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GLOBAL EXISTENCE OF SMALL SOLUTIONS TO THE DAVEY-STEWARTSON AND THE ISHIMORI SYSTEMS

Nakao Hayashi*
Department of Mathematics, Faculty of Engineering, Gunma University, Kiryu 376, Japan

Jean-Claude Saut
Laboratoire d'Analyse Numérique, CNRS et Université de Paris-Sud, Bâtiment 425, 91405 Orsay, France

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Abstract. We study the initial-value problems for the Davey-Stewartson systems and the Ishimori equations. Elliptic-hyperbolic and hyperbolic-elliptic cases were treated by the inverse scattering techniques ([2–4, 10, 13–15, 32] for the Davey-Stewartson systems and [28, 29, 33] for the Ishimori equations). Elliptic-elliptic and hyperbolic-elliptic cases were studied (in [16, 17] for the Davey-Stewartson systems and [31] for the Ishimori equations) without the use of the inverse scattering techniques. Existence of a weak solution to the Davey-Stewartson systems for the elliptic-hyperbolic case is also obtained in [16] with a smallness condition on the data in L^2 and a blow-up result was also obtained for the elliptic-elliptic case. By using the sharp smoothing property of solutions to the linear Schrödinger equations the local existence of a unique solution to the Davey-Stewartson systems for the elliptic-hyperbolic and hyperbolic-hyperbolic cases was established in [30] in the usual Sobolev spaces with a smallness condition on the data. We prove the local existence of a unique solution to the Davey-Stewartson systems for the elliptic-hyperbolic and hyperbolic-hyperbolic cases in some analytic function spaces without a smallness condition on the data. Furthermore we prove existence of global small solutions of these equations for the elliptic-hyperbolic and hyperbolic-hyperbolic cases in some analytic function spaces.

1. Introduction. In this paper we study the initial-value problems for the Davey-Stewartson (D-S) systems

$$\begin{cases} i\partial_t u + c_0\partial_x^2 u + \partial_y^2 u = c_1|u|^2u + c_2 u\partial_x\varphi, \quad t,x,y\in\mathbb{R}, \\ \partial_x^2\varphi + c_3\partial_y^2\varphi = \partial_x|u|^2, \\ u(x,y,0) = u_0(x,y) \end{cases} \tag{1.1}$$

and the Ishimori equations

$$\begin{cases} i\partial_t u + \partial_x^2 u + c_4\partial_y^2 u \\ = c_5\frac{\bar{u}}{1+|u|^2}((\partial_x u)^2 + c_6(\partial_y u)^2) + c_7(\partial_x u\partial_y\varphi + \partial_y u\partial_x\varphi), \quad t,x,y\in\mathbb{R}, \\ \partial_x^2\varphi + c_8\partial_y^2\varphi = c_9\frac{(\partial_x u)(\partial_y\bar{u}) - (\partial_x\bar{u})(\partial_y u)}{(1+|u|^2)^2}, \\ u(x,y,0) = u_0(x,y), \end{cases} \tag{1.2}$$

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*Present address: Department of Applied Mathematics, Science University of Tokyo, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162, Japan.
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where $c_0, c_3, c_4, c_8 \in \mathbb{R}$ and $c_1, c_2, c_5, c_6, c_7, c_9 \in \mathbb{C}$.

The (D-S) systems were first derived by Davey-Stewartson ([11]) Benney-Roskes ([7]) and Djordjevic-Redekopp ([12]) and model the evolution of weakly nonlinear water waves that travel predominantly in one direction, but in which the wave amplitude is modulated slowly in horizontal directions. Independently Ablowitz and Haberman ([1]) and Cornille ([10]) obtained a particular form of (1.1) as an example of a completely integrable model which generalizes the one-dimensional Schrödinger equation. In [12] it was shown that the parameter c_3 can become negative when capillary effects are important.

When $(c_0, c_1, c_2, c_3) = (1, -1, 2, -1)$, $(-1, -2, 1, 1)$ or $(-1, 2, -1, 1)$ the system in (1.1) is referred to in the inverse scattering literature as the DSI , DSII defocusing and DSII focusing respectively. In these cases several results concerning the existence of solitons or lump solutions to the Cauchy problem have been established ([2–4, 10, 13–15, 32]) by the inverse scattering techniques. Using the inverse scattering method, Sung ([32]) has proved the global existence and uniqueness for the Cauchy problem associated to the DSII systems (for any initial data u_0 such that $\hat{u}_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ in the defocusing case, for small data in the same space in the focusing case). He proved moreover many regularity or qualitative properties, for instance that $u(\cdot, t)$ disperses to 0 in $L^\infty(\mathbb{R}^2)$ as $t \to \infty$. Beals and Coifman ([8]) have proved the global well-posedness of the Cauchy problem for the DSII defocusing system in the Schwartz class. The same result has been obtained by Fokas and Sung ([15]) for the DSI system.

As a matter of fact, cases where (1.1) is of inverse scattering type are exceptional. In [16] the IVP (initial value problem) (1.1) was studied and classified as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic according to the respective sign of (c_0, c_3): $(+,+), (+,-), (-,+)$ and $(-,-)$. For the elliptic-elliptic and hyperbolic-elliptic cases, local and global properties of solutions were studied in the usual Sobolev spaces L^2, H^1, H^2. They also established the global existence of a weak solution of the IVP (1.1) under a smallness condition on the data in L^2 norm for elliptic-hyperbolic case and a blow-up result for elliptic-elliptic case.

In [30] Linares and Ponce obtained results concerning local existence and uniqueness of solutions for the elliptic-hyperbolic and hyperbolic-hyperbolic cases.

In these cases one has to assume that $\varphi(\cdot)$ satisfies the radiation condition; i.e.,

$$\varphi(x, y, t) \to 0 \quad \text{as} \quad x + y \quad \text{and} \quad x - y \to +\infty$$

(without loss of generality we have taken $c_3 = -1$ in (1.1) (or $c_8 = -1$ in (1.2))). This guarantees that if $F \in L^1(\mathbb{R}^2)$, $\mathcal{K}^{-1}F$ is well defined, where $\mathcal{K}\varphi = (\partial_x^2 - \partial_y^2)\varphi = F$.

Thus, the IVP (1.1) is equivalent to

$$i\partial_t u + c_0 \partial_x^2 u + \partial_y^2 u = c_1 |u|^2 u + c_2 u \partial_x^2 \mathcal{K}^{-1} |u|^2, \quad u(x, y, 0) = u_0(x, y). \tag{1.3}$$

In the hyperbolic-hyperbolic case ($c_0 = -1$) after a rotation in the xy-plane and rescaling, the system (1.3) can be written as

$$\begin{aligned} &i\partial_t u + \partial_x \partial_y u = c_1 |u|^2 u + c_2 u \int_y^\infty \partial_x |u|^2(x, y')\, dy' + c_3 u \int_x^\infty \partial_y |u|^2(x', y)\, dx', \\ &u(x, y, 0) = u_0(x, y), \end{aligned} \tag{1.4}$$

where c_1, c_2, c_3 are arbitrary constants.

In the elliptic-hyperbolic case ($c_0 = 1$) after a rotation in the xy-plane and rescaling, the system (1.3) can be written as

$$\begin{aligned} &i\partial_t u + (\partial_x^2 + \partial_y^2)u = c_1|u|^2 u + c_2 u \int_y^\infty \partial_x |u|^2(x, y')\, dy' + c_3 u \int_x^\infty \partial_y |u|^2(x', y)\, dx', \\ &u(x, y, 0) = u_0(x, y), \end{aligned} \tag{1.5}$$

where c_1, c_2, c_3 are arbitrary constants.

Linares and Ponce ([30]) proved local well-posedness results for the IVP (1.4) (or (1.5)) for small data by making use of the sharp version of the Kato smoothing effect obtained in [23] for the group $\{e^{it\partial_x\partial_y}\}_{-\infty}^{\infty}$ and $\{e^{it(\partial_x^2+\partial_y^2)}\}_{-\infty}^{\infty}$.

We turn now to the Ishimori system. Y. Ishimori ([21]) proposed the following system:

$$\begin{cases} \partial_t S = S \wedge (\partial_x^2 S + c_0 \partial_y^2 S) + c_1(\partial_x \varphi \partial_y S + \partial_y \varphi \partial_x S) \\ \partial_x^2 \varphi + c_2 \partial_y^2 \varphi = c_3 S \cdot (\partial_x S \wedge \partial_y S) \\ S(x, y, 0) = S_0(x, y), \end{cases} \tag{1.6}$$

where $(c_0, c_1, c_2, c_3) = (-1, c_1, 1, 2)$ or $(1, c_1, -1, -2)$, $S(\cdot, t) : \mathbb{R}^2 \to \mathbb{R}^3$, $|S|^2 = 1$, $S \to (0, 0, 1)$ as $\sqrt{x^2 + y^2} \to \infty$ and $\wedge$ denotes the wedge product in $\mathbb{R}^3$. The IVP (1.6) is considered as a two-dimensional generalization of the Heisenberg equation in ferromagnetism. We put

$$S = (S_1, S_2, S_3) = \frac{1}{1 + |u|^2}(u + \bar{u}, -i(u - \bar{u}), 1 - |u|^2),$$

where $u : \mathbb{R}^2 \to \mathbb{C}$. Then it is clear that $S(\cdot, t) : \mathbb{R}^2 \to \mathbb{R}^3$, $|S|^2 = 1$, $u = (S_1 + iS_2)/(1 + S_3)$ and $S \to (0, 0, 1)$ as $\sqrt{x^2 + y^2} \to \infty$ if $u \to 0$ as $\sqrt{x^2 + y^2} \to \infty$. When $c_0 = 1, c_1 = 0$ (1.6) is reduced to the two-dimensional Heisenberg equation, which was studied by [5]. When $c_1 = 1$, (1.6) was studied formally in [28] and [29] by using the inverse scattering transform. By using the new variable u the Ishimori equations (1.6) can be written as

$$\begin{cases} i\partial_t u + \partial_x^2 u + c_0 \partial_y^2 u = 2\frac{\bar{u}}{1+|u|^2}((\partial_x u)^2 + c_0(\partial_y u)^2) + ic_1(\partial_x u \partial_y \varphi + \partial_y u \partial_x \varphi) \\ \partial_x^2 \varphi + c_2 \partial_y^2 \varphi = 2ic_3 \frac{(\partial_x u)(\partial_y \bar{u}) - (\partial_x \bar{u})(\partial_y u)}{(1+|u|^2)^2} \\ u(x, y, 0) = u_0(x, y) \end{cases} \tag{1.7}$$

which is a special case of (1.2).

Following the classification of the Davey-Stewartson systems used in [16], we classify the Ishimori equations (1.2) as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic according to the respective sign of

$(c_4, c_8) : (+, +), (+, -), (-, +)$ and $(-, -)$.

In [31] Soyeur studied the case $(c_4, c_5, c_6, c_8, c_9) = (-1, 2, -1, 1, 4i)$ for (1.2) which corresponds to a hyperbolic-elliptic case and obtained local well-posedness results and a global existence of small solutions. The method used in [31] can be applicable to the cases $(c_4, c_6, c_8) = (c, c, b)$ in (1.2) with $c \in \mathbb{R}, c \neq 0, b > 0$ which correspond to a hyperbolic-elliptic case ($c < 0$) and a elliptic-elliptic case ($c > 0$). However it seems that his method does not work for the case $c_4 \neq c_6$ in (1.2) because he used a multiplication factor $1/(1+|u|^2)$ to handle the nonlinear term $\bar{u}((\partial_x u)^2 + c_6(\partial_y u)^2)/(1+|u|^2)$ to which we can not apply a classical energy method directly. More precisely a multiplication factor $1/(1+|u|^2)$ translates (1.2) with $c_4 = c_6, c_8 > 0$ to another system to which we can apply a classical energy method.

In [33] Sung used the gauge equivalence between the integrable Ishimori system and the focusing DSII system to prove the global existence and uniqueness for the Cauchy problem associated to the integrable Ishimori system for small initial data and show regularity results (including existence in the Schwartz class).

In the same way as in the derivations of (1.4) and (1.5) we see that by using a rotation in the xy-plane and rescaling, the system (1.2) can be written as

$$\begin{cases} i\partial_t u + \partial_x\partial_y u = \frac{\bar{u}}{1+|u|^2}(c_1(\partial_x u)^2 + c_2\partial_x u\partial_y u + c_3(\partial_y u)^2) \\ \qquad +(c_4\partial_x u\partial_x\varphi + c_5\partial_x u\partial_y\varphi + c_6\partial_y u\partial_x\varphi + c_7\partial_y u\partial_y\varphi), \\ \partial_x\partial_y\varphi = c_8\frac{(\partial_x u)(\partial_y\bar{u}) - (\partial_x\bar{u})(\partial_y u)}{(1+|u|^2)^2} \end{cases} \tag{1.8}$$

in the hyperbolic-hyperbolic case ($c_4 = c_8 = -1$ in (1.2)),

$$\begin{cases} i\partial_t u + (\partial_x^2 + \partial_y^2) u = \frac{\bar{u}}{1+|u|^2}(c_1(\partial_x u)^2 + c_2\partial_x u\partial_y u + c_3(\partial_y u)^2) \\ \qquad +(c_4\partial_x u\partial_x\varphi + c_5\partial_x u\partial_y\varphi + c_6\partial_y u\partial_x\varphi + c_7\partial_y u\partial_y\varphi), \\ \partial_x\partial_y\varphi = c_8\frac{(\partial_x u)(\partial_y\bar{u}) - (\partial_x\bar{u})(\partial_y u)}{(1+|u|^2)^2} \end{cases} \tag{1.9}$$

in the elliptic-hyperbolic case ($c_4 = 1, c_8 = -1$ in (1.2)), where $c_1, \dots, c_8$ are arbitrary constants.

Thus (1.2) is equivalent to

$$i\partial_t u + \partial_x\partial_y u = F(u) + G(u), \quad u(x, y, 0) = u_0(x, y), \tag{1.10}$$

where

$$\begin{aligned} F(u) &= \frac{\bar{u}}{1+|u|^2}(c_1(\partial_x u)^2 + c_2\partial_x u\partial_y u + c_3(\partial_y u)^2) \\ G(u) &= c_4\partial_x u \int_y^\infty K(u, \partial_x u, \partial_y u)(x, y')\, dy' + c_5\partial_x u \int_x^\infty K(u, \partial_x u, \partial_y u)(x', y)\, dx' \\ &\quad + c_6\partial_y u \int_y^\infty K(u, \partial_x u, \partial_y u)(x, y')\, dy' + c_7\partial_y u \int_x^\infty K(u, \partial_x u, \partial_y u)(x', y)\, dx' \end{aligned}$$

with

$$K(u, \partial_x u, \partial_y u) = \frac{(\partial_x u)(\partial_y\bar{u}) - (\partial_x\bar{u})(\partial_y u)}{(1+|u|^2)^2}$$

in the hyperbolic-hyperbolic case, and

$$i\partial_t u + (\partial_x^2 + \partial_y^2)u = F(u) + G(u), \quad u(x, y, 0) = u_0(x, y) \tag{1.11}$$

in the case of elliptic-hyperbolic case, where $c_1, \dots, c_7$ are arbitrary constants. We note here that the L^p-L^q estimates, energy method which are used in nonlinear Schrödinger equations do not apply in (1.4), (1.5), (1.8), and (1.9) because these equations can be seen as nonlinear Schrödinger equations involving derivatives and nonlocal terms in the nonlinearity.

The purpose of this paper is to prove local existence results without a smallness assumption on the data for the IVP (1.4) and (1.5) and global existence of small solutions for the IVP (1.4), (1.5), (1.10), and (1.11) in some analytic function spaces.

To state our results precisely, we introduce

Notation and function spaces. Let X be a Banach space with norm $\|\cdot\|_X$ and $B = (B_1, \dots, B_j)$ be a vector field of derivations. The generalized Sobolev space $B^{m,p}$ is defined by

$$B^{m,p} = \{f \in L^p : \|f\|_{B^{m,p}} = \sum_{|\alpha| \le m} \|B^\alpha f\|_{L^p} < \infty\},$$

where $B^\alpha = B_1^{\alpha_1} \cdots B_j^{\alpha_j}$, $|\alpha| = \sum_{1 \le k \le j} \alpha_k, \alpha_k \in \mathbb{N} \cup \{0\}$. Let $A > 0$. We define a generalized analytic function space as follows:

$$G^A(B; X) = \{f \in X : \|f\|_{G^A(B;X)} = \sum_{\beta \in (\mathbb{N} \cup \{0\})^j} \frac{A^{|\beta|}}{\beta!} \|B^\beta f\|_X < \infty\}.$$

In order to state the results we introduce the first-order differential operators $J_x = x + 2it\partial_x$, $J_y = y + 2it\partial_y$, $J_1 = y + it\partial_x$, $J_2 = x + it\partial_y$, $\Omega_{xy} = x\partial_y - y\partial_x$ and $\Omega_{12} = x\partial_x - y\partial_y$.

By using these operators we define

$$\partial = (\partial_x, \partial_y), \quad R = (\partial, J_1, J_2), \quad \Gamma = (R, \Omega_{12}), \quad \tilde{R} = (\partial, J_x, J_y), \quad \tilde{\Gamma} = (\tilde{R}, \Omega_{xy}).$$

These operators together with the identity form a Lie algebra. It is also useful to rewrite J_x, J_y, J_1 and J_2 as follows:

$$\begin{aligned} J_x &= 2ite^{\frac{i|\tilde{x}|^2}{4t}} \partial_x e^{-\frac{i|\tilde{x}|^2}{4t}}, \quad & J_y &= 2ite^{\frac{i|\tilde{x}|^2}{4t}} \partial_y e^{-\frac{i|\tilde{x}|^2}{4t}}, \\ J_1 &= ite^{i\frac{xy}{t}} \partial_x e^{-i\frac{xy}{t}}, & J_2 &= ite^{i\frac{xy}{t}} \partial_y e^{-i\frac{xy}{t}}, \end{aligned}$$

where $|\tilde{x}|^2 = x^2 + y^2$. By a direct calculation we have the following commutation relations:

$$\begin{cases} [J_x, i\partial_t + \Delta] = [J_y, i\partial_t + \Delta] = [\Omega_{xy}, i\partial_t + \Delta] = 0, \quad \Delta = \partial_x^2 + \partial_y^2 \\ [J_1, i\partial_t + \partial_x\partial_y] = [J_2, i\partial_t + \partial_x\partial_y] = [\Omega_{12}, i\partial_t + \partial_x\partial_y] = 0. \end{cases}$$

We use the standard notation $H^{m,p}$ instead of $\partial^{m,p}$. With this notation we state the main results of this paper.

Theorem 1. *We assume that* $u_0 \in G^A(\partial; H^{m,2})$, *where* $A > 0$ *and* $m \geq 3$. *Then there exists a unique local solution* u *of* (1.4) (*or* (1.5)) *and a positive constant* T *such that*

$$u(t,x) \in C([-T,T]; G^{A_1}(\partial; H^{m,2})),$$

where $A_1 < A$.

Remark 1. Typical examples of data satisfying the condition of Theorem 1 are given by $\frac{1}{1+x^2+y^2}$, $e^{-x^2-y^2}$.

Theorem 2. *We assume that* $u_0 \in G^A(\partial; H^{m,2})$, $\|u_0\|_{G^A(\partial;H^{m,2})} < \frac{1}{2}$, *where* $A > 0$ *and* $m \geq 4$. *Then the same result as in Theorem* 1 *holds for the IVP* (1.10) (*or* (1.11)).

Theorem 3. *We assume that* $u_0 \in G^A(R(0); R^{m,2}(0))$, $\|u_0\|_{G^A(R(0);R^{m,2}(0))} < \epsilon$, *where* $A > 0$, $m \geq 3$ *and* ϵ *is a sufficiently small positive constant. Then there exists a unique global solution* u *of* (1.4) *such that*

$$u(t,x) \in G^{A_1}(R(t); R^{m,2}(t)) \quad \textit{for any } t,$$

where $A_1 < A$.

Theorem 3′. *The result of Theorem* 3 *holds true for the IVP* (1.5) *under the hypotheses of Theorem* 3 *with* R *replaced by* $\tilde{R}$.

Theorem 4. *The result of Theorem* 3 *holds true with* $m \geq 5$ *for the IVP* (1.10) (*respectively* (1.11)), *provided we replace in the hypotheses,* R *by* Γ (*respectively* R *by* $\tilde{\Gamma}$).

Remark 2. A typical example of data satisfying the conditions of Theorems 3, 3′ and 4 is given by $\epsilon e^{-x^2-y^2}$. Thus our global results require an exponential decay condition on the data. Our results also imply a smoothing effect of solutions in some sense. We explain this point herein by using Theorem 3. We take the initial function as follows:

$$u_0(x,y) = \frac{1}{1+x^2+y^2} \epsilon e^{-x^2-y^2}$$

which satisfies the condition of Theorem 3 if ϵ is sufficiently small and $A < 1$. On the other hand the solution u constructed in Theorem 3 satisfies the estimate

$$\begin{aligned}\|e^{-ixy/t}u(t)\|_{G^{|t|A_1}(\partial;L^2)} &= \|e^{-ixy/t}u(t)\|_{G^{A_1}(|t|\partial;L^2)} \leq \|u(t)\|_{G^{A_1}(J;L^2)} \\ &\leq \|u(t)\|_{G^{A_1}(R;L^2)} < \infty.\end{aligned}$$

Hence the analyticity domain of the solution determined by $|t|A_1$ exceeds the analyticity domain of the initial function determined by A when $|t|A_1 > A$. This type of smoothing effect was stated in [19].

Corollary 5. *Let u be the solution constructed in Theorem $3, 3'$ and 4. Then we have*

$$\|u(t)\|_{G^{A_1}(\partial;L^\infty)} \le C(1+|t|)^{-1}\|u_0\|_{G^A(B(0);B^{m,2}(0))},$$

where $B = R$, $\tilde{R}$, Γ, or $\tilde{\Gamma}$. Moreover, for any u_0 as above there exist u^-, u^+ such that

$$\|u(t) - U(t)u^\mp\|_{G^{A_1}(\partial;L^2)} \to 0 \quad \text{as} \quad t \to \mp\infty,$$

where $U(t) = e^{it\partial_x\partial_y}$ or $e^{it(\partial_x^2+\partial_y^2)}$.

Remark 3. The rate of decay obtained in Corollary 5 is the same as that of solutions to linear Schrödinger equations. Time decay of solutions for the Davey-Stewartson systems in the elliptic-elliptic and hyperbolic-elliptic cases was obtained in [9] and [35] with the decay rate $(1+|t|)^{-1}$. For the Ishimori equations in the hyperbolic-elliptic case, the time decay of solutions was obtained in [31] with the decay rate $(1+|t|)^{-2/3}$. The time decay estimates of solutions to these equations for the hyperbolic-hyperbolic and elliptic-hyperbolic cases seem to be new.

Remark 4. For general c_0 or c_4, after a rotation in xy-plane and rescaling, the linear part of the IVP (1.1) or (1.2) can be written as, when $c_3, c_8 < 0$,

$$i\partial_t u - \frac{1}{4}((b+1)(\partial_x^2+\partial_y^2) + 2(b-1)\partial_x\partial_y)u, \quad \partial_y\partial_x\varphi,$$

where $b = c_0$ or $b = c_4$. The results mentioned above are still valid if we use

$$J_{bx} = ((b+1)x - (b-1)y) - 2itb\partial_x, \quad J_{by} = ((b+1)y - (b-1)x) - 2itb\partial_y,$$

$$\Omega_{bxy} = (b+1)(x\partial_y - y\partial_x) + (b-1)(x\partial_x - y\partial_y)$$

instead of J_x, J_y, Ω_{xy} or J_1, J_2, Ω_{12}. It is easy to check the following computation properties:

$$[J_{bx}, L] = [J_{by}, L] = [\Omega_{bxy}, L] = 0$$

with

$$L = i\partial_t - \frac{1}{4}((b+1)(\partial_x^2+\partial_y^2) + 2(b-1)\partial_x\partial_y)$$

and J_{bx}, J_{by} are rewritten as follows:

$$J_{bx} = -2itb\exp(-\frac{i}{4t}\phi(x,y))\partial_x\exp(\frac{i}{4t}\phi(x,y)),$$
$$J_{by} = -2itb\exp(-\frac{i}{4t}\phi(x,y))\partial_y\exp(\frac{i}{4t}\phi(x,y)),$$

where

$$\phi(x,y) = (b+1)(x^2+y^2) - 2(b-1)xy.$$

In order to obtain the global existence of solutions to (1.4), (1.5), (1.10) and (1.11), the form of nonlinear terms is important since they involve nonlocal nonlinear terms.

It is worth noticing that two-dimensional nonlinear Schrödinger equations with a cubic nonlocal nonlinear term of the form

$$u\int_y^\infty |u|^2(x,y')\,dy'$$

is similar to one-dimensional nonlinear Schrödinger equations with a cubic nonlinear term $|u|^2u$. We explain this point herein by considering the nonlinear Schrödinger equations

$$i\partial_t u+\Delta u=u\int_y^\infty |u|^p(x,y')\,dy',\quad u(x,y,0)=u_0(x,y),\quad t,x,y\in\mathbb{R}. \tag{1.12}$$

Global existence of solutions to (1.12) can be obtained if $p>2$ since the nonlinear term $F(u,\bar u)$ satisfies the condition that

$$F(e^{i\theta}u,\overline{e^{i\theta}u})=e^{i\theta}F(u,\bar u) \tag{1.13}$$

for any $\theta\in\mathbb{R}$, which allows us to use the chain rule for the operators J_x, J_y, which behave like usual derivatives. We have by Hölder's and Sobolev's inequalities

$$\begin{aligned}\|u\int_y^\infty |u|^p(x,y')\,dy'\|_{L_y^2L_x^2}&\le \|u\|^2_{L_y^2L_x^\infty}\|u\|^{p-2}_{L^\infty}\|u\|_{L^2}\le C|t|^{-1}\|J_xu\|_{L^2}\|u\|^{p-2}_{L^\infty}\|u\|^2_{L^2}\\&\le C|t|^{-(p-1)}\|J_xu\|_{L^2}\sum_{|\alpha|\le 2}\|J^\alpha u\|^{p-2}_{L^2}\|u\|^2_{L^2}\end{aligned} \tag{1.14}$$

and

$$\|u\int_y^\infty |u|^p(x,y')\,dy'\|_{L_y^2L_x^2}\le C\sum_{|\alpha|\le 2}\|\partial^\alpha u\|^{p+1}_{L^2},$$

where we have used the notation

$$\|u\|_{L_y^pL_x^q}=\Big(\int_{\mathbb{R}}\|u(\cdot,y)\|^p_{L^q(\mathbb{R})}\,dy\Big)^{1/p}$$

and we also have used the following Sobolev-type inequalities:

$$\|u\|_{L_y^2L_x^\infty}\le C|t|^{-1/2}\|u\|^{1/2}_{L^2}\|J_xu\|^{1/2}_{L^2}\quad\text{and}\quad \|u\|_{L^\infty}\le C|t|^{-1}\|u\|^{1/2}_{L^2}\Big(\sum_{|\alpha|=2}\|J^\alpha u\|_{L^2}\Big)^{1/2}$$

which are obtained by applying the usual Sobolev's inequality to $e^{-\frac{i|\tilde x|^2}{4t}}u$.

When $p=2$ the above time decay estimate (1.14) corresponds to the estimate of the cubic nonlinear term $|u|^2u$ in one space dimension, namely

$$\||u|^2u\|_{L_x^2}\le\|u\|^2_{L_x^\infty}\|u\|_{L_x^2}\le C|t|^{-1}\|J_xu\|_{L_x^2}\|u\|^2_{L_x^2}.$$

Thus we can not expect global results for the nonlinear Schrödinger equation with nonlinearity

$$u\int_y^\infty |u|^2(x,y')\,dy' \quad \text{or} \quad \partial_x u\int_y^\infty \frac{(\partial_x u)(\partial_y \bar{u})}{1+|u|^2}(x,y')\,dy'.$$

The reason why we can expect the global existence of solutions to nonlinear Schrödinger equations with nonlinear terms

$$u\int_y^\infty \partial_x|u(x,y')|^2\,dy' \ \text{ and } \ \partial_x u\Big(\int_y^\infty \frac{(\partial_x u)(\partial_y \bar{u})-(\partial_x \bar{u})(\partial_y u)}{1+|u|^2}(x,y')\,dy'\Big)$$

comes from the following identities:

$$u\int_y^\infty \partial_x|u|^2(x,y')\,dy' = \begin{cases} \frac{1}{it}u\int_y^\infty((J_1u)\bar{u}-(\overline{J_1u})u)(x,y')\,dy' \\ \frac{1}{2it}u\int_y^\infty((J_xu)\bar{u}-(\overline{J_xu})u)(x,y')\,dy' \end{cases}$$

and

$$\partial_x u\Big(\int_y^\infty \frac{(\partial_x u)(\partial_y \bar{u})-(\partial_x \bar{u})(\partial_y u)}{1+|u|^2}(x,y')\,dy'\Big)$$
$$= \begin{cases} \frac{1}{it}\partial_x u(\int_y^\infty \frac{(\partial_x u)(\overline{J_2u})-(\overline{J_1u})(\partial_y u)+(\Omega_{12}u)\bar{u}}{1+|u|^2}(x,y')\,dy') \\ \frac{1}{2it}\partial_x u(\int_y^\infty \frac{(\partial_x u)(\overline{J_yu})-(\overline{J_xu})(\partial_y u)+(\Omega_{xy}u)\bar{u}}{1+|u|^2}(x,y')\,dy'). \end{cases}$$

From these identities we see that we can get the time decay of order one by expressing ∂_x, ∂_y in terms of $J_1, J_2, J_x, J_y, \Omega_{12}$ and Ω_{xy}. These identities remind us of the null condition introduced by S. Klainerman ([27]) to prove a global existence theorem to quadratic nonlinear wave equations in three space dimensions. We will use these expressions and analytical conditions on the initial data to get our results. We note here that analytic function spaces similar to $G^A(\partial; H^{m,2})$ were used in [6], [18] and [24] to prove the local existence of solutions to various nonlinear evolution equations with nonlinear terms involving first-order space derivatives of unknown functions. Their method and our method do not apply to the higher order nonlinear dispersive equations considered in [25] and [26].

The null gauge condition for nonlinear Schrödinger equations was introduced in [34] in the case of one space dimension. In the usual weighted Sobolev spaces, global existence of small solutions to nonlinear Schrödinger equations with cubic power nonlinearities,

$$i\partial_t u + \partial_x^2 u = (\partial_x|u|^2)(\lambda u + \mu\partial_x u), \quad \lambda,\mu\in\mathbb{C},$$

was obtained in [22] in one space dimension.

The nonlocal nonlinear term appearing (1.4) or (1.5) behaves like the nonlinear term $(\partial_x|u|^2)u$ since $(\partial_x|u|^2)u$ can be written as $\frac{1}{it}(J_xu\cdot\bar{u}-\overline{J_xu}\cdot u)u$.

The proof given by [22], [34] cannot be applied to systems of nonlinear Schrödinger equations with cubic nonlinearities such as

$$i\partial_t u_j + \partial_x^2 u_j = \Big(\partial_x \sum_{k=1}^{n} \lambda_k |u_k|^2 (\sum_{k=1}^{n} (\lambda_k u_k + \mu_k \partial_x u_k)\Big), \quad t, x \in \mathbb{R},$$

where $\lambda_j, \mu_j \in \mathbb{C}, j = 1, 2, \dots, n$. On the other hand, our proof of Theorem 3 could be applied to such systems.

For the Ishimori equations we have to assume a smallness condition on the initial data, because $1/(1+x^2)$ has a singularity at $x = i$. More precisely, the analytic continuation of the term $1/(1+|u|^2)$ occurring in the nonlinear terms of the Ishimori equations (1.10) (or (1.11)), can be written as $1/(1+U(z)\overline{U(\bar{z})})$ which may have a singularity. Hence a smallness condition on the data is needed if we consider the problem in analytic function spaces. The problem of local existence of solutions for (1.10) (or (1.11)) without a smallness condition on the data is still open.

2. Preliminaries.

In this section we prepare important estimates of the main nonlinear terms which are needed to obtain our results.

For simplicity we use the following notation:

$$f_1(u) = u \int_y^\infty \partial_x |u|^2 (x, y')\, dy', \quad f_2(u) = \frac{\bar{u}}{1+|u|^2} (\partial_x u)^2,$$
$$f_3(u) = \partial_x u \int_y^\infty \frac{(\partial_x u)(\partial_y \bar{u}) - (\partial_x \bar{u})(\partial_y u)}{(1+|u|^2)^2} (x, y')\, dy'.$$

Lemma 2.1. *We have for* $j = 1, 2, 3$

$$\|f_j(u)\|_{G^A(B(t);B^{m,2}(t))} \le C \|u\|^2_{G^A(B(t);B^{m,2}(t))} \|u\|_{G^A(B(t);B^{m+1,2}(t))}$$

provided that

$$\|u\|_{G^A(B(t);B^{m,2}(t))} < 1, \quad j = 2, 3,$$

where $m \ge 3$ *for* $j = 1$, $m \ge 4$ *for* $j = 2, 3$, $B(t)$ *is one of the operators* $\partial, R, \Gamma, \tilde{R}$ *and* $\tilde{\Gamma}$.

Proof. Since $f_j(u)$ satisfies the gauge condition (1.13), the operators J_x, J_y, J_1 and J_2 act on $f_j(u)$ like a derivative ∂_x or ∂_y. By Lemma A.1 and Leibniz's formula

$$\|f_1(u)\|_{G^A(B(t);B^{m,2}(t))} \le C \sum_{|\alpha| \le m} \|B^\alpha \big(u \int_y^\infty \partial_x |u|^2\, dy'\big)\|_{G^A(B(t);L^2)}$$
$$\le C \sum_{|\alpha| \le m} \sum_{\substack{\beta \le \alpha \\ \gamma \le \beta}} \binom{\alpha}{\beta} \binom{\beta}{\gamma} \big(\|(B^{\alpha-\beta} u) \int_y^\infty (B^{\beta-\gamma} \partial_x u)(\overline{B^\gamma u})\, dy'\|_{G^A(B(t);L^2)}$$
$$+ \|(B^{\alpha-\beta} u) \int_y^\infty (\overline{B^{\beta-\gamma} \partial_x u})(B^\gamma u)\, dy'\|_{G^A(B(t);L^2)}\big).$$

In what follows we will suppress t in $B(t)$ and $B^{m,2}(t)$ throughout the proofs of the results. We apply Lemma A.2 to the previous inequality to get

$$\|f_1(u)\|_{G^A(B,B^{m,2})} \leq C\sum_{|\alpha|\leq m}\sum_{\substack{\beta\leq\alpha\\ \gamma\leq\beta}}\|B^{\alpha-\beta}u\|_{G^A(B;L^2_yL^\infty_x)}\|B^{\beta-\gamma}\partial_x u\|_{G^A(B;L^2)}\|B^\gamma u\|_{G^A(B;L^2_yL^\infty_x)}. \tag{2.1}$$

By Sobolev's inequality and Lemma A.1 the right-hand side of (2.1) is bounded by

$$C\sum_{|\alpha|\leq m}\sum_{\substack{\beta\leq\alpha\\ \gamma\leq\beta}}\Big(\sum_{|\delta|\leq 1}\|B^{\alpha-\beta+\delta}u\|_{G^A(B;L^2)}\Big)\Big(\sum_{|\delta|\leq 1}\|B^{\beta-\gamma+\delta}u\|_{G^A(B;L^2)}\Big)\Big(\sum_{|\delta|\leq 1}\|B^{\gamma+\delta}u\|_{G^A(B;L^2)}\Big). \tag{2.2}$$

If $a,b,c\in\mathbb{N}$ and $a+b+c\leq m$, then at least two of them are less than or equal to $m/2$. Therefore we have by (2.1) and (2.2)

$$\|f_1(u)\|_{G^A(B;B^{m,2})}\leq C\|u\|^2_{G^A(B;B^{m,2})}\|u\|_{G^A(B;B^{m+1,2})}. \tag{2.3}$$

This implies the lemma for $j=1$.

We next prove the lemma for $j=2$. By the Taylor expansion we have for $|a|<1$

$$\frac{1}{1+a}=\sum_{n=0}^\infty(-1)^n a^n.$$

In the same way as in the proof of (2.3) we have by Lemma A.3

$$\begin{aligned}
\|f_2(u)\|_{G^A(B;B^{m,2})} &\leq C\sum_{\substack{\beta\leq\alpha\\ \gamma\leq\beta}}\|(\overline{B^{\alpha-\beta}}\frac{\bar u}{1+|u|^2})(B^{\beta-\gamma}\partial_x u)(B^\gamma\partial_x u)\|_{G^A(B;L^2)}\\
&\leq C\|u\|_{G^A(B;B^{m,2})}(\|\frac{\bar u}{1+|u|^2}\|_{G^A(\bar B;B^{m,2})}\|u\|_{G^A(B;B^{m,2})}\\
&\quad+\|\frac{\bar u}{1+|u|^2}\|_{G^A(\bar B;B^{m-1,2})}\|u\|_{G^A(B;B^{m+1,2})})\\
&\leq C\|u\|_{G^A(B;B^{m,2})}(\sum_{n=0}^\infty\|\bar u|u|^{2n}\|_{G^A(\bar B;B^{m,2})}\|u\|_{G^A(B;B^{m,2})}\\
&\quad+\sum_{n=0}^\infty\|\bar u|u|^{2n}\|_{G^A(\bar B;B^{m-1,2})}\|u\|_{G^A(B;B^{m+1,2})})\\
&\leq C\|u\|^2_{G^A(B;B^{m,2})}\|u\|_{G^A(B;B^{m+1,2})}\sum_{n=0}^\infty\|u\|^{2n}_{G^A(B;B^{m,2})}
\end{aligned}$$

from which the lemma for $j=2$ follows.

The lemma for $j=3$ follows by combining the proof for $j=1$ and the proof for $j=2$.

We will need the following time decay estimates.

Lemma 2.2. *We have for* $j = 1, 2, 3$

$$\|f_j(u)\|_{G^A(B(t);B^{m,2}(t))} \le C(1+|t|)^{-2}\|u\|^2_{G^A(B(t);B^{m,2}(t))}\|u\|_{G^A(B(t);B^{m+1,2}(t))}$$

provided that

$$\|u\|_{G^A(B(t);B^{m,2}(t))} < 1 \quad \text{for} \quad j = 2, 3,$$

where $m \ge 3$ *for* $j = 1$, $m \ge 5$ *for* $j = 2, 3$, *and* $B(t)$ *is one of the operators* $R, \Gamma, \tilde{R}$ *and* $\tilde{\Gamma}$.

Proof. By Lemma 2.1 it is sufficient to prove that

$$\|f_j(u)\|_{G^A(B;B^{m,2})} \le Ct^{-2}\|u\|^2_{G^A(B;B^{m,2})}\|u\|_{G^A(B;B^{m+1,2})}. \tag{2.4}$$

We have

$$\partial_x|u|^2 = \begin{cases} \frac{1}{it}(\bar{u}J_1u - u\overline{J_1u}), \\ \frac{1}{2it}(\bar{u}J_xu - u\overline{J_xu}). \end{cases} \tag{2.5}$$

Hence in the same way as in the proof of (2.1)

$$\begin{aligned} \|f_1(u)\|_{G^A(B;B^{m,2})} \le C|t|^{-1} &\sum_{|\alpha|\le m}\sum_{\substack{\beta\le\alpha \\ \gamma\le\beta}} \|B^{\alpha-\beta}u\|_{G^A(B;L^2_yL^\infty_x)} \\ &\times \Big(\sum_{|\delta|\le 1}\|B^{\beta-\gamma+\delta}u\|_{G^A(B;L^2)}\Big)\|B^\gamma u\|_{G^A(B;L^2_yL^\infty_x)}. \end{aligned} \tag{2.6}$$

By Sobolev's inequality we can see that

$$\|u\|_{L^2_yL^\infty_x} \le C|t|^{-1/2}\begin{cases} \|J_xu\|^{1/2}_{L^2}\|u\|^{1/2}_{L^2}, \\ \|J_1u\|^{1/2}_{L^2}\|u\|^{1/2}_{L^2}. \end{cases} \tag{2.7}$$

We apply (2.7) and Lemma A.1 to the right-hand side of (2.6) to get (2.4) for $j = 1$.

In the same manner as in the proof of Lemma 2.1 for $j = 2$ we have (2.4) for $j = 2$ by using

$$\|u\|_{G^A(B;B^{m-1,\infty})} \le C|t|^{-1}\|u\|_{G^A(B;B^{m+1,2})}$$

which is obtained by Sobolev's inequality

$$\|u\|_{L^\infty} \le C|t|^{-1}\sum_{|\alpha|\le 2}\begin{cases} \|R^\alpha u\|_{L^2}, \\ \|\tilde{R}^\alpha u\|_{L^2}. \end{cases}$$

The proof for $j = 3$ follows from the identity

$$\partial_xu\partial_y\bar{u} - \partial_x\bar{u}\partial_yu = \begin{cases} \frac{1}{it}((\partial_xu)(\overline{J_2u}) - (\partial_yu)(\overline{J_1u}) + (\Omega_{12}u)\bar{u}), \\ \frac{1}{2it}((\partial_xu)(\overline{J_yu}) - (\partial_yu)(\overline{J_xu}) + (\Omega_{xy}u)\bar{u}) \end{cases}$$

and the same arguments as those used in the proof of (2.4) for j=1. This completes the proof of Lemma 2.2. □

By proofs similar to that of Lemma 2.1 and Lemma 2.2 we have

Lemma 2.3. *We have for* $j = 1, 2$

$$\begin{aligned}&\|F_j(u_1) - F_j(u_2)\|_{G^A(B(t);B^{m,2}(t))}\\ &\le C(1+|t|)^{-2}\sum_{k=1}^{2}\|u_k\|_{G^A(B(t);B^{m,2}(t))}\big(\|u_1-u_2\|_{G^A(B(t);B^{m,2}(t))}\sum_{k=1}^{2}\|u_k\|_{G^A(B(t);B^{m+1,2}(t))}\\ &\quad + \|u_1-u_2\|_{G^A(B(t);B^{m+1,2}(t))}\sum_{k=1}^{2}\|u_k\|_{G^A(B(t);B^{m,2}(t))}\big)\end{aligned}$$

provided that

$$\|u_k\|_{G^A(B(t);B^{m,2}(t))} \le 1 \quad \textit{for} \quad j = 2,$$

where

$$\begin{aligned}&F_1(u) = c_1|u|^2u + c_2u\int_y^\infty \partial_x|u|^2(x,y')\,dy' + c_3u\int_x^\infty \partial_y|u|^2(x',y)\,dx',\\ &F_2(u) = F(u) + G(u) \quad (\textit{see (1.10)}),\end{aligned}$$

$m \ge 3$ *for* $j = 1$, $m \ge 5$ *for* $j = 2$ *and* $B(t)$ *is one of the operators* $R, \Gamma, \tilde{R}, \tilde{\Gamma}$.

Lemma 2.4. *The result of Lemma* 2.3 *holds true for* $B(t) = \partial$, *when* $(1+|t|)^{-2}$ *is replaced by* 1.

3. Proofs of theorems. In this section we prove the main results of this paper. First we introduce the function spaces

$$X_T = \{f \in C([0,T]; H^{m,2}) : |||f|||_{X_T} < \infty\},$$

where

$$|||f|||_{X_T} = \sup_{t\in[0,T]}\|f(t)\|_{G^{A(t)}(B(t);B^{m,2}(t))} - \sum_{|\alpha|=1}\int_0^T \|B^\alpha f(t)\|_{G^{A(t)}(B(t);B^{m,2}(t))}A'(t)\,dt,$$

and where $B(t) = \partial, R, \tilde{R}, \Gamma$ or $\tilde{\Gamma}$ and $A(t)$ is a decreasing function with respect to t to be determined later.

Proof of Theorem 1. We prove Theorem 1 by using the classical contraction mapping principle in X_T with $B(t) = \partial$. For any $v \in X_T$ we consider the linearized equation of (1.4) (or (1.5))

$$i\partial_t u + \partial_x\partial_y u = F_1(v), \quad u(x,y,0) = u_0(x,y), \quad t,x,y \in \mathbb{R}, \tag{3.1}$$

or

$$i\partial_t u + \Delta u = F_1(v), \quad u(x,y,0) = u_0(x,y), \quad t,x,y \in \mathbb{R}, \tag{3.2}$$

where F_1 was defined in Lemma 2.3. We define the mapping M by $u = Mv$. It is sufficient to prove that M is a contraction mapping from $X_{T,\rho} = \{f \in X_T : |||f|||_{X_T} \le$

$\rho\}$ into itself for some time T. Applying ∂^α to both sides of (3.1), using the usual energy method we obtain

$$\frac{d}{dt}\|\partial^\alpha u(t)\|_{H^{m,2}} \le \|\partial^\alpha F_1(v)\|_{H^{m,2}}. \tag{3.3}$$

We multiply both sides of (3.3) by $A(t)^{|\alpha|}/\alpha!$ and make a summation to get

$$\frac{d}{dt}\|u(t)\|_{G^{A(t)}(\partial;H^{m,2})} - A'(t)\sum_{|\alpha|=1}\|\partial^\alpha u(t)\|_{G^{A(t)}(\partial;H^{m,2})} \le \|F_1(v)\|_{G^{A(t)}(\partial;H^{m,2})}, \tag{3.4}$$

where $A(t) = A\exp(-t/\epsilon)$ and ϵ is a sufficiently small positive constant. Integrating in t and using Lemma 2.4, we get

$$\begin{aligned} |||u|||_{X_T} &\le \|u_0\|_{G^A(\partial;H^{m,2})} + C\rho^2\int_0^T \|u(t)\|_{G^{A(t)}(\partial;H^{m+1,2})}\,dt \\ &\le \|u_0\|_{G^A(\partial;H^{m,2})} + C\rho^3(T + \frac{\epsilon}{A}e^{\frac{T}{\epsilon}}) \le \rho \end{aligned} \tag{3.5}$$

if we take

$$\|u_0\|_{G^A(\partial;H^{m,2})} \le \frac{\rho}{2}, \quad C\rho^2 T \le \frac{1}{4}, \quad T \le \epsilon \le \frac{A}{4C\rho^2 e}.$$

In the same way as in the proof of (3.5) we get, by Lemma 2.4,

$$|||u_1 - u_2|||_{X_T} \le C\rho^2(T + \frac{\epsilon}{A}e^{\frac{T}{\epsilon}})|||v_1 - v_2|||_{X_T} \le \frac{1}{2}|||v_1 - v_2|||_{X_T}, \tag{3.6}$$

where $u_j = Mv_j$. From (3.5) and (3.6) we have Theorem 1.

We omit the proof of Theorem 2 since it is similar to that of Theorem 1.

Proof of Theorem 3. As noted in the introduction, the operators R and $\tilde{R}$ commute with the linear part of (3.1) and (3.2), respectively. Hence in the same way as in the proof of (3.4) we see that there exists a positive constant C such that

$$\frac{d}{dt}\|u(t)\|_{G^{A(t)}(R;R^{m,2}(t))} - CA'(t)\sum_{|\alpha|=1}\|R^\alpha u(t)\|_{G^{A(t)}(R;R^{m,2}(t))} \le \|F_1(v)\|_{G^{A(t)}(R;R^{m,2}(t))}, \tag{3.7}$$

where

$$A(t) = A_1(1 + \frac{A - A_1}{A_1}(\log(e+t))^{-\epsilon})$$

and we have used Lemma A.1 to derive the second term of the left-hand side. We integrate (3.7) with respect to t and use Lemma 2.3 to get

$$\begin{aligned} |||u|||_{X_\infty} &\le C\|u_0\|_{G^A(R(0):R^{m,2}(0))} + C\rho^2\int_0^\infty (1+t)^{-2}\|u(t)\|_{G^{A(t)}(R(t);R^{m+1,2}(t))}\,dt \\ &\le C(\|u_0\|_{G^A(R(0):R^{m,2}(0))} + \rho^3(\sup_{t\in\mathbb{R}^+}(1+t)^{-2}(-A'(t))^{-1} + 1)) \\ &\le C(\|u_0\|_{G^A(R(0):R^{m,2}(0))} + \rho^3) \le \rho \end{aligned} \tag{3.8}$$

if we take

$$C\|u_0\|_{G^A(R(0):R^{m,2}(0))} \le \frac{\rho}{2}, \quad C\rho^2 \le \frac{1}{2}.$$

In the same way as in the proof of (3.8) we have by Lemma 2.3

$$|||u_1 - u_2|||_{X_\infty} \le C\rho^2 |||v_1 - v_2|||_{X_\infty} \le \frac{1}{2}|||v_1 - v_2|||_{X_\infty}, \tag{3.9}$$

where $u_j = Mv_j$. From (3.8) and (3.9) Theorem 3 follows.

The proofs of Theorems 3′ and 4 are similar to that of Theorem 3 and will be omitted.

Proof of Corollary 5. By Sobolev's inequality and Lemma A.1 we get

$$\begin{aligned}\|u(t)\|_{G^{A_1}(\partial;L^\infty)} &\le C(1+|t|)^{-1} \sum_{|\alpha|+|\beta|\le 2} \|J^\alpha \partial^\beta u(t)\|_{G^{A_1}(\partial;L^2)} \\ &\le C(1+|t|)^{-1}\|u(t)\|_{G^{A_1}(\partial;B^{2,2}(t))},\end{aligned}$$

where $B = R$ or $\tilde{R}$. This yields the time-decay estimate in Corollary 5. Now we define

$$u^\mp = u_0 + \int_0^{\mp\infty} U(-\tau)F_j(u(\tau))\,d\tau.$$

Then

$$U(t)u^\mp = u(t) + \int_t^{\mp\infty} U(t-\tau)F_j(u(\tau))\,d\tau$$

from which and Lemma 2.3 it follows that

$$\|u(t) - U(t)u^\mp\|_{G^{A_1}(\partial;L^2)} \le C\left|\int_t^{\mp\infty} (1+\tau)^{-2}\,d\tau\right|,$$

and the second part of Corollary 5 is proved.

Remark 5. For the sake of simplicity, we have only considered the case when φ satisfies the homogeneous boundary conditions:

$$\lim_{x\to\infty} \varphi(t,x,y) = \lim_{y\to\infty} \varphi(t,x,y) = 0.$$

However our results except for the global result for the IVP (1.11) in Theorem 4 are still valid for the general boundary condition on φ. We explain this point herein. We let $\varphi_1(t,x)$ and $\varphi_2(t,y)$ be such that

$$\lim_{y\to\infty} \varphi(t,x,y) = \varphi_1(t,x), \quad \lim_{x\to\infty} \varphi(t,x,y) = \varphi_2(t,y).$$

Under these boundary conditions the IVP (1.4) or (1.5) can be written as

$$\begin{aligned} i\partial_t u + Hu &= c_1|u|^2 u + c_2 u \int_y^\infty \partial_x |u|^2(x,y')\,dy' \\ &\quad + c_3 u \int_x^\infty \partial_y |u|^2(x',y)\,dx' + c_2 u\partial_x\varphi_1 + c_3 u \partial_y \varphi_2, \\ u(x,y,0) &= u_0(x,y), \end{aligned} \tag{3.10}$$

where $H = \partial_x\partial_y$ or $\partial_x^2 + \partial_y^2$, and the IVP (1.10) or (1.11) can be written as

$$i\partial_t u + Hu = F(u) + G(u) + G_1(u), \quad u(x,y,0) = u_0(x,y), \tag{3.11}$$

where $F(u)$ and $G(u)$ are the same ones as those defined in the introduction (see (1.10)), and

$$G_1(u) = c_4\partial_x u\partial_x\varphi_1 + c_5\partial_x u\partial_y\varphi_2 + c_6\partial_y u\partial_x\varphi_1 + c_7\partial_y u\partial_y\varphi_2.$$

Theorem 1 and Theorem 2 are valid for (3.10) provided

$$\partial_x\varphi_1(t,x) \in G^A(\partial_x; H^{m,2}), \quad \partial_y\varphi_2(t,y) \in G^A(\partial_y; H^{m,2}).$$

Theorem 3 and Theorem 3′ are valid for (3.10) if

$$\partial_x\varphi_1(t,x) \in G^A(\partial_x; G^{A|t|}(\partial_x; H^{m,2})), \quad \partial_y\varphi_2(t,y) \in G^A(\partial_y; G^{A|t|}(\partial_y; H^{m,2}))$$

satisfy

$$\|\partial_x\varphi_1(t)\|_{G^A(\partial_x; G^{A|t|}(\partial_x; H^{m,\infty}))} \le C(1+|t|)^{-a}$$

and

$$\|\partial_y\varphi_2(t)\|_{G^A(\partial_y; G^{A|t|}(\partial_y; H^{m,\infty}))} \le C(1+|t|)^{-a}$$

with $a > 1$.

Theorem 4 for (1.10) is valid for (3.11) with $H = \partial_x\partial_y$ if

$$\partial_x\varphi_1(t,x) \in G^A(\partial_x; G^{A|t|}(\partial_x; G^A(x\partial_x; H^{m,2}))),$$

$$\partial_y\varphi_2(t,y) \in G^A(\partial_y; G^{A|t|}(\partial_y; G^A(y\partial_y; H^{m,2})))$$

satisfy

$$\|\partial_x\varphi_1(t)\|_{G^A(\partial_x; G^{A|t|}(\partial_x; G^A(x\partial_x; H^{m,\infty})))} \le C(1+|t|)^{-a}$$

and

$$\|\partial_y\varphi_2(t)\|_{G^A(\partial_y; G^{A|t|}(\partial_y; G^A(y\partial_y; H^{m,\infty})))} \le C(1+|t|)^{-a}$$

with $a > 1$ and $A < 1$.

Typical examples of φ_1 and φ_2 satisfying the previous conditions are

$$\varphi_1(t,x) = \frac{1}{1+t^2+x^2}, \quad \varphi_2(t,y) = \frac{1}{1+t^2+y^2}.$$

For the IVP (1.11), we have used the operator $\Omega_{xy} = x\partial_y - y\partial_x$ in the proof of Theorem 4. The use of the operator Ω_{xy} requires the condition

$$\partial_x\varphi_1 \in G^A(\partial_x; G^{A|t|}(\partial_x; G^A(y\partial_x; H^{m,\infty})))$$

and

$$\partial_y\varphi_2 \in G^A(\partial_y; G^{A|t|}(\partial_y; G^A(x\partial_y; H^{m,\infty}))).$$

This condition implies that $\partial_x\varphi_1$ and $\partial_y\varphi_2$ are depending only on t. Hence Theorem 4 for the IVP (1.11) is valid for (3.11) if $\partial_x\varphi_1, \partial_y\varphi_2 = C(1+|t|)^{-a}$ with $a > 1$.

Appendix.

Lemma A.1. *There exist positive constants C_1 and C_2 such that*

$$C_1\|u\|_{G^A(B(t);B^{m,p}(t))} \le \sum_{|\alpha|\le m} \|B^\alpha u\|_{G^A(B(t);L^p)} \le C_2\|u\|_{G^A(B(t);B^{m,p}(t))},$$

where $B(t)$ is one of the operators $\partial, R, \Gamma, \tilde{R}$ and $\tilde{\Gamma}$.

Proof. We only prove the case $B(t) = R = (\partial, J)$, where $J = (J_1, J_2)$. By the definition of $G^A(R : R^{m,p}(t))$

$$\|u\|_{G^A(R:R^{m,p}(t))} = \sum_{|\alpha|+|\beta|\le m} \sum_{\gamma,\delta} \frac{A^{|\gamma|+|\delta|}}{\gamma!\delta!} \|\partial^\alpha J^\beta \partial^\gamma J^\delta u\|_{L^p}.$$

By [20, Lemma 2.6] we see that the right-hand side is estimated by

$$\begin{aligned} &C \sum_{|\alpha|+|\beta|\le m} \sum_{\gamma,\delta} \frac{A^{|\gamma|+|\delta|}}{\gamma!\delta!} \|\partial^\gamma J^\delta \partial^\alpha J^\beta u\|_{L^p} \\ =&C \sum_{|\alpha|+|\beta|\le m} \|\partial^\alpha J^\beta u\|_{G^A(R;L^p)} = C \sum_{|\alpha|\le m} \|R^\alpha u\|_{G^A(R;L^p)}. \end{aligned}$$

This implies the first inequality of the lemma. The second inequality of the lemma is obtained by [20, Lemma 2.6], similarly.

Lemma A.2. *We have*

$$\|f \int_y^\infty g\bar{h}(x,y')\,dy'\|_{G^A(B(t);L^2)} \le \|f\|_{G^A(B(t);L^2_yL^\infty_x)} \|g\|_{G^A(B(t);L^2_yL^\infty_x)} \|h\|_{G^A(B(t);L^2)},$$

where $B(t)$ is one of the operators $\partial, R, \Gamma, \tilde{R}$ and $\tilde{\Gamma}$.

Proof. We only prove the case $B = R$. We have

$$\begin{aligned} &\|f \int_y^\infty g\bar{h}(x,y')\,dy'\|_{G^A(B;L^2)} = \sum_{\alpha,\beta} \frac{A^{|\alpha|+|\beta|}}{\alpha!\beta!} \|\partial^\alpha J^\beta \big(f \int_y^\infty g\bar{h}(x,y')\,dy'\big)\|_{L^2} \\ &\le \sum_{\alpha,\beta} \sum_{\substack{\alpha_1\le\alpha \\ \beta_1\le\beta}} \binom{\alpha}{\alpha_1} \binom{\beta}{\beta_1} \frac{A^{|\alpha|+|\beta|}}{\alpha!\beta!} \|(\partial^{\alpha-\alpha_1} J^{\beta-\beta_1} f)\big(\int_y^\infty (\partial^{\alpha_1}(it\partial)^{\beta_1}(g\bar{h})(x,y')\,dy'\big)\|_{L^2} \end{aligned}$$

(by Leibniz' formula)

$$\begin{aligned} &\le \sum_{\alpha,\beta} \sum_{\substack{\alpha_1\le\alpha \\ \beta_1\le\beta}} \frac{A^{|\alpha|+|\beta|}}{(\alpha-\alpha_1)!(\beta-\beta_1)!\alpha_1!\beta_1!} \|\partial^{\alpha-\alpha_1} J^{\beta-\beta_1} f\|_{L^2_yL^\infty_x} \\ &\quad \times \|\int_y^\infty \partial^{\alpha_1}(it\partial)^{\beta_1}(g\bar{h})(x,y')\,dy'\|_{L^\infty_yL^2_x} \quad \text{(by Hölder's inequality)} \\ &\le \|f\|_{G^A(B;L^2_yL^\infty_x)} \sum_{\alpha,\beta} \frac{A^{|\alpha|+|\beta|}}{\alpha!\beta!} \|\int_y^\infty \partial^\alpha(it\partial)^\beta(g\bar{h})(x,y')\,dy'\|_{L^\infty_yL^2_x}. \end{aligned}$$

We again use Leibniz' formula and Hölder's inequality to obtain the result.

Lemma A.3. *We have for $m \geq 3$ and $1 \leq p \leq \infty$*

$$\sum_{n=0}^{\infty} \| |u|^{2n} u \|_{G^A(B(t);B^{m,p}(t))} \leq C \|u\|_{G^A(B(t);B^{m,p}(t))} \sum_{n=0}^{\infty} \|u\|^{2n}_{G^A(B(t);B^{m,2}(t))}.$$

Proof. In the same way as in the proof of Lemma A.2 we have

$$\| |u|^2 u \|_{G^A(B;L^p)} \leq \|u\|^2_{G^A(B;L^\infty)} \|u\|_{G^A(B;L^p)}, \quad 1 \leq p \leq \infty.$$

By iteration of this inequality it follows that

$$\| |u|^{2n} u \|_{G^A(B;L^p)} \leq \|u\|^{2n}_{G^A(B;L^\infty)} \|u\|_{G^A(B;L^p)}.$$

Hence Lemma A.1 gives for $m \geq 3$

$$\| |u|^{2n} u \|_{G^A(B;B^{m,p})} \leq C \|u\|^{2n}_{G^A(B;B^{m-2,\infty})} \|u\|_{G^A(B;B^{m,p})}, \tag{a.1}$$

where C is a constant independent of n. We again use Lemma A.1 and Sobolev's inequality in the right-hand side of (a.1) to obtain the claim.

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