_{\beta\varepsilon}(x+t(\xi-\xi_*),\xi_*)d\xi_* d\xi dx,$$

$$\mathcal{D}(f_\varepsilon(t)) \equiv \sum_{\alpha,\beta\in\{A,B\}} \mathcal{D}^{\alpha\beta}(f_\varepsilon(t)), \qquad \Lambda(f_\varepsilon(t)) \equiv \sum_{\alpha,\beta\in\{A,B\}} \Lambda^{\alpha\beta}(f_\varepsilon(t)).$$

Before we study the main result of this section, we need some technical lemmata.

Lemma 5.2. Suppose the main assumptions $\mathcal{A}$ in Section 1 hold, and let $(F_\alpha, F_\beta) \in \mathcal{S}_\alpha \times \mathcal{S}_\beta$ be mild solutions of (1.1). Then we have the following estimates:

$$(1)\ \int_0^\infty \int_{\mathbb{R}^3} |\xi-\xi_*|^\gamma f^\sharp_{\alpha\varepsilon}(x+(t+\tau)(\xi-\xi_*),\xi_*)d\xi_* d\tau \le \mathcal{O}(1),$$

$$(2)\ \int_{\mathbb{R}^3\times\mathbb{R}^3} (\widehat{Q}^{\alpha\beta}_{+\varepsilon} + \widehat{Q}^{\alpha\beta}_{-\varepsilon})(f_\alpha, f_\beta)d\xi dx \le \mathcal{O}(1)\Lambda^{\alpha\beta}(f_\varepsilon(t)).$$

Proof. (1) Note that

$$\begin{aligned} &f^\sharp_{\alpha\varepsilon}(x+(t+\tau)(\xi-\xi_*),\xi_*)\\ &\quad\le \varphi_{\alpha,r}(\xi) F^\sharp_{A\varepsilon}(x+(t+\tau)(\xi-\xi_*),\xi_*)\\ &\quad\le \varphi_{\alpha,r}\delta_\alpha e^{-m_\alpha(\frac{p}{2}|x+(t+\tau)(\xi-\xi_*)|^2+q|\xi_*|^2)}\\ &\quad\le \mathcal{O}(1)\delta_\alpha e^{-m_\alpha(\frac{p}{2}|x+(t+\tau)(\xi-\xi_*)|^2+\frac{q}{2}|\xi_*|^2)}. \end{aligned}$$

Then the first estimate follows by the same estimate as in Lemma 4.5.
(2) We use Fubini's theorem to get

$$\begin{aligned} \text{(LHS)} &\le \mathcal{O}(1)\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbf{S}^2_+} |\xi-\xi_*|^\gamma (f'_{\alpha\varepsilon} f'_{\beta*\varepsilon} + f_{\alpha\varepsilon} f_{\beta*\varepsilon})d\omega d\xi_* d\xi dx\\ &\le \mathcal{O}(1)\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |\xi-\xi_*|^\gamma f_{\alpha\varepsilon} f_{\beta*\varepsilon} d\xi_* d\xi dx\\ &\le \mathcal{O}(1)\Lambda^{\alpha\beta}(f_\varepsilon(t)). \end{aligned}$$

□

We now set

$$\mathcal{D}(f(t)) \equiv \lim_{\varepsilon\to 0}\mathcal{D}(f_\varepsilon(t)), \qquad \Lambda(f(t)) \equiv \lim_{\varepsilon\to 0}\Lambda(f_\varepsilon(t)).$$

Lemma 5.3. Let $f = (f_A, f_B)$ be the mild solution of (5.1) corresponding to the initial data $f_0 = (f_{A0}, f_{B0})$. Then $\mathcal{D}(f(t))$ satisfies

$$\mathcal{D}(f(t)) + C\int_0^t \Lambda(f(s))ds \le \mathcal{D}(f_0), \qquad t \ge 0,$$

where C is a positive constant independent of time t.

Proof. For $\alpha, \beta \in \{A, B\}$, we first estimate the integrand in $\mathcal{D}^{\alpha\beta}_{\varepsilon}(t)$:

$$\partial_t f^{\sharp}_{\alpha\varepsilon}(x,\xi) = [\widehat{Q}^{\alpha\alpha\sharp}(f_{\alpha\varepsilon}, f_{\alpha\varepsilon}) + \widehat{Q}^{\alpha\beta\sharp}(f_{\alpha\varepsilon}, f_{\beta\varepsilon})](x,\xi) + [\widehat{P}^{\alpha\alpha\sharp}_{\varepsilon} + \widehat{P}^{\alpha\beta\sharp}_{\varepsilon}](x,\xi), \tag{5.3}$$

$$\begin{aligned}\partial_t[f^{\sharp}_{\beta\varepsilon}(x+(t+\tau)(\xi-\xi_*),\xi_*)] &= \partial_\tau[f^{\sharp}_{\beta\varepsilon}(x+(t+\tau)(\xi-\xi_*),\xi_*)] \\ &\quad + [\widehat{Q}^{\beta\beta\sharp}(f_{\beta\varepsilon}, f_{\beta\varepsilon}) + \widehat{Q}^{\beta\alpha\sharp}(f_{\beta\varepsilon}, f_{\alpha\varepsilon})](x+(t+\tau)(\xi-\xi_*),\xi_*) \\ &\quad + [\widehat{P}^{\beta\beta\sharp}_{\varepsilon} + \widehat{P}^{\beta\alpha\sharp}_{\varepsilon}](x+(t+\tau)(\xi-\xi_*),\xi_*).\end{aligned} \tag{5.4}$$

We now combine (5.3) and (5.4) to obtain

$$\begin{aligned}&\partial_t\Big[|\xi-\xi_*|^\gamma f^{\sharp}_{\alpha\varepsilon}(x,\xi) f^{\sharp}_{\beta\varepsilon}(x+(t+\tau)(\xi-\xi_*),\xi_*)\Big] \\ &\quad = \partial_\tau\Big[|\xi-\xi_*|^\gamma f^{\sharp}_{\alpha\varepsilon}(x,\xi) f^{\sharp}_{\beta\varepsilon}(x+(t+\tau)(\xi-\xi_*),\xi_*)\Big] \\ &\quad + |\xi-\xi_*|^\gamma f^{\sharp}_{\alpha\varepsilon}(x,\xi)\Big(\widehat{Q}^{\beta\beta\sharp}_{\varepsilon}(f_\beta,f_\beta)+\widehat{Q}^{\beta\alpha\sharp}_{\varepsilon}(f_\beta,f_\alpha)+\widehat{P}^{\alpha\alpha\sharp}_{\varepsilon}+\widehat{P}^{\alpha\beta\sharp}_{\varepsilon}\Big)(x+(t+\tau)(\xi-\xi_*),\xi_*) \\ &\quad + |\xi-\xi_*|^\gamma f^{\sharp}_{\beta\varepsilon}(x+(t+\tau)(\xi-\xi_*),\xi_*) \\ &\quad \times\Big(\widehat{Q}^{\alpha\alpha\sharp}_{\varepsilon}(f_\alpha,f_\alpha)+\widehat{Q}^{\alpha\beta\sharp}_{\varepsilon}(f_\alpha,f_\beta)(x,\xi)+\widehat{P}^{\alpha\alpha\sharp}_{\varepsilon}+\widehat{P}^{\alpha\beta\sharp}_{\varepsilon}\Big)(x,\xi).\end{aligned} \tag{5.5}$$

We integrate (5.5) over $\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+$ with respect to (x,ξ,ξ_*,τ) to get

$$\begin{aligned}&\frac{d}{dt}\mathcal{D}^{\alpha\beta}(f_\varepsilon(t)) \\ &\quad = -\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |\xi-\xi_*|^\gamma f^{\sharp}_{\alpha\varepsilon}(x,\xi) f^{\sharp}_{\beta\varepsilon}(x+t(\xi-\xi_*),\xi_*)\,d\xi d\xi_* dx \\ &\qquad + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+} |\xi-\xi_*|^\gamma f^{\sharp}_{\beta\varepsilon}(x+(t+\tau)(\xi-\xi_*),\xi_*) \\ &\qquad\quad \times\Big[\widehat{Q}^{\alpha\alpha\sharp}_{\varepsilon}(f_\alpha,f_\alpha)+\widehat{Q}^{\alpha\beta\sharp}_{\varepsilon}(f_\alpha,f_\beta)+\widehat{P}^{\alpha\alpha\sharp}_{\varepsilon}+\widehat{P}^{\alpha\beta\sharp}_{\varepsilon}\Big](x,\xi)\,d\tau d\xi_* d\xi dx \\ &\qquad + \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+} |\xi-\xi_*|^\gamma f^{\sharp}_{\alpha\varepsilon}(x,\xi) \\ &\qquad\quad \times\Big[\widehat{Q}^{\beta\beta\sharp}_{\varepsilon}(f_\beta,f_\beta)+\widehat{Q}^{\beta\alpha\sharp}_{\varepsilon}(f_\beta,f_\alpha)+\widehat{P}^{\beta\beta\sharp}_{\varepsilon}+\widehat{P}^{\beta\alpha\sharp}_{\varepsilon}\Big](x+(t+\tau)(\xi-\xi_*),\xi_*)\,d\tau d\xi_* d\xi dx \\ &\quad \equiv -\Lambda^{\alpha\beta}(f_\varepsilon(t)) + \mathcal{K}^{\alpha\beta}_{1\varepsilon}(f(t)) + \mathcal{K}^{\alpha\beta}_{2\varepsilon}(f(t)).\end{aligned}$$

Next we estimate $\mathcal{K}^{\alpha\beta}_{1\varepsilon}(t)$ and $\mathcal{K}^{\alpha\beta}_{2\varepsilon}(t)$ separately. First, by Lemma 5.1 we have

$$\begin{aligned}&\mathcal{K}^{\alpha\beta}_{1\varepsilon}(f(t)) \\ &\quad = \int_{\mathbb{R}^3\times\mathbb{R}^3}\Big[(\widehat{Q}^{\alpha\alpha\sharp}_{\varepsilon}(f_\alpha,f_\alpha)+\widehat{Q}^{\alpha\beta\sharp}_{\varepsilon}(f_\alpha,f_\beta))(x,\xi,t)+\widehat{P}^{\alpha\alpha}_{\varepsilon}+\widehat{P}^{\alpha\beta}_{\varepsilon}\Big] \\ &\qquad \times\Big(\int_{\mathbb{R}^3\times\mathbb{R}_+}|\xi-\xi^*|^\gamma f^{\sharp}_{\beta\varepsilon}(x+(t+\tau)(\xi-\xi_*),\xi_*,t)\,d\tau d\xi_*\Big)d\xi dx \\ &\quad \le \mathcal{O}(1)\delta_\beta\int_{\mathbb{R}^3\times\mathbb{R}^3}\Big[(Q^{\alpha\alpha\sharp}_{+\varepsilon}+Q^{\alpha\alpha\sharp}_{-\varepsilon})(f_\alpha,f_\alpha)+(Q^{\alpha\beta\sharp}_{+\varepsilon}+Q^{\alpha\beta\sharp}_{-\varepsilon})(f_\alpha,f_\beta)+|P^{\alpha\alpha\sharp}_{\varepsilon}+P^{\alpha\beta\sharp}_{\varepsilon}|\Big]d\xi dx \\ &\quad \le \mathcal{O}(1)\delta_\beta[\Lambda^{\alpha\alpha}(f_\varepsilon(t))+\Lambda^{\alpha\beta}(f_\varepsilon(t))]+\mathcal{O}(1)\delta_\beta\Big(\int_{\mathbb{R}^3\times\mathbb{R}^3}|P^{\alpha\alpha}_{\varepsilon}+P^{\alpha\beta}_{\varepsilon}|dxd\xi\Big).\end{aligned}$$

Similarly we obtain

$$\mathcal{K}^{\alpha\beta}_{2\varepsilon}(f(t)) \le \mathcal{O}(1)\delta_\alpha\Big[\Lambda^{\beta\beta}(f_\varepsilon(t))+\Lambda^{\beta\alpha}(f_\varepsilon(t))\Big]+\mathcal{O}(1)\delta_\alpha\Big(\int_{\mathbb{R}^3\times\mathbb{R}^3}|P^{\beta\beta\sharp}_{\varepsilon}+P^{\beta\alpha\sharp}_{\varepsilon}|dxd\xi\Big).$$

We combine the above estimates to get

$$\frac{d}{dt}D^{\alpha\beta}(f_\varepsilon(t)) \le -\Lambda^{\alpha\beta}(f_\varepsilon(t)) + \mathcal{O}(1)\delta_\beta\Big[\Lambda^{\alpha\alpha}(f_\varepsilon(t)) + \Lambda^{\alpha\beta}(f_\varepsilon(t))\Big] + \mathcal{O}(1)\delta_\alpha\Big[\Lambda^{\beta\beta}(f_\varepsilon(t)) + \Lambda^{\alpha\beta}(f_\varepsilon(t))\Big] + \mathcal{O}(1)\sum_{\alpha\beta\in\{A,B\}}\int_{\mathbb{R}^3\times\mathbb{R}^3}|P^{\alpha\beta\sharp}_{\pm\varepsilon}|dxd\xi.$$

Summing up over $\alpha,\beta\in\{A,B\}$ we find

$$\begin{aligned}\frac{d}{dt}D(f_\varepsilon(t)) &= \sum_{\alpha,\beta\in\{A,B\}}\frac{d}{dt}D^{\alpha\beta}(f_\varepsilon(t))\\ &\le \Big(-1+\mathcal{O}(1)(\delta_A+\delta_B)\Big)\Lambda(f_\varepsilon(t))\\ &\le -C\Lambda(f_\varepsilon(t)) + \mathcal{O}(1)\int_{\mathbb{R}^3\times\mathbb{R}^3}\sum_{\alpha,\beta\in\{A,B\}}|P^{\alpha\beta\sharp}_{\pm\varepsilon}|dxd\xi.\end{aligned}$$

We integrate the above inequality from $s=0$ to $s=t$ to obtain

$$\mathcal{D}(f_\varepsilon(t)) + C\int_0^t\Lambda(f_\varepsilon(s))ds \le \mathcal{D}(f_\varepsilon(0)) + \mathcal{O}(1)\int_0^t\int_{\mathbb{R}^3\times\mathbb{R}^3}\sum_{\alpha,\beta\in\{A,B\}}|P^{\alpha\beta\sharp}_{\pm\varepsilon}|dxd\xi ds.$$

Let $\varepsilon\to 0$. Then by Lemma 4.5 and the LDCT we obtain

$$\mathcal{D}(f(t)) + C\int_0^t\Lambda(f(s))ds \le \mathcal{D}(f_0), \qquad t\ge 0.$$

□

PROPOSITION 5.1. Let (F_A, F_B) be the mild solution of (1.1) corresponding to the initial data (F_{A0}, F_{B0}). Then there exist the unique time-asymptotic states $F^\infty = (F^\infty_A, F^\infty_B)$ such that

$$\lim_{t\to\infty}\sum_{\alpha\in\{A,B\}}\|F^\sharp_\alpha(t) - F^\infty_\alpha\|_{L^1_{\alpha,r}} = 0.$$

Proof. Note that

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^3\times\mathbb{R}^3}\sum_{\alpha,\beta\in\{A,B\}}|\widehat{Q}^{\alpha\beta\sharp}(f_\alpha,f_\beta)|(x,\xi)d\xi dxdt\\ &\le \mathcal{O}(1)\sum_{\alpha,\beta\in\{A,B\}}\int_0^\infty\int_{\mathbb{R}^3\times\mathbb{R}^3}(Q^{\alpha\beta\sharp}_+ + Q^{\alpha\beta\sharp}_-)(f_\alpha,f_\beta)(x,\xi)d\xi dxdt\\ &\le \mathcal{O}(1)\sum_{\alpha,\beta\in\{A,B\}}\int_0^\infty\Lambda^{\alpha\beta\sharp}(f(t))dt\\ &= \mathcal{O}(1)\int_0^\infty\Lambda(f(t))dt \le \mathcal{O}(1)\mathcal{D}(f_0).\end{aligned}$$

This yields

$$\lim_{t\to\infty}\sum_{\alpha,\beta\in\{A,B\}}\int_t^\infty\int_{\mathbb{R}^3\times\mathbb{R}^3}|\widehat{Q}^{\alpha\beta\sharp}(f_\alpha,f_\beta)|(x,\xi)d\xi dxds = 0.$$

On the other hand, we set

$$\begin{aligned} f_\alpha^\infty(x,\xi) &\equiv f_{\alpha 0}(x,\xi) + \int_0^\infty [\widehat{Q}^{\alpha\alpha\sharp}(f_\alpha, f_\alpha) + \widehat{Q}^{\alpha\beta\sharp}(f_\alpha, f_\beta)](x,\xi,s)ds, \\ F_\alpha^\infty(x,\xi) &\equiv \phi_{\alpha,r}^{-1}(\xi) f_\alpha^\infty(x,\xi). \end{aligned}$$

Then we can see that $F_\alpha^\infty \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and

$$\begin{aligned} &\sum_{\alpha\in\{A,B\}} \|F_\alpha(x+t\xi,\xi,t) - F_\alpha^\infty(x,\xi)\|_{L^1_{\alpha,r}} \\ &\quad \le \sum_{\alpha\in\{A,B\}} \|f_\alpha(x+t\xi,\xi,t) - f_\alpha^\infty(x,\xi)\|_{L^1_{\alpha,r}} \\ &\quad \le \int_t^\infty \int_{\mathbb{R}^3\times\mathbb{R}^3} \sum_{\alpha,\beta\in\{A,B\}} |\widehat{Q}^{\alpha\beta\sharp}(f_\alpha, f_\beta)|(x,\xi,s)d\xi dx ds \to 0 \qquad \text{as } t\to\infty. \end{aligned}$$

□

6. Uniform $L^1_{\alpha,r}$-stability estimate. In this section, we study the uniform stability estimate with respect to initial data. For the stability analysis, we use the weight $\varphi_{\alpha,r}$. As in [9, 10], we employ the modified nonlinear functional which incorporates the cross collisions between the A and B species. Define nonlinear functionals as follows:

$$\begin{aligned} \mathcal{D}_{d\varepsilon}^{\alpha\beta}(t) &\equiv \int_{\mathbb{R}^3\times\mathbb{R}^3} |f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}|^\sharp(x,\xi)\Big[\int_0^\infty \int_{\mathbb{R}^3} |\xi-\xi_*|^\gamma \\ &\qquad \times (f_{\beta\varepsilon}^\sharp + \bar{f}_{\beta\varepsilon}^\sharp)(x+(t+\tau)(\xi-\xi_*),\xi_*)d\xi_* d\tau\Big] d\xi dx, \\ \mathcal{D}_{d\varepsilon}(t) &\equiv \sum_{\alpha,\beta\in\{A,B\}} \mathcal{D}_{d\varepsilon}^{\alpha\beta}(t), \\ \mathcal{H}_\varepsilon(t) &\equiv \sum_{\alpha\in\{A,B\}} \|f_{\alpha\varepsilon}(t) - \bar{f}_{\alpha\varepsilon}(t)\|_{L^1} + K\mathcal{D}_{d\varepsilon}(t), \end{aligned}$$

where K is a positive constant to be determined later. Then it is easy to see that $\mathcal{H}_\varepsilon(t)$ is equivalent to L^1_φ-distance:

$$\begin{aligned} \|f_{\alpha\varepsilon}(t) - \bar{f}_{\alpha\varepsilon}(t)\|_{L^1} &\le \|f_{\alpha\varepsilon}(t) - \bar{f}_{\alpha\varepsilon}(t)\|_{L^1} + K(\mathcal{D}_{d\varepsilon}^{\alpha\alpha}(t) + \mathcal{D}_{d\varepsilon}^{\alpha\beta}(t)) \\ &= \int_{\mathbb{R}^3\times\mathbb{R}^3} |f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}|^\sharp(x,\xi)\Big[1 + K\sum_{\eta\in\{\alpha,\beta\}} \int_0^\infty \int_{\mathbb{R}^3} |\xi-\xi_*|^\gamma \\ &\qquad \times (f_{\eta\varepsilon}^\sharp + \bar{f}_{\eta\varepsilon}^\sharp)(x+(t+\tau)(\xi-\xi_*),\xi_*)d\xi_* d\tau\Big] d\xi dx \\ &\le (1 + K(C_\alpha + C_\beta))\|f_{\alpha\varepsilon}(t) - \bar{f}_{\alpha\varepsilon}(t)\|_{L^1}, \end{aligned}$$

where

$$C_\beta = 2q_*\pi^2\sqrt{\frac{2\pi}{pm_\beta}}\Big[\frac{4\pi}{\gamma+2} + \sqrt{\Big(\frac{4\pi}{qm_\beta}\Big)^3}\Big].$$

We next derive differential inequalities for the difference $|f_\alpha - \bar{f}_\alpha|$. Recall that $f_{\alpha\varepsilon}$ and $\bar{f}_{\alpha\varepsilon}$ satisfy

$$\begin{aligned} \partial_t f_{\alpha\varepsilon} + \xi\cdot\nabla_x f_{\alpha\varepsilon} &= \widehat{Q}^{\alpha\alpha}(f_{\alpha\varepsilon}, f_{\alpha\varepsilon}) + \widehat{Q}^{\alpha\beta}(f_{\alpha\varepsilon}, f_{\beta\varepsilon}) + \widehat{P}_\varepsilon^{\alpha\alpha} + \widehat{P}_\varepsilon^{\alpha\beta}, \\ \partial_t \bar{f}_{\alpha\varepsilon} + \xi\cdot\nabla_x \bar{f}_{\alpha\varepsilon} &= \widehat{Q}^{\alpha\alpha}(\bar{f}_{\alpha\varepsilon}, \bar{f}_{\alpha\varepsilon}) + \widehat{Q}^{\alpha\beta}(\bar{f}_{\alpha\varepsilon}, \bar{f}_{\beta\varepsilon}) + \widehat{\bar{P}}_\varepsilon^{\alpha\alpha} + \widehat{\bar{P}}_\varepsilon^{\alpha\beta}. \end{aligned}$$

Then we subtract the above equations and multiply by $sgn(f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon})$ to find

$$\partial_t |f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}| + \xi \cdot \nabla_x |f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}| \leq \mathcal{R}_\varepsilon^{\alpha\alpha} + \mathcal{R}_\varepsilon^{\alpha\beta} + [[P_\varepsilon^{\alpha\alpha}]] + [[P_\varepsilon^{\alpha\beta}]], \tag{6.1}$$

where

$$\begin{aligned}\mathcal{R}_\varepsilon^{\alpha\beta} \equiv \frac{1}{2}\int_{\mathbb{R}^3\times\mathbf{S}^2_+} & q^{\alpha\beta}(\xi-\xi_*,\omega)\Big(|f'_{\alpha\varepsilon} - \bar{f}'_{\alpha\varepsilon}|(f'_{\beta*\varepsilon} + \bar{f}'_{\beta*\varepsilon}) + |f'_{\beta*\varepsilon} - \bar{f}'_{\beta*\varepsilon}| \\ & \times (f'_{\alpha\varepsilon} + \bar{f}'_{\alpha\varepsilon}) + |f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}|(f_{\beta*\varepsilon} + \bar{f}_{\beta*\varepsilon}) + |f_{\beta*\varepsilon} - \bar{f}_{\beta*\varepsilon}|(f_{\alpha\varepsilon} + \bar{f}_{\alpha\varepsilon})\Big) d\omega d\xi_*,\\ [[P_\varepsilon^{\alpha\beta}]] \equiv |\widehat{P}_\varepsilon^{\alpha\beta} - \widehat{\bar{P}}_\varepsilon^{\alpha\beta}|, & \qquad P_d(x,\xi,\varepsilon) \equiv \sum_{\alpha,\beta\in\{A,B\}} [[P_\varepsilon^{\alpha\beta}]].\end{aligned}$$

We define collision production rates as

$$\begin{aligned}\Lambda_{d\varepsilon}(t) &\equiv \sum_{\alpha,\beta\in\{A,B\}} \Lambda_{d\varepsilon}^{\alpha\beta}(t),\\ \Lambda_{d\varepsilon}^{\alpha\beta}(t) &\equiv \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |\xi-\xi_*|^\gamma |f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}|^\sharp(x,\xi)(f^\sharp_{\beta\varepsilon} + \bar{f}^\sharp_{\beta\varepsilon})(x+t(\xi-\xi_*),\xi_*) d\xi_* d\xi dx.\end{aligned}$$

Lemma 6.1. Suppose the main assumptions $\mathcal{A}$ in Section 1 hold, and let f_α, $\bar{f}_\alpha$ be mild solutions of (5.1) corresponding to the initial data $f_{\alpha 0}$, $\bar{f}_{\alpha 0}$ respectively. Then we have

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \mathcal{R}_\varepsilon^{\alpha\beta\sharp} d\xi dx \leq 4\pi q_* \Lambda_{d\varepsilon}^{\alpha\beta}(t).$$

Proof. It follows from Fubini's theorem and symmetry of $R_\varepsilon^{\alpha\beta}$ that we have

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{R}^3} \mathcal{R}_\varepsilon^{\alpha\beta\sharp} d\xi dx\\ &\quad= \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbf{S}^2_+} q^{\alpha\beta}(\xi-\xi_*,\omega)\Big(|f'_{\alpha\varepsilon} - \bar{f}'_{\alpha\varepsilon}|(f'_{\beta*\varepsilon} + \bar{f}'_{\beta*\varepsilon}) + |f'_{\beta*\varepsilon} - \bar{f}'_{\beta*\varepsilon}|(f'_{\alpha\varepsilon} + \bar{f}'_{\alpha\varepsilon})\\ &\qquad\qquad + |f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}|(f_{\beta*\varepsilon} + \bar{f}_{\beta*\varepsilon}) + |f_{\beta*\varepsilon} - \bar{f}_{\beta*\varepsilon}|(f_{\alpha\varepsilon} + \bar{f}_{\alpha\varepsilon})\Big) d\omega d\xi_* d\xi dx\\ &\quad= 2\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbf{S}^2_+} q^{\alpha\beta}(\xi-\xi_*,\omega)|f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}|(f_{\beta*\varepsilon} + \bar{f}_{\beta*\varepsilon}) d\omega d\xi_* d\xi dx\\ &\quad\leq 4\pi q_* \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |\xi-\xi_*|^\gamma |f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}|(f_{\beta*\varepsilon} + \bar{f}_{\beta*\varepsilon}) d\xi_* d\xi dx.\end{aligned}$$

Hence we get the desired result. □

Lemma 6.2. Suppose that the main assumptions $\mathcal{A}$ hold, and let (f_A, f_B) and $(\bar{f}_A, \bar{f}_B)$ be two mild solutions of (5.1) corresponding to the initial data (f_{A0}, f_{B0}) and $(\bar{f}_{A0}, \bar{f}_{B0})$ respectively. Then we have

$$(1)\ \frac{d}{dt}\sum_{\alpha\in\{A,B\}} \|f_{\alpha\varepsilon}(t) - \bar{f}_{\alpha\varepsilon}(t)\|_{L^1} \leq C_1 \Lambda_{d\varepsilon}(t) + \mathcal{O}(1)\int_{\mathbb{R}^3\times\mathbb{R}^3} P_d^\sharp(x,\xi,\varepsilon) d\xi dx,$$

$$\begin{aligned}(2)\ \frac{d}{dt}\mathcal{D}_{d\varepsilon}(t) \leq -C_2\Lambda_{d\varepsilon}(t) + \frac{\mathcal{O}(1)}{(t+1)^{\gamma+3}} &\sum_{\alpha\in\{A,B\}} \|f_{\alpha\varepsilon}(t) - \bar{f}_{\alpha\varepsilon}(t)\|_{L^1}\\ &+ \mathcal{O}(1)\int_{\mathbb{R}^3\times\mathbb{R}^3} P_d^\sharp(x,\xi,\varepsilon) d\xi dx.\end{aligned}$$

Proof. (1) We integrate (6.1) over $\mathbb{R}^3 \times \mathbb{R}^3$ to find

$$\begin{aligned}
\frac{d}{dt} \sum_{\alpha \in \{A,B\}} &\|f_{\alpha\varepsilon}(t) - \bar{f}_{\alpha\varepsilon}(t)\|_{L^1} \\
&\le \sum_{\alpha,\beta \in \{A,B\}} \Big(\int_{\mathbb{R}^3 \times \mathbb{R}^3} \mathcal{R}_\varepsilon^{\alpha\beta\sharp} d\xi dx + \int_{\mathbb{R}^3 \times \mathbb{R}^3} [[P_\varepsilon^{\alpha\beta}]]^\sharp d\xi dx \Big) \\
&\le 4\pi q_* \Lambda_{d\varepsilon}(t) + \sum_{\alpha,\beta \in \{A,B\}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} [[P_\varepsilon^{\alpha\beta}]]^\sharp d\xi dx. \\
&\le 4\pi q_* \Lambda_{d\varepsilon}(t) + \int_{\mathbb{R}^3 \times \mathbb{R}^3} P_d^\sharp(x,\xi,\varepsilon) d\xi dx.
\end{aligned}$$

Now we set $C_1 = 4\pi q_*$ to get the desired result.

(2) Before we estimate the time-variation of the collision potential, we note that

$$\begin{aligned}
&\partial_t |f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}|^\sharp(x,\xi) \le \Big(R_\varepsilon^{\alpha\alpha\sharp} + R_\varepsilon^{\alpha\beta\sharp} + [[P_\varepsilon^{\alpha\alpha}]]^\sharp + [[P_\varepsilon^{\alpha\beta}]]^\sharp \Big)(x,\xi), \\
&\partial_t f_{\beta\varepsilon}^\sharp(x + (t+\tau)(\xi - \xi_*), \xi_*) \\
&\quad = \partial_\tau \Big(f_{\beta\varepsilon}^\sharp(x + (t+\tau)(\xi - \xi_*), \xi_*) \Big) \\
&\quad + \Big(\widehat{Q}^{\beta\beta\sharp}(f_{\beta\varepsilon}, f_{\beta\varepsilon}) + \widehat{Q}^{\beta\alpha\sharp}(f_{\beta\varepsilon}, f_{\alpha\varepsilon}) + \widehat{P}_\varepsilon^{\beta\beta\sharp} + \widehat{P}_\varepsilon^{\beta\alpha\sharp} \Big)(x + (t+\tau)(\xi - \xi_*), \xi_*), \\
&\partial_t \bar{f}_{\beta\varepsilon}^\sharp(x + (t+\tau)(\xi - \xi_*), \xi_*) \\
&\quad = \partial_\tau \Big(\bar{f}_{\beta\varepsilon}^\sharp(x + (t+\tau)(\xi - \xi_*), \xi_*) \Big) \\
&\quad + \Big(\widehat{Q}^{\beta\beta\sharp}(\bar{f}_{\beta\varepsilon}, \bar{f}_{\beta\varepsilon}) + \widehat{Q}^{\beta\alpha\sharp}(\bar{f}_{\beta\varepsilon}, \bar{f}_{\alpha\varepsilon}) + \widehat{\bar{P}}_\varepsilon^{\beta\beta\sharp} + \widehat{\bar{P}}_\varepsilon^{\beta\alpha\sharp} \Big)(x + (t+\tau)(\xi - \xi_*), \xi_*).
\end{aligned}$$

We use the above relations to obtain

$$\begin{aligned}
&\partial_t \Big(|\xi - \xi_*|^\gamma |f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}|^\sharp(x,\xi)(f_{\beta\varepsilon}^\sharp + \bar{f}_{\beta\varepsilon}^\sharp)(x + (t+\tau)(\xi - \xi_*), \xi_*) \Big) \\
&\le \partial_\tau \Big(|\xi - \xi_*|^\gamma |f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}|^\sharp(x,\xi)(f_\varepsilon^{\beta\sharp} + \bar{f}_\varepsilon^{\beta\sharp})(x + (t+\tau)(\xi - \xi_*), \xi_*) \Big) \\
&+ |\xi - \xi_*|^\gamma \Big(\mathcal{R}_\varepsilon^{\alpha\alpha\sharp} + \mathcal{R}_\varepsilon^{\alpha\beta\sharp} + [[P_\varepsilon^{\alpha\alpha}]]^\sharp + [[P_\varepsilon^{\alpha\beta}]]^\sharp \Big)(x,\xi)(f_\varepsilon^{\beta\sharp} + \bar{f}_\varepsilon^{\beta\sharp})(x + (t+\tau)(\xi - \xi_*), \xi_*) \\
&+ |\xi - \xi_*|^\gamma |f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}|^\sharp(x,\xi) \Big(\widehat{Q}^{\beta\beta\sharp}(f_{\beta\varepsilon}, f_{\beta\varepsilon}) + \widehat{Q}^{\beta\alpha\sharp}(f_{\beta\varepsilon}, f_{\alpha\varepsilon}) \\
&\qquad + \widehat{Q}^{\beta\beta\sharp}(\bar{f}_{\beta\varepsilon}, \bar{f}_{\beta\varepsilon}) + \widehat{Q}^{\beta\alpha\sharp}(\bar{f}_{\beta\varepsilon}, \bar{f}_{\alpha\varepsilon}) \Big)(x + (t+\tau)(\xi - \xi_*), \xi_*) \\
&+ |\xi - \xi_*|^\gamma |f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}|^\sharp(x,\xi)(\widehat{P}_\varepsilon^{\beta\beta\sharp} + \widehat{P}_\varepsilon^{\beta\alpha\sharp} + \widehat{\bar{P}}_\varepsilon^{\beta\beta\sharp} + \widehat{\bar{P}}_\varepsilon^{\beta\alpha\sharp})(x + (t+\tau)(\xi - \xi_*), \xi_*).
\end{aligned}$$

We integrate the above inequality over $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+$ with respect to (x, ξ, ξ_*, τ) to get

$$\begin{aligned}
&\frac{d\mathcal{D}_{d\varepsilon}^{\alpha\beta}(t)}{dt} \\
&\le -\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} |\xi - \xi_*|^\gamma |f_{\alpha\varepsilon} - \bar{f}_{\alpha\varepsilon}|^\sharp(x,\xi)(f_{\beta\varepsilon}^\sharp + \bar{f}_{\beta\varepsilon}^\sharp)(x + t(\xi - \xi_*), \xi_*) d\xi d\xi_* dx \\
&\quad + \int_0^\infty \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} |\xi - \xi_*|^\gamma \Big(\mathcal{R}_\varepsilon^{\alpha\alpha\sharp} + \mathcal{R}_\varepsilon^{\alpha\beta\sharp} + [[P_\varepsilon^{\alpha\alpha\sharp}]] + [[P_\varepsilon^{\alpha\beta\sharp}]] \Big)(x,\xi) \\
&\qquad\qquad \times (f_{\beta\varepsilon}^\sharp + \bar{f}_{\beta\varepsilon}^\sharp)(x + (t+\tau)(\xi - \xi_*), \xi_*) d\xi_* d\xi dx d\tau \\
&\quad + \int_0^\infty \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} |\xi - \xi_*|^\gamma \Big(\widehat{Q}^{\beta\beta\sharp}(f_{\beta\varepsilon}, f_{\beta\varepsilon}) + \widehat{Q}^{\beta\alpha\sharp}(f_{\beta\varepsilon}, f_{\alpha\varepsilon})
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \widehat{Q}^{\beta\beta\sharp}(\bar f_{\beta\varepsilon}, \bar f_{\beta\varepsilon}) + \widehat{Q}^{\beta\alpha\sharp}(\bar f_{\beta\varepsilon}, \bar f_{\alpha\varepsilon})\Big)(x+(t+\tau)(\xi-\xi_*),\xi_*),\xi_*,t)\\
&\qquad\qquad \times |f_{\alpha\varepsilon} - \bar f_{\alpha\varepsilon}|^\sharp(x,\xi) d\xi_* d\xi dx d\tau\\
&\quad + \int_0^\infty \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |\xi-\xi_*|^\gamma |f_{\alpha\varepsilon} - \bar f_{\alpha\varepsilon}|^\sharp(x,\xi)\\
&\qquad\qquad \times (\widehat{P}^{\beta\beta\sharp}_\varepsilon + \widehat{P}^{\beta\alpha\sharp}_\varepsilon + \widehat{P}^{\beta\beta\sharp}_\varepsilon + \widehat{P}^{\beta\alpha\sharp}_\varepsilon)(x+(t+\tau)(\xi-\xi_*),\xi_*) d\xi_* d\xi dx d\tau\\
&\equiv -\Lambda^{\alpha\beta}_{d\varepsilon}(t) + \mathcal{N}_1(t) + \mathcal{N}_2(t) + \mathcal{N}_3(t).
\end{aligned}
$$

Next we estimates $\mathcal{N}_i$ as follows.
Case 1 ($\mathcal{N}_1$): We use Lemma 6.1 to obtain

$$
\begin{aligned}
\mathcal{N}_1(t) &= \int_{\mathbb{R}^3\times\mathbb{R}^3} \Big(R^{\alpha\alpha\sharp} + R^{\alpha\beta\sharp} + [[P^{\alpha\alpha\sharp}]] + [[P^{\alpha\beta\sharp}]]\Big)(x,\xi)\\
&\qquad\qquad \times\Big[\int_0^\infty \int_{\mathbb{R}^3} |\xi-\xi_*|^\gamma (f^\sharp_{\beta\varepsilon} + \bar f^\sharp_{\beta\varepsilon})(x+(t+\tau)(\xi-\xi_*),\xi_*) d\xi_* d\tau\Big] d\xi dx\\
&\le \delta_\beta C_\beta\Big(\Lambda^{\alpha\alpha}_\varepsilon(t) + \Lambda^{\alpha\beta}_\varepsilon(t)\Big) + \mathcal{O}(1)\int_{\mathbb{R}^3\times\mathbb{R}^3} ([[P^{\alpha\alpha}_\varepsilon]]^\sharp + [[P^{\alpha\beta}_\varepsilon]]^\sharp) d\xi dx\\
&\le (\delta_\alpha+\delta_\beta)(C_\alpha + C_\beta)\Lambda_{d\varepsilon}(t) + \mathcal{O}(1)\sum_{\alpha,\beta\in\{A,B\}} \int_{\mathbb{R}^3\times\mathbb{R}^3} [[P^{\alpha\beta}_\varepsilon]]^\sharp d\xi dx\\
&= (\delta_\alpha+\delta_\beta)(C_\alpha + C_\beta)\Lambda_{d\varepsilon}(t) + \mathcal{O}(1)\int_{\mathbb{R}^3\times\mathbb{R}^3} P^\sharp_d(x,\xi,\varepsilon) d\xi dx,
\end{aligned}
$$

where $C_\beta \equiv 4q_*\pi^2\sqrt{\frac{2\pi}{pm_\beta}}\Big[\frac{4\pi}{\gamma+2} + \sqrt{\big(\frac{4\pi}{qm_\beta}\big)^3}\Big]$.
Case 2 ($\mathcal{N}_2$): By Lemma 4.6, we have

$$
\begin{aligned}
\mathcal{N}_2(t) &\le \int_{\mathbb{R}^3\times\mathbb{R}^3} |f_{\alpha\varepsilon} - \bar f_{\alpha\varepsilon}|^\sharp(x,\xi)\Big[\int_0^\infty\int_{\mathbb{R}^3} |\xi-\xi_*|^\gamma\Big(\ Q^{\beta\beta\sharp}_+(f_{\beta\varepsilon}, f_{\beta\varepsilon}) + Q^{\beta\alpha\sharp}_+(f_{\beta\varepsilon}, f_{\alpha\varepsilon})\\
&\qquad\qquad + Q^{\beta\beta\sharp}_+(\bar f_{\beta\varepsilon}, \bar f_{\beta\varepsilon}) + Q^{\beta\alpha\sharp}_+(\bar f_{\beta\varepsilon}, \bar f_{\alpha\varepsilon})\ \Big)(x+(t+\tau)(\xi-\xi_*),\xi_*) d\xi_* d\tau\Big] d\xi dx\\
&\le \Big(\mathcal{E}^{\alpha\alpha}_\varepsilon(t) + \mathcal{E}^{\alpha\beta}_\varepsilon(t) + \bar{\mathcal{E}}^{\beta\beta}_\varepsilon(t) + \bar{\mathcal{E}}^{\beta\alpha}_\varepsilon(t)\Big)\sum_{\alpha\in\{A,B\}} \|f_{\alpha\varepsilon}(t) - \bar f_{\alpha\varepsilon}(t)\|_{L^1}\\
&\le \frac{\mathcal{O}(1)}{(t+1)^{\gamma+3}}\sum_{\alpha\in\{A,B\}} \|f_{\alpha\varepsilon}(t) - \bar f_{\alpha\varepsilon}(t)\|_{L^1}.
\end{aligned}
$$

Case 3 ($\mathcal{N}_3$): Similar to Case 2, we have

$$
\begin{aligned}
\mathcal{N}_3(t) &\le \int_{\mathbb{R}^3\times\mathbb{R}^3} |f_{\alpha\varepsilon} - \bar f_{\alpha\varepsilon}|^\sharp(x,\xi)\Big(\int_0^\infty\int_{\mathbb{R}^3} |\xi-\xi_*|^\gamma\\
&\qquad\qquad \times (\widehat{P}^{\alpha\alpha\sharp}_\varepsilon + \widehat{P}^{\alpha\beta\sharp}_\varepsilon + \widehat{\bar{P}}^{\beta\beta\sharp}_\varepsilon + \widehat{\bar{P}}^{\beta\alpha\sharp}_\varepsilon)(x+(t+\tau)(\xi-\xi_*),\xi_*) d\xi_* d\tau\Big) d\xi dx\\
&\le \mathcal{O}(1)\Big(\mathcal{G}^{\alpha\alpha}_\varepsilon(t) + \mathcal{G}^{\alpha\beta}_\varepsilon(t) + \bar{\mathcal{G}}^{\beta\beta}_\varepsilon(t) + \bar{\mathcal{G}}^{\beta\alpha}_\varepsilon(t)\Big)\sum_{\alpha\in\{A,B\}} \|f_{\alpha\varepsilon}(t) - \bar f_{\alpha\varepsilon}(t)\|_{L^1}\\
&\le \frac{\mathcal{O}(1)}{(t+1)^{\gamma+3}}\sum_{\alpha\in\{A,B\}} \|f_{\alpha\varepsilon}(t) - \bar f_{\alpha\varepsilon}(t)\|_{L^1}.
\end{aligned}
$$

We combine Cases 1–3 to obtain

$$
\frac{d}{dt}\mathcal{D}_{d\varepsilon}(t) \quad = \quad \sum_{\alpha,\beta\in\{A,B\}} \frac{d}{dt}\mathcal{D}^{\alpha\beta}_{d\varepsilon}(t)
$$

$$
\begin{aligned}
&\le \Big[-1 + (\delta_\alpha + \delta_\beta)(C_\alpha + C_\beta) \Big] \Lambda_{d\varepsilon}(t) + \mathcal{O}(1) \int_{\mathbb{R}^3 \times \mathbb{R}^3} P_d^\sharp(x, \xi, \varphi) d\xi dx \\
&+ \frac{\mathcal{O}(1)}{(t+1)^{\gamma+3}} \sum_{\alpha \in \{A,B\}} \|f_{\alpha\varepsilon}(t) - \bar{f}_{\alpha\varepsilon}(t)\|_{L^1}.
\end{aligned}
$$

Take δ_α and δ_β small enough so that $-1 + (\delta_\alpha + \delta_\beta)(C_\alpha + C_\beta) \le -C_2$ for some positive constant C_2. Then we have the desired result. □

The proof of Theorem 1.2. By definition of $\mathcal{H}_\varepsilon(t)$ and Lemma 6.2, we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H}_\varepsilon(t) &= \frac{d}{dt} \sum_{\alpha \in \{A,B\}} \|f_{\alpha\varepsilon}(t) - \bar{f}_{\alpha\varepsilon}(t)\|_{L^1} + K \frac{d}{dt} \mathcal{D}_{d\varepsilon}(t) \\
&\le \Big(C_1 - K C_2 \Big) \Lambda_{d\varepsilon}(t) + \frac{\mathcal{O}(1)}{(t+1)^{\gamma+3}} \sum_{\alpha \in \{A,B\}} \|f_{\alpha\varepsilon}(t) - \bar{f}_{\alpha\varepsilon}(t)\|_{L^1} \\
&+ K \int_{\mathbb{R}^3 \times \mathbb{R}^3} P_d^\sharp(x, \xi, \varepsilon) d\xi dx.
\end{aligned}
$$

We now choose K large enough so that $C_1 - KC_2 \le -C_3$ for some positive constant C_3 and use

$$
\sum_{\alpha \in \{A,B\}} \|f_{\alpha\varepsilon}(t) - \bar{f}_{\alpha\varepsilon}(t)\|_{L^1} \le \mathcal{H}_\varepsilon(t)
$$

to get

$$
\frac{d\mathcal{H}_\varepsilon(t)}{dt} + C_3 \, \Lambda_{d\varepsilon}(t) \le \frac{\mathcal{O}(1)}{(t+1)^{\gamma+3}} \mathcal{H}_\varepsilon(t) + K \int_{\mathbb{R}^3 \times \mathbb{R}^3} P_d^\sharp(x, \xi, \varepsilon) d\xi dx.
$$

We use Gronwall's Lemma to find

$$
\mathcal{H}_\varepsilon(t) + C_3 \int_0^t \Lambda_{d\varepsilon}(s) ds \le \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} P_d^\sharp(x, \xi, \varepsilon) d\xi dx ds + C_4 \mathcal{H}_\varepsilon(0).
$$

We let $\varepsilon \to 0$ and apply Lemma 4.5 to find

$$
\mathcal{H}(t) + C_3 \int_0^t \Lambda_d(s) ds \le C_4 \mathcal{H}(0).
$$

The L^1_a-stability of mild solutions follows from the equivalence between $\mathcal{H}(t)$ and $\|\cdot\|_{L^1}$:

$$
\sum_{\alpha \in \{A,B\}} \|f_\alpha(t) - \bar{f}_\alpha(t)\|_{L^1} \le \mathcal{H}(t) \le C_4 \mathcal{H}(0) \le C_0 C_4 \sum_{\alpha \in \{A,B\}} \|f_{\alpha 0} - \bar{f}_{\alpha 0}\|_{L^1}.
$$

Finally we set

$$
G \equiv C_0 C_4,
$$

and use the fact that $\sum_{\alpha \in \{A,B\}} \|F_\alpha(t) - \bar{F}_\alpha(t)\|_{L^1_{\alpha,r}} = \sum_{\alpha \in \{A,B\}} \|f_\alpha(t) - \bar{f}_\alpha(t)\|_{L^1}$ to obtain the desired result. □

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References

[1] Andries, P., Aoki, K., Perthame, B.: A consistent BGK-type model for gas mixtures. J. Stat. Phys. **106**, 993-1018 (2002). MR1889599 (2002k:82062)

[2] Aoki, K., Bardos, C., Takata, S.: Knudsen layer for gas mixtures. J. Stat. Phys. **112**, 629-655 (2003). MR1997264 (2004f:82065)

[3] Bellomo, N., Palczewski. A., Toscani, G.: Mathematical topics in nonlinear kinetic theory. World Scientific Publishing Co., Singapore, 1988. MR996631 (90m:82030)

[4] Bellomo, N., Toscani, G.: On the Cauchy problem for the nonlinear Boltzmann equation: Global existence, uniqueness and asymptotic behavior. J. Math. Phys. **26**, 334-338 (1985). MR776503 (86h:82038)

[5] Chapman, S., Cowling, T.G.: The mathematical theory of nonuniform gases. Cambridge University Press, London, 1952.

[6] Chae, M., Ha, S.-Y.: Stability estimates of the Boltzmann equation with quantum effects. Contin. Mech. Thermodyn. **17**, 511-524 (2006) MR2240606

[7] DiPerna, R., Lions, P. L.: On the Cauchy problem for Boltzmann equations: Global existence and weak stability. Ann Math. **130**, 321-366 (1989). MR1014927 (90k:82045)

[8] Garzo, V., Santos, A., Brey, J.J.: A kinetic model for a multi-component gas. Phys. Fluids A **1**, 380-383 (1989).

[9] Ha, S.-Y.: Nonlinear functionals of the Boltzmann equation and uniform stability estimates. J. Differential Equations **215**, 178-205 (2005). MR2146347 (2006b:35033)

[10] Ha, S.-Y., Yun, S.-B.: Uniform L^1-stability estimate of the Boltzmann equation near a local Maxwellian. Phys. D **220**, 79-97 (2006). MR2252152 (2007f:82082)

[11] Hamel, B.: Kinetic model for binary gas mixtures. Phys. Fluids **8**, 418-425 (1965).

[12] Illner, R., Shinbrot, M.: Global existence for a rare gas in an infinite vacuum. Commun. Math. Phys. **95**, 217-226 (1984). MR760333 (86a:82019)

[13] Kaniel, S., Shinbrot, M.: The Boltzmann equation 1: Uniqueness and local existence. Commun. Math. Phys. **58**, 65-84 (1978). MR0475532 (57:15133)

[14] Polewczak, J.: Classical solution of the nonlinear Boltzmann equation in all R^3: Asymptotic behavior of solutions. J. Stat. Phys. **50**, 611-632 (1988). MR939503 (89h:82021)

[15] Sirovich, L.: Kinetic modeling of gas mixtures. Phys. Fluids **5**, 908-918 (1962). MR0144719 (26:2260)

[16] Sone, Y.: Kinetic theory and fluid dynamics. Birkhäuser, Boston 2002. MR1919070 (2003h:76113)

[17] Sone, Y.: Molecular gas dynamics. Birkhäuser, Boston 2006. MR2274674

[18] Sotirov, A., Yu, S.-H.: On the Boltzmann diffusion of two gases. Submitted.

[19] Takata, S.: Half space problem of weak evaporation and condensation of a binary mixture of vapors, in: M. Capitelli (Ed.), Rarefied Gas Dynamics, AIP, New York, 503-508 (2005).

[20] Takata, S.: Kinetic theory analysis of the two-surface problem of a vapor-vapor mixture in the continuum limit. Phys. Fluids **16**, 2182-2188 (2004).

[21] Toscani, G.: H-thoerem and asymptotic trend of the solution for a rarefied gas in a vacuum. Arch. Rational Mech. Anal. **100**, 1-12 (1987). MR906131 (89b:82053)

[22] Toscani, G.: On the nonlinear Boltzmann equation in unbounded domains. Arch. Rational Mech. Anal. **95**, 37-49 (1986). MR849403 (87j:82060)

[23] Yasuda, S., Takata, S., Aoki, K.: Evaporation and condensation of a binary mixture of vapors on a plane condensed phase: Numerical analysis of the linearized Boltzmann equation. Phys. Fluids **17**, 047105 (2005).

[24] Yasuda, S., Takata, S. Aoki, K.: Numerical analysis of the shear flow of a binary mixture of hard-sphere gases over a plane wall, Phys. Fluids **16**, 1989-2003 (2004).