HIROSHIMA MATH. J.
13 (1983), 65–81

Global existence of solutions to nonautonomous differential equations in Banach spaces

Toshiyuki IWAMIYA

(Received May 10, 1982)

1. Introduction

Let X be a Banach space over the real field $\boldsymbol{R}$. Let Ω be a subset of $[a, b) \times X (a < b \leqq \infty)$ and A a continuous function from Ω into X. In this paper we study the initial-value problem for a nonautonomous differential equation in X

(1.1) $$u' = A(t, u), \qquad u(\tau) = z,$$

where (τ, z) is given in Ω.

This problem has been studied by many authors and the present paper is related to the works of Crandall [1], Deimling [2], Kato [3], [4], Kenmochi and Takahashi [5], Lakshmikantham, Mitchell and Mitchell [7], Lovelady and Martin [8], Martin [9], [10], and Pavel and Vrabie [11]. In the works [1], [7] and [9] the problem is treated in cace of cylindrical domain Ω (i.e., Ω is of the form $[a, b) \times D$); and in [5], [4], Kenmochi and Takahashi, and then Kato, generalized the results as obtained in the works [7] and [9] to allow the Ω to be genuinely noncylindrical.

Our purpose of this paper is to establish an existence and uniqueness theorem for solutions of (1.1) under three general conditions (called herein (Ω2), (Ω3) and (Ω4)) in addition to the condition (Ω1) that A is continuous. Although precise statements of these conditions are given in Section 2, we here make some mention of them in order to illustrate features of this paper. Condition (Ω2) imposes on the domain Ω a closedness condition in a certain sense. For instance, if $\Omega = [a, b) \times D$ and D is a closed subset of X then condition (Ω2) is satisfied. Condition (Ω3) is the so-called subtangential condition (cf. [5]) which is utilized to construct approximate solutions for (1.1). Condition (Ω4) is a relaxation of dissipativity conditions as employed in the papers cited above and guarantees the unicity of solutions to (1.1). Accordingly, condition (Ω4) generalizes most of conditions which are usually treated in the theory of ordinary differential equations. Under these conditions, we first investigate the local existence of solutions to (1.1). We then give a global existence theorem via the local existence result as well as the continuous dependence of solutions on initial data. Our result on the global existence is obtained under general conditions as mentioned above. Hence it extends most of the known results concerning the global existence of solutions of nonautonomous equations of the form (1.1).

In Section 2 some basic notation and terminologies are introduced and the uniqueness of solutions to (1.1) is discussed. Section 3 is devoted to construct approximate solutions for (1.1). The approximate solutions are constructed in a way similar to, but more refined than, that of [5]. In Section 4 it is verified that the approximate solutions converge to solutions to (1.1). This section contains a local existence theorem for solutions of (1.1). Finally, in Section 5, the global existence of solutions is discussed. This section contains the main result of this paper (Theorem 2).

2. The initial-value problem

Let X be a Banach space over $\boldsymbol{R}=(-\infty, +\infty)$ with norm $\|\ \|$. Given a subset Q of $\boldsymbol{R}\times X$ we denote by $Q(t)$ the section of Q at $t\in\boldsymbol{R}$, i.e., $Q(t)=\{x\in X; (t, x)\in Q\}$. In what follows, let $[a, b)$ be a fixed subinterval of $\boldsymbol{R}$ and Ω a fixed subset of $[a, b)\times X$ such that $\Omega(t)\neq\phi$ for all $t\in[a, b)$. By A we mean a given function from Ω into X.

Given $(\tau, z)\in\Omega$, we consider the initial-value problem

$$\text{(IVP}; \tau, z) \qquad \begin{cases} u'(t) = A(t, u(t)), & \tau \leqq t < b, \\ u(\tau) = z. \end{cases}$$

Let J be a subinterval of $[a, b)$ written in the from $[\tau, c]$ or $[\tau, c)$ and u a continuous function from J into X. We say that u is a *solution to (IVP*; τ, z) if $u(\tau)=z$, $(t, u(t))\in\Omega$ for all $t\in J$, u is differentiable on J, and u satisfies $u'(t)=A(t, u(t))$ for all $t\in J$. (If t is an endpoint of J, $u'(t)$ is understood to be the associated one-sided derivative of u at t.)

For each $x, y\in X$ we define

$$(2.1) \qquad \begin{aligned}[x, y]_- &= \lim_{h\uparrow 0} h^{-1}(\|x+hy\|-\|x\|) \\ &= \sup_{h<0} h^{-1}(\|x+hy\|-\|x\|).\end{aligned}$$

Note that $\|x\|\leqq\|x-hy\|+h[x, y]_-$ for all $h>0$. For each $(t, x)\in\boldsymbol{R}\times X$ and $r>0$ we define

$$S_r(t, x) = \{(s, y)\in\boldsymbol{R}\times X; |s-t| < r, \|y-x\| < r\}.$$

Moreover, if $x\in X$ and D is a subset of X we define the distance between $\{x\}$ and D by

$$d(x, D) = \inf\{\|x-y\|; y\in D\}.$$

Let g be a function from $[a, b)\times\boldsymbol{R}$ into $\boldsymbol{R}$ satisfying the following conditions:

(g1) $g(t, w)$ is continuous in w for each fixed t and Lebesgue measurable in t for each fixed w; and for each $r>0$, there is a locally integrable function $M_r(t)$

defined on $[a, b)$ such that $|g(t, w)| \leqq M_r(t)$ for $t \in [a, b)$ and for w with $|w| \leqq r$.

(g2) $g(t, 0) \equiv 0$; and $w(t) \equiv 0$ is the maximal solution to the initial value problem for the ordinary differential equation:

$$\begin{cases} w'(t) = g(t, w(t)), & a \leqq t < b, \\ w(a) = 0. \end{cases}$$

Given $(\tau, \delta) \in [a, b) \times \boldsymbol{R}$, we denote by $m(t; \tau, \delta)$ the maximal solution of the initial-value problem

$$\begin{cases} w'(t) = g(t, w(t)) & \tau \leqq t, \\ w(\tau) = \delta. \end{cases}$$

REMARK 1. Condition (g1) is often called the Caratheodory's condition. Condition (g2) states that for all $\tau \in [a, b)$, a maximal solution $m(t; \tau, 0)$ exists on $[\tau, b)$ and $m(t; \tau, 0) \equiv 0$.

In the following, we impose four conditions below on the function $A: \Omega \to X$.

(Ω1) A is continuous from Ω into X.

(Ω2) If $(t_n, x_n) \in \Omega$, $t_n \uparrow t \in [a, b)$ in $\boldsymbol{R}$ and $x_n \to x$ in X as $n \to \infty$, then $(t, x) \in \Omega$.

(Ω3) $\liminf_{h \downarrow 0} h^{-1} d(x + hA(t, x), \Omega(t+h)) = 0$ for all $(t, x) \in \Omega$.

(Ω4) $[x-y, A(t, x) - A(t, y)]_- \leqq g(t, \|x-y\|)$ for all $(t, x), (t, y) \in \Omega$.

We first cite the following well-known results (for the proofs, see e.g. Lakshmikantham and Leela [6]).

LEMMA 1. *Let $\tau \in [a, b)$ and let $[\tau, c]$ be a compact subinterval of $[\tau, b)$. Then there is a number $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0)$, a maximal solution $m(t; \tau, \delta)$ exists on $[\tau, c]$ and*

$$\lim_{\delta \downarrow 0} m(t; \tau, \delta) = 0 \quad \textit{uniformly on } [\tau, c].$$

LEMMA 2. *Let $[\tau, c]$ be a subinterval of $[a, b)$ and α an absolutely continuous function from $[\tau, c]$ into $\boldsymbol{R}$. Suppose that α satisfies*

$$\alpha(t_2) - \alpha(t_1) \leqq \int_{t_1}^{t_2} g(t, \alpha(t)) dt$$

for $\tau \leqq t_1 < t_2 \leqq c$. If $m(t; \tau, \delta)$ exists on $[\tau, c]$, then $\alpha(\tau) \leqq \delta$ implies

$$\alpha(t) \leqq m(t; \tau, \delta) \qquad \textit{for all} \quad t \in [\tau, c].$$

Using the above lemmas we now show the uniqueness of solutions.

PROPOSITION 1. *Suppose that condition (Ω4) is satisfied. Let $[\tau, c)$ be a subinterval of $[a, b)$ and $(\tau, z_i) \in \Omega$, $i = 1, 2$. Let u_i be solutions to $(IVP; \tau, z_i)$*

on $[\tau, c)$, *respectively. If* $m(t; \tau, \delta)$ *is defied on* $[\tau, c)$, *then* $\|z_1 - z_2\| \leqq \delta$ *implies*

$$\|u_1(t) - u_2(t)\| \leqq m(t; \tau, \delta) \qquad \text{for all} \quad t \in [\tau, c).$$

In particular, $(IVP; \tau, z)$ *has at most one solution on* $[\tau, c)$ *for all* $(\tau, z) \in \Omega$.

PROOF. Let $\alpha(t) = \|u_1(t) - u_2(t)\|$ for each $t \in [\tau, c)$. Then α is absolutely continuous on each compact subinterval of $[\tau, c)$ and left-differentiable on (τ, c). It follows from (2.1) and (Ω4) that

$$\begin{aligned}(d/dt)^- \alpha(t) &= [u_1(t) - u_2(t), u_1'(t) - u_2'(t)]_- \\ &= [u_1(t) - u_2(t), A(t, u_1(t)) - A(t, u_2(t))]_- \\ &\leqq g(t, \alpha(t))\end{aligned}$$

for all $t \in (\tau, c)$. Hence we have

$$\alpha(t_2) - \alpha(t_1) \leqq \int_{t_1}^{t_2} g(t, \alpha(t)) dt$$

for $\tau \leqq t_1 < t_2 < c$ (cf. [10]). The first assertion then follows from Lemma 2. The second assertion follows from the first assertion and the fact that $m(t; \tau, 0) \equiv 0$ on $[\tau, c)$. The proof is complete.

3. Approximate solutions

This section is devoted to construct an approximating family for the solution of the initial-value problem $(IVP; \tau, z)$.

PROPOSITION 2. *Suppose that conditions* (Ω1)–(Ω3) *are satisfied. Let* $(\tau, z) \in \Omega$. *Let* $R > 0$ *and* $M > 0$ *be such that* $\tau + R < b$ *and* $\|A(t, x)\| \leqq M$ *for* $(t, x) \in \Omega \cap S_R(\tau, z)$. *Then for each* $T \in (0, R/(M+1)]$ *and for each* $\varepsilon \in (0, 1)$, *there exists an X-valued function u on the interval* $[\tau, \tau + T]$ *and a partition* $\{t_i\}_{0 \leqq i \leqq N}$ *of* $[\tau, \tau + T]$ *with the properties listed below*:

(i) $t_{i+1} - t_i \leqq \varepsilon$ *for* $0 \leqq i \leqq N - 1$,

(ii) $u(\tau) = z$ *and* $\|u(t) - u(s)\| \leqq |t - s|(M + \varepsilon)$ *for* $t, s \in [\tau, \tau + T]$,

(iii) $(t_i, u(t_i)) \in \Omega \cap S_R(\tau, z)$ *for* $0 \leqq i \leqq N$,

(iv) *u is linear on* $[t_i, t_{i+1}]$ *and*

$$\|u(s) - u(t_i) - (s - t_i)A(t_i, u(t_i))\| \leqq (s - t_i)\varepsilon$$

for $s \in [t_i, t_{i+1}]$, $0 \leqq i \leqq N - 1$,

(v) *if* $(s_1, y_1), (s_2, y_2) \in \Omega \cap S_{r_i}(t_i, u(t_i))$, *where* $r_i = (t_{i+1} - t_i)(M + 1)$, *then* $\|A(s_1, y_1) - A(s_2, y_2)\| \leqq \varepsilon$ *for* $0 \leqq i \leqq N-1$.

In this case, u has the following additional property:

(vi) *for each* $\delta > 0$ *such that* $\delta < t_{i+1} - t_i$ *for* $0 \leqq i \leqq N-1$, *there exists an X-valued, strongly measurable function v on* $[\tau, \tau+T]$ *such that*

(a) $v(t_i) = u(t_i)$ *for* $0 \leqq i \leqq N$ *and* $(s, v(s)) \in \Omega \cap S_R(\tau, z)$,

(b) $\|u(s) - v(s)\| \leqq 3\varepsilon^2$ *for* $s \in [\tau, \tau+T]$,

(c) $\int_{\tau+\delta}^{\tau+T} \|v(s) - v(s-\delta) - \delta A(s, v(s))\| ds \leqq 14T\delta\varepsilon$.

The family of functions u obtained through the above proposition provides the aimed family approximating the solution of $(IVP; \tau, z)$. In the following we call the function u given for a positive number ε an *ε-approximate solution for* $(IVP; \tau, z)$.

In what follows, we assume that conditions (Ω1)–(Ω3) are satisfied. We begin by preparing a few lemmas. The first lemma is due to Kenmochi and Takahashi [5, Lemma 1].

LEMMA 3. *Let* $(t, x) \in \Omega$ *and* $\varepsilon > 0$. *Let* $r > 0$ *be such that* $\|A(s, y) - A(t, x)\| \leqq \varepsilon$ *for* $(s, y) \in \Omega \cap S_r(t, x)$. *Also let* $M > 0$ *be such that* $\|A(s, y)\| \leqq M$ *for* $(s, y) \in \Omega \cap S_r(t, x)$ *and set* $h_0 = \min\{r, r/M, b-t\}$. *Then for each* $h \in (0, h_0)$ *there exists an element* $y \in \Omega(t+h)$ *such that* $(t+h, y) \in \Omega \cap S_r(t, x)$ *and* $\|y - x - hA(t, x)\| \leqq h\varepsilon$.

REMARK 2. The above lemma seems to be a refinement of Lemma 1 of [5] since the latter lemma only asserts that $d(x + hA(t, x), \Omega(t+h)) \leqq h\varepsilon$ for $h \in (0, h_0)$. But the proof involves the verification of the last half of the above lemms.

LEMMA 4. *Let* $(t, x) \in \Omega$ *and* $\varepsilon \in (0, 1)$. *Let* $r > 0$ *satisfy* $t + r < b$ *and* $\|A(s_1, y_1) - A(s_2, y_2)\| \leqq \varepsilon$ *for* $(s_1, y_1), (s_2, y_2) \in \Omega \cap S_r(t, x)$. *Also, let* $M > 0$ *be such that* $\|A(s, y)\| \leqq M$ *for* $(s, y) \in \Omega \cap S_r(t, x)$. *Let* $h \in [0, r/(M+1))$, $y \in \Omega(t+h)$, *and* $\|y - x\| \leqq h(M+1)$. *Then for each* $\hat{h} \in (h, r/(M+1)]$ *there exists an element* $\hat{y} \in \Omega(t+\hat{h})$ *such that* $(t+\hat{h}, \hat{y}) \in \Omega \cap S_r(t, x)$ *and*

$$\|\hat{y} - y - (\hat{h} - h)A(t+h, y)\| \leqq (\hat{h} - h)\varepsilon.$$

PROOF. Let $\hat{h} \in (h, r/(M+1)]$. Set $\hat{r} = (\hat{h} - h)(M+1)$. Then $S_{\hat{r}}(t+h, y) \subset S_r(t, x)$. Hence $\|A(s', y') - A(t+h, y)\| \leqq \varepsilon$ and $\|A(s', y')\| \leqq M$ for all $(s', y') \in \Omega \cap S_{\hat{r}}(t+h, y)$. Hence, by Lemma 3, there exists an element $\hat{y} \in \Omega(t+\hat{h})$ such that $(t+\hat{h}, \hat{y}) \in \Omega \cap S_{\hat{r}}(t+h, y) \subset \Omega \cap S_r(t, x)$ and $\|\hat{y} - y - (\hat{h} - h)A(t+h, y)\| \leqq$

$(\hat{h}-h)\varepsilon$. This completes the proof.

LEMMA 5. *Let* $(t, x)\in\Omega$, $\varepsilon\in(0, 1)$ *and let* r *be a positive number such that* $t+r<b$ *and* $\|A(s_1, y_1)-A(s_2, y_2)\|\leqq\varepsilon$ *for all* $(s_1, y_1), (s_2, y_2)\in\Omega\cap S_r(t, x)$. *Also, let* $M>0$ *be such that* $\|A(s, y)\|\leqq M$ *for all* $(s, y)\in\Omega\cap S_r(t, x)$. *Let* $\hat{t}$ *and* δ *be such that* $\delta>0$ *and* $t+\delta<\hat{t}\leqq t+r/(M+1)$. *Then there exists an* X*-valued, strongly measurable function* v *on* $[t, \hat{t})$ *with the following properties*:

(i) $v(t)=x$,

(ii) $(s, v(s))\in\Omega\cap S_r(t, x)$ *for* $s\in[t, \hat{t})$,

(iii) $\|v(s)-x-(s-t)A(t, x)\|\leqq 2(s-t)\varepsilon$ *for* $s\in[t, \hat{t})$,

(iv) $\|v(s)-v(s-\delta)-\delta A(s, v(s))\|\leqq 7\delta\varepsilon$ *for* $s\in[t+\delta, \hat{t})$.

PROOF. For each nonnegative integer n, let $N(n)$ be an integer satisfying $t+N(n)\delta/2^n<\hat{t}\leqq t+(N(n)+1)\delta/2^n$. Set $t_k^n=t+k\delta/2^n$ for $n\geqq 0$, $0\leqq k\leqq N(n)$ and $I_n=\{t_k^n;\ 0\leqq k\leqq N(n)\}$ for $n\geqq 0$. Note that $I_n\subset I_{n+1}$ and $t_k^{n+1}\in I_n$ for k even. Now, for each $n\geqq 0$, we define a step function γ_n on $[t, \hat{t})$ with values in I_n by

$$\gamma_n(s)=\begin{cases} t_k^n & \text{for } s\in[t_k^n, t_{k+1}^n),\quad 0\leqq k\leqq N(n)-1,\\ t_{N(n)}^n & \text{for } s\in[t_{N(n)}^n, \hat{t}).\end{cases}$$

It is easy to see that the sequence $\{\gamma_n\}_{n\geqq 0}$ satisfies the following:

(3.1) $\gamma_n(s)=s$ for $s\in I_n$, $n\geqq 0$;

(3.2) $\gamma_m(\gamma_n(s))=\gamma_n(\gamma_m(s))=\gamma_m(s)$ for $0\leqq m\leqq n$, $s\in[t, \hat{t})$;

(3.3) $\gamma_n(s-\delta)=\gamma_n(s)-\delta$ for $n\geqq 0$, $s\in[t+\delta, \hat{t})$; and

(3.4) the sequence $\{\gamma_n(s)\}_{n\geqq 0}$ is monotone increasing for each $s\in[t, \hat{t})$ and $\gamma_n(s)\uparrow s$ as $n\to\infty$.

We then aim to construct a sequence $\{v_n\}_{n\geqq 0}$ of X-valued step functions on $[t, \hat{t})$ which satisfies the following:

(3.5) $v_n(t)=x$ and $(\gamma_n(s), v_n(s))\in\Omega\cap S_r(t, x)$ for $n\geqq 0$, $s\in[t, \hat{t})$;

(3.6) $\|v_0(s)-v_0(s-\delta)-\delta A(\gamma_0(s-\delta), v_0(s-\delta))\|\leqq\delta\varepsilon$ for $s\in[t+\delta, \hat{t})$;

(3.7) $\|v_n(s)-x\|\leqq(\gamma_n(s)-t)(M+1)$ for $n\geqq 0$, $s\in[t, \hat{t})$;

(3.8) $\|v_n(s)-x-(\gamma_n(s)-t)A(t, x)\|\leqq 2(\gamma_n(s)-t)\varepsilon$ for $n\geqq 0$, $s\in[t, \hat{t})$;

(3.9) $\|v_n(s)-v_{n-1}(s)-(\gamma_n(s)-\gamma_{n-1}(s))A(\gamma_{n-1}(s), v_{n-1}(s))\|$
$\leqq(\gamma_n(s)-\gamma_{n-1}(s))\varepsilon$ for $n\geqq 1$, $s\in[t, \hat{t})$ and

(3.10) $\|v_n(s)-v_0(s)-(\gamma_n(s)-\gamma_0(s))A(\gamma_0(s), v_0(s))\|$

$$\leqq 2(\gamma_n(s)-\gamma_0(s))\varepsilon \qquad \text{for} \quad n \geqq 1,\ s \in [t, \hat{t}).$$

To this end, we begin by choosing a sequence $\{v_0(t_k^0)\}_{0 \leqq k \leqq N(0)}$ of elements in X such that

(3.11) $(t_k^0, v_0(t_k^0)) \in \Omega \cap S_r(t, x) \qquad \text{for} \quad 0 \leqq k \leqq N(0),$

(3.12) $\|v_0(t_k^0)-v_0(t_{k-1}^0)-\delta A(t_{k-1}^0, v_0(t_{k-1}^0))\| \leqq \delta\varepsilon \qquad \text{for} \quad 1 \leqq k \leqq N(0),$

(3.13) $\|v_0(t_k^0)-x\| \leqq (t_k^0-t)(M+1) \qquad \text{for} \quad 0 \leqq k \leqq N(0),$

(3.14) $\|v_0(t_k^0)-x-(t_k^0-t)A(t, x)\| \leqq 2(t_k^0-t)\varepsilon \qquad \text{for} \quad 1 \leqq k \leqq N(0).$

This is accomplished induction on k. Set $v_0(t_0^0)=x$. Let j be an integer such that $0 \leqq j \leqq N(0)-1$ and assume that a sequence $\{v_0(t_k^0)\}_{0 \leqq k \leqq j}$ has been chosen so that (3.11)–(3.14) may hold for $0 \leqq k \leqq j$. Applying Lemma 3 with $h=t_j^0-t$ and $\hat{h}=t_{j+1}^0-t=h+\delta$, one can choose an element $v_0(t_{j+1}^0) \in X$ such that $(t_{j+1}^0, v_0(t_{j+1}^0)) \in \Omega \cap S_r(t, x)$ and

$$\|v_0(t_{j+1}^0) - v_0(t_j^0) - \delta A(t_j^0, v_0(t_j^0))\| \leqq \delta\varepsilon.$$

We have

$$\begin{aligned} \|v_0(t_{j+1}^0)-x\| &\leqq \|v_0(t_{j+1}^0) - v_0(t_j^0)-\delta A(t_j^0, v_0(t_j^0))\| \\ &\quad + \|v_0(t_j^0)-x\| + \delta\|A(t_j^0, v_0(t_j^0))\| \\ &\leqq \delta\varepsilon + (t_j^0-t)(M+1) + \delta M \\ &\leqq (t_{j+1}^0-t)(M+1) \end{aligned}$$

and

$$\begin{aligned} &\|v_0(t_{j+1}^0) - x - (t_{j+1}^0-t)A(t, x)\| \\ &\leqq \|v_0(t_{j+1}^0) - v_0(t_j^0) - \delta A(t_j^0, v_0(t_j^0))\| \\ &\quad + \|v_0(t_j^0) - x - (t_j^0-t)A(t, x)\| \\ &\quad + \delta\|A(t_j^0, v_0(t_j^0)) - A(t, x)\| \\ &\leqq \delta\varepsilon + 2(t_j^0-t)\varepsilon + \delta\varepsilon \\ &= 2(t_{j+1}^0-t)\varepsilon. \end{aligned}$$

This means that the desired sequence $\{v_0(t_k)\}_{0 \leqq k \leqq N(0)}$ can be constructed such that (3.11)–(3.14) hold. Set $v_0(s)=v_0(\gamma_0(s))$ for $s \in [t, \hat{t})$. Then it follows from (3.11)–(3.14) that v_0 satisfies (3.5) through (3.8) (and (3.9), (3.10) in a trivial sense) for $n=0$ and $s \in [t, \hat{t})$. This completes the first stage of our construction. Next we define a sequence $\{v_n\}_{n \geqq 1}$ in the following way. Let j be a positive integer and assume that a sequence $\{v_n\}_{0 \leqq n \leqq j}$ has been defined in such a way that (3.5)–(3.10) hold for $0 \leqq n \leqq j$ and $s \in [t, \hat{t})$. To construct the $(j+1)$th function v_{j+1} on $[t, \hat{t})$, we first specify the values of v_{j+1} on the set of mesh points I_{j+1}. Let $s \in I_{j+1}$. If $s \in I_j$, we set $v_{j+1}(s)=v_j(s)$. If $s \in I_{j+1}-I_j$, then by use of Lemma 3 with

$h=\gamma_j(s)-t$, $y=v_j(s)$ and $\hat{h}=s-t$, one finds an element, say $v_{j+1}(s)$, such that $(s, v_{j+1}(s))\in\Omega\cap S_r(t, x)$ and

$$\|v_{j+1}(s)-v_j(s)-(s-\gamma_j(s))A(\gamma_j(s), v_j(s))\| \leqq (s-\gamma_j(s))\varepsilon.$$

We now set $v_{j+1}(s)=v_{j+1}(\gamma_{j+1}(s))$ for $s\in[t, \hat{t})$. Then we infer from the definition of v_{j+1} that $(\gamma_{j+1}(s), v_{j+1}(s))\in\Omega\cap S_r(t, x)$ and

$$\|v_{j+1}(s)-v_j(s)-(\gamma_{j+1}(s)-\gamma_j(s))A(\gamma_j(s), v_j(s))\| \leqq (\gamma_{j+1}(s)-\gamma_j(s))\varepsilon.$$

Moreover we have

$$\begin{aligned}
&\|v_{j+1}(s)-x\|\\
&\quad\leqq \|v_{j+1}(s)-v_j(s)-(\gamma_{j+1}(s)-\gamma_j(s))A(\gamma_j(s), v_j(s))\|\\
&\qquad+\|v_j(s)-x\|+(\gamma_{j+1}(s)-\gamma_j(s))\|A(\gamma_j(s), v_j(s))\|\\
&\quad\leqq (\gamma_{j+1}(s)-\gamma_j(s))\varepsilon+(\gamma_j(s)-t)(M+1)+(\gamma_{j+1}(s)-\gamma_j(s))M\\
&\quad\leqq (\gamma_{j+1}(s)-t)(M+1),\\
&\|v_{j+1}(s)-x-(\gamma_{j+1}(s)-t)A(t, x)\|\\
&\quad\leqq \|v_{j+1}(s)-v_j(s)-(\gamma_{j+1}(s)-\gamma_j(s))A(\gamma_j(s), v_j(s))\|\\
&\qquad+\|v_j(s)-x-(\gamma_j(s)-t)A(t, x)\|\\
&\qquad+(\gamma_{j+1}(s)-\gamma_j(s))\|A(\gamma_j(s), v_j(s))-A(t, x)\|\\
&\quad\leqq 2(\gamma_{j+1}(s)-t)\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\|v_{j+1}(s)-v_0(s)-(\gamma_{j+1}(s)-\gamma_0(s))A(\gamma_0(s), v_0(s))\|\\
&\quad\leqq \|v_{j+1}(s)-v_j(s)-(\gamma_{j+1}(s)-\gamma_j(s))A(\gamma_j(s), v_j(s))\|\\
&\qquad+\|v_j(s)-v_0(s)-(\gamma_j(s)-\gamma_0(s))A(\gamma_0(s), v_0(s))\|\\
&\qquad+(\gamma_{j+1}(s)-\gamma_j(s))\|A(\gamma_j(s), v_j(s))-A(\gamma_0(s), v_0(s))\|\\
&\quad\leqq 2(\gamma_{j+1}(s)-\gamma_0(s))\varepsilon.
\end{aligned}$$

This means that v_{j+1} satisfies (3.5) through (3.10) with $n=j+1$. Thus a sequence $\{v_n\}_{n\geqq 0}$ of functions satisfying (3.5)–(3.10) is constructed.

We now find an X-valued function v on $[t, \hat{t})$ possessing the properties (i)–(iv) as listed in the statement of the lemma. Since (3.9) implies

$$\|v_n(s)-v_{n-1}(s)\| \leqq (\gamma_n(s)-\gamma_{n-1}(s))(M+1)$$

for $n\geqq 1$ and $s\in[t, \hat{t})$ and $\lim_{n\to\infty}\gamma_n(s)=s$, the sequence $\{v_n(s)\}_{n\geqq 0}$ is a Cauchy sequence for each $s\in[t, \hat{t})$. Hence one can define a function v on $[t, \hat{t})$ by $v(s)=$

$\lim_{n\to\infty} v_n(s)$. v is clearly strongly measurable. Since $(\gamma_n(s), v_n(s)) \in \Omega$, $\gamma_n(s) \uparrow s$ in $\boldsymbol{R}$ and $v_n(s) \to v(s)$ in X, it follows from (Ω2) that $(s, v(s)) \in \Omega$. By (3.7) we have

$$\|v(s) - x\| \leqq (s-t)(M+1) < r \qquad \text{for} \quad s \in [t, \hat{t}),$$

and hence $(s, v(s)) \in \Omega \cap S_r(t, x)$. Moreover, by use of (3.8) and (3.9), we obtain

$$\|v(s) - x - (s-t)A(t, x)\| \leqq 2(s-t)\varepsilon \qquad \text{for} \quad s \in [t, \hat{t}),$$

and

$$\|v(s) - v_0(s) - (s-\gamma_0(s))A(\gamma_0(s), v_0(s))\| \leqq 2(s-\gamma_0(s))\varepsilon$$

for $s \in [t, \hat{t})$.
Hence it follows that

$$\begin{aligned}
&\|v(s) - v(s-\delta) - \delta A(s, v(s))\| \\
&\quad \leqq \|v(s) - v_0(s) - (s-\gamma_0(s))A(\gamma_0(s), v_0(s))\| \\
&\qquad + \|v(s-\delta) - v_0(s-\delta) - (s-\gamma_0(s))A(\gamma_0(s-\delta), v_0(s-\delta))\| \\
&\qquad + \|v_0(s) - v_0(s-\delta) - \delta A(\gamma_0(s-\delta), v_0(s-\delta))\| \\
&\qquad + (s-\gamma_0(s))\|A(\gamma_0(s), v_0(s)) - A(\gamma_0(s-\delta), v_0(s-\delta))\| \\
&\qquad + \delta\|A(\gamma_0(s-\delta), v_0(s-\delta)) - A(s, v(s))\| \\
&\quad \leqq 2(s-\gamma_0(s))\varepsilon + 2(s-\gamma_0(s))\varepsilon + \delta\varepsilon + (s-\gamma_0(s))\varepsilon + \delta\varepsilon \\
&\quad \leqq 7\delta\varepsilon
\end{aligned}$$

for all $s \in [t, \hat{t})$. This completes the proof of Lemma 5.

PROOF of Proposition 2. The existence of ε-approximate solutions u for $(IVP; \tau, z)$ satisfying (i)–(v) of Proposition 2 was already verified in Kenmochi and Takahashi [5; Proposition 2]. So it suffices to show that the ε-approximate solution u has the last property (vi) of Proposition 2.

Let $0 \leqq i \leqq N-1$. We first observe that if $(s, y) \in \Omega$, $|s-t_i| < (t_{i+1}-t_i) \times (M+1)$ and $\|y-u(t_i)\| < (t_{i+1}-t_i)(M+1)$, then $(s, y) \in \Omega \cap S_R(\tau, z)$ and hence $\|A(s, y)\| \leqq M$. Using this fact together with the property (v) and applying Lemma 5 with $t = t_i$ and $\hat{t} = t_{i+1}$, we see that there exists an X-valued measurable function v_i on $[t_i, t_{i+1})$ satisfying:

(3.15) $v_i(t_i) = u(t_i)$,

(3.16) $(s, v_i(s)) \in \Omega \cap S_R(\tau, z)$ for $s \in [t_i, t_{i+1})$,

(3.17) $\|v_i(s) - u(t_i) - (s-t_i)A(t_i, u(t_i))\| \leqq 2(s-t_i)\varepsilon$ for $s \in [t_i, t_{i+1})$,

(3.18) $\|v_i(s) - v_i(s-\delta) - \delta A(s, v_i(s))\| \leqq 7\delta\varepsilon$ for $s \in [t_i+\delta, t_{i+1})$.

For each $s\in[\tau, \tau+T]$, we set

$$\begin{cases} v(s) = v_i(s) & \text{if} \quad s\in[t_i, t_{i+1}), \\ v(\tau+T) = u(\tau+T) \end{cases}$$

Then it is clear that the function v has the property (a). Let $s\in[t_i, t_{i+1})$. It follows from (iv) and (3.17) that

$$\begin{aligned} \|u(s) - v(s)\| &\leqq \|u(s) - u(t_i) - (s-t_i)A(t_i, u(t_i))\| \\ &\quad + \|v(s) - u(t_i) - (s-t_i)A(t_i, u(t_i))\| \\ &\leqq 3(s-t_i)\varepsilon \\ &\leqq 3\varepsilon^2 \end{aligned}$$

which is nothing but (b). To see that v has the property (c), we estimate the norms $\|v(s)-v(s-\delta)-\delta A(s, v(s))\|$, $s\in[\tau+\delta, \tau+T]$. If $t_i+\delta\leqq s<t_{i+1}$ for some i with $0\leqq i\leqq N-1$, then (3.18) yields

$$\|v(s) - v(s-\delta) - \delta A(s, v(s))\| \leqq 7\delta\varepsilon.$$

If $t_i\leqq s<t_i+\delta$ for some i with $1\leqq i\leqq N-1$, then (iv), (v) and (3.17) together imply

$$\begin{aligned} &\|v(s) - v(s-\delta) - \delta A(s, v(s))\| \\ &\leqq \|v(s) - u(t_i) - (s-t_i)A(t_i, u(t_i))\| \\ &\quad + \|u(t_i) - u(t_{i-1}) - (t_i - t_{i-1})A(t_{i-1}, u(t_{i-1}))\| \\ &\quad + \|v(s-\delta) - u(t_{i-1}) - (s-\delta-t_{i-1})A(t_{i-1}, u(t_{i-1}))\| \\ &\quad + (t_i+\delta-s)\|A(t_i, u(t_i)) - A(t_{i-1}, u(t_{i-1}))\| \\ &\quad + \delta\|A(s, v(s)) - A(t_i, u(t_i))\| \\ &\leqq 2(s-t_i)\varepsilon + (t_i-t_{i-1})\varepsilon + 2(s-\delta-t_{i-1})\varepsilon + (t_i+\delta-s)\varepsilon + \delta\varepsilon \\ &\leqq 7(t_i-t_{i-1})\varepsilon. \end{aligned}$$

Hence we have

$$\begin{aligned} &\int_{\tau+\delta}^{\tau+T} \|v(s) - v(s-\delta) - \delta A(s, v(s))\| ds \\ &= \textstyle\sum_{i=1}^{N-1} \displaystyle\int_{t_i+\delta}^{t_{i+1}} \|v(s) - v(s-\delta) - \delta A(s, v(s))\| ds \\ &+ \textstyle\sum_{i=1}^{N-1} \displaystyle\int_{t_i}^{t_i+\delta} \|v(s) - v(s-\delta) - \delta A(s, v(s))\| ds \\ &\leqq \textstyle\sum_{i=0}^{N-1} 7(t_{i+1}-t_i)\delta\varepsilon + \sum_{i=1}^{N-1} 7(t_i-t_{i-1})\delta\varepsilon \end{aligned}$$

$$\leqq 14T\delta\varepsilon,$$

which completes the proof.

4. Local existence

In this section we establish a result on the local existence of solutions to $(IVP; \tau, z)$.

THEOREM 1. *Suppose that conditions* $(\Omega 1)$–$(\Omega 4)$ *are satisfied. Let* $(\tau, z) \in \Omega$. *Let* $R>0$ *and* $M>0$ *satisfy* $\tau+R<b$ *and* $\|A(t, x)\| \leqq M$ *for* $(t, x) \in \Omega \cap S_R(\tau, z)$. *If* $T \in (0, R/(M+1)]$, *then* $(IVP; \tau, z)$ *has a unique solution* u *on* $[\tau, \tau+T]$ *such that* $\|u(t)-u(s)\| \leqq M|t-s|$ *for all* $t, s \in [\tau, \tau+T]$.

PROOF. Let $T \in (0, R/(M+1)]$ and $\{\varepsilon_n\}_{n \geqq 1}$ a null-sequence in $(0, 1)$. Then, by Proposition 2, there is an ε_n-approximate solution u_n for $(IVP; \tau, z)$ on $[\tau, \tau+T]$ for each $n \geqq 1$. We denote by $\{t_i^n\}_{0 \leqq i \leqq N(n)}$ the partition of $[\tau, \tau+T]$ associated with u_n. Let m and n be positive integers. Let $\delta>0$ be such that $\delta<t_{i+1}^m - t_i^m$ for $0 \leqq i \leqq N(m)-1$ and $\delta < t_{j+1}^n - t_j^n$ for $0 \leqq j \leqq N(n)-1$. Then Proposition 2 implies that to u_m and u_n, there correspond X-valued strongly measurable functions v_m and v_n having the properties (a)–(c) as mentioned in (vi) of Proposition 2, respectively. By $(\Omega 4)$, we have

$$\begin{aligned} (4.1) \qquad & \|v_m(s) - v_n(s)\| \\ & \leqq \|v_m(s) - v_n(s) - \delta(A(s, v_m(s)) - A(s, v_n(s)))\| \\ & \quad + \delta g(s, \|v_m(s) - v_n(s)\|) \\ & \leqq \|v_m(s-\delta) - v_n(s-\delta)\| \\ & \quad + \|v_m(s) - v_m(s-\delta) - \delta A(s, v_m(s))\| \\ & \quad + \|v_n(s) - v_n(s-\delta) - \delta A(s, v_n(s))\| \\ & \quad + \delta g(s, \|v_m(s) - v_n(s)\|) \end{aligned}$$

for $s \in [\tau+\delta, \tau+T]$. Let t_1 and t_2 be such that $\tau \leqq t_1 < t_1+\delta < t_2 \leqq \tau+T$. Integrating both sides of (4.1) from $t_1+\delta$ to t_2, we obtain

$$\begin{aligned} (4.2) \qquad & \int_{t_2-\delta}^{t_2} \|v_m(s) - v_n(s)\| ds - \int_{t_1}^{t_1+\delta} \|v_m(s) - v_n(s)\| ds \\ & \leqq \delta \int_{t_1+\delta}^{t_2} g(s, \|v_m(s) - v_n(s)\|) ds \\ & \quad + \int_{t_1+\delta}^{t_2} \|v_m(s) - v_m(s-\delta) - \delta A(s, v_m(s))\| ds \end{aligned}$$

$$+ \int_{t_1+\delta}^{t_2} \|v_n(s) - v_n(s-\delta) - \delta A(s, v_n(s))\| ds$$

$$\leqq \delta \int_{t_1+\delta}^{t_2} g(s, \|v_m(s) - v_n(s)\|) ds + 14T\delta(\varepsilon_m + \varepsilon_n).$$

Set $U_{m,n}(s) = \|u_m(s) - u_n(s)\|$ and $V_{m,n}(s) = \|v_m(s) - v_n(s)\|$ for $s \in [\tau, \tau+T]$. Then (4.2) is written as

$$(4.3) \qquad \int_{t_2-\delta}^{t_2} V_{m,n}(s) ds - \int_{t_1}^{t_1+\delta} V_{m,n}(s) ds$$
$$\leqq \delta \int_{t_1+\delta}^{t_2} g(s, V_{m,n}(s)) ds + 14T\delta(\varepsilon_m + \varepsilon_n).$$

Since $|U_{m,n}(s) - V_{m,n}(s)| \leqq \|u_m(s) - v_m(s)\| + \|u_n(s) - v_n(s)\| \leqq 3(\varepsilon_m^2 + \varepsilon_n^2)$ and $|U_{m,n}(t) - U_{m,n}(s)| \leqq \|u_m(t) - u_m(s)\| + \|u_n(t) - u_n(s)\| \leqq 2|t-s|(M+1)$ for $t, s \in [\tau, \tau+T]$, we obtain

$$\delta U_{m,n}(t_2) = \int_{t_2-\delta}^{t_2} U_{m,n}(t_2) ds$$
$$\leqq \int_{t_2-\delta}^{t_2} V_{m,n}(s) ds + 3\delta(\varepsilon_m^2 + \varepsilon_n^2) + 2\delta^2(M+1),$$

and

$$\delta U_{m,n}(t_1) = \int_{t_1}^{t_1+\delta} U_{m,n}(t_1) ds$$
$$\geqq \int_{t_1}^{t_1+\delta} V_{m,n}(s) ds - 3\delta(\varepsilon_m^2 + \varepsilon_n^2) - 2\delta^2(M+1).$$

Hence

$$(4.4) \qquad \delta\{U_{m,n}(t_2) - U_{m,n}(t_1)\} \leqq \int_{t_2-\delta}^{t_2} V_{m,n}(s) ds - \int_{t_1}^{t_1+\delta} V_{m,n}(s) ds$$
$$+ 6\delta(\varepsilon_m^2 + \varepsilon_n^2) + 4\delta^2(M+1).$$

Combining (4.3) with (4.4) yields

$$(4.5) \qquad U_{m,n}(t_2) - U_{m,n}(t_1) \leqq \int_{t_1+\delta}^{t_2} g(s, V_{m,n}(s)) ds + C(\varepsilon_m + \varepsilon_n)$$

for some constant $C > 0$ which depends only on R and M. Since $|U_{m,n}(t) - U_{m,n}(s)| \leqq 2|t-s|(M+1)$ and $|U_{m,n}(s)| \leqq 2(s-\tau)(M+1)$ for $t, s \in [\tau, \tau+T]$, the family of functions $\{U_{m,n}\}_{m \geqq 1, n \geqq 1}$ is equicontinuous and uniformly bounded.

We now claim that

$$\lim_{m \to 0, n \to 0} U_{m,n}(s) = 0 \qquad \text{for all} \quad s \in [\tau, \tau+T].$$

If this were not true, there would be an $s_0 \in [\tau, \tau+T]$ and subsequences

$\{m(k)\}_{k\geqq 1}$, $\{n(k)\}_{k\geqq 1}$ such that $m(k)\to\infty$, $n(k)\to\infty$ as $k\to\infty$, and $\lim_{k\to\infty} U_{m(k),n(k)}(s_0)\neq 0$. Since $\{U_{m(k),n(k)}\}_{k\geqq 1}$ is also equicontinuous and uniformly bounded, we can assume with the aid of Ascoli-Arzela's theorem that

$$\lim_{k\to\infty} U_{m(k),n(k)}(s) = U(s) \tag{4.6}$$

for some continuous function U on $[\tau, \tau+T]$, $U(s_0)\neq 0$, where the convergence is uniform on $[\tau, \tau+T]$. On the other hand, $|U_{m,n}(s)-V_{m,n}(s)|\leqq 3(\varepsilon_m^2+\varepsilon_n^2)$, and so we see that

$$\lim_{k\to\infty} V_{m(k),n(k)}(s) = U(s) \tag{4.7}$$

holds uniformly on $[\tau, \tau+T]$. It, thus, follows from (4.5), (4.6), (4.7) and the Lebesgue convergence theorem that

$$U(t_2) - U(t_1) \leqq \int_{t_1}^{t_2} g(s, U(s))ds \text{ whenever } \tau \leqq t_1 < t_2 \leqq \tau + T.$$

However, we must have $U(s)\equiv 0$ by Lemma 2, which is a contradiction. Thus $\lim_{m\to\infty, n\to\infty} U_{m,n}(s)=0$ holds for $s\in[\tau, \tau+T]$. Since $\{U_{m,n}\}_{m\geqq 1, n\geqq 1}$ is equicontinuous, the convergence is uniform on $[\tau, \tau+T]$. This means that $\{u_n\}_{n\geqq 1}$ is uniformly Cauchy on $[\tau, \tau+T]$.

We then define $u(t)=\lim_{n\to\infty} u_n(t)$ for each $t\in[\tau, \tau+T]$. Then $u(\tau)=z$ and $\|u(t)-u(s)\|\leqq M|t-s|$ for $t, s\in[\tau, \tau+T]$. Since $(t_i^n, u_n(t_i^n))\in\Omega$, it follows from ($\Omega$2) that $(t, u(t))\in\Omega$ for all $t\in[\tau, \tau+T]$. Also, we infer from (Ω1) and the property (iv) as mentioned in Proposition 2 that

$$\lim_{n\to\infty} u_n'(t) = A(t, u(t)) \qquad \text{for a.e.} \quad t\in[\tau, \tau+T].$$

Therefore, the application of the Lebesgue convergence theorem yields

$$u(t) = \lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} \{z + \int_\tau^t u_n'(s)ds\} = z + \int_\tau^t A(s, u(s))ds$$

for all $t\in[\tau, \tau+T]$. This shows that u is a solution to $(IVP;\, \tau, z)$ on $[\tau, \tau+T]$. Since the uniqueness follows from Proposition 1, the proof of the theorem is now complete.

COROLLARY 1. *Suppose that conditions* (Ω1)–(Ω4) *are satisfied. Let* $(\tau, z)\in\Omega$. *Then there is a number* c *with the following properties*:

(i) $\tau<c<b$ *and* $(IVP;\, \tau, z)$ *has a unique solution* u *on* $[\tau, c]$.

(ii) *Let* $\varepsilon>0$, *then there is a number* $r>0$ *such that* $\tau+r<c$ *and for every* $(t, x)\in\Omega\cap S_r(\tau, z)$, $(IVP;\, t, x)$ *has a unique solution* v *on* $[t, c]$ *with* $\|v(c)-u(c)\|\leqq\varepsilon$.

PROOF. Let $R>0$ and $M>0$ be such that $\tau+R<b$ and $\|A(t, x)\|\leqq M$ for

$(t, x) \in \Omega \subset S_R(\tau, z)$. We shall show that any number c in the interval $(\tau, \tau+R/(M+1))$ is the desired one. The first property follows from Theorem 1. To show that c has the second property, let $\varepsilon>0$. By Lemma 1 $\delta>0$ can be found such that the maximal solution $m(s; \tau, \delta)$ exists on $[\tau, c]$ and $m(c; \tau, \delta) \leqq \varepsilon$. Let $r>0$ be such that $\tau+r<c$, $\tau-r+(R-r)/(M+1) \geqq c$ and $r(M+1) \leqq \inf_{0 \leqq \sigma \leqq r} m(\tau+\sigma; \tau, \delta)$. Take any $(t, x) \in \Omega \cap S_r(\tau, z)$ and set $\hat{r}=R-r$. Since $\|A(s, y)\| \leqq M$ for all $(s, y) \in \Omega \cap S_{\hat{r}}(t, x)$, $(IVP; t, x)$ has a unique solution v on $[t, t+\hat{r}/(M+1)]$ by Theorem 1. Since $t+\hat{r}/(M+1)>\tau-r+(R-r)/(M+1) \geqq c$, we infer that v is defined on $[\tau, c]$. If $t \leqq \tau$, then

$$\begin{aligned} \|v(\tau) - u(\tau)\| &\leqq \|v(\tau) - x\| + \|x - z\| \\ &< M(\tau-t) + r \\ &< (M+1)r \\ &\leqq m(\tau; \tau, \delta). \end{aligned}$$

Hence $\|v(c)-u(c)\| \leqq m(c; \tau, \delta) \leqq \varepsilon$ by Proposition 1. If $t>\tau$, then we have

$$\begin{aligned} \|v(t) - u(t)\| &\leqq \|x - z\| + \|u(t) - z\| \\ &< r + M(t-\tau) \\ &< (M+1)r \\ &\leqq m(t; \tau, \delta). \end{aligned}$$

Hence $\|v(c)-u(c)\| \leqq m(c; \tau, \delta) \leqq \varepsilon$ by Proposition 1. Thus the proof is complete.

A concluding REMARK. As is easily seen from the above arguments, the conclusion of Theorem 1 remains valid even if $(\Omega 2)$ is replaced by the following condition (which is a localized form of $(\Omega 2)$):

$(\Omega 2)_{\mathrm{loc}}$ For each $(\tau, z) \in \Omega$, there is a number $R>0$ such that whenever $(t_n, x_n) \in \Omega \cap S_R(\tau, z)$, $t_n \uparrow t \in [a, b)$ and $x_n \to x$ as $n \to \infty$, we have $(t, x) \in \Omega$.

Hence in particular, if Ω is locally closed and $(\Omega 1)$, $(\Omega 3)$ and $(\Omega 4)$ are satisfied, the conclusion of Theorem 1 is valid.

5. Existence in the large

In this section we examine the maximal interval of existence of solutions to $(IVP; \tau, z)$ and give our main result on the global existence.

Suppose that conditions $(\Omega 1)$–$(\Omega 4)$ are satisfied. Let $(\tau, z) \in \Omega$ and let u be a solution to $(IVP; \tau, z)$ that is noncontinuable to the right. We denote its final time by $T(\tau, z)$; hence $\tau < T(\tau, z) < b$ and u is a solution to $(IVP; \tau, z)$ on $[\tau, T(\tau, z))$. Since $(IVP; \tau, z)$ has a unique solution, $T(\tau, z)$ is well-defined for every $(\tau, z) \in \Omega$. We consider T as a function from the metric space Ω into the extended real line $\boldsymbol{R} \cup \{\infty\}$ endowed with the usual topology.

We first show that T is continuous.

LEMMA 6. *Suppose that conditions* $(\Omega 1)$–$(\Omega 4)$ *are satisfied. Let* $(\tau, z) \in \Omega$. *Let* $d \in (\tau, b)$ *be any number such that* $(IVP; \tau, z)$ *has a solution* u *on* $[\tau, d]$. *Then there eixsts a number* $r > 0$ *with* $\tau + r < d$ *such that for any* $(t, x) \in \Omega \cap S_r(\tau, z)$, $(IVP; t, x)$ *has a solution on* $[t, d]$.

PROOF. Since the set $\{(s, u(s));\ s \in [\tau, d]\}$ is compact in Ω, $(\Omega 1)$ ensures that there are numbers $R > 0$ and $M > 0$ such that $\|A(t, x)\| \leqq M$ for $(t, x) \in \Omega \cap \bigcup_{s \in [\tau, d]} S_R(s, u(s))$. Let c be any number with the properties (i) and (ii) as mentioned in Corollary 1; we may assume that $c < d$. Let ε be a positive number such that the maximal solution $m(s; c, \varepsilon)$ exists on $[c, d]$ and $m(s; c, \varepsilon) < R$ on $[c, d]$, r a positive number satisfying (ii) as mentioned in Corollary 1 for the ε, $(t, x) \in \Omega \cap S_R(\tau, z)$, and let v be a noncontinuable solution to $(IVP; t, x)$. Clearly $T(t, x) > c$. We then demonstrate that $T(t, x) > d$. Assume that $T(t, x) \leqq d$. Then, by Proposition 1, $\|v(s) - u(s)\| \leqq m(s; c, \varepsilon) < R$ on $[c, T(t, x))$. This implies that $\|A(s, v(s))\| \leqq M$ for $s \in [c, T(t, x))$ and hence $\|v(s_1) - v(s_2)\| \leqq M|s_1 - s_2|$ for $s_1, s_2 \in [c, T(t, x))$. Therefore, $v_0 = \lim_{s \uparrow T(t,x)} v(s)$ exists in X and $(T(t, x), v_0) \in \Omega$ by $(\Omega 2)$. But, in view of Theorem 1, this contradicts the fact that v is noncontinuable to the right of $T(t, x)$ and the proof is complte.

LEMMA 7. *Suppose that conditions* $(\Omega 1)$–$(\Omega 4)$ *are satisfied. Let* $\{(t_n, x_n)\}_{n \geqq 1}$ *be a sequence in* Ω *converging to* $(\tau, z) \in \Omega$ *and let* $d \in (\tau, b)$. *Assume that* $(IVP; t_n, x_n)$ *has a solution* u_n *on* $[t_n, d]$ *for each* $n \geqq 1$. *Then* $(IVP; \tau, z)$ *has also a solution* u *on* $[\tau, d]$.

PROOF. Let u be a noncontinuable solution to $(IVP; \tau, z)$. We aim to show that u is defined on $[\tau, d]$. To this end, let c be a number with the properties (i) and (ii) as mentioned in Corollary 1; we may assume that $t_n < c < d$ for all $n \geqq 1$. Let ε be a positive number for which the maximal solution $m(t; c, 2\varepsilon)$ exists on $[c, d]$. (The existence of such ε is guaranteed by Lemma 1.) Let r be a number satisfying (ii) (mentioned in Corollary 1) for the ε. Let $(t_m, x_m), (t_n, x_n) \in \Omega \cap S_r(\tau, z)$. Then

$$\|u_m(c) - u_n(c)\| \leqq \|u_m(c) - u(c)\| + \|u_n(c) - u(c)\| \leqq 2\varepsilon,$$

and hence

$$\|u_m(t) - u_n(t)\| \leqq m(t; c, 2\varepsilon) \quad \text{on} \quad [c, d].$$

In view of Lemma 1, this means that the sequence $\{u_n\}_{n \geqq 1}$ of functions is uniformly

Cauchy on $[c, d]$. Set $\bar{u}(t)=\lim_{n\to\infty} u_n(t)$ for all $t\in[c, d]$. Since $u_n(t)=u_n(c)+\int_c^t A(s, u_n(s))ds$ for all $t\in[c, d]$ and $\bar{u}(c)=\lim_{n\to\infty} u_n(c)=u(c)$, we have

$$\bar{u}(t) = u(c) + \int_c^t A(s, \bar{u}(s))ds \qquad \text{for all} \quad t\in[c, d]$$

by the Lebesgue convergence theorem; hence $\bar{u}$ is a solution to $(IVP; c, u(c))$. Since $(IVP; \tau, z)$ has a unique solution u, $\bar{u}$ must coincide on $[c, d]$ with u. This means that $T(\tau, z)>d$; and the proof is complete.

PROPOSITION 3. *Suppose that conditions* $(\Omega 1)$–$(\Omega 4)$ *are satisfied. Given a* $(\tau, z)\in\Omega$, *let* $T(\tau, z)$ *be a final time of a noncontinualbe solution to* $(IVP; \tau, z)$. *Then* T *is a continuous function from* Ω *into* $\boldsymbol{R}\cup\{\infty\}$.

PROOF. Let $\{(t_n, x_n)\}_{n\geqq 1}$ be a sequence in Ω converging to $(\tau, z)\in\Omega$. We wish to show that $\lim_{n\to\infty} T(t_n, x_n)=T(\tau, z)$. To this end, let $d\in[\tau, T(\tau, z))$. By Lemma 6, $d<T(t_n, x_n)$ for sufficiently large n. Hence $d\leqq \liminf_{n\to\infty} T(t_n, x_n)$. Since d was arbitrarily chosen, we have

$$T(\tau, z) \leqq \liminf_{n\to\infty} T(t_n, x_n)\,. \tag{5.1}$$

Let $d\in(\tau, \limsup_{n\to\infty} T(t_n, x_n))$. Then there is a subsequence $\{(t_{n(k)}, x_{n(k)})\}_{k\geqq 1}$ of $\{(t_n, x_n)\}_{n\geqq 1}$ such that $t_{n(k)}<d<T(t_{n(k)}, x_{n(k)})$ for all $k\geqq 1$ and $(t_{n(k)}, x_{n(k)})\to(\tau, z)$ as $k\to\infty$. By Lemma 7, $(IVP; \tau, z)$ has a solution on $[\tau, d]$ and hence $d<T(\tau, z)$. Since d was arbitrarily chosen, we conclude that

$$\limsup_{n\to\infty} T(t_n, x_n) \leqq T(\tau, z)\,. \tag{5.2}$$

Combining (5.1) and (5.2) we obtain $\lim_{n\to\infty} T(t_n, x_n)=T(\tau, z)$ and the proof is complete.

We are now in a position to state our main result of this paper.

THEOREM 2. *Suppose that conditions* $(\Omega 1)$–$(\Omega 4)$ *are satisfied. Let* C *be a connected component of* Ω *and set* $d=\sup\{t\in[a, b);\ C(t)\neq\phi\}$. *Then for each* $(\tau, z)\in C$, $(IVP; \tau, z)$ *has a unique solution on* $[\tau, d)$ *and the interval* $[\tau, d)$ *is the maximal interval of existence of solution. In particular, if* Ω *is connected, then for each* $(\tau, z)\in\Omega$ $(IVP; \tau, z)$ *has a unique solution on* $[\tau, b)$.

PROOF. We show that T takes the constant value d on C. Let $c, c'\in T(C)$. We may assume that $c\leqq c'$. Set $C_1=\{(t, x)\in C;\ T(t, x)\leqq c\}$ and $C_2=\{(t, x)\in C;\ T(t, x)>c\}$. Since T is a continuous function by Proposition 3, C_2 is an open subset of C. Let $\{(t_n, x_n)\}_{n\geqq 1}$ be a sequence in C_2 converging to $(t, x)\in C$. Then $T(t, x)=\lim_{n\to\infty} T(t_n, x_n)\geqq c$. Assume that $T(t, x)=c>t$. Then $t_n<c<T(t_n, x_n)$ for n sufficiently large and $(IVP; t_n, x_n)$ has a solution on $[t_n, c]$ for n sufficiently

large. Therefore, it follows from Lemma 7 that $(IVP; t, x)$ has a solution on $[t, c]$. However, in view of Theorem 1, this contradicts the assumption. Therefore $T(t, x)>c$, which means that C_2 is a closed subset of C. Since C is connected, $C=C_1 \cup C_2$ (disjoint union) and $C_1 \neq \phi$, we conclude that $C_2=\phi$. Thus $c' \leqq c$; and hence $c=c'$. It turns out that $T(C)$ is a singleton set $\{c\}$. Since $t<T(t, x)=c$ for all $(t, x) \in C$, we have $d=\sup\{t; C(t) \neq \phi\} \leqq c$. On the other hand, it is clear that $c=T(t, x) \leqq d$ for $(t, x) \in C$. Consequently, we have $T(C)=\{d\}$ and the proof is complete.

Finally, we consider the case where Ω is a cylindrical domain. In this case it is natural to assume $b=+\infty$. Then $d=+\infty$ and Theorem 2 may be restated in the following form.

COROLLARY 2. *Let D be a closed subset of X and suppose that conditions* (Ω1), (Ω3) *and* (Ω4) *are satisfied with $\Omega=[0, +\infty) \times D$. Then for each $z \in D$, $(IVP; 0, z)$ has a unique solution on $[0, +\infty)$.*

References

[1] M. G. Crandall, Differential equation on convex sets, J. Math. Soc. Japan, **22** (1970), 443–445.

[2] K. Deimling, Ordinary differential equations in Banach spaces, Lecture Notes in Math., **596** (1977), Springer-Verlag.

[3] S. Kato, On the global existence of unique solutions of differential equations in a Banach space, Hokkaido Math. J., **7** (1978), 58–73.

[4] S. Kato, Some remarks on nonlinear ordinary differential equations in a Banach space, Nonlinear Analysis TMA, **5** (1981), 81–93.

[5] N. Kenmochi and T. Takahashi, Nonautonomous differential equations in Banach spaces, Nonlinear Analysis TMA, **4** (1980), 1109–1121.

[6] V. Lakshmikantham and S. Leela, Differential and integral inequalities, Vol. 1. Academic Press, New York, 1969.

[7] V. Lakshmikantham, A. R. Mitchell and R. W. Mitchell, Differential equations on closed subsets of a Banach space, Trans. Amer. Math. Soc., **220** (1976), 103–113.

[8] D. I. Lovelady and R. H. Martin Jr., A global existence theorem for a nonautonomous differential equation in a Banach space, Proc. Amer. Math. Soc., **35** (1972), 445–449.

[9] R. H. Martin Jr., Differential equations on closed subsets of a Banach space, Trans. Amer. Math. Soc., **179** (1973), 399–414.

[10] R. H. Martin Jr., Nonlinear operators and differential equations in Banach spaces, Wiley-Interscience, New York, 1976.

[11] N. H. Pavel and I. I. Vrabie, Differential equations associated with continuous and dissipative time-dependent domain operators, Lecture Notes in Math., **727** (1979), Springer-Verlag.

National Aerospace Laboratory,
Jindaiji, Chofu,
Tokyo, Japan