Global Existence for Three-Dimensional Incompressible Isotropic Elastodynamics

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Abstract

The existence of global-in-time classical solutions to the Cauchy problem for incompressible, nonlinear, isotropic elastodynamics for small initial displacements is proved. The generalized energy method is used to obtain strong dispersive estimates that are needed for long-time stability. This requires the use of weighted local decay estimates for the linearized equations, which are obtained as a special case of a new general result for certain isotropic symmetric hyperbolic systems. In addition, the pressure that arises as a Lagrange multiplier to enforce the incompressibility constraint is estimated as a nonlinear term. The incompressible elasticity equations are inherently linearly degenerate in the isotropic case; i.e., the equations satisfy a null condition necessary for global existence in three dimensions.

1 Introduction

The behavior of elastic waves in a three-dimensional isotropic incompressible material is studied. Unlike compressible elastodynamics, where there are nonlinear interactions of shear and pressure waves, with incompressible elastodynamics the only waves present are shear waves. In an isotropic system, shear waves are linearly degenerate, and therefore global solutions to the perturbative incompressible equations can be expected via the generalized energy method. This article confirms this intuitive idea.

Strong dispersive estimates are needed for long-time stability of the solutions for incompressible elastodynamics. The generalized energy method based on the Lorentz invariance of the wave equation combines energy and decay estimates that, together with a null condition, provide global existence results (see [6, 7]). However, the equations of motion for elasticity are only Galilean invariant and the Lorentz rotations cannot be used. As Klainerman and Sideris observed in [8], however, Lorentz invariance is not necessary to obtain almost global existence in three dimensions for isotropic systems such as the equations of elasticity. Global small

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solutions to the equations for compressible elasticity were obtained in [12, 13] with the addition of a null condition for pressure waves; see also [1, 2].

With the observation that the null condition is inherently satisfied by shear waves, the authors were able to show global existence for incompressible elasticity for small data as a limit of slightly compressible materials in [14]. The key step was that the pressure waves vanish in the limit and the shear waves are already null. For this work it was convenient to consider the equations of elasticity as a first-order system in Eulerian coordinates. In this frame the singular term is linear. We find this same set of variables works well for the incompressible equations as the constraints are most naturally posed in that frame. The system can be viewed as an extension of the incompressible Euler equations where the inverse deformation gradient is coupled with the velocity and pressure. It also shares common features with viscoelastic theories, in particular for the Oldroyd-B system for viscoelastic materials [3, 9, 10, 11].

The argument requires the use of weighted local decay estimates for the linearized incompressible equations. We obtain these as a special case of a new general result for certain isotropic symmetric hyperbolic systems presented in a separate paper [15]. This article contains further examples, including compressible elasticity.

One must also handle the pressure that arises as a Lagrange multiplier that enforces the incompressibility constraint. Although it involves nonlocal operators, the pressure term is compatible with our weighted estimate. It can be treated as a quadratic nonlinearity and hence can be bounded by the energy.

The global existence of small solutions to the three-dimensional incompressible and isotropic elasticity equations was announced by Ebin [5]. His direct argument relies on the Lorentz invariance of the wave equation, the linearized operator in the incompressible case; however, in our view insufficient attention is paid to the incompressibility constraint, which is incompatible with the Lorentz rotations. The special case of incompressible neo-Hookean materials was studied in [4].

The main result, Theorem 2.2, is stated on page 1711 after introducing the equations under consideration. In Section 3 we show that the equations satisfy the hypotheses of the local decay theorem in [15], and we describe the notation to be used throughout. In Section 4, we state the local decay estimates from [15], adapt these estimates to our argument, and highlight the Sobolev estimates we will need. In Section 5 we complete the proof of the main result via energy estimates.

2 Preliminaries

Classically the motion of an elastic body is described as a second-order evolution equation in Lagrangian coordinates. In [14] it was shown that one can write these equations as a first-order system with constraints in Eulerian coordinates. This simplifies the incompressibility constraint. Here we will use the same variables as in [14]. We start with a time-dependent family of orientation-preserving diffeomorphisms $x(t,\cdot)$, $0 \le t < T$. Material points X in the reference configuration are deformed to the spatial position $x(t,X)$ at time t. We also write $X(t,x)$ for the inverse transformation. Derivatives with respect to the material coordinates will be written as (D_t, D) and with respect to the spatial coordinate as $(\partial_t, \partial_1, \partial_2, \partial_3) = (\partial_t, \nabla) = \partial$.

LEMMA 2.1 *Given a family of deformations* $x(t,X)$ *with inverse* $X(t,x)$*, define the velocity and inverse deformation gradient as follows:*

$$v(t,x) = D_t x(t, X(t,x)), \tag{2.1a}$$

$$H(t,x) = \nabla X(t,x). \tag{2.1b}$$

Then for $(t,x) \in [0,T) \times \mathbb{R}^3$,

$$\partial_t H + \nabla(Hv) = \partial_t H + v \cdot \nabla H + H \nabla v = 0, \tag{2.2a}$$

$$\partial_j H^i_k(t,x) = \partial_k H^i_j(t,x). \tag{2.2b}$$

If, in addition, $\det H(0,x) = 1$ *and* $\nabla \cdot v(t,x) = 0$*, then we have*

$$\det H(t,x) = 1. \tag{2.2c}$$

PROOF: Since $X(t, x(t,X)) = X$, we see that $X(t,x)$ is constant along particle trajectories. This means that

$$\partial_t X + v \cdot \nabla X = 0.$$

Taking the gradient with respect to x yields (2.2a). As soon as H satisfies (2.1b), (2.2b) follows. Using (2.2a), one can write down the continuity equation for $\det H$ given by

$$\partial_t \det H + v \cdot \nabla \det H + \nabla \cdot v \;\det H = 0.$$

If $\nabla \cdot v(t,x) = 0$, then we see that $\det H$ is constant along particle paths. Thus if $\det H(0,x) = 1$, then $\det H(t,x) = 1$ for all $t \ge 0$. □

The equations of motion for incompressible elasticity can be derived from the formal variational problem

$$\delta \iint \Big[\frac{1}{2}|v|^2 - W(F) + \lambda(\det H - 1) + \mu^i_\ell(\partial_t H^i_\ell + v \cdot \nabla H^i_\ell + H^i_k \partial_\ell v^k) + \eta^i_{\ell m}(\partial_\ell H^i_m - \partial_m H^i_\ell) \Big] \det H \, dx\, dt = 0,$$

with $H = F^{-1}$, in which the isotropic strain energy function $W(F) \in C^\infty(\mathrm{GL}_3, \mathbb{R})$ depends on F through the principal invariants of the strain matrix $F^\mathsf{T} F$. Summation over repeated indices will always be understood. We use the notation GL_3 for the group of invertible 3×3 matrices over $\mathbb{R}$ with positive determinant. The quantities λ, μ, and η are Lagrange multipliers.

The Piola-Kirchhoff stress has the form $S(F) = \frac{\partial W}{\partial F}$. We assume that the material is stress free at the identity, i.e.,

$$S(I) = 0, \tag{2.3}$$

and we define the elasticity tensor

$$A_{ij}^{\ell m}(F) = \frac{\partial S_i^\ell}{\partial F_m^j}(F) = \frac{\partial^2 W}{\partial F_\ell^i \partial F_m^j}(F). \tag{2.4a}$$

In the isotropic case (and under the Legendre-Hadamard ellipticity condition), the linearized elasticity tensor takes the form

$$A_{ij}^{\ell m}(I) = (c_1^2 - 2c_2^2)\delta_i^\ell \delta_j^m + c_2^2(\delta^{\ell m}\delta_{ij} + \delta_j^\ell \delta_i^m) \quad \text{with } c_1 > c_2 > 0. \tag{2.4b}$$

The parameters c_1 and c_2 depend only on W, and they represent the speeds of propagation of pressure and shear waves, respectively. As we will see later, the pressure waves are degenerate for incompressible motions. Note that the hydrodynamical case $W \equiv W(\det F)$ is ruled out by the condition $c_2 > 0$ in (2.4b). Then for $(t, x) \in [0, T) \times \mathbb{R}^3$, the equations of motion for the system of incompressible elasticity are

$$\partial_t H_\ell^i + v \cdot \nabla H_\ell^i + H_p^i \partial_\ell v^p = 0, \tag{2.5a}$$

$$\partial_t v^i + v \cdot \nabla v^i + \hat{A}_{pj}^{\ell m}(H) H_i^p \partial_\ell H_m^j + \partial_i p = 0, \tag{2.5b}$$

with the constraints

$$\partial_\ell H_m^i = \partial_m H_\ell^i, \tag{2.5c}$$

$$\det H = 1, \tag{2.5d}$$

$$\nabla \cdot v = 0. \tag{2.5e}$$

We use definitions (2.4a) and (2.4b) and the chain rule to relate the elasticity tensor $\hat{A}(H)$ to the tensor $A(F)$ coming from the strain energy function $W(F)$. In order to ensure that the energy density for our system is positive definite, we add the null Lagrangian $c_2^2(\delta_i^\ell \delta_j^m - \delta_j^\ell \delta_i^m)$ in the definition of $\hat{A}(H)$. This will not change the equations as long as we consider solutions that satisfy the constraint (2.5c). We have the following:

$$\hat{A}_{ij}^{\ell m}(H) = A_{IJ}^{LM}(F) F_i^I F_j^J F_L^\ell F_M^m \big|_{F=H^{-1}} + c_2^2(\delta_i^\ell \delta_j^m - \delta_j^\ell \delta_i^m), \tag{2.6a}$$

$$\hat{A}_{ij}^{\ell m}(H) = \hat{A}_{ji}^{m\ell}(H), \tag{2.6b}$$

$$\hat{A}_{ij}^{\ell m}(I) = (c_1^2 - c_2^2)\delta_i^\ell \delta_j^m + c_2^2 \delta^{\ell m}\delta_{ij}, \tag{2.6c}$$

and, in addition, for $|\dot{H}| \equiv |H - I|$ sufficiently small,

$$\hat{A}_{ij}^{\ell m}(H) \dot{H}_\ell^i \dot{H}_m^j \geq c_2^2 |\dot{H}|^2. \tag{2.6d}$$

We will be studying global existence for the system of incompressible elastodynamics with small initial data. We consider small perturbations from the background state $(H, v) = (I, 0)$, and the linearized equations for $\dot{U} = (\dot{H}, \dot{v}) = (H - I, v)$ are given below,

$$\partial_t \dot{H} + \nabla \dot{v} = N^H(\dot{U}), \tag{2.7a}$$

$$\partial_t \dot{v} + \nabla \cdot T\dot{H} + \nabla p = N^v(\dot{U}), \tag{2.7b}$$

where $T \in \mathcal{L}(\mathbb{R}^3 \otimes \mathbb{R}^3, \mathbb{R}^3 \otimes \mathbb{R}^3)$ is defined by

$$T\dot{H} = c_2^2 \dot{H} + (c_1^2 - c_2^2) \operatorname{tr} \dot{H} I \tag{2.7c}$$

and

$$(\nabla \cdot T\dot{H})^i = \partial_\ell (T\dot{H})^i_\ell.$$

Furthermore,

$$(N^H)^i_\ell(\dot{U}) = -\dot{v} \cdot \nabla \dot{H}^i_\ell - \dot{H}^i_p \partial_\ell \dot{v}^p \tag{2.7d}$$

and

$$(N^v)^i(\dot{U}) = -\dot{v} \cdot \nabla \dot{v}^i - \hat{A}^{\ell m}_{pj}(H) \dot{H}^p_i \partial_\ell \dot{H}^j_m. \tag{2.7e}$$

Since

$$\det(H) = \det(I + \dot{H}) = 1 + \operatorname{tr} \dot{H} + \frac{1}{2}((\operatorname{tr} \dot{H})^2 - \operatorname{tr} \dot{H}^2) + \det \dot{H},$$

we can express the constraints appearing in Lemma 2.1 as

$$\partial_\ell H^i_m - \partial_m H^i_\ell = 0, \tag{2.7f}$$

$$\nabla \operatorname{tr} H = M^H(\dot{U}), \tag{2.7g}$$

$$\nabla \cdot v = 0, \tag{2.7h}$$

with

$$M^H(\dot{U}) = -\nabla \left[\frac{1}{2} \left((\operatorname{tr} \dot{H})^2 - \operatorname{tr} \dot{H}^2 \right) + \det \dot{H} \right]. \tag{2.7i}$$

We write condition (2.7g) as a derivative so it will fit in the form of the theorem in [15].

We now state the global existence result. For expository reasons, we postpone the definition of the generalized energy norm $E_\kappa[U(t)]$ and the generalized Sobolev spaces H^κ_Λ and $H^\kappa_\Gamma(T)$ until Section 3.3. In what follows we use the notation $\langle f \rangle = (1 + |f|^2)^{1/2}$.

THEOREM 2.2 *Let $W(F)$ be an isotropic strain energy function satisfying* (2.4b). *Let $X_0(x)$ be an orientation- and volume-preserving diffeomorphism on $\mathbb{R}^3$, and let $v_0(x)$ be a divergence-free vector field on $\mathbb{R}^3$. Define*

$$U_0 = (H_0, v_0) = (\nabla X_0, v_0),$$
$$\dot{U}_0 = (\dot{H}_0, \dot{v}_0) = (H_0 - I, v_0).$$

Suppose that $\dot{U}_0 \in H^\kappa_\Lambda$, *with* $\kappa \geq 8$, *and that*

$$E^{1/2}_\kappa[U_0] < C, \quad E^{1/2}_{\kappa-2}[U_0] < \varepsilon, \tag{2.8}$$

for uniform constants C and ε.

If ε *is sufficiently small, then the initial value problem for* (2.7a)–(2.7b) *with initial data* $U(0) = U_0$ *has a unique solution* $U(t) \in H^\kappa_\Gamma(T)$ *for all* T *that satisfies the constraints* (2.7f)–(2.7h) *and the estimates*

$$E^{1/2}_{\kappa-2}[U(t)] \leq C' E^{1/2}_{\kappa-2}[U_0] \leq C'\varepsilon, \tag{2.9a}$$

$$E^{1/2}_\kappa[U(t)] \leq C' E^{1/2}_\kappa[U_0]\langle t\rangle^{C'\varepsilon}, \tag{2.9b}$$

for all $t \in (0, \infty)$, *where* C' *is a uniform constant.*

The proof of this theorem will be given in Section 5.

3 General Framework

3.1 Structure of Equations

In order to obtain local energy decay, we first put the system (2.5a)–(2.5b) with constraints (2.5c)–(2.5e) in the framework of the theorem from [15]. The goal is to write the problem as a symmetric hyperbolic system

$$L(\partial)\dot{U} \equiv \partial_t \dot{U} - A(\nabla)\dot{U} = N(\dot{U}) - \nabla P, \tag{3.1a}$$

together with a system of constraints

$$B(\nabla)\dot{U} = M(\dot{U}). \tag{3.1b}$$

The natural vector space in which to work is $\mathcal{V} = (\mathbb{R}^3 \otimes \mathbb{R}^3) \times \mathbb{R}^3$, and we will write $U = (H, v)$, $\bar{U} = (\bar{H}, \bar{v})$, etc., for elements in $\mathcal{V}$.

In (3.1a), we have $\dot{U} = (H - I, v)$, $P = (0, p)$, and $A(\nabla) = A_k \partial_k$. Here the coefficients $A_k \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ are defined by

$$A_k U = A_k(H, v) = (-v \otimes e_k, -THe_k),$$

where T is as in (2.7c). Using (2.7d) and (2.7e), the nonlinearity is given by $N(\dot{U}) = (N^H(\dot{U}), N^v(\dot{U}))$. The system given in (2.7a)–(2.7b) is equivalent to equation (3.1a).

Next, we reformulate the system of constraints. Let $\{e_i\}_{i=1}^3$ be the standard basis on $\mathbb{R}^3$, and define the antisymmetric maps

$$S_{ij} = e_i \otimes e_j - e_j \otimes e_i, \quad 1 \leq i < j \leq 3. \tag{3.2}$$

With

$$\mathcal{W} = (\mathbb{R}^3 \otimes \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R},$$

we define $B_k \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ by

$$\begin{aligned} B_k U = B_k(H, v) = \sum_{\ell,m,n} &\langle H, e_n \otimes S_{\ell m} e_k\rangle_{\mathbb{R}^3 \otimes \mathbb{R}^3}(S_{\ell m}, e_n, 0, 0) \\ &+ \operatorname{tr} H(0, 0, e_k, 0) + \langle v, e_k\rangle_{\mathbb{R}^3}(0, 0, 0, 1). \end{aligned} \tag{3.3}$$

Then the constraint system (2.7f)–(2.7h) can be written in the general form (3.1b) with $B(\nabla) = B_k \partial_k$ and $M(\dot{U}) = (0, M^H(\dot{U}), 0)$ by using (2.7i).

To discuss symmetry, the appropriate inner product on $\mathcal{V}$ is

$$\begin{aligned}\langle U, \bar{U}\rangle_{\mathcal{V}} &= \langle (H, v), (\bar{H}, \bar{v})\rangle_{\mathcal{V}} \\ &= \langle TH, \bar{H}\rangle_{\mathbb{R}^3\otimes\mathbb{R}^3} + \langle v, \bar{v}\rangle_{\mathbb{R}^3} \\ &= \operatorname{tr}(TH)\bar{H}^* + \langle v, \bar{v}\rangle_{\mathbb{R}^3}.\end{aligned}$$

According to the result in [15], there are three conditions that our system (3.1a)–(3.1b) must satisfy. The first is the symmetry condition, namely, as elements of $\mathcal{L}(\mathcal{V}, \mathcal{V})$,

$$A_k = A_k^*, \quad k = 1, 2, 3, \tag{3.4}$$

which clearly holds with the defined inner product.

Associated to the differential operators $A(\nabla)$ and $B(\nabla)$, define the symbols

$$A(\xi) = A_k \xi^k \quad \text{and} \quad B(\xi) = B_k \xi^k, \quad \xi \in \mathbb{R}^3.$$

The second assumption is that

$$\ker B(\xi) \cap \ker A(\xi) = \{0\} \quad \text{for every } 0 \neq \xi \in \mathbb{R}^3. \tag{3.5}$$

To verify this condition, we show that $\ker A(\omega) \subset \ker B(\omega)^\perp$ for $\omega \in \mathbb{S}^2$. Take

$$U \in \ker A(\omega) = \{(H, v) \in \mathcal{V} : TH\omega = 0,\ v = 0\},$$

and, using the notation

$$P_1(\omega) = \omega \otimes \omega, \quad P_2(\omega) = I - \omega \otimes \omega, \tag{3.6}$$

take

$$\bar{U} \in \ker B(\omega) = \{(\bar{H}, \bar{v}) \in \mathcal{V} : H = \bar{H} P_1,\ \operatorname{tr} \bar{H} = 0,\ \langle \bar{v}, \omega\rangle_{\mathbb{R}^3} = 0\}.$$

Then

$$\begin{aligned}\langle U, \bar{U}\rangle_{\mathcal{V}} &= \langle TH, \bar{H}\rangle_{\mathbb{R}^3\otimes\mathbb{R}^3} + \langle v, \bar{v}\rangle_{\mathbb{R}^3} \\ &= \langle TH, \bar{H} P_1\rangle_{\mathbb{R}^3\otimes\mathbb{R}^3} = \langle TH\omega, \bar{H}\omega\rangle_{\mathbb{R}^3} = 0.\end{aligned}$$

Note that the inclusion is strict: $\ker A \subsetneq \ker B^\perp$; this is different from the example of compressible elasticity considered in [14, 15].

The third assumption is that there exist smooth maps carrying the identity to the identity such that

$$V : \mathrm{SO}(\mathbb{R}^3) \to \mathrm{SO}(\mathcal{V}) \quad \text{and} \quad W : \mathrm{SO}(\mathbb{R}^3) \to \mathcal{L}(\mathcal{W}, \mathcal{W})$$

such that for every $\xi \in \mathbb{R}^3$ and $R \in \mathrm{SO}(\mathbb{R}^3)$

$$A(R\xi) = V(R)A(\xi)V(R)^* \tag{3.7a}$$

and

$$B(R\xi) = W(R)B(\xi)V(R)^*. \tag{3.7b}$$

The maps V and W will be used to define commuting vector fields. For our situation we define the mapping $V : \mathrm{SO}(\mathbb{R}^3) \to \mathcal{L}(\mathcal{V}, \mathcal{V})$ for pairs $U = (H, v)$ by

$$V(R)U = (RHR^*, Rv). \tag{3.8}$$

This map, in fact, takes values in $\mathrm{SO}(\mathcal{V})$, and it also satisfies conditions (3.7a) and (3.7b). We will verify (3.7b). By the definitions

$$\begin{aligned} B(R\omega)V(R)U &= \sum_{\ell,m,n} \langle RHR^*, e_n \otimes S_{\ell m} R\omega\rangle_{\mathbb{R}^3\otimes\mathbb{R}^3}(S_{\ell m}, e_n, 0, 0) \\ &\qquad + \operatorname{tr} RHR^*(0, 0, R\omega, 0) + \langle Rv, R\omega\rangle_{\mathbb{R}^3}(0, 0, 0, 1) \\ &= \sum_{\ell,m,n} \langle H, Re_n \otimes R^* S_{\ell m} R\omega\rangle_{\mathbb{R}^3\otimes\mathbb{R}^3}(S_{\ell m}, e_n, 0, 0) \\ &\qquad + \operatorname{tr} H(0, 0, R\omega, 0) + \langle v, \omega\rangle_{\mathbb{R}^3}(0, 0, 0, 1). \end{aligned}$$

Since $R^* S_{\ell m} R$ is antisymmetric, it lies in the span of the S_{ij}, and so the last expression above depends linearly on the coordinates of BU. This implies the existence of a map $W(R) \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ for which (3.7b) is valid. The result from [15] gives local energy decay for a hyperbolic system satisfying (3.4), (3.5), (3.7a), and (3.7b). This decay will be discussed further in Section 4.

3.2 Vector Fields

Using the antisymmetric maps S_{ij} defined in (3.2) let $R_{ij}(\tau) = \exp(\tau S_{ij})$ be a smooth, one-parameter family in $\mathrm{SO}(\mathbb{R}^3)$. The perturbed angular momentum operators that we will use arise as infinitesimal generators

$$\widetilde{\Omega}_{ij} U(x) \equiv \Omega_{ij} U(x) + Z_{ij} U(x) = \frac{d}{d\tau} V(R_{ij}(\tau)) U(R_{ij}(\tau)^* x)\Big|_{\tau=0},$$

where $\Omega_{ij} = x^i \partial_j - x^j \partial_i$ are the standard angular momentum operators and

$$Z_{ij} = \frac{d}{d\tau} V(R_{ij}(\tau))\Big|_{\tau=0} \in \mathcal{L}(\mathcal{V}, \mathcal{V}).$$

Furthermore, we define

$$Y_{ij} = \frac{d}{d\tau} W(R_{ij}(\tau))\Big|_{\tau=0} \in \mathcal{L}(\mathcal{W}, \mathcal{W}).$$

We shall also make use of the scaling vector field

$$S = t\partial_t + r\partial_r.$$

The vector fields will be abbreviated as Γ. We let

$$\Gamma = (\Gamma_1, \dots, \Gamma_8) = (\partial, \tilde{\Omega}, S).$$

Hence by ΓU we mean any one of ∂U, $\tilde{\Omega} U$, or SU. By Γ^a, $a = (a_1, \dots, a_\kappa)$, we denote an ordered product of $\kappa = |a|$ vector fields $\Gamma_{a_1} \cdots \Gamma_{a_\kappa}$. We note that the commutator of any two Γ's is again a Γ.

In order to characterize the initial data, we introduce the time-independent analogue of Γ. Set

$$\Lambda = (\Lambda_1, \dots, \Lambda_7) = (\nabla, \tilde{\Omega}, x \cdot \nabla).$$

Then the commutator of any two Λ's is again a Λ.

3.3 Spaces and Norms

In the following $\|\cdot\|$ and $|\cdot|$ will always denote the norms in $L^2(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$, respectively. Let SL_3 be the group of 3×3 matrices in $\mathbb{R}$ with determinant 1. Define

$$H^\kappa_\Lambda = \{U = (H, v) : \mathbb{R}^3 \to \mathrm{SL}_3 \times \mathbb{R}^3 : \Lambda^a U \in L^2(\mathbb{R}^3),\ |a| \le \kappa\},$$

with the norm

$$\|U\|^2_{H^\kappa_\Lambda} = \sum_{|a| \le \kappa} \|\Lambda^a U\|^2.$$

We now define the energy norm associated with the first-order system. Given $U = (H, v) \in \mathrm{SL}_3 \times \mathbb{R}^3$ and $\dot{U} = (\dot{H}, \dot{v}) = (H - I, v) \in (\mathbb{R}^3 \otimes \mathbb{R}^3) \times \mathbb{R}^3$, define

$$e_U(\dot{U}) = \frac{1}{2}[\hat{A}^{\ell m}_{ij}(H)\dot{H}^i_\ell \dot{H}^j_m + |\dot{v}|^2]. \tag{3.9}$$

Solutions will be constructed in the space

$$H^\kappa_\Gamma(T) \equiv \Big\{ U = (H, v) : [0, T) \times \mathbb{R}^3 \to \mathrm{SL}_3 \times \mathbb{R}^3 \mid \dot{U} = (\dot{H}, \dot{v}) \equiv (H - I, v) \in \bigcap_{j=0}^{\kappa} C^j([0, T), H^{\kappa - j}_\Lambda) \Big\}.$$

Given $U \in H^\kappa_\Gamma(T)$, define

$$E_\kappa[U(t)] = \sum_{|a| \le \kappa} \int e_{U(t)}(\Gamma^a \dot{U}(t))dx. \tag{3.10}$$

By (2.6d), for $|\dot{U}(t)| < \varepsilon$,

$$E^{1/2}_\kappa[U(t)] \sim \sum_{|a| \le \kappa} \|\Gamma^a \dot{U}(t)\|. \tag{3.11}$$

We caution the reader that $\dot{U}$ denotes a perturbation from the background state and not a derivative.

3.4 The Null Condition

Here we formulate the null condition that arises in our specific situation, restricting the nonlinear interaction of shear waves. It is important to note that these conditions are inherent properties of the PDEs and do not involve additional assumptions.

Recalling the projection matrices (3.6) $P_1 = \omega \otimes \omega$ and $P_2 = I - P_1$, a general 6-tensor $\mathcal{B}$ will be said to satisfy the null condition if

$$\mathcal{B}^{\ell mn}_{ijk}(P_1)^L_\ell (P_1)^M_m (P_1)^N_n (P_2)^i_I (P_2)^j_J (P_2)^k_K = 0 \tag{3.12}$$

for all $\omega \in \mathbb{R}^3$ and all I, J, K, L, M, and N.

There are three separate instances where this condition holds. The first involves the main coefficients in the nonlinearity

$$B^{\ell mn}_{ijk} = \frac{\partial^3 W}{\partial F^i_\ell \partial F^j_m \partial F^k_n}(I) = \frac{\partial A^{\ell m}_{ij}}{\partial F^k_n}(I), \tag{3.13a}$$

as was verified in [13]. As in [14], two other sets of coefficients appear that also satisfy the null condition (3.12). These are

$$\hat{B}^{\ell mn}_{ijk}(H) = \frac{\partial \hat{A}^{\ell m}_{ij}}{\partial H^k_n}(H) \quad \text{and} \quad \tilde{B}^{\ell mn}_{ijk} = \hat{A}^{\ell m}_{kj}\delta^n_i. \tag{3.13b}$$

3.5 Commutation

As was shown in [15] we have the following commutation properties for our vector fields Γ:

$$\begin{aligned} L(\partial)\tilde{\Omega}_{ij}\dot{U} &= \tilde{\Omega}_{ij}N(\dot{U}) - \nabla\tilde{\Omega}_{ij}P, \\ B(\nabla)\tilde{\Omega}_{ij}\dot{U} &= (\Omega_{ij} + Y_{ij})M(\dot{U}), \end{aligned} \tag{3.14a}$$

as well as

$$\begin{aligned} L(\partial)S\dot{U} &= (S+1)N(\dot{U}) - \nabla SP, \\ B(\nabla)S\dot{U} &= (S+1)M(\dot{U}). \end{aligned} \tag{3.14b}$$

Using Lemma 4.7 in Section 4, we can make sense of $\Gamma f(\dot{U}(t,x))$ for a nonlinear function f that vanishes to at least quadratic order. Hence we will use the notation $\Gamma f(\dot{U}(t,x))$ with the understanding that these terms will be handled with this lemma. The only difference in our situation from the general case considered in [15] is the pressure, but it is easy to check that the vector fields commute as above when we define ΓP via $\tilde{\Omega}P = (0, \Omega p)$, $SP = (0, Sp)$, and $\partial P = (0, \partial p)$. This leads to the following proposition:

PROPOSITION 3.1 *For any solution* $U = (H, v) \in H^{\kappa}_{\Gamma}(T)$ *of the PDEs* (2.7a)–(2.7b) *and the constraints* (2.7f)–(2.7h)*, we have*

$$\begin{aligned}(3.15\text{a})\qquad &\partial_t \Gamma^a \dot{H} + v \cdot \nabla \Gamma^a \dot{H} + H \nabla \Gamma^a \dot{v} \\ &\quad + \sum_{\substack{b+c=a \\ c\neq a}} [\Gamma^b \dot{v} \cdot \nabla \Gamma^c \dot{H} + \Gamma^b \dot{H} \nabla \Gamma^c \dot{v}] = 0,\end{aligned}$$

$$\begin{aligned}(3.15\text{b})\qquad &\partial_t (\Gamma^a \dot{v})^i + v \cdot \nabla (\Gamma^a \dot{v})^i + \hat{A}^{\ell m}_{pj}(H) H^p_i \partial_\ell (\Gamma^a \dot{H})^j_m + \partial_i \Gamma^a p \\ &\quad + \sum_{\substack{b+c=a \\ c\neq a}} \{\Gamma^b \dot{v} \cdot \nabla (\Gamma^c \dot{v})^i + \Gamma^b [\hat{A}(H) H]^{\ell m}_{ij} \partial_\ell (\Gamma^c \dot{H})^j_m\} = 0,\end{aligned}$$

in which the sums extend over all ordered partitions of the sequence a, with $|a| \le \kappa$*. In addition, the following constraints hold*:

$$\partial_j (\Gamma^a H)^i_k = \partial_k (\Gamma^a H)^i_j, \tag{3.15c}$$

$$\nabla \operatorname{tr} \Gamma^a H = \Gamma^a (M(\dot{U})), \tag{3.15d}$$

$$\nabla \cdot \Gamma^a v = 0. \tag{3.15e}$$

The proof of the commutation comes directly from [15].

3.6 Spectral Projections

For a system of the form (3.1a), consider the eigensystem corresponding to the symbol $A(\omega)$ for each $\omega \in \mathbb{S}^2$. Let $\mathcal{P}_\beta(\omega)$ be the orthogonal projection of $\mathcal{V}$ onto the eigenspace of $A(\omega)$ corresponding to the eigenvalue λ_β. We regard the projections $\{\mathcal{P}_\beta(\omega)\}$ as homogeneous functions of degree 0 on $\mathbb{R}^n$ by setting $\omega = x/|x|$. We have the following:

LEMMA 3.2 *The orthogonal projections* $\mathcal{P}_\beta(\omega)$ *are smooth functions of* $\omega = x/|x|$ *on* $\mathbb{S}^{n-1}$ *that satisfy the commutation property* $[\widetilde{\Omega}_{ij}, \mathcal{P}_\beta(\omega)] = 0$.

The proof is straightforward and given in [15].

4 Decay Estimates

4.1 Local Energy Decay

The local energy decay result from [15] is:

THEOREM 4.1 *Assume that conditions* (3.4) *and* (3.5) *hold. There are positive constants* α *and* C*, depending on the coefficients* A_k *and* B_k*, such that all sufficiently regular solutions of* (3.1a) *and* (3.1b) *satisfy the estimate*

$$\begin{aligned}(4.1)\qquad \alpha t \|\partial_j U\|_{L^2(\{r\le \alpha t\}, \mathcal{V})} \le{}& C\|U\|_{L^2(\mathbb{R}^3, \mathcal{V})} + \|SU\|_{L^2(\mathbb{R}^3, \mathcal{V})} \\ &+ t\|N(\dot{U})\|_{L^2(\mathbb{R}^3, \mathcal{V})} + t\|M(\dot{U})\|_{L^2(\mathbb{R}^3, \mathcal{W})}.\end{aligned}$$

If, in addition, conditions (3.7a) *and* (3.7b) *hold, then*

$$\|(\lambda_\beta t - r)\mathcal{P}_\beta \partial_j U\|_{L^2(\{r\geq\alpha t\},\mathcal{V})} \leq C[\|\widetilde{\Omega} U\|_{L^2(\mathbb{R}^3,\mathcal{V})} + \|U\|_{L^2(\mathbb{R}^3,\mathcal{V})}] + \|SU\|_{L^2(\mathbb{R}^3,\mathcal{V})} + t\|N(\dot{U})\|_{L^2(\mathbb{R}^3,\mathcal{V})} \tag{4.2}$$

and

$$\|rB\partial_j U\|_{L^2(\{r\geq\alpha t\},\mathcal{W})} \leq C[\|\widetilde{\Omega} U\|_{L^2(\mathbb{R}^3,\mathcal{V})} + \|U\|_{L^2(\mathbb{R}^3,\mathcal{V})}] + \|rM(\dot{U})\|_{L^2(\mathbb{R}^3,\mathcal{W})}. \tag{4.3}$$

Having verified conditions (3.4), (3.5), (3.7a), and (3.7b), we apply Theorem 4.1 to the system (2.7a)–(2.7b) with constraints (2.7f)–(2.7h). Consider the eigensystem for $A(\omega)$,

$$A(\omega)U = A(\omega)(H, v) = (-v \otimes \omega, -TH\omega),$$

where T is defined in (2.7c). In this case, the nonzero eigenvalues of $A(\omega)$, representing slow and fast propagation speeds, are

$$\lambda_s^\pm = \pm c_2, \quad \lambda_f^\pm = \pm c_1.$$

Using the notation (3.6) the spectral projections are

$$\mathcal{P}_s^\pm U = \frac{1}{2}(P_2 H P_1 \pm c_2^{-1} P_2 v \otimes \omega, \pm c_2 P_2 H\omega + P_2 v),$$

$$\mathcal{P}_f^\pm U = \langle U, z^\pm\rangle_{\mathcal{V}}\, z^\pm, \quad z^\pm = \frac{1}{\sqrt{2}c_1}(P_1, \pm c_1\omega).$$

Let $\mathcal{P}_s = \mathcal{P}_s^+ + \mathcal{P}_s^-$. Theorem 4.1, via (4.1) and (4.2), gives a bound for

$$\|(\lambda_s t - r)\mathcal{P}_s \partial_j U\|_{L^2(\mathbb{R}^3,\mathcal{V})} = \|(\lambda_s t - r)(P_2\partial_j H P_1, P_2 \partial_j v)\|_{L^2(\mathbb{R}^3,\mathcal{V})} \tag{4.4}$$

in terms of

$$C\left[\|\Gamma U\|_{L^2(\mathbb{R}^3,\mathcal{V})} + \|U\|_{L^2(\mathbb{R}^3,\mathcal{V})} + t\|\nabla p\|_{L^2(\mathbb{R}^3,\mathcal{V})} + t\|N(\dot{U})\|_{L^2(\mathbb{R}^3,\mathcal{V})} + t\|M(\dot{U})\|_{L^2(\mathbb{R}^3,\mathcal{V})}\right]. \tag{4.5}$$

The fast components turn out to be anomalous because the additional constraints strengthen the estimates.

The projection onto ker A is

$$\mathcal{P}_0 U = (HP_2, 0) - \frac{c_1^2 - c_2^2}{c_1^2}\,\mathrm{tr}\, HP_2(P_1, 0).$$

Since

$$|\mathcal{P}_0 U|_{\mathcal{V}}^2 = c_2^2 |HP_2|^2_{\mathbb{R}^3\otimes\mathbb{R}^3} + (c_1^2 - c_2^2)\frac{c_2^2}{c_1^2}(\mathrm{tr}\, HP_2)^2,$$

we get a bound for

$$\|(t + r)\partial_j HP_2\|^2_{L^2(\mathbb{R}^3)} \tag{4.6}$$

in terms of (4.5).

Combining (4.4) and (4.6), we have that

$$\|(\lambda_s t - r)(P_2\partial_j H,\ P_2\partial_j v)\|_{L^2(\mathbb{R}^3,\mathcal{V})} \tag{4.7}$$

is bounded by (4.5).

In view of (3.3), we get from (4.1) and (4.3) that

$$\|(t+r)\,\mathrm{tr}\,\partial_j H\|_{L^2(\mathbb{R}^3)} + \|(t+r)\langle\omega,\partial_j v\rangle\|_{L^2(\mathbb{R}^3)} \tag{4.8}$$

is bounded by the expression (4.5) plus $\|rM(\dot{U})\|_{L^2(\mathbb{R}^n,\mathcal{W})}$. Since

$$\begin{aligned} |P_1 H|^2_{\mathbb{R}^3\otimes\mathbb{R}^3} &= |P_1 H P_1|^2_{\mathbb{R}^3\otimes\mathbb{R}^3} + |P_1 H P_2|^2_{\mathbb{R}^3\otimes\mathbb{R}^3} \\ &= (\mathrm{tr}\,HP_1)^2 + |P_1 H P_2|^2_{\mathbb{R}^3\otimes\mathbb{R}^3} \\ &= (\mathrm{tr}\,H - \mathrm{tr}\,HP_2)^2 + |P_1 H P_2|^2_{\mathbb{R}^3\otimes\mathbb{R}^3} \end{aligned}$$

and also $|P_1 v|^2_{\mathbb{R}^3} = \langle\omega, v\rangle^2_{\mathbb{R}^3}$, we obtain from (4.6) and (4.8) that

$$\|(t+r)(P_1\partial_j H,\ P_1\partial_j v)\|_{L^2(\mathbb{R}^3,\mathcal{V})} \tag{4.9}$$

has the same bound as (4.8).

We may therefore conclude with the following corollary:

COROLLARY 4.2 *A C^1 solution U of* (2.7a)–(2.7b) *and* (2.7f)–(2.7h) *satisfies the estimate*

$$\begin{aligned} &\|(\lambda_s t - r)(P_2\partial_j H,\ P_2\partial_j v)\|_{L^2(\mathbb{R}^3,\mathcal{V})} + \|(t+r)(P_1\partial_j H,\ P_1\partial_j v)\|_{L^2(\mathbb{R}^3,\mathcal{V})} \\ &\quad\le C\big[\|\Gamma U\|_{L^2(\mathbb{R}^3,\mathcal{V})} + \|U\|_{L^2(\mathbb{R}^3,\mathcal{V})} + t\|\nabla p\|_{L^2(\mathbb{R}^3,\mathcal{V})} \\ &\qquad + t\|N(U)\|_{L^2(\mathbb{R}^3,\mathcal{V})} + \|(t+r)M(U)\|_{L^2(\mathbb{R}^3,\mathcal{V})}\big]. \end{aligned}$$

4.2 Bound for Pressure

The following simple lemma is important because it shows that the gradient of the pressure can be treated as a nonlinear term in $H^k_\Gamma(T)$.

LEMMA 4.3 *Let $U \in H^\kappa_\Gamma(T)$ solve the equations* (2.7a)–(2.7b) *and the constraints* (2.7f)–(2.7h). *Then we have for $|a| \le \kappa - 1$,*

$$\|\nabla\Gamma^a p\| \le C[\|\Gamma^a N(\dot{U})\| + \|\Gamma^a M(\dot{U})\|]. \tag{4.10}$$

PROOF: We begin by taking the divergence of (2.7b), which, with the use of the constraints (2.7f)–(2.7h), gives

$$\triangle p = \nabla\cdot N^v(\dot{U}) - c_1^2\nabla\cdot M^H(\dot{U}).$$

This holds for higher derivatives as well and hence for any a, $|a| \le \kappa - 1$,

$$\triangle\Gamma^a p = \nabla\cdot[\Gamma^a(N^v(\dot{U})) - c_1^2\Gamma^a(M^H(\dot{U}))]. \tag{4.11}$$

If we temporarily call the right-hand side of (4.11) $\nabla \cdot \Phi$, we integrate by parts to get

$$\|\nabla\Gamma^a p\|^2 = \int \partial_i \Gamma^a p \partial_i \Gamma^a p\, dx = -\int \triangle\Gamma^a p \Gamma^a p\, dx$$
$$= -\int \nabla\cdot\Phi\Gamma^a p\, dx = \int \Phi\nabla\Gamma^a p\, dx \le \|\Phi\|\cdot\|\nabla\Gamma^a p\|,$$

which implies the result. □

4.3 Weighted L^2 Estimates

The main decay result we need can now be stated by combining Corollary 4.2 with Lemma 4.3. First we will introduce some notation. Define the weights

$$\mathcal{W}_1 = \langle t + r\rangle \quad \text{and} \quad \mathcal{W}_2 = \langle c_2 t - r\rangle. \tag{4.12}$$

Using the projections defined in (3.6) (and abusing notation), we define the projection of U by

$$P_1 U = (P_1 H, P_1 v) \quad \text{and} \quad P_2 U = (P_2 H, P_2 v). \tag{4.13}$$

Further, define

$$\mathcal{X}_\kappa[U(t)] = \sum_{|a|\le\kappa-1} \sum_{i=1}^{3} \sum_{\alpha=1,2} \|\mathcal{W}_\alpha P_\alpha \partial_i \Gamma^a U(t)\|. \tag{4.14}$$

PROPOSITION 4.4 *Let $U \in H^\kappa_\Gamma(T)$ solve equations* (2.7a)–(2.7b) *and constraints* (2.7f)–(2.7h). *Then we have*

$$\begin{aligned}\mathcal{X}_\sigma[U(t)] \le\ & C E^{1/2}_\sigma[U(t)] \\ & + C\sum_{|a|\le\sigma}\left(\langle t\rangle\|\Gamma^a(N(\dot U))\| + \|\langle t+r\rangle\Gamma^a(M(\dot U))\|\right).\end{aligned} \tag{4.15}$$

PROOF: We can obtain the decay results for higher derivatives by first taking Γ^a derivatives of the equations (2.7a)–(2.7b) and constraints (2.7f)–(2.7h) and using the commutation properties given in Proposition 3.1. In this way we can obtain Corollary 4.2 for higher derivatives. The result follows upon applying Lemma 4.3. □

4.4 Sobolev Estimates

The following two results appeared in [14]; we will omit the proofs.

PROPOSITION 4.5 *Let $\dot U \in H^\kappa_\Gamma(T)$, with $\mathcal{X}_\kappa[U(t)] < \infty$ and $|\dot U| < \delta$ small. Then for $i = 1, 2$,*

$$\langle r\rangle|\Gamma^a \dot U(t,x)| \le C E^{1/2}_\kappa[U(t)], \qquad |a| + 2 \le \kappa, \tag{4.16a}$$

$$\langle r\rangle\mathcal{W}_i^{1/2}|P_i\Gamma^a \dot U(t,x)| \le C[E^{1/2}_\kappa[U(t)] + \mathcal{X}_\kappa[U(t)]], \qquad |a| + 2 \le \kappa, \tag{4.16b}$$

$$\langle r\rangle\mathcal{W}_i|P_i\nabla\Gamma^a \dot U(t,x)| \le C\mathcal{X}_\kappa[U(t)], \qquad |a| + 3 \le \kappa. \tag{4.16c}$$

LEMMA 4.6 *Suppose that* $f : \mathbb{R}^3 \to \mathbb{R}^3$, $\sum_{|a|\le 2} \|\Omega^a f\|_{W^{2-|a|,2}} < \infty$, *and* $\nabla \wedge f = 0$. *Then*

$$|x|^{3/2}|P_2 f(x)| \le C \sum_{|a|\le 2} \|\Omega^a f\|. \tag{4.17}$$

4.5 Bootstrapping the Nonlinearity

Here we show that the $\mathcal{X}$ norm can be bounded by the energy for small initial data. The factor $\langle t+r\rangle$ can be absorbed using the estimates from Proposition 4.5. We will also need some facts about the weights W_α as defined in (4.12), which are collected below.

We observe that $r \sim \langle c_2 t\rangle$ on $\mathcal{C}_{c_2} \equiv \{|t - r/c_2| < t/2\}$ and $\langle c_2 t - r\rangle \ge C\langle t\rangle$ on $\mathcal{C}_{c_2}^c$, which implies that

$$C\langle t\rangle^{-1}\langle r\rangle\langle c_2 t - r\rangle \ge 1. \tag{4.18}$$

Additionally, since

$$\langle t+r\rangle \le C\langle r\rangle \mathcal{W}_\alpha \quad \text{for } \alpha = 1, 2,$$

we have that

$$\begin{aligned}\langle t+r\rangle|U| = \langle t+r\rangle|P_1 U + P_2 U| &\le \langle t+r\rangle \sum_{\alpha=1,2} |P_\alpha U| \\ &\le C\langle r\rangle \sum_{\alpha=1,2} \mathcal{W}_\alpha |P_\alpha U|\end{aligned} \tag{4.19}$$

for all $U \in (\mathbb{R}^3 \otimes \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}$.

Next, we include a technical result that allows us to handle the cubic and higher-order terms.

LEMMA 4.7 *Suppose that* $U \in H^\kappa_\Gamma(T)$ *with* $\kappa \ge 3$. *Set* $\kappa' = [\kappa/2] + 2$ *(so that* $\kappa' \le \kappa$*). Suppose that* $E_{\kappa'}[U(t)] < 1$ *and* $|\dot U(t)| \le \delta$, $0 \le t < T$, *with* δ *sufficiently small. Consider a smooth mapping* $f : (\mathbb{R}^3 \otimes \mathbb{R}^3) \times \mathbb{R}^3 \to \mathbb{R}^d$ *for any* d. *If* f *vanishes to order* p *at the origin, then we have the pointwise estimate*

$$|\Gamma^b f(\dot U(t,x))| \le C \sum_{|b_1|+\cdots+|b_p|\le |b|} |\Gamma^{b_1}\dot U(t,x)| \cdots |\Gamma^{b_p}\dot U(t,x)|, \quad |b| \le \kappa.$$

PROOF: Using the chain rule, we write

$$\Gamma^b f(\dot U)(t,x) = \sum_{j\le |b|} \sum_{b_1+\cdots+b_j=b} f^{(j)}(\dot U(t,x))\Gamma^{b_1}\dot U(t,x)\cdots\Gamma^{b_j}\dot U(t,x). \tag{4.20}$$

At most one derivative above can exceed order $[\kappa/2]$, since $|b| \le \kappa$. Since $\dot U$ is small and $E_{\kappa'}[U(t)] < 1$, we have by the Sobolev lemma and (3.11) that

$$|\Gamma^c \dot U(t,x)| \le C E^{1/2}_{\kappa'}[U(t)] \le C, \quad |c| \le [\kappa/2].$$

The result now follows from (4.20) since by the mean value theorem

$$|f^{(j)}(\dot U)| \le C|\dot U|^{p-j}, \quad j \le p,$$

for $|\dot U| \le 1$. □

We are now ready to move to the main results of this section.

LEMMA 4.8 *Let $U \in H^\mu_\Gamma(T)$, $\mu \ge 3$, be a solution of the PDEs* (2.7a)–(2.7b) *and the constraints* (2.7f)–(2.7h). *Set $\mu' = [\mu/2]+2$, and assume that $E_{\mu'}[U(t)] < 1$ and $|\dot U(t)| < \delta$ throughout* $[0,T)$, *with δ sufficiently small. Then we have*

$$\mathcal{X}_\mu[U(t)] \le C\big[E_\mu^{1/2}[U(t)] + \mathcal{X}_{\mu'}[U(t)]E_\mu^{1/2}[U(t)] + \mathcal{X}_\mu[U(t)]E_{\mu'}^{1/2}[U(t)]\big].$$

PROOF: Using Proposition 4.4 we have

$$\mathcal{X}_\mu[U(t)] \le CE_\sigma^{1/2}[U(t)] + C\sum_{\substack{|a|\le\mu\\|b|+|c|=|a|}}\big(\langle t\rangle\|\Gamma^a(N(\dot U))\| + \|\langle t+r\rangle\Gamma^a(M(\dot U))\|\big).$$

Here the form of $\Gamma^a(N(\dot U))$ and $\Gamma^a(M(\dot U))$ can be given by

$$\sum_{|a|=|b|+|c|\le\mu-1}\Gamma^b f(\dot U)\nabla\Gamma^c\dot U,$$

where f vanishes to order $p=1$. This can be checked from the definitions of N and M. Applying Lemma 4.7, we have the pointwise estimate

$$\sum_{|a|\le\mu-1}|\Gamma^a N(\dot U(t,x))| \le C\sum_{|b|+|c|\le\mu-1}|\Gamma^b\dot U(t,x)||\nabla\Gamma^c\dot U(t,x)|,$$

and similarly for $\Gamma^a M(\dot U)$.

With this and (4.19), we obtain

$$\begin{aligned}\sum_{|a|\le\mu-1}&\big[\langle t\rangle\|\Gamma^a N(\dot U)\| + \|\langle t+r\rangle\Gamma^a M(\dot U)\|\big]\\ &\le C\sum_{|b|+|c|\le\mu-1}\|\langle t+r\rangle|\Gamma^b\dot U|\,|\nabla\Gamma^c\dot U|\,\|\\ &\le C\sum_{\substack{\alpha=1,2\\ i=1,2,3\\|b|+|c|\le\mu-1}}\|\langle r\rangle|\Gamma^b\dot U|\,\mathcal{W}_\alpha|P_\alpha\partial_i\Gamma^c\dot U|\,\|.\end{aligned}$$

In the sum, either $|b| \le [\mu/2]$ or $|c|+1 \le [\mu/2]$, according to which we estimate as follows:

$$\|\langle r\rangle|\Gamma^b\dot U|\,\mathcal{W}_\alpha\,|P_\alpha\partial_i\Gamma^c\dot U|\,\| \le C\begin{cases}\|\langle r\rangle\mathcal{W}_\alpha P_\alpha\partial_i\Gamma^c\dot U\|_\infty\,\|\Gamma^b\dot U\| & \text{if } |c|+1\le[\mu/2],\\ \|\mathcal{W}_\alpha P_\alpha\partial_i\Gamma^c\dot U\|\,\|\langle r\rangle\Gamma^b\dot U\|_\infty & \text{if } |b|\le[\mu/2].\end{cases}$$

In the first case, using (4.16c), we get the upper bound

$$C\mathcal{X}_{\mu'}[U(t)]E_{\mu}^{1/2}[U(t)],$$

and in the second case, using (4.16a), we get the upper bound

$$C\mathcal{X}_{\mu}[U(t)]E_{\mu'}^{1/2}[U(t)]. \qquad \square$$

The next step is to bootstrap the preceding result to bound $\mathcal{X}$ by the energy.

PROPOSITION 4.9 *Let* $U \in H_\Gamma^\kappa(T)$, $\kappa \geq 8$, *be a solution of* (2.7a)–(2.7b). *If* $E_\mu[U(t)] < \varepsilon'$, $\mu = \kappa - 2$, *remains sufficiently small on* $[0, T)$, *then for either* $\nu = \kappa$ *or* $\nu = \mu$,

$$\mathcal{X}_\nu[U(t)] \leq C E_\nu^{1/2}[U(t)]. \tag{4.21}$$

PROOF: Since we have $\mu \geq 6$, it follows that $\mu' = [\mu/2] + 2 \leq \mu$. We ensure that ε' is small enough so that for $t \in [0, T)$, $E_\mu[U(t)] < \varepsilon' < 1$, and also $|\dot{U}(t)| < \delta$, which is possible by Sobolev embedding. Hence we can apply Lemma 4.8 to obtain

$$\mathcal{X}_\mu[U(t)] \leq C\big[E_\mu^{1/2}[U(t)] + \mathcal{X}_\mu[U(t)]E_\mu^{1/2}[U(t)]\big].$$

Since $E_\mu^{1/2}[U(t)] < \varepsilon'$, we have for ε' small enough that the bound (4.21) holds for $\nu = \mu$.

Since $\kappa \geq 8$, we have that $\kappa' = [\kappa/2]+2 \leq \kappa-2 = \mu$. So again by Lemma 4.8, we may write

$$\mathcal{X}_\kappa[U(t)] \leq C\big[E_\kappa^{1/2}[U(t)] + \mathcal{X}_\mu[U(t)]E_\kappa^{1/2}[U(t)] + \mathcal{X}_\kappa[U(t)]E_\mu^{1/2}[U(t)]\big].$$

If $E_\mu^{1/2}[U(t)] < \varepsilon'$ is small, then this implies that

$$\mathcal{X}_\kappa[U(t)] \leq C E_\kappa^{1/2}[U(t)][1 + \mathcal{X}_\mu[U(t)]].$$

Thus we obtain (4.21) for $\nu = \kappa$ from this, (4.21) for $\nu = \mu$, and the fact that $E_\mu[U(t)]$ is small for $t \in [0, T)$. $\square$

5 Energy Estimates

We now have the decay estimates needed to complete the argument for global existence.

PROOF OF THEOREM 2.2: Let $U \in H_\Gamma^\kappa(T)$ be a local solution of (2.7a)–(2.7b) and the constraints (2.7f)–(2.7h). Suppose that $E_\mu[U(t)] < \varepsilon'$ and $\mu = \kappa - 2$ for $0 \leq t < T$, where ε' is sufficiently small, determined by Proposition 4.9. In order to prove the global bounds in Theorem 2.2, it is sufficient to prove the following inequalities:

$$\frac{d}{dt}E_\kappa[U(t)] \leq C\langle t\rangle^{-1}E_\mu^{1/2}[U(t)]E_\kappa[U(t)], \tag{5.1a}$$

$$\frac{d}{dt}E_\mu[U(t)] \leq C\langle t\rangle^{-3/2}E_\kappa^{1/2}[U(t)]E_\mu[U(t)]. \tag{5.1b}$$

We will use the generalized energy method. Start by applying the derivative Γ^a, $|a| \le \kappa$, to the system (2.7a)–(2.7b), according to Proposition 3.1. We then symmetrize the system by multiplying by the tensor $\hat{A}$. This results in

$$\hat{A}^{\ell m}_{pj}(H)\big[\partial_t(\Gamma^a\dot{H})^p_\ell + v\cdot\nabla\Gamma^a\dot{H}^p_\ell + H^p_k\partial_\ell\Gamma^a\dot{v}^k\big] = \hat{N}^H_a, \tag{5.2a}$$

$$\partial_t(\Gamma^a\dot{v})_i + v\cdot\nabla(\Gamma^a\dot{v})_i + H^p_i\hat{A}^{\ell m}_{pj}(H)\partial_\ell(\Gamma^a\dot{H})^j_m + \partial_i\Gamma^a p = \hat{N}^v_a. \tag{5.2b}$$

From (3.15a)–(3.15b) we have

$$\hat{N}_a(\dot{U}) = (\hat{N}^H_a, \hat{N}^v_a)$$

defined as follows:

$$\hat{N}^H_a = -\hat{A}^{\ell m}_{pj}(H)\sum_{\substack{b+c=a\\ c\ne a}}\big[\Gamma^b\dot{v}\cdot\nabla(\Gamma^c\dot{H})^p_\ell + (\Gamma^b\dot{H})^p_k\partial_\ell(\Gamma^c\dot{v})^k\big], \tag{5.3a}$$

$$\hat{N}^v_a = -\sum_{\substack{b+c=a\\ c\ne a}}\big\{\Gamma^b\dot{v}\cdot\nabla(\Gamma^c\dot{v})_i + \Gamma^b[\hat{A}(H)H]^{\ell m}_{ij}\partial_\ell(\Gamma^c\dot{H})^j_m\big\}. \tag{5.3b}$$

It is important to notice that $\hat{N}_a(\dot{U})$ will never have more than κ derivatives falling on a single term.

Next we proceed with the energy method by taking the L^2 inner product of (5.2a)–(5.2b) with $\Gamma^a\dot{U}$ and sum over $|a| \le \nu$. Because the system has been symmetrized, after integrating by parts and using the constraint $\nabla\cdot v = 0$ and summing over $|a| \le \nu$, we obtain

$$\begin{aligned}\frac{d}{dt}E_\nu[U(t)] = \sum_{|a|\le\nu}\bigg[\frac{1}{2}\int &\partial_t\hat{A}^{\ell m}_{pj}(H)(\Gamma^a\dot{H})^p_\ell(\Gamma^a\dot{H})^j_m\,dx\\ &+\int\partial_k\hat{A}^{\ell m}_{pj}(H)v^k(\Gamma^a\dot{H})^p_\ell(\Gamma^a\dot{H})^j_m dx\\ &+\int\partial_\ell(H^p_i\hat{A}^{\ell m}_{pj}(H))(\Gamma^a\dot{H})^j_m(\Gamma^a\dot{v})^i\,dx\\ &+\int\langle\hat{N}_a(\dot{U}),\ \Gamma^a\dot{U}\rangle dx\bigg].\end{aligned} \tag{5.4}$$

In order to estimate the terms with the time derivatives, we use equation (2.5a) and (3.13b). This gives

$$\begin{aligned}\partial_t\hat{A}^{\ell m}_{pj}(H) &= \hat{B}^{\ell mn}_{pjk}(H)\partial_t\dot{H}^k_n\\ &= -\hat{B}^{\ell mn}_{pjk}(H)\big[v\cdot\nabla H^k_n + H^k_q\partial_n v^q\big].\end{aligned}$$

We substitute this into (5.4), resulting in the energy identity

$$
\begin{aligned}
&\frac{d}{dt}E_\nu[U(t)]\\
&\quad=\sum_{|a|\le\nu}\Big[-\frac12\int \hat B^{\ell mn}_{pjk}(H)[\dot v\cdot\nabla \dot H^k_n+H^k_q\partial_n\dot v^q](\Gamma^a\dot H)^p_\ell(\Gamma^a\dot H)^j_m\,dx\\
&\qquad\qquad+\int \partial_k\hat A^{\ell m}_{pj}(H)v^k(\Gamma^a\dot H)^p_\ell(\Gamma^a\dot H)^j_m\,dx\\
&\qquad\qquad+\int \partial_\ell(H^p_i\hat A^{\ell m}_{pj}(H))(\Gamma^a\dot H)^j_m(\Gamma^a\dot v)^i\,dx\\
&\qquad\qquad+\int\langle \hat N_a(\dot U),\ \Gamma^a\dot U\rangle dx\Big].
\end{aligned}
\tag{5.5}
$$

For the higher-energy estimate (5.1a), we will start with (5.5) with $\nu=\kappa$. Since $\kappa\ge 8$, notice that we have $[\kappa/2]+2\le\mu$. Using the smallness condition we may apply Lemma 4.7 to (5.5), which gives

$$
\frac{d}{dt}E_\kappa[U(t)]\le C\sum_{\substack{|b|+|c|\le|a|\\ c\ne a\\ |a|\le\kappa}}\big\|\,|\Gamma^b\dot U|\,|\nabla\Gamma^c\dot U|\,\big\|\,\|\Gamma^a\dot U\|.
\tag{5.6a}
$$

Set $m=[(\kappa+1)/2]$. Using the property (4.19) for the weights, the Sobolev inequalities (4.16a) and (4.16c), and the bootstrapping lemma, we have the following bound for the norms on the right:

$$
\begin{aligned}
&\|\,|\Gamma^b\dot U|\,|\nabla\Gamma^c\dot U|\,\|\\
&\quad\le C\langle t\rangle^{-1}\sum_{\alpha=1,2}\sum_{i=1}^{3}\|\langle r\rangle|\Gamma^b\dot U|\,\mathcal W_\alpha\,|P_\alpha\partial_i\Gamma^c\dot U|\,\|\\
&\quad\le C\langle t\rangle^{-1}\begin{cases}\sum_{\alpha=1,2}\sum_{i=1}^3\|\langle r\rangle\Gamma^b\dot U\|_\infty\,\|\mathcal W_\alpha P_\alpha\partial_i\Gamma^c\dot U\|, & |b|\le m,\\ \sum_{\alpha=1,2}\sum_{i=1}^3\|\Gamma^b\dot U\|\,\|\langle r\rangle\mathcal W_\alpha P_\alpha\partial_i\Gamma^c\dot U\|_\infty, & |c|\le m-1,\end{cases}\\
&\quad\le C\langle t\rangle^{-1}\begin{cases}E^{1/2}_{|b|+2}[U(t)]\mathcal X_{|c|+1}[U(t)], & |b|\le m,\\ E^{1/2}_{|b|}[U(t)]\mathcal X_{|c|+3}[U(t)], & |c|\le m-1,\end{cases}\\
&\quad\le C\langle t\rangle^{-1}\big(E^{1/2}_{m+2}[U(t)]E^{1/2}_\kappa[U(t)]+E^{1/2}_\kappa[U(t)]E^{1/2}_{m+2}[U(t)]\big).
\end{aligned}
\tag{5.6b}
$$

Now $\kappa\ge 8$, so $m+2\le\kappa-2=\mu$. Therefore, inequality (5.1a) follows from (5.6a) and (5.6b).

For the lower energy ($\nu=\mu=\kappa-2$ in (5.5)) we are looking for the sharp estimate (5.1b). It is necessary at this stage to separate the quadratic portion of the nonlinear terms in (5.5). Referring to (5.3a)–(5.3b), we use $\nu = \kappa - 2 = \mu$ in (5.5) to obtain

$$\frac{d}{dt} E_\mu[U(t)] = \sum_{|a|\le\mu}\left[\int \langle \bar{Q}_a(\dot U, \nabla \dot U), \Gamma^a \dot U\rangle dx + \int \langle C_a(\dot U),\ \Gamma^a \dot U\rangle dx\right], \tag{5.7a}$$

in which $\bar{Q}_a(\dot U, \nabla \dot U)$ and $C_a(\dot U)$ represent quadratic and higher-order terms, respectively. The precise form of the quadratic terms in (5.7a) is

$$\begin{aligned}
&\langle \bar{Q}_a(\dot U, \nabla \dot U), \Gamma^a \dot U\rangle \\
&\quad = -\frac{1}{2}\hat B^{\ell mn}_{pjk}(I)\partial_n \dot v^k (\Gamma^a \dot H)^p_\ell (\Gamma^a \dot H)^j_m + \hat B^{\ell mn}_{ijk}(I)\partial_\ell \dot H^k_n (\Gamma^a \dot H)^j_m (\Gamma^a \dot v)^i \\
&\qquad + \hat A^{\ell m}_{pj}(I)\partial_\ell \dot H^p_i (\Gamma^a \dot H)^j_m (\Gamma^a \dot v)^i \\
&\qquad - \sum_{\substack{b+c=a\\ c\neq a}} \big\{\hat A^{\ell m}_{pj}(I)[\Gamma^b \dot v\cdot\nabla(\Gamma^c \dot H)^p_\ell + (\Gamma^b \dot H)^p_i \partial_\ell (\Gamma^c \dot v)^i](\Gamma^a \dot H)^j_m \\
&\qquad\qquad + [\Gamma^b \dot v \cdot \nabla (\Gamma^c \dot v)^i + \hat A^{\ell m}_{pj}(I)(\Gamma^b \dot H)^p_i \partial_\ell (\Gamma^c \dot H)^j_m \\
&\qquad\qquad + \hat B^{\ell mn}_{ijk}(I)(\Gamma^b \dot H)^k_n \partial_\ell (\Gamma^c \dot H)^j_m](\Gamma^a \dot v)^i\big\}.
\end{aligned} \tag{5.7b}$$

Before confronting these crucial terms, let us first examine the highest-order terms in (5.7a). Using Lemma 4.7, we have

$$\begin{aligned}
\int \langle C_a(\dot U), \Gamma^a \dot U\rangle dx &\le \|C_a(\dot U)\|\,\|\Gamma^a \dot U\| \\
&\le C \sum_{\substack{|b_1|+|b_2|+|b_3|\le|a|\\ |b_3|\neq|a|}} \|\,|\Gamma^{b_1}\dot U|\,|\Gamma^{b_2}\dot U|\,|\nabla\Gamma^{b_3}\dot U|\,\|\,\|\Gamma^a \dot U\|.
\end{aligned}$$

Without loss of generality assume that $|b_1| \ge |b_2|$. Since we are considering higher-order terms, we have more flexibility with our weights, and using (4.18) we see that one can bound $\langle t\rangle^{3/2} \le C\langle r\rangle^2 \mathcal{W}_\alpha^{1/2}\mathcal{W}_\beta$ for any $\alpha, \beta = 1, 2$; thus we have

$$\begin{aligned}
&\|\,|\Gamma^{b_1}\dot U|\,|\Gamma^{b_2}\dot U|\,|\nabla\Gamma^{b_3}\dot U|\,\| \\
&\quad \le C\langle t\rangle^{-3/2}\sum_{\alpha,\beta=1,2} \|\langle r\rangle^2 |\Gamma^{b_1}\dot U|\,|\mathcal{W}_\alpha^{1/2} P_\alpha \Gamma^{b_2}\dot U|\,|W_\beta P_\beta \nabla \Gamma^{b_3}\dot U|\,\| \\
&\quad \le C\langle t\rangle^{-3/2}\sum_{\alpha,\beta=1,2} \|\langle r\rangle \Gamma^{b_1}\dot U\|_\infty\, \|\langle r\rangle \mathcal{W}_\alpha^{1/2} P_\alpha \Gamma^{b_2}\dot U\|_\infty\, \|W_\beta P_\beta \nabla\Gamma^{b_3}\dot U\|.
\end{aligned}$$

With the aid of (4.16a) and (4.16b), this in turn is bounded by

$$C\langle t\rangle^{-3/2} E^{1/2}_{|b_1|+2}[U(t)]\big(E^{1/2}_{|b_2|+2}[U(t)] + \mathcal{X}_{|b_2|+2}[U(t)]\big)\mathcal{X}_{|b_3|}[U(t)].$$

Now $2|b_2| \leq |b_1| + |b_2| \leq |a| \leq \mu$. Thus, $|b_2| + 2 \leq [\mu/2] + 2 \leq \mu$, since $\mu \geq 6$. We also have $|b_1| + 2 \leq \kappa$. Therefore, by the smallness assumption and Proposition 4.9, all of the cubic and higher-order terms on the right-hand side of (5.7a) are bounded by

$$C\langle t\rangle^{-3/2}\, E_\kappa^{1/2}[U(t)]\, E_\mu[U(t)],$$

as required for (5.1b).

It remains to bound the terms in (5.7a) arising from the quadratic part of the nonlinearity. It is necessary to partition the domain of integration into two components: $\mathcal{R} \equiv \{r \leq \langle c_2 t/2\rangle\}$ and its complement. On $\mathcal{R}$ we can still work generally, and so using the fact that $\langle c_2 t - r\rangle \sim \langle t\rangle$, along with (4.16b) we can bound

$$\begin{aligned}
&\int_{\mathcal{R}} \langle \bar{Q}_a(\dot{U}, \nabla\dot{U}), \Gamma^a\dot{U}\rangle dx \\
&\leq \|\, |\Gamma^b\dot{U}|\, |\nabla\Gamma^c\dot{U}|\, \|_{L^2(\mathcal{R})}\, \|\Gamma^a\dot{U}\| \\
&\leq C\langle t\rangle^{-3/2} \sum_{\alpha,\beta=1,2} \|\mathcal{W}_\alpha^{1/2}|P_\alpha\Gamma^b U|\, \mathcal{W}_\beta\, |P_\beta\nabla\Gamma^c U|\, \|_{L^2(\mathcal{R})}\, \|\Gamma^a\dot{U}\| \qquad (5.8)\\
&\leq C\langle t\rangle^{-3/2} \sum_{\alpha,\beta=1,2} \big|\mathcal{W}_\alpha^{1/2}|P_\alpha\Gamma^b U|\,\big|_{L^\infty(\mathcal{R})}\, \|\mathcal{W}_\beta|P_\beta\nabla\Gamma^c U|\, \|_{L^2(\mathcal{R})}\, \|\Gamma^a\dot{U}\| \\
&\leq C\langle t\rangle^{-3/2}\big[E_\kappa^{1/2}[U(t)] + \mathcal{X}_\kappa[U(t)]\big]\mathcal{X}_\mu[U(t)]E_\mu^{1/2}[U(t)].
\end{aligned}$$

Recall that we are summing over $|a| \leq \mu$ and $|c| < \mu$; also $|b| + 2 \leq \mu + 2 = \kappa$. Plugging this into (5.7a) will lead to the bound given in (5.1b) upon application of Proposition 4.9.

Next we consider the region $\mathcal{R}^c$. Using the fact that $U = P_1U + P_2U$, as defined in (4.13), we can introduce the projection matrices into our quadratic terms from (5.7a). All of the terms will be of the form

$$\int_{\mathcal{R}^c} \langle \bar{Q}_a(P_\alpha\dot{U}, P_\beta\nabla\dot{U}), P_\gamma\Gamma^a\dot{U}\rangle dx, \qquad (5.9)$$

for $|a| \leq \mu$ and $\alpha, \beta, \gamma = 1, 2$. As long as one of α, β, or γ is equal to 1, the decay will come from at least one factor of $\langle t + r\rangle$ along with $\langle r\rangle$ decay since on $\mathcal{R}^c$ we are away from the origin.

For example, suppose $\alpha = 1$; then we have $\mathcal{W}_\alpha = \langle t+r\rangle$, and since $r \geq \langle c_2 t\rangle/2$ we can use (4.16b) to obtain

$$\begin{aligned}
&\|\, |P_1\Gamma^b\dot{U}|\, |P_\beta\nabla\Gamma^c\dot{U}|\, \|_{L^2(\mathcal{R}^c)} \\
&\quad\leq C\langle t\rangle^{-3/2}|\langle r\rangle\langle t+r\rangle^{1/2}P_1\Gamma^b\dot{U}|_{L^\infty(\mathcal{R}^c)}\, \|\nabla\Gamma^c\dot{U}\,\|_{L^2(\mathcal{R}^c)} \\
&\quad\leq C\langle t\rangle^{-3/2}\big[E_\kappa^{1/2}[U(t)] + \mathcal{X}_\kappa[U(t)]\big]\mathcal{X}_\mu[U(t)]. \qquad (5.10)
\end{aligned}$$

The other cases with at least one of α, β, or γ equal to 1 are similar. Note that unlike the case of compressible elasticity [13], there is no need for a null condition for the case $(\alpha, \beta, \gamma) = (1, 1, 1)$. This is the result of the stronger estimates that are available due to the constraint equations. Recall that we did not use the estimates obtained for $\mathcal{P}_f$, but rather the constraint equations were sufficient for proving estimates for $\|\langle t+r\rangle P_1 \nabla U\|$.

We are now left with verifying the case where $(\alpha, \beta, \gamma) = (2, 2, 2)$ on $\mathcal{R}^{\mathsf{c}}$. We must look carefully at the form of the nonlinear terms that appear explicitly in (5.7b). In particular, we have two types of terms: one type involving convective derivatives and the other involving terms satisfying the null condition given in (3.12). First, we consider the terms in (5.7b) containing a convective derivative, $P_2\Gamma^b v \cdot \nabla$. Since $\langle P_2\Gamma^b v, \omega\rangle = 0$, we split the gradient into radial and angular components,

$$\nabla = \omega\partial_r - \frac{1}{r}(\omega \wedge \Omega) \quad \text{where } \omega = \frac{x}{r} \text{ and } r = |x|, \tag{5.11}$$

to obtain

$$P_2\Gamma^b v \cdot \nabla = -r^{-1} P_2\Gamma^b v \cdot (w \wedge \Omega),$$

and so by (4.16b) we have

$$\begin{aligned}\|P_2\Gamma^b v \cdot \nabla\Gamma^c \dot{U}\|_{L^2(\mathcal{R}^{\mathsf{c}})} &\le \|r^{-1}|P_2\Gamma^b v|\,|\Gamma^{c+1}\dot{U}|\,\|_{L^2(\mathcal{R}^{\mathsf{c}})}\\ &\le C\langle t\rangle^{-2}\|\langle r\rangle|P_2\Gamma^b v|\,|\Gamma^{c+1}\dot{U}|\,\|_{L^2(\mathcal{R}^{\mathsf{c}})}\\ &\le C\langle t\rangle^{-2}\|\langle r\rangle P_2\Gamma^b v\|_\infty\,\|\Gamma^{c+1}\dot{U}\|\\ &\le C\langle t\rangle^{-2}\big[E_\kappa^{1/2}[U(t)] + \mathcal{X}_\kappa[U(t)]\big]E_\mu^{1/2}[U(t)].\end{aligned}$$

After applying Proposition 4.9 we have the estimate for the terms with convective derivatives.

The remaining terms in (5.7a) have one of the following forms:

$$\int \mathcal{B}_{ijk}^{\ell mn}(P_2)_I^i(P_2)_J^j(P_2)_K^k(\Gamma^b\dot{H})_m^J\partial_\ell(\Gamma^c\dot{H})_n^K(\Gamma^a\dot{v})^I\,dx, \tag{5.12a}$$

$$\int \mathcal{B}_{ijk}^{\ell mn}(P_2)_I^i(P_2)_J^j(P_2)_K^k(\Gamma^a\dot{H})_m^J(\Gamma^b\dot{H})_n^K\partial_\ell(\Gamma^c\dot{v})^I\,dx, \tag{5.12b}$$

$$\int \mathcal{B}_{ijk}^{\ell mn}(P_2)_I^i(P_2)_J^j(P_2)_K^k(\partial_t\dot{H})_n^K(\Gamma^a\dot{H})_\ell^I(\Gamma^a\dot{H})_m^J\,dx, \tag{5.12c}$$

in which $\mathcal{B}_{ijk}^{\ell mn}$ is either $\hat{B}_{ijk}^{\ell mn}(I)$, $\hat{B}_{ikj}^{\ell nm}(I)$, $\hat{B}_{kji}^{nm\ell}(I)$, $\tilde{B}_{ijk}^{\ell mn}$, or $\tilde{B}_{ikj}^{\ell nm}$, as defined in (3.13b), Thus, the coefficients $\mathcal{B}$ satisfy the null condition for shear waves (3.12). As usual, the derivatives are constrained by the relations $b + c = a$, $c \ne a$, and $|a| \le \mu$. All three terms can be handled in the same manner, and so we will outline the procedure only for the first group (5.12a). Thus, we are faced with estimating

$$\int_{\mathcal{R}^{\mathsf{c}}} \mathcal{B}_{ijk}^{\ell mn}(P_2)_I^i(P_2)_J^j(P_2)_K^k(\Gamma^b\dot{H})_m^J\partial_\ell(\Gamma^c\dot{H})_n^K(\Gamma^a\dot{v})^I\,dx. \tag{5.13}$$

We can further introduce projections in the remaining indices,

$$\mathcal{B}^{\ell mn}_{ijk}(P_2)^i_I(P_2)^j_J(P_2)^k_K = \sum_{\alpha,\beta,\gamma} \mathcal{B}^{\ell mn}_{ijk}(P_\alpha)^L_\ell(P_\beta)^M_m(P_\gamma)^N_n(P_2)^i_I(P_2)^j_J(P_2)^k_K.$$

Thanks to the null condition (3.12), we can rule out $(\alpha,\beta,\gamma) = (1,1,1)$ in the sum, and so we need only consider the three possibilities that either α, β, or γ is equal to 2.

Now if $\alpha = 2$, then we use (5.11) to write $(P_2)^L_\ell \partial_L = -r^{-1}(\omega\wedge\Omega)_\ell$. Thus, this piece of our integral (5.13) is controlled by

$$\int_{\mathcal{R}^c} r^{-1}|\Gamma^b \dot{H}|\,|\Gamma^{c+1}\dot{H}|\,|\Gamma^a \dot{v}|dx.$$

Recall that on $\mathcal{R}^c$ we have have $r \geq C\langle t\rangle$. Hence, using (4.16a), we find the upper bound

$$\langle t\rangle^{-2}E^{1/2}_\kappa[U(t)]E_\mu[U(t)].$$

If $\beta = 2$, then thanks to the constraint (2.5c), we can use Lemma 4.6 to see that

$$\|r^{3/2}(P_2)^M_m(\Gamma^b\dot{H})^j_M\|_\infty \leq CE^{1/2}_\kappa[U(t)].$$

Also, when $\gamma = 2$ we get

$$\|r^{3/2}(P_2)^N_n\partial_\ell(\Gamma^c\dot{H})^k_N\|_\infty \leq CE^{1/2}_\kappa[U(t)].$$

In either case, this again leads to the bound

$$\langle t\rangle^{-3/2}E^{1/2}_\kappa[U(t)]E_\mu[U(t)]$$

for the remainder of (5.13). □

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