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GLOBAL EXISTENCE, LARGE TIME BEHAVIOR AND LIFE SPAN OF SOLUTIONS OF A SEMILINEAR PARABOLIC CAUCHY PROBLEM

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ABSTRACT. We investigate the behavior of the solution $u(x, t)$ of

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u^p & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, 0) = \varphi(x) & \text{in } \mathbb{R}^n, \end{cases}$$

where $\Delta = \sum_{i=1}^n \partial^2/\partial_{x_i}^2$ is the Laplace operator, $p > 1$ is a constant, $T > 0$, and φ is a nonnegative bounded continuous function in $\mathbb{R}^n$. The main results are for the case when the initial value φ has polynomial decay near $x = \infty$. Assuming $\varphi \sim \lambda(1 + |x|)^{-a}$ with $\lambda, a > 0$, various questions of global (in time) existence and nonexistence, large time behavior or life span of the solution $u(x, t)$ are answered in terms of simple conditions on λ, a, p and the space dimension n.

1. INTRODUCTION

In this paper we shall consider the following Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u^p & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, 0) = \varphi(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^n \partial^2/\partial_{x_i}^2$ is the Laplace operator, $p > 1$ is a constant, $T > 0$, and φ is a nonnegative bounded continuous function in $\mathbb{R}^n$. Due to the possible nonuniqueness of solutions of (1.1), we shall restrict our attention to a certain class of solutions $u(x, t; \varphi)$ of (1.1); namely, those with the following properties:

(i) $u(x, t; \varphi) \geq 0$ in $\mathbb{R}^n \times (0, T)$,
(ii) u satisfies the integral equation in $\mathbb{R}^n \times [0, T)$,

$$\begin{aligned} u(x, t; \varphi) = {} & \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} \varphi(y)\, dy \\ & + \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{-|x-y|^2/4(t-s)} u^p(y, s; \varphi)\, dy\, ds. \end{aligned} \tag{1.2}$$

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On the other hand, it is proved in [F, Proposition A.4] that if u satisfies (1.2) and is bounded in $\mathbb{R}^n \times [0, T)$ then u is unique and is a classical solution of (1.1); i.e. $u \in C^{2,1}(\mathbb{R}^n \times (0, T)) \cap C(\mathbb{R}^n \times [0, T))$ and u satisfies (1.1). Since all the solutions we are concerned with in this paper will be bounded in $[0, T']$ for all $T' < T[\varphi]$ where

(1.3) $T[\varphi] = \sup\{T > 0 | (1.2) \text{ possesses a nonnegative solution in } \mathbb{R}^n \times [0, T)\}$

is the *life span* of the solution $u(x, t; \varphi)$, *we shall not distinguish the solution of* (1.1) *and* (1.2).

In 1966, Fujita [F] proved that if $p < (n+2)/n$, then $T[\varphi] < \infty$ for all $\varphi \geq 0$ and $\not\equiv 0$ in $\mathbb{R}^n$, and in case $p > (n+2)/n$ then $T[\varphi] = \infty$ (i.e. global existence in time) if φ is bounded by $\varepsilon \exp(-|x|^2)$ where ε is a small positive number. The case $p = (n+2)/n$ belongs to global nonexistence and was settled later by Hayakawa [H], and Kobayashi, Sirao and Tanaka [KST]. Different proofs have been given by various authors including, for instance, Aronson and Weinberger [AW] and Weissler [We]. Weissler also treated (1.1) in L^p-spaces. We refer the interested readers to a recent survey by Levine [L] for other related results.

In this paper an attempt to understand the behavior of the solution $u(x, t; \varphi)$ while the initial value φ is not so small near $x = \infty$ is made. For instance in case φ has polynomial decay near $x = \infty$, say, $\varphi \sim \lambda(1 + |x|)^{-a}$ where both λ and a are positive, we are interested in the question of global existence and nonexistence, large time behavior or life span of the solution $u(x, t; \varphi)$ in terms of λ and a. Theorem 3.2 below gives a necessary condition for global existence in terms of a and p and a sharp estimate of $T[\varphi]$ in terms of λ and p as $\lambda \to \infty$. In Theorem 3.8 a precise large time behavior, in terms of a and n, of the solution $u(x, t; \varphi)$ is obtained for λ sufficiently small when global existence prevails. Finally, the behavior of $T[\varphi]$ as $\lambda \to 0$ is obtained in terms of p, n, a and λ in case of finite time blow-up. We hope that with the aid of those results the "transition" from fast decay (in time) of $u(x, t; \varphi)$ to slow decay as $t \to \infty$, from global existence to finite time blow-up, and from long life span to short lift span, is better understood.

Our main results are stated and proved in §3. Notations and technical lemmas are included in §2, and §4 contains some concluding remarks.

2. Preliminaries

The following notations will be used throughout the rest of this paper. First, we denote by μ_R the first eigenvalue of $-\Delta$ in B_R, the ball of radius R in $\mathbb{R}^n$, with zero Dirichlet boundary value, and ρ_R the corresponding positive eigenfunction with $\int_{B_R} \rho_R = 1$. Then we set $C_b(\mathbb{R}^n)$ to be the space of all bounded continuous functions in $\mathbb{R}^n$ and, for $a \geq 0$,

$$I^a = \left\{ \psi \in C_b(\mathbb{R}^n) | \psi \geq 0 \text{ and } \limsup_{|x| \to \infty} |x|^a \psi(x) < \infty \right\},$$

$$I_a = \left\{ \psi \in C_b(\mathbb{R}^n) | \psi \geq 0 \text{ and } \liminf_{|x| \to \infty} |x|^a \psi(x) > 0 \right\}.$$

For two functions $f(r)$ and $g(r)$, we say that $f \sim g$ near $r = 0$ (∞ respectively) if there exists two positive constants C_1, C_2 such that $C_1 f(r) \geq g(r) \geq C_2 f(r)$ near $r = 0$ (∞ respectively). (Note that the variable r could also represent either $x \in \mathbb{R}^n$ or $t > 0$.) The letter C denotes a positive generic constant which may vary from line to line. We shall also use the usual notation $e^{t\Delta}\varphi$ to represent the solution of the heat equation with initial value φ; i.e. if $K(x, t) = (4\pi t)^{-n/2}e^{-|x-y|^2/4t}$ denotes the heat kernel, then

$$(2.1)\qquad \begin{aligned}(e^{t\Delta}\varphi)(x) &= \int_{\mathbb{R}^n} K(x-y, t)\varphi(y)\,dy \\ &= \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-|x-y|^2/4t}\varphi(y)\,dy.\end{aligned}$$

Then (1.2) takes the following form

$$(2.2)\qquad u(x, t) = e^{t\Delta}\varphi + \int_0^t e^{(t-s)\Delta}u^p(y, s)\,ds.$$

The first preliminary result we need is a standard comparison principle which will be used frequently in this paper.

Lemma 2.3. *Suppose that $f \in C^1(\mathbb{R})$ and $\overline{u}(x, t)$, $\underline{u}(x, t) \in C^{2,1}(\mathbb{R}^n \times (0, T)) \cap C(\mathbb{R}^n \times [0, T))$ are bounded in $\mathbb{R}^n \times [0, T']$ for all $T' < T$. If $\overline{u}(x, 0) \geq \underline{u}(x, 0)$ for all $x \in \mathbb{R}^n$ and*

$$(2.4)\qquad \begin{cases} \overline{u}_t - \Delta\overline{u} \geq f(\overline{u}) & \text{in } \mathbb{R}^n \times (0, T), \\ \underline{u}_t - \Delta\underline{u} \leq f(\underline{u}) & \text{in } \mathbb{R}^n \times (0, T), \end{cases}$$

then $\overline{u}(x, t) \geq \underline{u}(x, t)$ for all $(x, t) \in \mathbb{R}^n \times [0, T)$. Furthermore, for any $\varphi \in C_b(\mathbb{R}^n)$ with $\overline{u}(x, 0) \geq \varphi(x) \geq \underline{u}(x, 0)$ in $\mathbb{R}^n$, there exists a unique solution $u(x, t) \in C^{2,1}(\mathbb{R}^n \times (0, T)) \cap C(\mathbb{R}^n \times [0, T))$ of the problem

$$(2.5)\qquad \begin{cases} u_t = \Delta u + f(u) & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, 0) = \varphi(x) & \text{in } \mathbb{R}^n, \end{cases}$$

with the property that $\overline{u} \geq u \geq \underline{u}$ in $\mathbb{R}^n \times [0, T)$.

Remark 2.6. Lemma 2.3 is well known (see e.g. [AW] for the first half of this lemma). It also holds in a more general setting allowing f to depend on x, t and u, and for weak super- and sub-solutions of (2.4) which are unbounded but satisfy certain growth conditions near $x = \infty$. We refer the interested readers to [W] in which Lemma 2.3 is proved as a special case by using a maximum principle in [Fr, Chapter 2, Theorem 9] and the monotone iteration method in [S, Theorem 3.1].

Our next lemma is a variant of a well-known result of Kaplan [K]. The difference here is that we impose no boundary condition on u.

Lemma 2.7. *Let $u(x, t)$ be a nonnegative global solution of the equation $u_t = \Delta u + u^p$ in $\Omega \times [0, \infty)$ where Ω is a bounded smooth domain in $\mathbb{R}^n$. Suppose that $\mu > 0$ and $\rho(x) > 0$ in Ω are respectively the first eigenvalue and the first normalized (i.e. $\int_\Omega \rho = 1$) eigenfunction of $-\Delta$ on Ω with zero Dirichlet boundary condition, then*

$$(2.8)\qquad \int_\Omega u(x, t)\rho(x)\,dx \leq \mu^{1/(p-1)} \quad \textit{for all } t \geq 0.$$

Proof. The proof is almost identical to the original one. We include a sketch with the necessary modifications here. As in [K], we set

$$w(t) = \int_\Omega u(x, t)\rho(x)\,dx.$$

Then by Green's identity and Jensen's inequality we derive

$$w_t \geq w^p - \mu w \quad \text{in } t > 0. \tag{2.9}$$

Note that we have used the normalization of ρ, the nonnegativity of u and the fact that $\partial\rho/\partial\nu < 0$ on the boundary $\partial\Omega$ (where ν is the unit outer normal of $\partial\Omega$). Then simple arguments show that global existence of $w(t)$, hence that of $u(x, t)$, requires that (2.8) holds for every $t \geq 0$. Q.E.D.

We shall also need the following estimates for the solution of the linear heat equation. To simplify the notation we set

$$l(t; \psi) = \|e^{t\Delta}\psi\|_{L^\infty(\mathbb{R}^n)} \tag{2.10}$$

and

$$q(t; a, n) = \begin{cases} t^{-\frac{1}{2}\min\{a, n\}} & \text{if } a \neq n, \\ t^{-\frac{n}{2}} \log t & \text{if } a = n. \end{cases} \tag{2.11}$$

Lemma 2.12. (i) *$l(t; \psi) = O(q(t; a, n))$ near $t = \infty$ for every $\psi \in I^a$.*

(ii) *$q(t; a, n) = O(l(t; \psi))$ near $t = \infty$ for every $\psi \in I_a$.*

(iii) *$t^{-n/2} = O(l(t; \psi))$ near $t = \infty$ for every nonnegative $\psi \not\equiv 0$ in $C_b(\mathbb{R}^n)$.*

Proof. Since $l(t; \psi)$ is bounded in $t > 0$ if $\psi \in C_b(\mathbb{R}^n)$, the comparison principle Lemma 2.3 applies. To prove (i), we may assume, without loss of generality, that $\psi(x) = (1 + |x|^2)^{-a/2}$. It is then not hard to see that $e^{t\Delta}\psi$ is radially symmetric in x and $(e^{t\Delta}\psi)(0) = l(t; \psi)$ for $t \geq 0$. (The last assertion follows from a symmetric rearrangement argument.) Thus straightforward computation shows that

$$\begin{aligned} l(t; \psi) = (e^{t\Delta}\psi)(0) &= (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|y|^2/4t}(1 + |y|^2)^{-a/2}\, dy \\ &= Ct^{-n/2} \int_0^\infty e^{-r/4t}(1 + r)^{-a/2} r^{n/2-1}\, dr \sim q(t; a, n) \end{aligned}$$

as $t \to \infty$, where the last estimate is obtained by decomposing the integral from 0 to ∞ into two integrals—one from 0 to 1 and the other from 1 to ∞—and estimating them separately.

To establish (ii), again by a simple comparison argument, it suffices to consider, for some R large,

$$\psi(x) = \begin{cases} (1 + |x|^2)^{-a/2}, & |x| \geq R, \\ 0, & |x| \leq R - 1, \end{cases} \tag{2.13}$$

and $\psi(x) = \psi(|x|)$ is linear in $(R - 1, R)$ so that ψ is continuous in $\mathbb{R}^n$. Observe that by a similar computation as above, it holds that

$$\begin{aligned} l(t; \psi) \geq (e^{t\Delta}\psi)(0) &\geq Ct^{-n/2} \int_{R^2}^\infty e^{-r/4t}(1 + r)^{-a/2} r^{n/2-1}\, dr \\ &\sim t^{-n/2} \int_{R^2}^\infty e^{-r/4t} r^{n/2-a/2-1}\, dr \sim q(t; a, n) \end{aligned} \tag{2.14}$$

as $t \to \infty$.

To prove (iii), observe that after a translation we may assume without loss of generality that $\psi > 0$ in a neighborhood of the origin, say, in $B_{2\delta}(0)$. Then

$$l(t; \psi) \geq (e^{t\Delta}\psi)(0) \geq Ct^{-n/2} \int_{B_\delta(0)} e^{-|y|^2/4t}\, dy \geq Ct^{-n/2}$$

for t large. Q.E.D.

Lemma 2.15. *Let $\psi \in C_b(\mathbb{R}^n)$ be nonnegative and not identically zero. Then there exists a positive constant δ such that*

$$(e^{2\Delta}\psi)(x) \geq \delta K(x, 1) \tag{2.16}$$

for all $x \in \mathbb{R}^n$. Moreover, for $\psi \in I_n$, there exists a positive constant δ such that

$$(e^{(t+2)\Delta}\psi)(x) \geq \delta K\left(x, \frac{t+1}{4}\right)\log(1+t) \tag{2.17}$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$.

Proof. First we prove (2.16). Suppose that $\psi \geq \delta_1 > 0$ in $B_\varepsilon(x_0)$, a ball of radius $\varepsilon > 0$ centered at x_0, for some $x_0 \in \mathbb{R}^n$. Let

$$\delta_2 = \inf\left\{ \frac{K(x-y, 2)}{K(x, 1)} \middle| x \in \mathbb{R}^n, \ y \in B_\varepsilon(x_0) \right\}.$$

It is easily seen that $\delta_2 > 0$, and that for every $x \in \mathbb{R}^n$, we have

$$(e^{2\Delta}\psi)(x) \geq \delta_1 \int_{B_\varepsilon(x_0)} K(x-y, 2)\, dy \geq \delta_1\delta_2 K(x, 1)|B_\varepsilon(x_0)|,$$

where $|B_\varepsilon(x_0)|$ is the measure of $B_\varepsilon(x_0)$. Thus (2.16) holds.

We now turn to (2.17). For $\psi \in I_n$, by a comparison argument we may assume without loss of generality that for some $R > 0$ large

$$\psi(x) = \begin{cases} |x|^{-n} & \text{for } |x| \geq R, \\ 0 & \text{for } |x| \leq R-1, \end{cases}$$

and, ψ is radially symmetric and is linear in $(R-1, R)$ so that ψ is continuous in $\mathbb{R}^n$. Using a similar computation as in (2.14) in the proof of Lemma 2.12(ii) we conclude that there is a positive constant δ such that (2.17) holds for all $t \geq 0$ *and for all* $|x| \leq 2R$. It remains to consider (2.17) for $|x| > 2R$. For $|x| > 2R$ we have

$$\begin{aligned}
(e^{(t+2)\Delta}\psi)(x) &= [4\pi(t+2)]^{-n/2}\left(\int_{R\leq|y|\leq|x|} + \int_{|x|<|y|}\right)(e^{-|x-y|^2/4(t+2)}|y|^{-n}\, dy) \\
&\geq C(t+2)^{-n/2}\left(e^{-|x|^2/(t+2)}\int_{R\leq|y|\leq|x|}|y|^{-n}\, dy\right. \\
&\qquad\qquad \left. + \int_{|x|<|y|} e^{-|y|^2/(t+2)}|y|^{-n}\, dy\right) \\
&\geq C(t+2)^{-n/2}e^{-|x|^2/(t+2)}\left(\int_{R\leq|y|\leq|x|}|y|^{-n}\, dy\right. \\
&\qquad\qquad \left. + \int_0^{t+2} e^{-r/(t+2)}(r+|x|^2)^{-1}\, dr\right)
\end{aligned}$$

where $r = |y|^2 - |x|^2$. Since $e^{-r/(t+2)} \geq e^{-1}$ for $r \in (0, t+2)$, we obtain

$$\begin{aligned}
(e^{(t+2)\Delta}\psi)(x) &\geq C(t+2)^{-n/2}e^{-|x|^2/(t+2)}\left(\log\frac{|x|^2}{R^2} + \log\frac{t+2+|x|^2}{|x|^2}\right) \\
&\geq C(t+2)^{-n/2}e^{-|x|^2/(t+2)}\log\left(4 + \frac{t+2}{R^2}\right)
\end{aligned}$$

since $|x| > 2R$. Our assertion then follows from the observation that there exists a positive constant γ such that $4 + (t+2)/R^2 \geq (t+1)^\gamma$ for all $t \geq 0$. Q.E.D.

We shall conclude this section by two technical lemmas. Let $p > 1$, $\alpha > 0$ be two constants. We define a_k and $A_k(x, t)$, $k = 0, 1, \dots$, recursively as follows: $a_0 = 1$.

$$a_{k+1} = a_k^p (4\pi)^{-1} p^{k(1-n/2)-n}(p-1)(p^{k+1}-1)^{-1}, \tag{2.18}$$

and

$$A_k(x, t) = a_k \alpha^{p^k} K(x, (t+1)p^{-k})[\log(t+1)]^{(p^k-1)(p-1)^{-1}}. \tag{2.19}$$

Lemma 2.20. (i) *There exists $\beta > 0$ such that $a_k \geq \beta^{p^k}$ for all $k \geq 0$.*

(ii) *If $p = (n+2)/n$ then we have*

$$\int_0^t \int_{\mathbb{R}^n} K(x-y, t-s) A_k^p(y, s)\, dy\, ds \geq A_{k+1}(x, t). \tag{2.21}$$

Proof. (i) First observe that $a_k < 1$ for all $k \geq 1$. Setting $\xi_k = p^{-k}(-\log a_k)$, we assert that $\sup_{k \geq 0} \xi_k < \infty$. Since

$$-\log a_{k+1} = -p \log a_k + \log[4\pi p^{n+(n/2-1)k}(p^{k+1}-1)(p-1)^{-1}],$$

it follows that

$$\xi_{k+1} - \xi_k = p^{-(k+1)} \log[4\pi p^{n+(n/2-1)k}(p^{k+1}-1)(p-1)^{-1}].$$

Thus there exist two positive constants C_1, C_2 such that $0 > \xi_{k+1} - \xi_k < p^{-(k+1)}(C_1 k + C_2)$ for all $k \geq 0$. This implies that

$$\xi_{k+1} - \xi_0 = \sum_{m=0}^{k} (\xi_{m+1} - \xi_m) < \sum_{m=0}^{\infty} p^{-(m+1)}(C_1 m + C_2) < \infty,$$

and our assertion is established.

(ii) Straightforward computation shows that for $k \geq 0$ we have

$$\begin{aligned}
&\int_0^t K(\cdot, t-s) * A_k^p(\cdot, s)\, ds \\
&\quad = a_k^p \alpha^{p^{k+1}} \int_0^t K(\cdot, t-s) * K^p(\cdot, (s+1)p^{-k})[\log(s+1)]^{p(p^k-1)(p-1)^{-1}}\, ds \\
&\quad = (4\pi)^{-1} p^{k-n/2} a_k^p \alpha^{p^{k+1}} \int_0^t K(\cdot, t-s) \\
&\qquad\qquad * K\left(\cdot, \frac{s+1}{p^{k+1}}\right) \frac{[\log(s+1)]^{p(p^k-1)(p-1)^{-1}}}{s+1}\, ds.
\end{aligned}$$

Next, note that

$$t + p^{-k-1} \geq t - s + p^{-k-1}(s+1) \geq p^{-k-1}(t+1)$$

holds for $s \in [0, t]$, it follows that

$$\begin{aligned}
K(\cdot, t-s) * K(\cdot, (s+1)p^{-k-1}) &= K(\cdot, t-s+p^{-k-1}(s+1)) \\
&\geq \left[\frac{p^{-k-1}(t+1)}{t+p^{-k-1}}\right]^{n/2} K(\cdot, (t+1)p^{-k-1}) \\
&\geq p^{-n(k+1)/2} K(\cdot, (t+1)p^{-k-1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_0^t K(\cdot\,, t-s) * A_k^p(\cdot\,, s)\, ds \\
&\quad \geq (4\pi)^{-1} p^{k(1-n/2)-n} a_k^p \alpha^{p^{k+1}} K(\cdot\,, (t+1)p^{-k-1}) \\
&\quad\quad \cdot \int_0^t (s+1)^{-1} [\log(s+1)]^{p(p^k-1)(p-1)^{-1}}\, ds \\
&\quad = (4\pi)^{-1} p^{k(1-n/2)-n} a_k^p \alpha^{p^{k+1}} K\left(\cdot\,, \frac{t+1}{p^{k+1}}\right) (p-1)(p^{k+1}-1)^{-1} \\
&\quad\quad \cdot [\log(t+1)]^{(p^{k+1}-1)(p-1)^{-1}} \\
&\quad = A_{k+1}(\cdot\,, t). \quad \text{Q.E.D.}
\end{aligned}$$

Our second technical lemma is similar to Lemma 2.20. Again, let $p > 1$, $\alpha > 0$ be two constants. We define d_k and $D_k(x, t)$, $k = 0, 1, \ldots$, recursively as follows: $d_0 = 1$,

$$d_{k+1} = d_k^p 4^{-n/2} \pi^{-1} p^{k(1-n/2)-n} (p-1)(p^{k+2}-1)^{-1}, \tag{2.22}$$

and

$$D_k(x, t) = d_k \alpha^{p^k} K\left(x, \left(\frac{t+1}{4}\right) p^{-k}\right) [\log(t+1)]^{(p^{k+1}-1)(p-1)^{-1}}. \tag{2.23}$$

And, we have

Lemma 2.24. (i) *There exists* $\eta > 0$ *such that* $d_k \geq \eta^{p^k}$ *for all* $k \geq 0$.

(ii) *If* $p = (n+2)/n$, *then we have*

$$\int_0^t \int_{\mathbb{R}^n} K(x-y, t-s) D_k^p(y, s)\, dy\, ds \geq D_{k+1}(x, t). \tag{2.25}$$

Proof. The proof of (i) is almost identical to that of part (i) of Lemma 2.20, hence is omitted here. Part (ii) can also be proved in a similar fashion as we did in Lemma 2.20(ii). We shall include a sketch below. Since $p = (n+2)/n$, straightforward computation gives that

$$\begin{aligned}
&\int_0^t K(\cdot\,, t-s) * D_k^p(\cdot\,, s)\, ds \\
&\quad = \alpha^{p^{k+1}} d_k^p \int_0^t K(\cdot\,, t-s) * \left[\frac{p^{k-n/2}}{\pi} K\left(\cdot\,, \frac{s+1}{4p^{k+1}}\right)\right] \\
&\qquad\qquad \cdot \frac{[\log(s+1)]^{p(p^{k+1}-1)(p-1)^{-1}}}{s+1}\, ds \\
&\quad \geq \alpha^{p^{k+1}} d_k^p \frac{p^{k-n/2}}{\pi} \left[4^{-n/2} p^{-n(k+1)/2} K\left(\cdot\,, \frac{t+1}{4p^{k+1}}\right)\right] \\
&\qquad\qquad \cdot \int_0^t \frac{[\log(s+1)]^{p(p^{k+1}-1)(p-1)^{-1}}}{s+1}\, ds \\
&\quad = \alpha^{p^{k+1}} d_k^p \frac{p^{k(1-n/2)-n}(p-1)}{4^{n/2}\pi(p^{k+2}-1)} K\left(\cdot\,, \frac{t+1}{4p^{k+1}}\right) \\
&\quad\quad \cdot [\log(t+1)]^{(p^{k+2}-1)(p-1)^{-1}} \\
&\quad = D_{k+1}(\cdot\,, t).
\end{aligned}$$

Note that in the above derivation we have used the following estimates.

$$\begin{aligned}K(\cdot, t-s) * K(\cdot, 4^{-1}p^{-k-1}(s+1)) &= K(\cdot, t-s+4^{-1}p^{-k-1}(s+1)) \\ &\geq \left(\frac{4^{-1}p^{-k-1}(t+1)}{t+4^{-1}p^{-k-1}}\right)^{n/2} K(\cdot, 4^{-1}p^{-k-1}(t+1)) \\ &\geq 4^{-n/2}p^{-n(k+1)/2}K(\cdot, 4^{-1}p^{-k-1}(t+1)) \quad \text{for } s \in [0, t].\end{aligned}$$

This completes the proof of (2.25). Q.E.D.

3. Main results and their proofs

In this section, we consider the following Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, 0) = \lambda\psi(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $p > 1$, $\lambda > 0$ are two constants, $\psi \geq 0$ and $\psi \not\equiv 0$ is in $C_b(\mathbb{R}^n)$. As stated in the Introduction, we shall only consider the nonnegative solution satisfying (1.2) and denote it by $u(x, t; \lambda\psi)$. Our main concerns are the global existence or nonexistence, large time behavior, and life span $T[\lambda\psi]$ (defined by (1.3)) of the solution. The first result contains a sufficient condition for finite time blow-up in terms of the behavior of ψ at $x = \infty$ and a sharp estimate for the life span $T[\lambda\psi]$ as $\lambda \to \infty$.

Theorem 3.2. (i) $T[\psi] < \infty$ *(i.e. setting* $\lambda = 1$ *in* (3.1)) *if*

$$\liminf_{x\to\infty} |x|^{2/(p-1)}\psi(x) > \mu_1^{1/(p-1)}.$$

(ii) *There exists* $\Lambda \geq 0$, *depending on* p, n *and* ψ, *such that* $T[\lambda\psi] < \infty$ *for* $\lambda > \Lambda$ *and* $T[\lambda\psi] \sim \lambda^{-(p-1)}$ *as* $\lambda \to \infty$.

Proof. (i) We shall apply Lemma 2.7 with $\Omega = B_R$. To this end, we assume that $T[\psi] = \infty$. By the scaling property of eigenvalues and eigenfunctions we see that $\mu_R = \mu_1 R^{-2}$ and $\rho_R(x) = R^{-n}\rho_1(xR^{-1})$, $x \in B_R$. It then follows from (2.8) that for every $0 < \varepsilon < 1$,

$$\begin{aligned}\mu_1^{1/(p-1)}R^{-2/(p-1)} &\geq \int_{B_R\setminus B_{\varepsilon R}} \psi\rho_R \geq \left(\inf_{R\geq|x|\geq\varepsilon R}\psi(x)\right)\int_{B_R\setminus B_{\varepsilon R}}\rho_R \\ &= \left(\inf_{R\geq|x|\geq\varepsilon R}\psi(x)\right)\int_{B_1\setminus B_\varepsilon}\rho_1.\end{aligned}$$

Thus

$$\mu_1^{1/(p-1)} \left(\inf_{R\geq|x|\geq\varepsilon R}|x|^{2/(p-1)}\psi(x)\right)\int_{B_1\setminus B_\varepsilon}\rho_1 \tag{3.3}$$

for every $R > 0$ and $\varepsilon \in (0, 1)$. Now, for a fixed integer k, let $R > k$ and $\varepsilon = k/R$. Then letting $R \to \infty$ we obtain from (3.3) that

$$\mu_1^{1/(p-1)} \geq \left(\inf_{|x|\geq k}|x|^{2/(p-1)}\psi(x)\right)\int_{B_1}\rho_1 = \inf_{|x|\geq k}|x|^{2/(p-1)}\psi(x).$$

Since this holds for every k, (i) is established.

(ii) It is clear that the existence of Λ is an immediate consequence of the estimate $T[\lambda\psi] \sim \lambda^{-(p-1)}$ whose proof we now turn to. The lower estimate

$T[\lambda\psi] \geq C\lambda^{-(p-1)}$ is easily obtained by applying Lemma 2.3 to the pair of sub- and super-solutions $\underline{u}(x, t) \equiv 0$ and

$$\overline{u}(x, t) = [(\lambda\|\psi\|_{L^\infty(\mathbb{R}^n)})^{-(p-1)} - (p-1)t]^{-1/(p-1)}. \tag{3.4}$$

To derive the upper estimate, we first choose R so large that $\psi \not\equiv 0$ in B_R. It then follows from Lemma 2.7 (with $\Omega = B_R$) that $T[\lambda\psi] < \infty$ if $\lambda > \mu_R^{1/(p-1)}(\int_{B_R}\psi\rho_R)^{-1}$. (Thus we may choose $\Lambda = \mu_R^{1/(p-1)}(\int_{B_R}\psi\rho_R)^{-1}$ for instance.) Next, we set

$$w(t) = \int_{B_R} u(x, t; \lambda\psi)\rho_R(x)\,dx$$

for $0 \leq t < T[\lambda\psi]$. By Green's identity and Jensen's inequality (as in the proof of Lemma 2.7) we obtain

$$\begin{cases} w_t \geq w^p - \mu_R w & \text{for } t < T[\lambda\psi], \\ w(0) = \lambda\int_{B_R}\psi\rho_R > 0. \end{cases} \tag{3.5}$$

Since it is clear that $w_t \geq w^p - \mu_R w > 0$ for all $t \in [0, T[\lambda\psi])$ if λ is large, we deduce from (3.5) that

$$\frac{T[\lambda\psi]}{2} \leq \int_{w(0)}^{w(T[\lambda\psi]/2)} \frac{dw}{w^p - \mu_R w} \leq \frac{C}{w^{p-1}(0)} = \frac{C}{\lambda^{p-1}}.$$

This finishes the proof. Q.E.D.

Remark 3.6. (i) If $p \leq (n+2)/n$, then we may take $\Lambda = 0$ in part (ii) of Theorem 3.2 by previous works of Fujita, and others (see for instance [We]).

(ii) As a consequence of part (i) of Theorem 3.2 we see that all positive self-similar solutions $t^{-1/(p-1)}v(xt^{-1/2})$ of the equation in (3.1) (see [HW] for the existence) must satisfy

$$\liminf_{x\to\infty} |x|^{2/(p-1)}v(x) \leq \mu_1^{1/(p-1)}, \tag{3.7}$$

and so do all the positive steady states of (3.1).

Our next result concerns the large time behavior of solutions of (3.1).

Theorem 3.8. *Suppose that $p > (n+2)/n$ and $a \geq 2/(p-1)$. Then the following conclusions hold.*

(i) *For every $\psi \in I^a$, there exists $\Lambda_0 > 0$, depending on p, n and ψ, such that $T[\lambda\psi] = \infty$ for every $\lambda < \Lambda_0$. Moreover, for $\lambda < \Lambda_0$ we have*

$$\|u(\cdot, t; \lambda\psi)\|_{L^\infty(\mathbb{R}^n)} = O(q(t; a, n)) \quad \text{as } t \to \infty. \tag{3.9}$$

(ii) *For every $\psi \in I_a$ with $T[\psi] = \infty$, we have*

$$q(t; a, n) = O(\|u(\cdot, t; \psi)\|_{L^\infty(\mathbb{R}^n)}) \quad \text{as } t \to \infty. \tag{3.10}$$

Proof. (i) We shall construct $\underline{u} \equiv 0$ and $\overline{u}$ with the following property

$$\begin{cases} \overline{u}_t \geq \Delta\overline{u} + \overline{u}^p & \text{in } \mathbb{R}^n \times (0, \infty), \\ \overline{u}(x, 0) \geq \lambda\psi(x) & \text{in } \mathbb{R}^n, \\ \|\overline{u}(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} = O(q(t; a, n)) & \text{as } t \to \infty. \end{cases} \tag{3.11}$$

Then Lemma 2.3 will guarantee that $\overline{u}(x, t) \geq u(x, t; \lambda\psi) \geq 0$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$ and the proof will then be complete. To construct such a $\overline{u}$, we need to distinguish two cases: $a > 2/(p-1)$ and $a = 2/(p-1)$.

Case I: $a = 2/(p-1)$. Since $p > (n+2)/n$, we have $n > 2/(p-1) = a$. Thus $q(t; a, n) \sim t^{-1/(p-1)}$ near $t = \infty$. It was proved in [HW, Theorem 5] that the equation in (3.1) possesses a positive self-similar solution $\tilde{u}(x, t) = t^{-1/(p-1)}v(xt^{-1/2})$ with $v(x) > 0$ on $\mathbb{R}^n$ and $\lim_{x\to\infty} |x|^{2/(p-1)}v(x) > 0$. Let Λ_0 be a positive constant such that $v(x) \geq \Lambda_0\psi(x)$ on $\mathbb{R}^n$. Then for $\lambda < \Lambda_0$, we have $\tilde{u}(x, 1) = v(x) \geq \lambda\psi(x)$ and it follows that $\tilde{u}(x, t+1) \geq u(x, t; \lambda\psi)$ for all $x \in \mathbb{R}^n$ and $t \geq 0$. Since

$$\|\tilde{u}(\cdot, t+1)\|_{L^\infty(\mathbb{R}^n)} = (t+1)^{-1/(p-1)}v(0) \sim t^{-1/(p-1)} \quad \text{for } t \text{ large},$$

we only have to set $\overline{u}(x, t) = \tilde{u}(x, t+1)$ and $\overline{u}$ solves (3.11).

Case II. $a > 2/(p-1)$. In this case $\overline{u}(x, t)$ takes the form $\overline{u}(x, t) = h_\lambda(t)e^{t\Delta}\psi$ where

$$h_\lambda(t) = \left[\lambda^{1-p} - (p-1)\int_0^t \|e^{s\Delta}\psi\|^{p-1}_{L^\infty(\mathbb{R}^n)}\,ds\right]^{1/(1-p)}. \tag{3.12}$$

Since $\min\{a, n\} > 2/(p-1)$,

$$\int_0^\infty \|e^{s\Delta}\psi\|^{p-1}_{L^\infty(\mathbb{R}^n)}\,ds < \infty$$

in view of Lemma 2.12(i). Therefore, for $\lambda < \Lambda_0$ where

$$\Lambda_0 = \left[2(p-1)\int_0^\infty \|e^{s\Delta}\psi\|^{p-1}_{L^\infty(\mathbb{R}^n)}\,ds\right]^{1/(1-p)},$$

$h_\lambda(t)$ is well defined in $[0, \infty)$ and there exists a constant ε such that $h_\lambda(t) \geq \varepsilon > 0$ for all $t > 0$. This implies that $\overline{u}(x, t)$ is defined on $\mathbb{R}^n \times [0, \infty)$ with

$$\|\overline{u}(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \sim \|e^{t\Delta}\psi\|_{L^\infty(\mathbb{R}^n)} = O(q(t; a, n))$$

near $t = \infty$. It is clear that $\overline{u}(x, 0) = \lambda\psi(x)$. Finally, the conclusion that $\overline{u}$ satisfies the differential inequality in (3.11) follows immediately from the fact that

$$\frac{dh_\lambda}{dt} = \|e^{t\Delta}\psi\|^{p-1}_{L^\infty(\mathbb{R}^n)}h_\lambda^p \quad \text{in } t > 0,$$

which in turn may be verified by straightforward computation.

(ii) We set $\overline{u}(x, t) = u(x, t; \psi)$ and $\underline{u}(x, t) = e^{t\Delta}\psi$ in $\mathbb{R}^n \times [0, \infty)$. Then Lemma 2.3 implies that $\overline{u} \geq \underline{u}$ in $\mathbb{R}^n \times (0, \infty)$, and our assertion (3.10) follows from Lemma 2.12(ii). Q.E.D.

Finally we come to the estimates of the life span $T[\lambda\psi]$ as λ approaches 0 in case $u(x, t; \lambda\psi)$ blows up at finite time for arbitrarily small $\lambda > 0$. First we make the following observation for general initial value $\psi \geq 0$ in $C_b(\mathbb{R}^n)$.

Remark 3.13. Let $0 \leq \psi \in C_b(\mathbb{R}^n)$ and $\psi \not\equiv 0$. Then from (3.11) it is clear that $T[\lambda\psi] \geq T_\lambda$ where $[0, T_\lambda)$ is the maximal time interval for the existence of $\overline{u}$, i.e. that of h_λ in (3.12). Thus T_λ is given by

$$(p-1)^{-1}\lambda^{1-p} = \int_0^{T_\lambda} \|e^{s\Delta}\psi\|^{p-1}_{L^\infty(\mathbb{R}^n)}\,ds. \tag{3.14}$$

Since

$$\|e^{s\Delta}\psi\|_{L^\infty(\mathbb{R}^n)} \leq \|\psi\|_{L^\infty(\mathbb{R}^n)}$$

for all $s \geq 0$, we deduce that $T_\lambda \geq C\lambda^{1-p} \to \infty$ as $\lambda \to 0$.

Theorem 3.15. (i) *Suppose that* $p = (n+2)/n$ *and* $a \geq 2/(p-1) = n$. *Then for every* $\psi \in I_a$, *there exists a constant* $C > 0$ *such that*

$$\log T[\lambda\psi] \leq \begin{cases} C(1/\lambda)^{p-1}, & \text{if } a > n, \\ C(1/\lambda)^{(p-1)/p}, & \text{if } a = n, \end{cases} \tag{3.16}$$

as $\lambda \to 0$.

(ii) *Suppose that* $p < (n+2)/n$ *or* $a < 2/(p-1)$. *Then for every* $\psi \in I_a$, *there exists a constant* $C > 0$ *such that*

$$T[\lambda\psi] \leq \begin{cases} C(1/\lambda)^{(1/(p-1)-\frac{1}{2}\min\{a,n\})^{-1}}, & \text{if } a \neq n, \\ C(\frac{1/\lambda}{\log(1/\lambda)})^{(1/(p-1)-n/2)^{-1}}, & \text{if } a = n, \end{cases} \tag{3.17}$$

as $\lambda \to 0$.

Proof. (i) First we treat the case $a > n$. Let $\psi \in I_a$ and $u(x, t; \lambda\psi)$ be the solution of (3.1). Setting $\overline{u}(x, t) = u(x, t+2; \lambda\psi)$ for $t \geq 0$, we see that $\overline{u}(x, 0) = u(x, 2; \lambda\psi) \geq (e^{2\Delta}\lambda\psi)(x) \geq \lambda\delta K(x, 1)$ for some $\delta > 0$ as guaranteed by Lemma 2.3 and (2.16). Thus if we set $\underline{u}(x, t) = \lambda\delta K(x, t+1)$ for $t \geq 0$, then it follows from Lemma 2.3 again that $u(x, t+2; \lambda\psi) \geq \lambda\delta K(x, t+1)$ in $\mathbb{R}^n \times [0, T[\lambda\psi]-2)$. Note that $A_0(x, t) = \lambda\delta K(x, t+1)$ if we choose $\alpha = \lambda\delta$ in (2.19). From the integral representation of $\overline{u}$ and (2.21) we obtain

$$\begin{aligned} u(x, t+2; \lambda\psi) &\geq \int_0^t \int_{\mathbb{R}^n} K(x-y, t-s) u^p(y, s+2; \lambda\psi)\, dy\, ds \\ &\geq \int_0^t \int_{\mathbb{R}^n} K(x-y, t-s) A_0^p(y, s)\, dy\, ds \\ &\geq A_1^p(x, t). \end{aligned}$$

Iterating this argument yields that

$$u(x, t+2; \lambda\psi) \geq A_k(x, t), \quad \text{for } k = 0, 1, 2, \ldots,$$

and therefore by Lemma 2.20(i) it follows that

$$\begin{aligned} &u(x, t+2; \lambda\psi) \\ &\qquad \geq \sup_k \{(\lambda\delta\beta)^{p^k} K(x, (t+1)p^{-k})[\log(t+1)]^{(p^k-1)(p-1)^{-1}}\} \end{aligned} \tag{3.18}$$

for $(x, t) \in \mathbb{R}^n \times [0, T[\lambda\psi]-2)$. For the right-hand side of (3.18) to be finite at $x = 0$ it is necessary to have

$$\lambda\delta\beta \log(t+1)^{1/(p-1)} \leq 1,$$

that is,

$$\log(t+1) \leq \left(\frac{1}{\delta\beta}\right)^{p-1} \left(\frac{1}{\lambda}\right)^{p-1},$$

and (3.16) is established in the case $a > n$.

In case $a = n = 2/(p-1)$, we set $\overline{u}(x, t) = u(x, t+2; \lambda\psi)$ and $\underline{u}(x, t) = (\lambda e^{(t+2)\Delta}\psi)(x)$. Then Lemma 2.3 and (2.17) together imply that

$$\begin{aligned} u(x, t+2; \lambda\psi) &\geq (\lambda e^{(t+2)\Delta}\psi)(x) \\ &\geq \lambda\delta K\left(x, \frac{t+1}{4}\right) \log(t+1) = D_0(x, t) \end{aligned}$$

in $\mathbb{R}^n \times [0, T[\lambda\psi]-2)$, since $\psi \in I_n$. Applying Lemma 2.24 instead of Lemma 2.20 in argument above leading to (3.18) yields

$$u(x, t+2; \lambda\psi) \geq \sup_k \left\{ (\lambda\delta\eta)^{p^k} K\left(x, \frac{t+1}{4}p^{-k}\right) \cdot [\log(t+1)]^{(p^{k+1}-1)(p-1)^{-1}} \right\}. \tag{3.19}$$

Similarly the finiteness of the right-hand side of (3.19) at $x = 0$ implies that $\lambda\delta\eta \log(t+1)^{p/(p-1)} \leq 1$ which in turn implies (3.16) in the case $a = n$. This completes the proof of part (i).

(ii) The proof of this part is easier. By Lemma 2.3, we may assume without loss of generality that ψ is given by (2.13). Setting $M = T[\lambda\psi]/2$, we define

$$w(t) = \int_{\mathbb{R}^n} K(x, M-t)u(x, t; \lambda\psi)\, dx \quad \text{for } t \in [0, M].$$

Since u must be radially symmetric and bounded, we have, for each $t \in [0, M]$, there exists a sequence $r_m \to \infty$ such that

$$\max\{K(x, M-t)|\nabla u(x, t; \lambda\psi)| \,|\, |x| = r_m\} \to 0$$

as $m \to \infty$. This, combined with Green's theorem and Jensen's inequality, implies that

$$w_t \geq w^p \quad \text{in } [0, M], \tag{3.20}$$

and, by the computation in (2.14),

$$w(0) = \lambda \int_{\mathbb{R}^n} K(x, M)\psi(x)\, dx = (\lambda e^{M\Delta}\psi)(0) \geq \lambda C q(M; a, n)$$

for some constant $C > 0$. (Note that $M \to \infty$ as $\lambda \to 0$ in view of Remark 3.13.) Integrating (3.20) from 0 to M, we deduce that

$$M \leq (p-1)^{-1} w^{1-p}(0) \leq C\left(\frac{1}{\lambda}\right)^{p-1} q^{1-p}(M; a, n).$$

Now (3.17) follows from the explicit form of $q(M; a, n)$ in (2.11) and the fact that the function $t(\log t)^{-1}$ is increasing for t large. Q.E.D.

We should point out that the method used in handling part (i) of Theorem 3.15 is similar to the one in [H].

Our last result establishes a converse of Theorem 3.15.

Theorem 3.21. (i) *Suppose that* $p = (n+2)/n$ *and* $a \geq 2/(p-1) = n$. *Then for every* $\psi \in I^a$, *there exists a constant* $C > 0$ *such that*

$$\log T[\lambda\psi] \geq \begin{cases} C(1/\lambda)^{p-1}, & \text{if } a > n, \\ C(1/\lambda)^{(p-1)/p}, & \text{if } a = n, \end{cases} \tag{3.22}$$

as $\lambda \to 0$.

(ii) *Suppose that* $p < (n+2)/n$ *or* $a < 2/(p-1)$. *Then for every* $\psi \in I^a$, *there exists a constant* $C > 0$ *such that*

$$T[\lambda\psi] \geq \begin{cases} C(1/\lambda)^{(1/(p-1)-\frac{1}{2}\min\{a,n\})^{-1}}, & \text{if } a \neq n, \\ C(\frac{1/\lambda}{\log(1/\lambda)})^{(1/(p-1)-n/2)^{-1}}, & \text{if } a = n, \end{cases} \tag{3.23}$$

as $\lambda \to 0$.

Proof. From Remark 3.13 we know that (3.14) holds. Since $\psi \in I^a$, and $T_\lambda \to \infty$ as $\lambda \to 0$, by Lemma 2.12(i), we have

$$\begin{aligned}
\left(\frac{1}{\lambda}\right)^{p-1} &\le C + C\int_1^{T_\lambda} \|e^{s\Delta}\psi\|^{p-1}_{L^\infty(\mathbb{R}^n)}\,ds \\
&\le C + C\int_1^{T_\lambda} q^{p-1}(s;a,n)\,ds \\
&\le \begin{cases} CT_\lambda^{1-((p-1)/2)\min\{a,n\}}, & \text{if } \min\{a,n\} < 2/(p-1) \\ & \text{and } a \ne n, \\ CT_\lambda^{1-n(p-1)/2}(\log T_\lambda)^{p-1}, & \text{if } a = n < 2/(p-1), \\ C\log T_\lambda, & \text{if } a > n = 2/(p-1), \\ C(\log T_\lambda)^p, & \text{if } a = n = 2/(p-1), \end{cases}
\end{aligned} \tag{3.24}$$

where C represents generic positive constants independent of λ. (Note that the first two cases in the last inequality correspond to (ii) while the last two correspond to (i).) Since $T[\lambda\psi] \ge T_\lambda$ (by Remark 3.13), our conclusions follow from (3.24) by straightforward computations. Q.E.D.

4. Concluding remarks

In this paper we have only considered global existence or nonexistence, large time behavior and life span of the solution $u(x, t; \lambda\psi)$ of (3.1) for ψ with polynomial decay and, for λ sufficiently large or sufficiently small. Due to the (possible) presence of steady states of (3.1) and their complicated structure which depends on p and n, the same questions considered here for λ of intermediate sizes are extremely interesting and must have very different answers. Good progress has been made recently by X. Wang [W].

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