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Global existence of classical solutions to systems of wave equations with critical nonlinearity in three space dimensions

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Abstract. We discuss the global existence of small solutions to the Cauchy problem for systems of quasilinear wave equations in three space dimensions, when their nonlinear terms have quadratic nonlinearity. A global existence theorem is established on the null condition which is extended to the condition for systems of wave equations with different propagation speeds.

1. Introduction.

We consider the Cauchy problem for systems of quasilinear wave equations

$$\Box_i u^i = F^i(\partial u, \partial^2 u) \quad \text{in } [0,\infty) \times \boldsymbol{R}^3, \tag{1.1}$$

$$u^i(0,\cdot) = \varepsilon f^i, \quad \partial_t u^i(0,\cdot) = \varepsilon g^i \quad \text{in } \boldsymbol{R}^3, \quad i = 1,\dots,m, \tag{1.2}$$

where $\Box_i = \partial_0^2 - c_i^2 \sum_{j=1}^3 \partial_j^2$, $c_i > 0$, $\partial_\alpha = \partial/\partial x^\alpha$, $t = x^0$, $x = (x^1, x^2, x^3)$. $u^i = u^i(t,x)$ are real-valued unknown functions, and F^i, f^i, g^i are given functions. We denote by ∂ the space-time derivatives, *i.e.*

$$\partial u = (\partial_\alpha u^i)_{\alpha,i}, \quad \partial^2 u = (\partial_\alpha \partial_\beta u^i)_{\alpha,\beta,i},$$

where α, β range over $0,1,2,3$ and i over $1,\dots,m$. Assume that f^i and g^i belong to $C_0^\infty(\boldsymbol{R}^3)$ and ε is a positive small parameter. We also assume that each $F^i(\partial u, \partial^2 u)$ takes the form

$$F^i(\partial u, \partial^2 u) = \sum_{j=1}^m \sum_{\alpha,\beta=0}^3 C_{\alpha\beta}^{ij}(\partial u) \partial_\alpha \partial_\beta u^j + D^i(\partial u), \tag{1.3}$$

$$C_{\alpha\beta}^{ij}(\partial u) = C_{\beta\alpha}^{ij}(\partial u), \tag{1.4}$$

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$$C^{ij}_{\alpha\beta}(0) = 0, \quad D^i(0) = \frac{\partial D^i}{\partial(\partial_\alpha u^j)}(0) = 0, \tag{1.5}$$

where $C^{ij}_{\alpha\beta}$ and D^i are C^∞-functions near $\partial u = 0$.

We give a condition to assure global existence for the Cauchy problem (1.1)–(1.2) in three space dimensions. Small solutions exist globally when $F^i(\partial u, \partial^2 u)$ do not have quadratic parts. But in general we cannot expect global solutions when they have quadratic parts even if ε is small (see, e.g., [**5**]). However, S. Klainerman [**8**] introduced the null condition to deal with the quadratic parts of single wave equations (or systems of wave equations with the same propagation speeds) and proved a global existence theorem on that condition. We extend the Klainerman's null condition to the case where the propagation speeds are different. In two space dimensions, it was shown in [**1**] that the null condition for the systems with different propagation speeds could be derived by applying John-Shatah observation, provided $D^i(\partial u) = 0$. We can derive the following null condition to our case, by applying the similar argument. That is,

$$\sum_{\alpha,\beta,\gamma=0}^{3} C^{iii}_{\alpha\beta\gamma} X^i_\alpha X^i_\beta X^i_\gamma = 0, \quad \sum_{\alpha,\beta=0}^{3} D^{iii}_{\alpha\beta} X^i_\alpha X^i_\beta = 0 \quad (i = 1, \dots, m)$$
$$\text{for all real vectors } X^i = (X^i_0, X^i_1, X^i_2, X^i_3) \text{ satisfying} \tag{1.6}$$
$$(X^i_0)^2 - c_i^2 \sum_{j=1}^{3} (X^i_j)^2 = 0.$$

Here we have set

$$C^{ijk}_{\alpha\beta\gamma} = \frac{\partial C^{ij}_{\alpha\beta}}{\partial(\partial_\gamma u^k)}(0), \quad D^{ijk}_{\alpha\beta} = \frac{\partial^2 D^i}{\partial(\partial_\alpha u^j)\partial(\partial_\beta u^k)}(0).$$

We remark that the null condition for the systems with different propagation speeds is partly suggested in [**2**]. The aim of this paper is to prove a global existence theorem under the null condition (1.6).

Theorem 1.1. *Let propagation speeds c_i be different from each other. Assume that the nonlinear terms $F^i(\partial u, \partial^2 u)$ given by* (1.3)–(1.5) *satisfy the null condition* (1.6) *and*

$$C^{ij}_{\alpha\beta}(\partial u) = C^{ji}_{\alpha\beta}(\partial u). \tag{1.7}$$

Then there exits a positive constant ε_0 such that the Cauchy problem (1.1)–(1.2) *has a unique C^∞-solution in $[0, \infty) \times \boldsymbol{R}^3$ for ε with $0 \le \varepsilon < \varepsilon_0$.*

In order to prove the theorem, we use the invariant Sobolev norms. Unlike one-speed cases where we can fully use generators of the Poincaré group, the boost operators are unavailable for systems with different propagation speeds, since they do not have good commutation relations with $\square_i$. To avoid this difficulty, some estimates with the use of only generators of translations, rotations and dilations are proved ([**3**], [**9**]). Especially in [**3**], A. Hoshiga and H. Kubo proved a global existence theorem corresponding to Theorem 1.1 in two space dimensions. However, comparing with the two spatial dimensions, we have more trouble with time decay estimates. In fact, even in the simple case where $C^{iii}_{\alpha\beta\gamma} = D^{iii}_{\alpha\beta} = 0$, our weighted L^∞-estimates involve $\log t$ (Proposition 3.1; see also [**3**], Proposition 4.3). We show that the null condition enables us to remove $\log t$ from our estimates (section 3.2). Further, by virtue of the null condition, we can prove L^2-boundedness of the derivatives of the solution (section 4), which leads to the global existence theorem.

2. Notation.

Set

$$\partial_\alpha = \partial/\partial x^\alpha, \quad x^0 = t,$$

$$\partial = (\partial_0, \partial_1, \partial_2, \partial_3),$$

$$\nabla = (\partial_1, \partial_2, \partial_3),$$

$$r = |x| \quad \text{for } x \in \boldsymbol{R}^3.$$

We denote by $\Gamma = (\Gamma_0, \dots, \Gamma_7)$ the collection of differential operators ∂, Ω, S where

$$\Omega = x \wedge \nabla, \tag{2.1}$$

$$\Gamma_7 = S = t\partial_t + r\partial_r, \tag{2.2}$$

$$\partial_r = \frac{x}{r} \cdot \nabla. \tag{2.3}$$

Then we find that the bracket $[\Gamma_\alpha, \Gamma_\beta]$ of any Γ_α and Γ_β is written by another Γ_γ. Moreover, we have

$$[\Gamma_\alpha, \square_i] = 0 \quad \text{for } 0 \le \alpha \le 6 \quad \text{and} \quad [\Gamma_7, \square_i] = -2\square_i. \tag{2.4}$$

We also note that

$$\nabla = \frac{x}{r}\partial_r - \frac{x}{r^2} \wedge \Omega. \tag{2.5}$$

For $a = (a_1, \dots, a_k)$ $(a_i \in \{0, \dots, 7\}, 1 \le i \le k)$ we define

$$\Gamma^a = \Gamma_{a_1} \cdots \Gamma_{a_k} \quad \text{and} \quad |a| = k. \tag{2.6}$$

Let $u = {}^t(u^1, \dots, u^m)$ be a vector and set

$$w_i(t,r) = (1+r)(1+|c_i t - r|) \quad (i = 1, \dots, m). \tag{2.7}$$

Then we define

$$[\partial u]_{k,t} = \sum_{|a| \le k} \sum_{i=1}^{m} \sum_{\alpha=0}^{3} \sup_{0 \le s \le t} \sup_{x \in \boldsymbol{R}^3} |w_i(s,|x|) \Gamma^a \partial_\alpha u^i(s,x)|, \tag{2.8}$$

$$\|\partial u(t)\|_k = \sum_{|a| \le k} \sum_{i=1}^{m} \sum_{\alpha=0}^{3} \|\Gamma^a \partial_\alpha u^i(t,\cdot)\|_{L^2(\boldsymbol{R}^3)}, \tag{2.9}$$

$$\|\partial u\|_{k,t} = \sup_{0 \le s \le t} \|\partial u(s)\|_k. \tag{2.10}$$

3. Weighted L^∞-estimates.

3.1. Estimates for solutions of scalar wave equations.

Let $v = v(t,x)$ be the smooth solution of the Cauchy problem

$$\partial_t^2 v - c_0^2 \Delta v = F \quad \text{in } [0,T) \times \boldsymbol{R}^3, \tag{3.1}$$

$$v(0,\cdot) = \partial_t v(0,\cdot) = 0 \quad \text{in } \boldsymbol{R}^3, \tag{3.2}$$

where $F \in C^\infty([0,T) \times \boldsymbol{R}^3)$ and $F(t,\cdot) \in C_0^\infty(\boldsymbol{R}^3)$ for each t. In this subsection we present the decay estimates for v that we will use later on.

Proposition 3.1. *Let v be the solution of* (3.1)–(3.2). *For $1 \le \mu$, $0 < \nu$ and $0 \le c$, we set*

$$z_{\mu,\nu}(s,\lambda) = (1 + |cs - \lambda|)^\mu (1 + s + \lambda)^\nu,$$

$$\Phi_\theta(t) = \begin{cases} \log(2+t) & (\theta = 0) \\ 1 & (\theta > 0), \end{cases} \tag{3.3}$$

$$M_{\mu,\nu;k}(F) = \sum_{|a| \le k} \sup_{0 \le s \le t} \sup_{y \in \boldsymbol{R}^3} |y| z_{\mu,\nu}(s,|y|) |\Gamma^a F(s,y)|. \tag{3.4}$$

Then we have

(i) $$|v(t,x)| \le C(1+t+|x|)^{-1} \Phi_{\mu-1}(t) \Phi_{\nu-1}(t) M_{\mu,\nu;0}(F) \tag{3.5}$$
for $1 \le \mu, 1 \le \nu$,

(ii) $$|\partial v(t,x)| \le C(1+|x|)^{-1}(1+|c_0 t - |x||)^{-\nu} \Phi_{\mu-1}(t) M_{\mu,\nu;1}(F) \tag{3.6}$$
for $1 \le \mu, 0 < \nu, c \ne c_0$, *and*

(iii) $$|\partial v(t,x)| \le C(1+|x|)^{-1}\{(1+|c_0t-|x||)^{-\nu}\Phi_{\mu-1}(t) + (1+|c_0t-|x||)^{-\mu}\Phi_{\nu-1}(t)\}M_{\mu,\nu;1}(F) \tag{3.7}$$

for $1 \le \mu, 1 \le \nu, c = c_0$.

Here, the constant C *depends on* c_0, c, μ *and* ν.

Proof. By a change of coodinates, the proof can be reduced to the case where $c_0 = 1$. So we let $c_0 = 1$. Set $|x| = r$. In appendix in [**4**], F. John showed that the solution of (3.1)–(3.2) could be expressed in the form

$$v(t,x) = (4\pi r)^{-1}\int_0^t ds \int_{|r-t+s|}^{r+t-s} \lambda\, d\lambda \int_0^{2\pi} F(s,\lambda\Theta)\, d\varphi, \tag{3.8}$$

where

$$\begin{aligned} &\Theta = \Theta(s,\lambda,\varphi) = R(\sin\psi\cos\varphi, \sin\psi\sin\varphi, \cos\psi), \\ &R \text{ is an orthogonal transformation with } R(0,0,r) = x, \\ &\cos\psi = (2r\lambda)^{-1}(r^2+\lambda^2-(t-s)^2), \quad \sin\psi = (1-\cos^2\psi)^{1/2}. \end{aligned} \tag{3.9}$$

Thus, v can be written as

$$v(t,x) = (4\pi r)^{-1}\int_D \lambda\, d\lambda\, ds \int_0^{2\pi} F(s,\lambda\Theta)\, d\varphi, \tag{3.10}$$

where

$$\begin{aligned} &D = \{(s,\lambda) | 0 < s < t, \lambda_1 < \lambda < \lambda_2\}, \\ &\lambda_1 = |r-t+s|, \quad \lambda_2 = r+t-s. \end{aligned} \tag{3.11}$$

We first prove (3.5). By (3.10),

$$|v(t,x)| \le CI_0 M_{\mu,\nu;0}(F)$$

where

$$I_0 = r^{-1}\int_D z_{\mu,\nu}(s,\lambda)^{-1}\, d\lambda ds. \tag{3.12}$$

We obtain (3.5) from

$$I_0 \le C(1+t+r)^{-1}\Phi_{\mu-1}(t)\Phi_{\nu-1}(t), \tag{3.13}$$

which we will prove now. In order to prove (3.13), four cases

1. $r \leq 1$,
2. $1 \leq r$ and $(2+c)t \leq r$,
3. $1 \leq r$ and $2^{-1}t \leq r \leq (2+c)t$,
4. $1 \leq r$ and $r \leq 2^{-1}t$

are considered separately.

In the first case, we have

$$\lambda_2 - \lambda_1 \leq 2r \ (\leq 2),$$

and hence

$$\begin{aligned} I_0 &\leq Cr^{-1} \int_0^t ds \int_{\lambda_1}^{\lambda_2} z_{\mu,\nu}(s,\lambda_2)^{-1}\, d\lambda \\ &\leq C \int_0^t (1+|cs-\lambda_2|)^{-\mu}(1+s+\lambda_2)^{-\nu}\, ds \\ &\leq C(1+t+r)^{-\nu}\Phi_{\mu-1}(t). \end{aligned}$$

Therefore, (3.13) is proved for the case 1.

In the second case, the inequalities

$$\lambda - cs \geq \min\{r-t, r-ct\} \geq (2+c)^{-1}(1+\min\{1,c\})r \tag{3.14}$$

hold for $(s,\lambda) \in D$. Hence it follows from

$$z_{\mu,\nu}(s,\lambda)^{-1} \leq C(1+r)^{-\mu-\nu} \tag{3.15}$$

that

$$\begin{aligned} I_0 &\leq Cr^{-1} \int_D (1+r)^{-\mu-\nu}\, d\lambda ds \\ &\leq C(1+t+r)^{1-\mu-\nu}. \end{aligned}$$

Consequently, we have (3.13) for the case 2.

In the third and the fourth case, we introduce the new variables of integration

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -c & 1 \end{pmatrix} \begin{pmatrix} s \\ \lambda \end{pmatrix}. \tag{3.16}$$

Set

$$\alpha_0 = 2^{-1}\{(1-c)\alpha + (1+c)(r-t)\}. \tag{3.17}$$

If $|t-r| < \alpha < t+r$, then $-c\alpha < \alpha_0 < \alpha$. Hence,

$$\int_D z_{\mu,\nu}(s,\lambda)^{-1}\,d\lambda ds \le \int_D (1+|cs-\lambda|)^{-\mu}(1+s+\lambda)^{-\nu}\,d\lambda ds$$

$$\le \int_{|r-t|}^{r+t}(1+\alpha)^{-\nu}\,d\alpha\int_{\alpha_0}^{\alpha}(1+|\beta|)^{-\mu}\,d\beta$$

$$\le \int_{|r-t|}^{r+t}(1+\alpha)^{-\nu}\Phi_{\mu-1}(\alpha)\,d\alpha.$$

Therefore in the third case, it follows that

$$I_0 \le C(1+r)^{-1}\Phi_{\mu-1}(t)\int_{|r-t|}^{r+t}(1+\alpha)^{-\nu}\,d\alpha$$

$$\le C(1+t+r)^{-1}\Phi_{\mu-1}(t)\Phi_{\nu-1}(t),$$

and in the fourth case, it follows that

$$I_0 \le Cr^{-1}(1+t-r)^{-\nu}\Phi_{\mu-1}(t)\int_{t-r}^{t+r}d\alpha$$

$$\le C(1+t+r)^{-\nu}\Phi_{\mu-1}(t).$$

Hence we have proved (3.13) in all cases.

We next prove (3.6) and (3.7). In order to prove (3.6) and (3.7), we give representation formulae for $\partial_\alpha v$.

Let $(s,\lambda)\in D$ and $0\le\varphi\le 2\pi$. By (2.5),

$$(\nabla F)(s,\lambda\Theta) = \Theta(\partial_r F)(s,\lambda\Theta) - \lambda^{-1}\Theta\wedge(\Omega F)(s,\lambda\Theta). \tag{3.18}$$

Since $\partial_\lambda\Theta\cdot\Theta = 0$, it follows that

$$\begin{aligned}(\partial_r F)(s,\lambda\Theta) &= \partial_\lambda\{F(s,\lambda\Theta)\} - \lambda\partial_\lambda\Theta\cdot(\nabla F)(s,\lambda\Theta)\\ &= \partial_\lambda\{F(s,\lambda\Theta)\} - \lambda\partial_\lambda\Theta\cdot\{\Theta(\partial_r F)(s,\lambda\Theta) - \lambda^{-1}\Theta\wedge(\Omega F)(s,\lambda\Theta)\}\\ &= \partial_\lambda\{F(s,\lambda\Theta)\} + \partial_\lambda\Theta\cdot(\Theta\wedge(\Omega F)(s,\lambda\Theta)).\end{aligned} \tag{3.19}$$

In a similar manner,

$$(\partial_t F)(s,\lambda\Theta) = \partial_s\{F(s,\lambda\Theta)\} + \partial_s\Theta\cdot(\Theta\wedge(\Omega F)(s,\lambda\Theta)). \tag{3.20}$$

If we substitute (3.19) into (3.18), we have

$$(\nabla F)(s,\lambda\Theta) = \Theta\partial_\lambda\{F(s,\lambda\Theta)\} + \Theta\{\partial_\lambda\Theta\cdot(\Theta\wedge(\Omega F)(s,\lambda\Theta))\}$$
$$-\lambda^{-1}\Theta\wedge(\Omega F)(s,\lambda\Theta). \tag{3.21}$$

We use these expressions in

$$\partial_\alpha v(t,x) = (4\pi r)^{-1}\int_D \lambda\,d\lambda ds\int_0^{2\pi}(\partial_\alpha F)(s,\lambda\Theta)\,d\varphi,$$

which are obtained from (3.10). We split the domain of integration D into D_1 and D_2, where

$$\begin{aligned} D_1 &= \{(s,\lambda)\in D\,|\,\lambda_1<\lambda<\lambda_1+\delta \text{ or } \lambda_2-\delta<\lambda<\lambda_2\},\\ D_2 &= D\backslash D_1,\\ \delta &= \min\{1,r\}. \end{aligned} \tag{3.22}$$

Using (3.20), (3.21) and integrating by parts on the domain D_2, we obtain the following representation formulae.

$$\begin{aligned} 4\pi r\partial_t v(t,x) &= \int_{D_1}\lambda\,d\lambda ds\int_0^{2\pi}(\partial_t F)(s,\lambda\Theta)\,d\varphi + \int_{\partial D_2} n_s\,d\sigma\int_0^{2\pi}\lambda F(s,\lambda\Theta)\,d\varphi\\ &\quad+\int_{D_2}\lambda\,d\lambda ds\int_0^{2\pi}\partial_s\Theta\cdot(\Theta\wedge(\Omega F)(s,\lambda\Theta))\,d\varphi \end{aligned} \tag{3.23}$$

$$\begin{aligned} 4\pi r\nabla v(t,x) &= \int_{D_1}\lambda\,d\lambda ds\int_0^{2\pi}(\nabla F)(s,\lambda\Theta)\,d\varphi + \int_{\partial D_2} n_\lambda\,d\sigma\int_0^{2\pi}\lambda\Theta F(s,\lambda\Theta)\,d\varphi\\ &\quad-\int_{D_2}d\lambda ds\int_0^{2\pi}\{\Theta F(s,\lambda\Theta)+\Theta\wedge(\Omega F)(s,\lambda\Theta)\}\,d\varphi\\ &\quad+\int_{D_2}\lambda\,d\lambda ds\int_0^{2\pi}[-\partial_\lambda\Theta F(s,\lambda\Theta)+\Theta\{\partial_\lambda\Theta\cdot(\Theta\wedge(\Omega F)(s,\lambda\Theta))\}]\,d\varphi \end{aligned} \tag{3.24}$$

Here, (n_s, n_λ) is the unit outer normal vector field on ∂D_2, and $d\sigma$ is the line element on ∂D_2.

Note that $D_2 = \varnothing$ if $r<1$. Therefore it follows from (3.23) and (3.24) that

$$|\partial v(t,x)| \le C\sum_{i=1}^{4} I_i M_{\mu,\nu;1}(F) \tag{3.25}$$

where

$$I_1 = r^{-1} \int_{D_1} z_{\mu,\nu}(s,\lambda)^{-1}\, d\lambda ds, \tag{3.26}$$

$$I_2 = (1+r)^{-1} \int_{\partial D_2} z_{\mu,\nu}(s,\lambda)^{-1}\, d\sigma, \tag{3.27}$$

$$I_3 = (1+r)^{-1} \int_{D_2} (1+\lambda)^{-1} z_{\mu,\nu}(s,\lambda)^{-1}\, d\lambda ds, \tag{3.28}$$

$$I_4 = (1+r)^{-1} \int_{D_2} \sup_{0 \le \varphi \le 2\pi} |\partial\Theta| z_{\mu,\nu}(s,\lambda)^{-1}\, d\lambda ds. \tag{3.29}$$

By the definition of the domain D_1, we can easily see

$$I_1 \le CI_2. \tag{3.30}$$

Concerning I_2, I_3 and I_4, we will prove

$$I_2 \le C(1+r)^{-1}(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) \tag{3.31}$$

for $1 \le \mu,\ 0 < \nu,\ c \ne 1$,

$$I_2 \le C(1+r)^{-1}\{(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) + (1+|t-r|)^{-\mu}\Phi_{\nu-1}(t)\} \tag{3.32}$$

for $1 \le \mu,\ 1 \le \nu,\ c = 1$,

$$I_3 \le C(1+r)^{-1}(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) \tag{3.33}$$

for $1 \le \mu,\ 0 < \nu$, and

$$I_4 \le C(1+r)^{-1}(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) \tag{3.34}$$

for $1 \le \mu,\ 0 < \nu,\ c \ne 1$ or $1 \le \mu,\ 1/2 < \nu,\ c = 1$. If we have proved (3.31)–(3.34), then (3.6) and (3.7) are obtained through (3.25), (3.30) and these estimates.

(a) Proof of (3.31) and (3.32).

If $r \ge (2+c)t$, (3.31) and (3.32) are easily obtained from (3.14). So we let $r \le (2+c)t$.

By the definition of I_2, we have

$$(1+r)I_2 \le C(I_2' + I_2'' + I_2'''),$$

where

$$I_2' = \int_0^t z_{\mu,\nu}(s,\lambda_1)^{-1}\,ds,$$

$$I_2'' = \int_0^t z_{\mu,\nu}(s,\lambda_2)^{-1}\,ds,$$

$$I_2''' = \int_{|t-r|}^{t+r} z_{\mu,\nu}(0,\lambda)^{-1}\,d\lambda.$$

If $c \neq 1$, it follows that

$$\begin{aligned} I_2' &\le \int_0^t (1+|cs-\lambda_1|)^{-\mu}(1+s+\lambda_1)^{-\nu}\,ds \\ &\le C(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t), \end{aligned}$$

because $s+\lambda_1 \ge |t-r|$. However, if $c=1$,

$$\begin{aligned} I_2' &\le \int_0^t (1+|s-\lambda_1|)^{-\mu}(1+s+\lambda_1)^{-\nu}\,ds \\ &= \int_0^{(t-r)_+} (1+|s-\lambda_1|)^{-\mu}(1+t-r)^{-\nu}\,ds \\ &\quad + \int_{(t-r)_+}^t (1+|t-r|)^{-\mu}(1+s+\lambda_1)^{-\nu}\,ds \\ &\le C(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) + C(1+|t-r|)^{-\mu}\Phi_{\nu-1}(t). \end{aligned}$$

As for I_2'' and I_2''', we obtain, straightforwardly,

$$I_2'' \le C(1+t+r)^{-\nu}\Phi_{\mu-1}(t),$$

$$I_2''' \le C(1+|t-r|)^{1-\mu-\nu}$$

for $1 \le \mu$ and $0 < \nu$. Thus we have proved (3.31) and (3.32).

(b) Proof of (3.33).

In case $r \ge (2+c)t$, (3.33) results from (3.14).

Let $r \le (2+c)t$. If $c = 0$, by the change of variables (3.16),

$$\begin{aligned} (1+r)I_3 &\le \int_{|t-r|}^{t+r} (1+\alpha)^{-\nu}\,d\alpha \int_{\alpha_0}^{\alpha} (1+\beta)^{-1-\mu}\,d\beta \\ &\le C\int_{|t-r|}^{t+r} (1+|t-r|)^{-\nu}(1+\alpha_0)^{-\mu}\,d\alpha \\ &\le C(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t). \end{aligned}$$

If $c > 0$, set

$$\begin{aligned} \Lambda_0 &= \{(s, \lambda) \mid 0 \le \lambda \le 2^{-1} cs\}, \\ \Lambda_1 &= \{(s, \lambda) \mid 0 \le 2^{-1} cs \le \lambda\}. \end{aligned} \tag{3.35}$$

Since

$$(s, \lambda) \in \Lambda_0 \Rightarrow (1+\lambda)^{-1} z_{\mu,\nu}(s,\lambda)^{-1} \le C(1+\lambda)^{-1}(1+s+\lambda)^{-\mu-\nu}, \tag{3.36}$$

$$(s, \lambda) \in \Lambda_1 \Rightarrow (1+\lambda)^{-1} z_{\mu,\nu}(s,\lambda)^{-1} \le C(1+|cs-\lambda|)^{-\mu}(1+s+\lambda)^{-1-\nu}, \tag{3.37}$$

the inequality

$$(1+\lambda)^{-1} z_{\mu,\nu}(s,\lambda)^{-1} \le C\{(1+\lambda)^{-\mu} + (1+|cs-\lambda|)^{-\mu}\}(1+s+\lambda)^{-1-\nu} \tag{3.38}$$

holds. Hence, adapting (3.16) for each term of (3.38), we obtain (3.33).

(c) Proof of (3.34).

We first need estimates of $|\partial\Theta|$. By the definition (3.9) of Θ, we have

$$|\partial_\lambda \Theta| = |\lambda^2 + \bar{\lambda}_1 \lambda_2| \lambda^{-1} \{(\lambda^2 - \lambda_1^2)(\lambda_2^2 - \lambda^2)\}^{-1/2},$$

$$|\partial_s \Theta| = (\bar{\lambda}_1 + \lambda_2)\{(\lambda^2 - \lambda_1^2)(\lambda_2^2 - \lambda^2)\}^{-1/2},$$

where $\bar{\lambda}_1 = t - s - r$. Noting

$$\bar{\lambda}_1 = \begin{cases} -\lambda_1 & \text{for } (t-r)_+ < s < t \\ \lambda_1 & \text{for } 0 < s < t - r, \end{cases}$$

it follows from

$$\lambda^2 + \bar{\lambda}_1 \lambda_2 = \lambda(\lambda + \bar{\lambda}_1) + \bar{\lambda}_1(\lambda_2 - \lambda),$$

$$\bar{\lambda}_1 + \lambda_2 = (\lambda + \bar{\lambda}_1) + (\lambda_2 - \lambda)$$

that

$$(t-r)_+ < s < t \Rightarrow |\partial\Theta| \le (\lambda^2 - \lambda_1^2)^{-1/2} + (\lambda_2^2 - \lambda^2)^{-1/2}, \tag{3.39}$$

$$0 < s < t - r \Rightarrow |\partial\Theta| \le (\lambda^2 - \lambda_1^2)^{-1/2} + \{(\lambda - \lambda_1)(\lambda_2 - \lambda)\}^{-1/2}. \tag{3.40}$$

As before, the case $r \ge (2+c)t$ is easy. So we consider $r \le (2+c)t$. Set

$$\begin{aligned} D_2^{(1)} &= \{(s, \lambda) \in D_2 \mid (t-r)_+ < s < t\}, \\ D_2^{(2)} &= \{(s, \lambda) \in D_2 \mid 0 < s < t - r\}. \end{aligned} \tag{3.41}$$

Then from (3.39) and (3.40),

$$(s,\lambda)\in D_2^{(1)} \Rightarrow |\partial\Theta| \le C\{(1+\lambda-\lambda_1)^{-1/2}+(1+\lambda_2-\lambda)^{-1/2}\}(1+\lambda)^{-1/2},$$

$$(s,\lambda)\in D_2^{(2)} \Rightarrow |\partial\Theta| \le C\{(1+\lambda)^{-1/2}+(1+\lambda_2-\lambda)^{-1/2}\}(1+\lambda-\lambda_1)^{-1/2}.$$

Hence, noting (3.36) and (3.37), we obtain the following estimates:

$$(s,\lambda)\in D_2^{(i)} \Rightarrow |\partial\Theta| z_{\mu,\nu}(s,\lambda)^{-1} \le C\{p_{\mu,\nu}^{(i)}(s,\lambda)+q_{\mu,\nu}^{(i)}(s,\lambda)\}, \quad i=1,2 \tag{3.42}$$

where

$$p_{\mu,\nu}^{(i)}(s,\lambda)=\xi_{\mu,\nu}^{(i),1}(s,\lambda)+\xi_{\mu,\nu}^{(i),2}(s,\lambda), \quad i=1,2, \tag{3.43}$$

$$q_{\mu,\nu}^{(i)}(s,\lambda)=\eta_{\mu,\nu}^{(i),1}(s,\lambda)+\eta_{\mu,\nu}^{(i),2}(s,\lambda), \quad i=1,2, \tag{3.44}$$

$$\xi_{\mu,\nu}^{(1),1}(s,\lambda)=(1+\lambda-\lambda_1)^{-1/2}(1+\lambda)^{-1/2-\mu}(1+s+\lambda)^{-\nu},$$

$$\xi_{\mu,\nu}^{(1),2}(s,\lambda)=(1+\lambda_2-\lambda)^{-1/2}(1+\lambda)^{-1/2-\mu}(1+s+\lambda)^{-\nu},$$

$$\eta_{\mu,\nu}^{(1),1}(s,\lambda)=(1+\lambda-\lambda_1)^{-1/2}(1+|cs-\lambda|)^{-\mu}(1+s+\lambda)^{-1/2-\nu},$$

$$\eta_{\mu,\nu}^{(1),2}(s,\lambda)=(1+\lambda_2-\lambda)^{-1/2}(1+|cs-\lambda|)^{-\mu}(1+s+\lambda)^{-1/2-\nu},$$

$$\xi_{\mu,\nu}^{(2),1}(s,\lambda)=(1+\lambda)^{-1/2-\mu}(1+\lambda-\lambda_1)^{-1/2}(1+s+\lambda)^{-\nu},$$

$$\xi_{\mu,\nu}^{(2),2}(s,\lambda)=(1+\lambda_2-\lambda)^{-1/2}(1+\lambda-\lambda_1)^{-1/2}(1+\lambda)^{-\mu}(1+s+\lambda)^{-\nu},$$

$$\eta_{\mu,\nu}^{(2),1}(s,\lambda)=\eta_{\mu,\nu}^{(1),1}(s,\lambda),$$

$$\eta_{\mu,\nu}^{(2),2}(s,\lambda)=(1+\lambda_2-\lambda)^{-1/2}(1+\lambda-\lambda_1)^{-1/2}(1+|cs-\lambda|)^{-\mu}(1+s+\lambda)^{-\nu}.$$

We change the variables of integration by (3.16). Here, we let $c=0$ to adapt (3.16) for $p_{\mu,\nu}^{(1)}(s,\lambda)$ and $p_{\mu,\nu}^{(2)}(s,\lambda)$. Then we can prove

$$\int_{D_2^{(i)}} \xi_{\mu,\nu}^{(i),j}(s,\lambda)\,d\lambda ds \le C(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) \tag{3.45}$$

for $1\le\mu,\ 0<\nu$,

$$\int_{D_2^{(i)}} \eta_{\mu,\nu}^{(i),j}(s,\lambda)\,d\lambda ds \le C(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) \tag{3.46}$$

for $1 \le \mu,\ 0 < \nu,\ (i,j) \neq (1,1)$, and

$$\int_{D_2^{(1)}} \eta_{\mu,\nu}^{(1),1}(s,\lambda)\,d\lambda ds \le C(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t) \tag{3.47}$$

for $1 \le \mu$, $0 < \nu$, $c \neq 1$ or $1 \le \mu$, $1/2 < \nu$, $c = 1$. We show the estimate of the integral of $\xi_{\mu,\nu}^{(1),1}(s,\lambda)$ and $\eta_{\mu,\nu}^{(1),1}(s,\lambda)$ here. The others are easy to treat.

We first prove

$$\int_{\alpha_0}^{\alpha} (1+\beta-\alpha_0)^{-1/2}(1+|\beta|)^{-\tau}\,d\beta$$
$$\le C\{(1+|\alpha_0|)^{1/2-\tau} + \chi(\alpha_0)(1+|\alpha_0|)^{-1/2}\Phi_{\tau-1}(\alpha)\}, \tag{3.48}$$

for $\tau \ge 1$ and $|t-r| < \alpha < t+r$, where χ is the characteristic function of the interval $(-\infty, 0)$. Let $(a,b) \subset (\alpha_0, \alpha)$ to be the interval where $1+\beta-\alpha_0 < 1+|\beta|$. If $\alpha_0 \ge 0$, then $(a,b) = (\alpha_0, \alpha)$, and if $\alpha_0 < 0$, then $(a,b) = (\alpha_0, \alpha_0/2)$. Integrating by parts,

$$\int_a^b (1+\beta-\alpha_0)^{-1/2}(1+|\beta|)^{-\tau}\,d\beta$$
$$= \int_a^b \partial_\beta\{2(1+\beta-\alpha_0)^{1/2}\}(1+|\beta|)^{-\tau}\,d\beta$$
$$\le 2(1+|b|)^{1/2-\tau} + 2\tau\int_a^b (1+|\beta|)^{-1/2-\tau}\,d\beta$$
$$\le C\{(1+|a|)^{1/2-\tau} + (1+|b|)^{1/2-\tau}\}$$
$$\le C(1+|\alpha_0|)^{1/2-\tau}.$$

Hence,

$$\int_{\alpha_0}^{\alpha} (1+\beta-\alpha_0)^{-1/2}(1+|\beta|)^{-\tau}\,d\beta$$
$$= \int_a^b + \int_b^\alpha$$
$$\le C(1+|\alpha_0|)^{1/2-\tau} + C\chi(\alpha_0)\int_b^\alpha (1+|\alpha_0|)^{-1/2}(1+|\beta|)^{-\tau}\,d\beta,$$

and we obtain (3.48).

Since $\lambda-\lambda_1=2(1+c)^{-1}(\beta-\alpha_0)$ in $D_2^{(1)}$, it follows from (3.48) that

$$\begin{aligned}
&\int_{D_2^{(1)}} \xi_{\mu,\nu}^{(1),1}(s,\lambda)\,d\lambda ds \\
&\quad\le C\int_{|t-r|}^{t+r}(1+\alpha)^{-\nu}\,d\alpha\int_{\alpha_0}^{\alpha}(1+\beta-\alpha_0)^{-1/2}(1+\beta)^{-1/2-\mu}\,d\beta \\
&\quad\le C\int_{|t-r|}^{t+r}(1+\alpha)^{-\nu}(1+\alpha_0)^{-\mu}\,d\alpha \\
&\quad\le C(1+|t-r|)^{-\nu}\Phi_{\mu-1}(t),
\end{aligned}$$

which proves (3.45) for $i=j=1$. Concerning $\eta_{\mu,\nu}^{(1),1}(s,\lambda)$, we have

$$\begin{aligned}
&\int_{D_2^{(1)}} \eta_{\mu,\nu}^{(1),1}(s,\lambda)\,d\lambda ds \\
&\quad\le C\int_{|t-r|}^{t+r}(1+\alpha)^{-1/2-\nu}\,d\alpha\int_{\alpha_0}^{\alpha}(1+\beta-\alpha_0)^{-1/2}(1+|\beta|)^{-\mu}\,d\beta \\
&\quad\le C\int_{|t-r|}^{t+r}(1+\alpha)^{-1/2-\nu}(1+|\alpha_0|)^{-1/2}\,d\alpha\Phi_{\mu-1}(t). \qquad (3.49)
\end{aligned}$$

If $c=1$, we note that $\alpha_0=r-t$. If $c\ne 1$, let $h(\alpha)$ be a primitive of $(1+|\alpha_0|)^{-1/2}$. Integrating by parts, it follows that

$$\begin{aligned}
&\int_{|t-r|}^{t+r}(1+\alpha)^{-1/2-\nu}(1+|\alpha_0|)^{-1/2}\,d\alpha \\
&\quad\le (1+\alpha)^{-1/2-\nu}|h(\alpha)|\Big|_{\alpha=|t-r|,\,t+r}+(1/2+\nu)\int_{|t-r|}^{t+r}(1+\alpha)^{-3/2}|h(\alpha)|\,d\alpha \\
&\quad\le C(1+|t-r|)^{-\nu},
\end{aligned}$$

since $|h(\alpha)|\le C(1+\alpha)^{1/2}$. Hence we obtain (3.47).

3.2. An estimate for the quasilinear system with the null condition.

Proposition 3.2. *Let $u=(u^1,\dots,u^m)$ be the smooth solution of*

$$\square_i u^i=F^i(\partial u,\partial^2 u) \quad in\ [0,T)\times \boldsymbol{R}^3, \qquad (3.50)$$

$$u^i(0,\cdot)=\varepsilon f^i, \quad \partial_t u^i(0,\cdot)=\varepsilon g^i \quad in\ \boldsymbol{R}^3, \qquad (3.51)$$

$$i=1,\dots,m.$$

Assume that F^i *satisfy* (1.6) *and* c_i *are different from each other. If* $\varepsilon < 1$ *and* $[\partial u]_{[(N+6)/2],\,t} < 1$, *then there exists a constant* C_N', *depending on* N, c_i $(1 \le i \le m)$ *and given functions, such that*

$$[\partial u]_{N,\,t} \le C_N'(\varepsilon + \|\partial u\|_{N+8,\,t}^6). \tag{3.52}$$

Proof. Let u_0^i be the solutions of the homogeneous equations

$$\Box_i u_0^i = 0 \quad \text{in } [0,\infty) \times \boldsymbol{R}^3, \tag{3.53}$$

$$u_0^i(0,\cdot) = \varepsilon f^i, \quad \partial_t u_0^i(0,\cdot) = \varepsilon g^i \quad \text{in } \boldsymbol{R}^3, \tag{3.54}$$

$$i = 1, \dots, m.$$

From (2.4), each $\Gamma^a u_0^i$ satisfies (3.53). Hence by the proof of Theorem 1 in [**7**] we have

$$\begin{aligned} |\Gamma^a u_0^i(t,x)| &\le C_N \varepsilon (1+r)^{-1}(1+|c_i t - r|)^{-1} \\ &\le C_N \varepsilon (1+t+r)^{-1}, \end{aligned} \tag{3.55}$$

$$|\partial \Gamma^a u_0^i(t,x)| \le C_N \varepsilon (1+r)^{-1}(1+|c_i t - r|)^{-1}, \tag{3.56}$$

for $|a| \le N$. Here and hereafter, we denote by C_N a various constant depending on N, c_i and given functions.

Set

$$u_1 = u - u_0. \tag{3.57}$$

Then, each $\Gamma^a u_1^i$ satisfies the equation of the form

$$\Box_i \Gamma^a u_1^i = \sum_{|b| \le |a|} C_{ab} \Gamma^b F^i(\partial u, \partial^2 u) \quad \text{in } [0,T) \times \boldsymbol{R}^3, \tag{3.58}$$

$$\Gamma^a u_1^i(0,\cdot) = \partial_t \Gamma^a u_1^i(0,\cdot) = 0 \quad \text{in } \boldsymbol{R}^3. \tag{3.59}$$

We apply Proposition 3.1 to a solution of (3.58)–(3.59) by replacing the weight $z_{\mu,\nu}(s,\lambda)$ with

$$z(s,\lambda) = \left\{ \sum_{i=1}^{m+1} z_{1,1}^{(i)}(s,\lambda)^{-1} \right\}^{-1},$$

where

$$z_{\mu,\nu}^{(i)}(s,\lambda) = (1+|c_i s - \lambda|)^{\mu}(1+s+\lambda)^{\nu}, \quad c_{m+1} = 0. \tag{3.60}$$

If we replace the weight $z_{\mu,\nu}(s,\lambda)$ in (3.4) with $z(s,\lambda)$, then each $z_{\mu,\nu}(s,\lambda)^{-1}$ in (3.12), (3.26)–(3.29) is also changed for $z(s,\lambda)^{-1} = \sum_{i=1}^{m+1} z_{1,1}^{(i)}(s,\lambda)^{-1}$. Hence we have

$$|\Gamma^a u_1^i(t,x)| \le C_N(1+t+r)^{-1}\{\log(2+t)\}^2 M_N(F^i), \tag{3.61}$$

$$|\partial\Gamma^a u_1^i(t,x)| \le C_N(1+r)^{-1}(1+|c_i t - r|)^{-1}\log(2+t) M_{N+1}(F^i), \tag{3.62}$$

for $|a| \le N$, where

$$M_k(F^i) = \sum_{|a|\le k} \sup_{0\le s\le t} \sup_{y\in \boldsymbol{R}^3} |y| z(s,|y|) |\Gamma^a F^i(\partial u, \partial^2 u)(s,y)|.$$

In order to estimate $M_k(F^i)$, we use the Sobolev inequality (Lemma 4.2 in [**9**]):

$$|y|\,|f(y)| \le C\left\{\sum_{|a|\le 2} \|\Omega^a f\|_{L^2(\boldsymbol{R}^3)} + \sum_{|a|\le 1} \|\partial_r \Omega^a f\|_{L^2(\boldsymbol{R}^3)}\right\}. \tag{3.63}$$

If $|b| + |c| \le k$ and $0 \le s \le t$, we have

$$|y| z(s,|y|) |\partial\Gamma^b u^j(s,y)|\,|\partial\Gamma^c u^l(s,y)| \le C_k [\partial u]_{[k/2],t} \|\partial u\|_{k+2,t},$$

since $z(s,|y|) \le C w_i(s,|y|)$ $(i = 1,\dots,m)$. Therefore,

$$M_k(F^i) \le C_k [\partial u]_{[(k+1)/2],t} \|\partial u\|_{k+3,t}, \tag{3.64}$$

provided $|\partial u|, |\partial^2 u| < 1$. Therefore, it follows from (3.55), (3.56), (3.61), (3.62) and (3.64) that

$$|\Gamma^a u^i(t,x)| \le C_N(1+t+r)^{-1}\{\log(2+t)\}^2(\varepsilon + [\partial u]_{[(N+1)/2],t}\|\partial u\|_{N+3,t}), \tag{3.65}$$

$$|\partial\Gamma^a u^i(t,x)| \le C_N(1+r)^{-1}(1+|c_i t - r|)^{-1}\log(2+t)(\varepsilon + [\partial u]_{[(N+2)/2],t}\|\partial u\|_{N+4,t}) \tag{3.66}$$

for $|a| \le N$.

Next, we estimate the nonlinear terms by making use of (3.65), (3.66). We separate F^i into three parts:

$$F^i(\partial u, \partial^2 u) = N^i(\partial u, \partial^2 u) + R^i(\partial u, \partial^2 u) + G^i(\partial u, \partial^2 u), \tag{3.67}$$

where

$$N^i(\partial u, \partial^2 u) = \sum_{0\le\alpha,\beta,\gamma\le 3} C^{iii}_{\alpha\beta\gamma}\partial_\gamma u^i \partial_\alpha\partial_\beta u^i + \sum_{0\le\alpha,\beta\le 3} D^{iii}_{\alpha\beta}\partial_\alpha u^i \partial_\beta u^i, \tag{3.68}$$

$$R^i(\partial u, \partial^2 u) = \sum_{(j,k)\ne(i,i)}\left(\sum_{0\le\alpha,\beta,\gamma\le 3} C^{ijk}_{\alpha\beta\gamma}\partial_\gamma u^k \partial_\alpha\partial_\beta u^j + \sum_{0\le\alpha,\beta\le 3} D^{ijk}_{\alpha\beta}\partial_\alpha u^j \partial_\beta u^k\right), \tag{3.69}$$

and $G^i(\partial u, \partial^2 u)$ are higher order terms. Moreover, by the null condition (1.6), $N^i(\partial u, \partial^2 u)$ can be expressed in the form

$$N^i(\partial u, \partial^2 u) = \sum_{0 \le \alpha \le 3} T^i_\alpha Q^i(u^i, \partial_\alpha u^i) + \sum_{0 \le \alpha, \beta, \gamma \le 3} T^i_{\alpha\beta\gamma} Q_{\alpha\beta}(u^i, \partial_\gamma u^i)$$

$$+ \sum_{0 \le \alpha \le 3} \bar{T}^i_\alpha \partial_\alpha u^i \Box_i u^i + T^i Q^i(u^i, u^i), \tag{3.70}$$

where

$$Q^i(u, v) = \partial_t u \partial_t v - c_i^2 \nabla u \cdot \nabla v, \tag{3.71}$$

$$Q_{\alpha\beta}(u, v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v. \tag{3.72}$$

These forms gain good decay near $c_i t - r = 0$ $(1 \le i \le m)$. Indeed, the following estimates hold for $|c_i t - r| < c_i t/2$:

$$|Q^i(u, v)| \le C|c_i t - r|(1 + t + r)^{-1}|\partial u|\,|\partial v| + C(1 + t + r)^{-1}(|\Gamma u|\,|\partial v| + |\partial u|\,|\Gamma v|), \tag{3.73}$$

$$|Q_{\alpha\beta}(u, v)| \le C(1 + t + r)^{-1}(|\partial u|\,|\Gamma v| + |\Gamma u|\,|\partial v|), \tag{3.74}$$

$$|\Box_i u| \le C|c_i t - r|(1 + t + r)^{-1}|\partial^2 u| + C(1 + t + r)^{-1}(|\partial u| + |\partial \Gamma u|). \tag{3.75}$$

Let us give a proof of (3.73)–(3.75). They trivially hold for $r \le 1$, so we suppose $r \ge 1$. Following [**3**], we define

$$S_i^\pm = \partial_t \pm c_i \partial_r. \tag{3.76}$$

Noting

$$S_i^+ u = t^{-1}(c_i t - r)\partial_r u + t^{-1} S u,$$

$$|r^{-1} \Omega u| \le C |\nabla u|,$$

(3.73) follows from the identity

$$Q^i(u, v) = 2^{-1}(S_i^+ u S_i^- v + S_i^- u S_i^+ v) - c_i^2 r^{-2}(\hat{x} \wedge \Omega u) \cdot (\hat{x} \wedge \Omega v), \quad \hat{x} = x/r,$$

which is derived from (2.5). If we rewrite $Q_{\alpha\beta}(u, v)$ by using (2.5) and

$$\partial_t u = -r t^{-1} \partial_r u + t^{-1} S u,$$

we can prove (3.74). Finally, (3.75) is the consequence of

$$\Box_i u = S_i^+ S_i^- u - c_i^2 (2 r^{-1} \partial_r u + r^{-2} \Omega u \cdot \Omega u).$$

Hence, it follows from (3.70), (3.73)–(3.75) that

$$|\Gamma^a N^i(\partial u, \partial^2 u)| \le C|c_i t - r|(1+t+r)^{-1} \sum_{|b|+|c|\le |a|+1} |\partial \Gamma^b u^i|\,|\partial \Gamma^c u^i| + C(1+t+r)^{-1} \sum_{\substack{|b|+|c|\le |a|+2 \\ |b|\ne 0}} |\Gamma^b u^i|\,|\partial \Gamma^c u^i| \tag{3.77}$$

for $|c_i t - r| < c_i t/2$. Here we have used Lemma 1.2 in [**8**]. Therefore, if $|c_i t - r| < c_i t/2$, the estimates (3.65), (3.66) and (3.77) yield

$$|\Gamma^a N^i(\partial u, \partial^2 u)| \le C_N (1+t+r)^{-3}(1+|c_i t - r|)^{-1}\{\log(2+t)\}^3(\varepsilon + [\partial u]_{[(N+3)/2],t} \|\partial u\|^2_{N+5,t}) \tag{3.78}$$

for $|a| \le N$, since $\varepsilon < 1$ and $[\partial u]_{[(N+3)/2],t} < 1$. In case $|c_i t - r| \ge c_i t/2$, we find from (3.66) that

$$\begin{aligned} |\Gamma^a N^i(\partial u, \partial^2 u)| &\le C_N \sum_{|b|+|c|\le N+1} |\partial \Gamma^b u^i|\,|\partial \Gamma^c u^i| \\ &\le C_N (1+r)^{-2}(1+t+r)^{-2}\{\log(2+t)\}^2(\varepsilon + [\partial u]_{[(N+3)/2],t} \|\partial u\|^2_{N+5,t}). \end{aligned} \tag{3.79}$$

Similarly, by (3.66)

$$\begin{aligned} |\Gamma^a R^i(\partial u, \partial^2 u)| &\le C_N \sum_{(j,k)\ne(i,i)} \sum_{|b|+|c|\le N+1} |\partial \Gamma^b u^j|\,|\partial \Gamma^c u^k| \\ &\le C_N\{(1+r)^{-2}(1+t+r)^{-2} \\ &\quad + \sum_{j\ne i}(1+t+r)^{-2}(1+|c_j t - r|)^{-2}\}\{\log(2+t)\}^2 \cdot \\ &\quad (\varepsilon + [\partial u]_{[(N+3)/2],t} \|\partial u\|^2_{N+5,t}), \end{aligned} \tag{3.80}$$

$$\begin{aligned} |\Gamma^a G^i(\partial u, \partial^2 u)| &\le C_N \sum_{1\le j,k,l\le m} \sum_{|b|+|c|+|d|\le N+1} |\partial \Gamma^b u^j|\,|\partial \Gamma^c u^k|\,|\partial \Gamma^d u^l| \\ &\le C_N\{(1+r)^{-3}(1+t+r)^{-3} \\ &\quad + \sum_{1\le j\le m}(1+t+r)^{-3}(1+|c_j t - r|)^{-3}\}\{\log(2+t)\}^3 \cdot \\ &\quad (\varepsilon + [\partial u]_{[(N+3)/2],t} \|\partial u\|^3_{N+5,t}), \end{aligned} \tag{3.81}$$

for $|a| \le N$.

Therefore, it follows from (3.78)–(3.81) that

$$|\Gamma^a F^i(\partial u, \partial^2 u)| \le C_N\{(1+t+r)^{-1} z^{(i)}_{1+\gamma,1+\kappa}(t,r)^{-1} + (1+t+r)^{-1}\sum_{j\neq i} z^{(j)}_{2,1-\rho}(t,r)^{-1} + (1+r)^{-1} z^{(m+1)}_{1+\gamma,1+\kappa}(t,r)^{-1}\}(\varepsilon + [\partial u]_{[(N+3)/2],t}\|\partial u\|^3_{N+5,t}) \tag{3.82}$$

for $0<\gamma<1$, $0<\kappa<1-\gamma$, $0<\rho<1$ and $|a|\le N$. (3.82) gives estimates of the right-hand side of the equations (3.58), and hence by Proposition 3.1 and (3.56),

$$|\partial\Gamma^a u^i(t,x)| \le C_N(1+r)^{-1}(1+|c_i t - r|)^{-1+\rho}(\varepsilon + [\partial u]_{[(N+4)/2],t}\|\partial u\|^3_{N+6,t}). \tag{3.83}$$

Using (3.83), we estimate $\Gamma^a R^i(\partial u, \partial^2 u)$ again. Then

$$|\Gamma^a R^i(\partial u, \partial^2 u)| \le C_N\{(1+r)^{-2}(1+t+r)^{-2+2\rho} + \sum_{j\neq i}(1+t+r)^{-2}(1+|c_j t - r|)^{-2+2\rho}\}(\varepsilon + [\partial u]_{[(N+5)/2],t}\|\partial u\|^6_{N+7,t}), \tag{3.84}$$

for $|a|\le N$. We take $\rho<1/2$ and replace the estimate (3.80) with (3.84). Then we have

$$|\Gamma^a F^i(\partial u, \partial^2 u)| \le C_N\{(1+t+r)^{-1} z^{(i)}_{1+\gamma,1+\kappa}(t,r)^{-1} + (1+t+r)^{-1}\sum_{j\neq i} z^{(j)}_{1+\gamma,1}(t,r)^{-1} + (1+r)^{-1} z^{(m+1)}_{1+\gamma,1+\kappa}(t,r)^{-1}\}(\varepsilon + [\partial u]_{[(N+5)/2],t}\|\partial u\|^6_{N+7,t}) \tag{3.85}$$

for some $0<\gamma$, $\kappa<1$. Then, applying Proposition 3.1 again, we have

$$|\partial\Gamma^a u^i(t,x)| \le C_N(1+r)^{-1}(1+|c_i t - r|)^{-1}(\varepsilon + [\partial u]_{[(N+6)/2],t}\|\partial u\|^6_{N+8,t}). \tag{3.86}$$

Consequently we have finished the proof.

4. Energy estimates.

PROPOSITION 4.1. *Let u be the solution of* (3.50)–(3.51). *Assume that $F^i(\partial u, \partial^2 u)$ satisfy* (1.6) *and c_i are different from each other. Assume moreover that $C^{ij}_{\alpha\beta}(\partial u)$ satisfy* (1.7) *and $[\partial u]_{[(N+9)/2],t} \le \omega_N$, where $\omega_N \le 1$ is a small number depending on N, c_i and given functions. Then there exists a constant C''_N, depending on N, c_i and given functions,* *such that*

$$\|\partial u\|_{N,t} \le C_N'' \varepsilon. \tag{4.1}$$

The proof of Theorem 1.1 is now easy. Since the Cauchy problem (1.1)–(1.2) can be converted to a Cauchy problem for a symmetric hyperbolic system, we know the local existence of the smooth solution to (1.1)–(1.2) (See Kato [**6**], Majda [**11**] Chapter 2). Uniqueness is proved by the application of the method in John [**5**], Appendix. Moreover, $u(t,\cdot)$ has compact support.

Take $C \ge \max\{C_{17}', C_{25}''\}$ large enough, so that $[\partial u]_{17,t}|_{t=0} \le C\varepsilon$. Then there exists $T > 0$ such that the solution to (1.1)–(1.2) in $[0,T]\times \boldsymbol{R}^3$ satisfies $[\partial u]_{17,T} \le 2C\varepsilon$ (Kato [**6**], Majda [**11**] Chapter 2). Set $\varepsilon_0 = \omega_{25}/2C^6$. Suppose that T_*, the maximal of T above, is finite for $0 \le \varepsilon < \varepsilon_0$. Then $[\partial u]_{17,t} \uparrow 2C\varepsilon$ as $t \uparrow T_*$. However, since $[\partial u]_{17,t} \le \omega_{25}$ for $t < T_*$, it follows from Proposition 3.2 and Proposition 4.1 that

$$\begin{aligned}[\partial u]_{17,t} &\le C\varepsilon + C^7\varepsilon^6\\ &\le C\varepsilon + C\varepsilon C^6\varepsilon_0\\ &\le (3/2)C\varepsilon\end{aligned}$$

for $t < T_*$. This is contradiction, and hence T_* cannot be finite. Therefore, by the corollaries to Theorem 2.2 in [**11**], the proof is complete.

Proof of Proposition 4.1. If $v = (v^1,\dots,v^m)$ satisfies

$$\sum_{0\le\alpha,\beta\le 3}\ \sum_{1\le j\le m} a_{\alpha\beta}^{ij}\partial_\alpha\partial_\beta v^j = b^i, \quad i = 1,\dots,m \tag{4.2}$$

with

$$a_{\alpha\beta}^{ij} = a_{\beta\alpha}^{ij} = a_{\alpha\beta}^{ji}, \tag{4.3}$$

then we have the energy identity

$$\begin{aligned}\sum_{0\le\alpha,\beta\le 3}\ \sum_{1\le i,j\le m} &\{\partial_\alpha(a_{\alpha\beta}^{ij}\partial_0 v^i\partial_\beta v^j) - \partial_\alpha a_{\alpha\beta}^{ij}\partial_0 v^i\partial_\beta v^j\\ &- 2^{-1}\partial_0(a_{\alpha\beta}^{ij}\partial_\alpha v^i\partial_\beta v^j) + 2^{-1}\partial_0 a_{\alpha\beta}^{ij}\partial_\alpha v^i\partial_\beta v^j\}\\ &= \sum_{1\le i\le m} b^i\partial_0 v^i,\end{aligned} \tag{4.4}$$

by multiplying both side of (4.2) by $\partial_0 v^i$. Integrating (4.4) on $[0,t]\times\boldsymbol{R}^3$, we obtain

$$2^{-1}\int_{\boldsymbol{R}^3}\langle \partial v(t),\partial v(t)\rangle\,dx-2^{-1}\int_{\boldsymbol{R}^3}\langle \partial v(0),\partial v(0)\rangle\,dx$$

$$=\int_0^t ds\int_{\boldsymbol{R}^3}\left\{\sum_{0\le\alpha,\beta\le3}\ \sum_{1\le i,j\le m}(\partial_\alpha a^{ij}_{\alpha\beta}\partial_0 v^i\partial_\beta v^j-2^{-1}\partial_0 a^{ij}_{\alpha\beta}\partial_\alpha v^i\partial_\beta v^j)+\sum_{1\le i\le m} b^i\partial_0 v^i\right\}dx, \tag{4.5}$$

where

$$\langle \partial v,\partial w\rangle=\sum_{1\le i,j\le m}\left(a^{ij}_{00}\partial_0 v^i\partial_0 w^j-\sum_{1\le k,l\le3}a^{ij}_{kl}\partial_k v^i\partial_l w^j\right). \tag{4.6}$$

We first set

$$a^{ij}_{\alpha\beta}=\eta^i_{\alpha\beta}\delta^{ij}-C^{ij}_{\alpha\beta}(\partial u), \tag{4.7}$$

$$b^i=\sum_{1\le j\le m}\ \sum_{0\le\alpha,\beta\le3}\{a^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^a u^j-\Gamma^a(a^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta u^j)\}+\Gamma^a D^i(\partial u), \tag{4.8}$$

where

$$(\eta^i_{\alpha\beta})_{0\le\alpha,\beta\le3}=\begin{pmatrix}1&0&0&0\\0&-c_i^2&0&0\\0&0&-c_i^2&0\\0&0&0&-c_i^2\end{pmatrix}. \tag{4.9}$$

Then each $a^{ij}_{\alpha\beta}$ satisfies (4.3), and $\Gamma^a u$ is a solution of (4.2). Therefore, it follows from (4.5) that

$$2^{-1}\int_{\boldsymbol{R}^3}\langle \partial\Gamma^a u(t),\partial\Gamma^a u(t)\rangle\,dx-2^{-1}\int_{\boldsymbol{R}^3}\langle \partial\Gamma^a u(0),\partial\Gamma^a u(0)\rangle\,dx$$

$$=\int_0^t ds\int_{\boldsymbol{R}^3}\left\{\sum_{0\le\alpha,\beta\le3}\ \sum_{1\le i,j\le m}(\partial_\alpha a^{ij}_{\alpha\beta}\partial_0\Gamma^a u^i\partial_\beta\Gamma^a u^j-2^{-1}\partial_0 a^{ij}_{\alpha\beta}\partial_\alpha\Gamma^a u^i\partial_\beta\Gamma^a u^j)\right.$$

$$\left.+\sum_{1\le i\le m} b^i\partial_0\Gamma^a u^i\right\}dx.$$

Noting

$$\sum_{1\le j\le m}\ \sum_{0\le\alpha,\beta\le3}\{a^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^a u^j-\Gamma^a(a^{ij}_{\alpha\beta}\partial_\alpha\partial_\beta u^j)\}$$

$$=[\Box_i,\Gamma^a]u^i-\sum_{1\le j,k\le m}\ \sum_{0\le\alpha,\beta\le3}\{C^{ij}_{\alpha\beta}(\partial u)\partial_\alpha\partial_\beta\Gamma^a u^j-\Gamma^a(C^{ij}_{\alpha\beta}(\partial u)\partial_\alpha\partial_\beta\Gamma^a u^j)\},$$

we obtain

$$2^{-1}\int_{\boldsymbol{R}^3}\langle\partial\Gamma^a u(t),\partial\Gamma^a u(t)\rangle\,dx-2^{-1}\int_{\boldsymbol{R}^3}\langle\partial\Gamma^a u(0),\partial\Gamma^a u(0)\rangle\,dx$$

$$\leq C_N\sum_{\substack{|b|+|c|\leq N+1\\|b|,|c|\neq 0}}\ \sum_{1\leq i,j,k\leq m}\int_0^t ds\int_{\boldsymbol{R}^3}|\Gamma^a\partial u^i|\,|\Gamma^b\partial u^j|\,|\Gamma^c\partial u^k|\,dx$$

$$\leq C_N[\partial u]_{[(N+1)/2],\,t}\int_0^t(1+s)^{-1}\|\partial u(s)\|_N^2\,ds$$

for $|a|\leq N$. Since

$$\sum_{|a|\leq N}\int_{\boldsymbol{R}^3}\langle\partial\Gamma^a u(t),\partial\Gamma^a u(t)\rangle\,dx\geq C_N\|\partial u(t)\|_N^2$$

if $[\partial u]_{0,\,t}$ is small, it follows that

$$\|\partial u(t)\|_N^2\leq C_N\left\{\varepsilon^2+[\partial u]_{[(N+1)/2],\,t}\int_0^t(1+s)^{-1}\|\partial u(s)\|_N^2\,ds\right\}.$$

Hence, by Gronwall's lemma,

$$\|\partial u(t)\|_N^2\leq C_N\varepsilon^2(1+t)^{C_N[\partial u]_{[(N+1)/2],\,t}}.\tag{4.10}$$

Next, we set

$$a^{ij}_{\alpha\beta}=\eta^i_{\alpha\beta}\delta^{ij},\tag{4.11}$$

$$b^i=[\Box_i,\Gamma^a]u^i+\Gamma^a F^i(\partial u,\partial^2 u).\tag{4.12}$$

Then by (4.5), we have

$$\|\partial u(t)\|_N^2\leq C_N\left(\varepsilon^2+\sum_{|a|,|b|\leq N}\ \sum_{1\leq i\leq m}\int_0^t ds\int_{\boldsymbol{R}^3}|\Gamma^b F^i(\partial u,\partial^2 u)|\,|\partial_0\Gamma^a u^i|\,dx\right).\tag{4.13}$$

Using (3.85) and (3.86), we have

$$|\Gamma^b F^i(\partial u,\partial^2 u)|\,|\partial_0\Gamma^a u^i|$$

$$\leq C_N\{(1+s+r)^{-1}z^{(i)}_{1+\gamma,\,1+\kappa}(s,r)^{-1}+(1+s+r)^{-1}\sum_{j\neq i}z^{(j)}_{1+\gamma,\,1}(s,r)^{-1}$$

$$+(1+r)^{-1}z^{(m+1)}_{1+\gamma,\,1+\kappa}(s,r)^{-1}\}(1+r)^{-1}(1+|c_is-r|)^{-1}(\varepsilon^2+[\partial u]_{[(N+6)/2],\,s}\|\partial u\|^{12}_{N+8,\,s})$$

$$\le C_N\{(1+s+r)^{-3-\kappa}(1+|c_is-r|)^{-2-\gamma}+(1+s+r)^{-4}\sum_{j\ne i}(1+|c_js-r|)^{-1-\gamma}$$

$$+(1+r)^{-3-\gamma}(1+s+r)^{-2-\kappa}\}(\varepsilon^2+[\partial u]_{[(N+6)/2],s}\|\partial u\|^{12}_{N+8,s})$$

$$\le C_N(1+s)^{-1-\kappa}\sum_{j=1}^{m+1}(1+|c_js-r|)^{-1-\gamma}r^{-2}(\varepsilon^2+[\partial u]_{[(N+6)/2],s}\|\partial u\|^{12}_{N+8,s}). \quad (4.14)$$

Moreover, by (4.10) and (4.14) it follows that

$$|\Gamma^b F^i(\partial u,\partial^2 u)|\,|\partial_0\Gamma^a u^i|$$

$$\le C_N(1+s)^{-1-\kappa}\sum_{j=1}^{m+1}(1+|c_js-r|)^{-1-\gamma}r^{-2}(\varepsilon^2+[\partial u]_{[(N+6)/2],s}\varepsilon^{12}(1+s)^{C_N[\partial u]_{[(N+9)/2],s}})$$

$$\le C_N\varepsilon^2(1+[\partial u]_{[(N+6)/2],s})(1+s)^{-1-\kappa+C_N[\partial u]_{[(N+9)/2],s}}\sum_{j=1}^{m+1}(1+|c_js-r|)^{-1-\gamma}r^{-2}. \quad (4.15)$$

Therefore, if $[\partial u]_{[(N+9)/2],t}\le 2^{-1}C_N^{-1}\kappa$, we obtain (4.1) from (4.13) and (4.15).

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