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Global Existence of Solutions to the Coupled Einstein and Maxwell-Higgs System in the Spherical Symmetry

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Abstract. We prove the global unique existence of classical solutions to the Einstein equations coupled with Maxwell-Higgs system for small initial data under the spherical symmetry. We also obtain the decay estimates of the solutions, and find that the corresponding space-time is time-like and null geodesically complete toward the future. For the proof we reduce the system to a single first order integro-differential equation, and use the contraction mapping theorem in the appropriate function spaces. We also obtain the completeness of space-time along the future directed time-like lines exterior to a region which resembles the even horizon of the Reissner-Nordström black hole.

Introduction

The Cauchy problem of the Einstein equations is initiated by Choquet-Bruhat[3], and the general type of local unique existence of solution is established using the harmonic coordinates (see [13], [16], [23] for updated surveys). Compared to the local existence results following [3], most of global existence theorems are based on the assumptions on the size of initial data and on symmetries of the Lorentzian geometries to construct. The most remarkable one of them is the global existence result for the vacuum Einstein equations for initial data close to the trivial one due to Christodoulou and Klainerman[11], the argument of which is being simplified in [12], [22]. We also refer the corresponding results for the spatially closed space-time due to Andersson and Moncrief[1]. For the Einstein-matter system there is a semi-global result for the Einstein-Maxwell-Yang-Mills system for small data due to Friedrich[14]. For the small data global existence of the Einstein equations coupled with the other matters we note the self-graviting scalar system by Christodoulou[6], and the self-graviting Vlasov-Poisson system by Rein and Rendall[24] both under the assumption of the spherical symmetry of the space-time.
In this paper we are concerned on the global existence problem for the Einstein equations coupled with the Maxwell-Higgs fields in the spherical symmetry. We note that our matter field is quite general in the sense described below. Our system reduces to the system of (massless) self-graviting charged scalar fields assuming the zero Higgs potential term, $V(|\phi|) = 0$, for which under the spherical symmetry there are numerical studies due to Hod and Piran([19], [20]) regarding to the critical behaviours during the collapse. Our result would be the first rigorous mathematical analysis incorporating their model as a special case. For the Maxwell-Higgs

equations on the background of Minkowski space we have global existence of solutions with arbitrary size of energy norm due to Ginbre and Velo[17] (two space dimension), and Erdley and Moncrief[14] (three space dimension). (See also previous small data global existence results due to Choquet-Bruhat[4], Choquet-Bruhat and Christodoulou[5].)

Also, if we assume $\phi = \phi^*$(real scalar field), $A_\mu = 0$ (no Maxwell field coupled), and $V(|\phi|) = \frac{1}{p+1}|\phi|^{p+1}$, then our system reduces to the coupled Einstein and the nonlinear Klein-Gordon equation, which, setting $V(|\phi|) = 0$, further reduces to the spherically symmetric self-graviting (neutral) scalar system previously studied by Christodoulou in a series of papers ([6], [7], [8], [9], [10]). We note that the nonlinear Klein-Gordon equation on the background of Minkowski space was studied by Struwe[25] in the spherical symmetry, and by Grillakis[18] without assumption of the symmetry. Our main purpose in this paper is that there exists a unique global classical solution to the coupled Einstein Maxwell-Higgs system for small initial data under the assumption of spherical symmetry. In the case we do not have assumption on the size of initial data it is expected that a black hole forms during evolution for large data. In this case we will show that the space-time is complete along the time-like lines exterior to a region determined by the final Bondi mass and the charge. The region resembles the Reissner-Nordström black hole. Our study is motivated mainly by the work of Christodoulou in [6].

The organization of this paper is the following. In Section 1 we introduce the coupled Einstein and Maxwell-Higgs equations, and reduce them to the single first order integro-differential equation. In Section 2 we state and prove the main theorem on the global unique existence of solution. In Section 3 we study the completeness of the space-time along the time-like lines.

We close this section by describing our metric under the assumption of spherically symmetric space-time.

We consider the space and time oriented Lorentzian manifold diffeomorphic to $\mathbb{R}^4$, on which the group $SO(3)$ acts as an isometry, and the group orbits are the metric spacelike 2-spheres. The invariants of the group form a time-like curve in the space-time, which is the world line of the center of the spheres. In this spherically symmetric space-time it is convenient to introduce the function r, defined by

$$r = \sqrt{\frac{A}{4\pi}},$$

where A is the area of the 2-sphere. Thus the metric on the 2-sphere is given by

$$ds^2 = r^2 d\Sigma^2 = r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

The quotient of the space-time by $SO(3)$ is the Lorentzian 2-manifold with signature 0. We will define a coordinate u, which is a constant on every future null cones with vertices on the centers of the 2-spheres. With this coordinate system we can represent the metric in the form

$$ds^2 = -g(u,r)\tilde{g}(u,r)du^2 - 2g(u,r)dudr + r^2 d\Sigma^2, \tag{0.1}$$

where g and $\tilde{g}$ tend to 1 as r goes to infinity. We will write $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ in the following sections.

1 The coupled Einstein and Maxwell-Higgs system

The Lagrangian of the Maxwell-Higgs fields in a given metric $g_{\mu\nu}$ is

$$\mathcal{L}_{MH} = -\frac{1}{8\pi}F_{\mu\alpha}F_{\nu\beta}g^{\mu\nu}g^{\alpha\beta} - g^{\mu\nu}D_\mu\phi(D_\nu\phi)^* - V(|\phi|), \tag{1.1}$$

where $\phi = \phi_1 + i\phi_2, \quad i = \sqrt{-1}$ is a complex scalar field, and $D_\mu\phi = \partial_\mu\phi + iA_\mu$ is the (gauge) covariant derivative, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. $V(\cdot)$ is a continuously differentiable function, further conditions on which will be imposed in the later sections.

The corresponding energy-momentum-stress tensor $T_{\mu\nu}$ is

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{4\pi}\left[F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} - \frac{1}{4}g_{\mu\nu}F_{\rho\alpha}F_{\sigma\beta}g^{\rho\sigma}g^{\alpha\beta}\right] \\ &\quad + Re\{D_\mu\phi(D_\nu\phi)^*\} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}D_\alpha\phi(D_\beta\phi)^* - g_{\mu\nu}V(|\phi|). \end{aligned} \tag{1.2}$$

The coupled Einstein and Maxwell-Higgs equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}. \tag{1.3}$$

This system is coupled with the matter field equations

$$\nabla_\mu T^{\mu\nu} = 0, \tag{1.4}$$

which corresponds to

$$g^{\mu\nu}D_\mu D_\nu\phi = \frac{\partial V(|\phi|)}{\partial\phi^*} \tag{1.5}$$

with its complex conjugate, and

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu, \qquad J^\mu = Im\{\phi(D^\mu\phi)^*\}. \tag{1.6}$$

Following [6], we introduce the new function

$$h = \frac{\partial(r\phi)}{\partial r}.$$

Then,

$$\phi = \bar{h} = \frac{1}{r}\int_0^r h(s)ds.$$

We introduce the local charge function

$$Q(u,r) = \int_{B(0,r)} J^0 dv = 4\pi\int_0^r J^0\sqrt{-\gamma}r^2 dr,$$

which, physically, represents the total charge inside $B(0,r)$, the ball of radius r, at the retarded time u. Here we use the notation $\gamma = det(g_{\mu\nu})$. In our choice of metric in (0.1) we can represent the charge function in terms of h as

$$Q(u,r) = 4\pi i \int_0^r s(\bar{h}^* h - \bar{h} h^*)ds. \tag{1.7}$$

In the spherical symmetry we assume $A_\theta = A_\varphi = 0$ as usual. Then, we can choose our gauge so that $A_r = 0$. Thus, among the four gauge field components we are left only with $A_u = A_u(u,r)$ as a nontrivial unknown. Then, the radial component of (1.6) can be integrated to give

$$A_u = \int_0^r \frac{Q}{s^2} g ds. \tag{1.8}$$

The $\{rr\}$ component of (1.3) is

$$\frac{2}{r}\frac{1}{g}\frac{\partial g}{\partial r} = 8\pi \left|\frac{\partial \phi}{\partial r}\right|^2. \tag{1.9}$$

This, combined with the formula

$$\frac{\partial \phi}{\partial r} = \frac{h - \bar{h}}{r}$$

gives

$$g = \exp\left[-4\pi \int_r^\infty \frac{|h-\bar{h}|^2}{s} ds\right]. \tag{1.10}$$

The $\{ur\}$ component of (1.3) is

$$\frac{1}{r^2}\tilde{g}\left[\frac{g}{\tilde{g}} + f\frac{\tilde{g}}{g}\frac{\partial}{\partial r}\left(\frac{g}{\tilde{g}}\right) - 1\right] = 8\pi\left[\frac{1}{2}\tilde{g}\left|\frac{\partial \phi}{\partial r}\right|^2 + \frac{1}{8\pi} g \frac{Q^2}{r^4} + gV(|\bar{h}|)\right],$$

which can be integrated with respect to $\tilde{g}$ as

$$\begin{aligned} \tilde{g} &= \frac{1}{r}\int_0^r \left(1 - \frac{Q^2}{s^2}\right) g ds - \frac{8\pi}{r}\int_0^r s^2 V(|\bar{h}|) g ds \\ &= \bar{g} - \frac{1}{r}\int_0^r \frac{Q^2}{s^2} g ds - \frac{8\pi}{r}\int_0^r s^2 V(|\bar{h}|) g ds, \end{aligned} \tag{1.11}$$

where we set $\bar{g} = \frac{1}{r}\int_0^r g ds$.

Using (1.9), (1.10) and (1.11), we can rewrite the equation (1.5) as

$$Dh = \frac{1}{2r}(g - \tilde{g})(h - \bar{h}) - \frac{Q^2}{2r^3}(h - \bar{h})g - \frac{iQ}{2r} g\bar{h} - ihA_0 - 4\pi g r \frac{\partial V(|\bar{h}|)}{\partial \bar{h}^*}, \tag{1.12}$$

where

$$D = \frac{\partial}{\partial u} - \frac{\tilde{g}}{2}\frac{\partial}{\partial r}.$$

2 Global existence of solutions

In this section we consider the initial value problem of (1.10)–(1.12) with the initial data $h_0(r) = h(0, r)$. Let us introduce the function space X defined by

$$X = \{\ h(\cdot, \cdot) \in C^1([0, \infty) \times [0, \infty)) \quad | \quad \|h\|_X < \infty\ \},$$

where

$$\|h\|_X := \sup_{u \geq 0} \sup_{r \geq 0} \left\{ (1 + r + u)^2 |h(u, r)| + (1 + r + u)^3 \left| \frac{\partial h}{\partial r}(u, r) \right| \right\}.$$

We also introduce

$$X_0 = \{\ h(\cdot) \in C^1([0, \infty)) \quad | \quad \|h\|_{X_0} < \infty\ \},$$

where

$$\|h\|_{X_0} := \sup_{r \geq 0} \left\{ (1 + r)^2 |h(r)| + (1 + r)^3 \left| \frac{\partial h}{\partial r}(r) \right| \right\}.$$

and denote $\|h_0\|_{X_0} = d$. Below we will also use the space Y containing X, and defined by

$$Y = \{h(\cdot, \cdot) \in C^1([0, \infty) \times [0, \infty)) \quad | \quad h(0, r) = h_0(r), \quad \|h\|_Y < \infty\},$$

where

$$\|h\|_Y = \sup_{u \geq 0} \sup_{r \geq 0} \left\{ (1 + r + u)^2 |h(u, r)| \right\}.$$

The purpose of this section is the proof of the following theorem.

Theorem 2.1 *Let us assume that the function $V(\cdot)$ in (1.1) is a twice continuous differentiable function, and there exists a constant $K_0 \geq 0$ such that*

$$|V(|\phi|)| + \left| \frac{\partial V(|\phi|)}{\partial \phi^*} \right| |\phi| + \left| \frac{\partial^2 V(|\phi|)}{\partial \phi \partial \phi^*} \right| |\phi|^2 \leq K_0 |\phi|^{p+1} \quad \forall \phi \in \mathbb{C} \tag{2.1}$$

for some $p \in [3, \infty)$. Suppose we have an initial data $h(0, r) \in C^1([0, \infty))$ such that $h(0, r) = O(r^{-2})$ and $\frac{\partial h}{\partial r}(0, r) = O(r^{-3})$. Let us put $d = \|h(0, \cdot)\|_{X_0}$. Then, there exists $\delta > 0$ such that if $d < \delta$, then there exists a unique global classical solution $h \in C^1([0, \infty) \times [0, \infty))$ of (1.10)–(1.12) with $h(0, r)$ as the initial data. This solution has the decay property:

$$|h(u, r)| \leq C(1 + u + r)^{-2}, \qquad \left| \frac{\partial h}{\partial r}(u, r) \right| \leq C(1 + u + r)^{-3}, \tag{2.2}$$

and the corresponding space-time is time-like and null geodesically complete toward the future.

Remark 2.1. In the case of self-graviting scalar fields as studied by Christodoulou in [6], the order of decays for initial data and solutions is higher than ours obtained in Theorem 1, namely he obtained $|h(u,r)| + r\left|\frac{\partial h(u,r)}{\partial r}\right| = O(r^{-3})$ for the solution $h(u,r)$ under the assumption on the initial data $|h(0,r)| + r\left|\frac{\partial h(0,r)}{\partial r}\right| = O(r^{-3})$. In our case we find difficulties in obtaining similar decay rates. Currently we do not know if this lower decay estimates is just from technical difficulty in the method of the proof of the theorem, or an essential obstacle.

Remark 2.2. If we set $V(|\phi|) = 0$ in (1.2), then the energy momentum tensor of Maxwell-Higgs fields reduces to that of (massless) self-graviting charged scalar fields, and the matter field equations become

$$g^{\mu\nu} D_\mu D_\nu \phi = 0, \quad \nabla_\nu F^{\mu\nu} = 4\pi Im\{\phi(D^\mu \phi)^*\}.$$

In this case we find that the reduced equation (1.12) becomes

$$Dh = \frac{1}{2r}(g - \tilde{g})(h - \bar{h}) - \frac{Q^2}{2r^3}(h - \bar{h})g - \frac{iQ}{2r}g\bar{h} - ihA_0, \tag{2.3}$$

where

$$\tilde{g} = \bar{g} - \frac{1}{r}\int_0^r \frac{Q^2}{s^2} g ds.$$

Thus Theorem 2.1 implies that we have the global existence of classical solution for the equations of (massless) self-graviting charged scalar fields with the similar decays as in (2.2) under the similar hypothesis on the initial data as in the theorem.

Remark 2.3. In the case $\phi = \phi^*$ (real scalar field) we have $Q = 0$ (see (1.7)), and thus $A_\mu = 0$ (see (1.8)), and $F_{\mu\nu} = 0$, and thus choosing $V(|\phi|) = \frac{|\phi|^{p+1}}{p+1}$, the matter field equations become

$$\Box_g \phi = \phi|\phi|^{p-1},$$

which is the nonlinear Klein-Gordon equation, and can be reduced to

$$Dh = \frac{1}{2r}(g - \tilde{g})(h - \bar{h}) - 4\pi g r \bar{h}|\bar{h}|^{p-1}, \tag{2.4}$$

where

$$\tilde{g} = \bar{g} - \frac{8\pi}{(p+1)r}\int_0^r s^2 |\bar{h}|^{p+1} g ds.$$

In this case Theorem 2.1 implies the global existence of classical solutions of the coupled Einstein and nonlinear Klein-Gordon system, which is obtained previously in [2].

Proof of Theorem 2.1. We consider the mapping $h \mapsto \mathcal{F}(h)$, which is defined as the solution of the first order linear partial differential equation

$$D\mathcal{F} = \frac{1}{2r}(g-\tilde{g})(\mathcal{F}-\bar{h}) - \frac{Q^2}{2r^3}(\mathcal{F}-\bar{h})g - \frac{iQ}{2r}g\bar{h} - i\mathcal{F}A_0 - 4\pi gr\frac{\partial V(|\bar{h}|)}{\partial \bar{h}^*}, \quad (2.5)$$

with the initial condition

$$\mathcal{F}(h)(0,r) = h(0,r) \quad (2.6)$$

Let us denote $B(0,x) = \{f \in X \mid \|f\|_X \leq x\}$. We will prove the theorem by verifying that for suitable x and $d = d(x)$ the mapping $\mathcal{F}(\cdot)$ satisfies:

(i) $\mathcal{F} : B(0,x) \to B(0,x)$, and

(ii) there exists $\lambda = \lambda(x) \in (0,1)$ such that

$$\|\mathcal{F}(h_1) - \mathcal{F}(h_2)\|_Y \leq \lambda \|h_1 - h_2\|_Y,$$

i.e., $\mathcal{F}$ contracts in Y.

Then, the standard contraction mapping theorem provides us the unique fixed point $h \in X$ such that $\mathcal{F}(h) = h$, which is the solution of the nonlinear problem (1.10)–(1.12). We set $\|h\|_X = x$.
Let $r(u) = \chi(u; r_0)$ be the solution of the ordinary differential equation

$$\frac{dr}{du} = -\frac{1}{2}\tilde{g}(u,r), \qquad r(0) = r_0. \quad (2.7)$$

We denote $r_1 = \chi(u_1; r_0)$, then (2.7) gives

$$r_1 = r_0 - \frac{1}{2}\int_0^{u_1} \tilde{g}(u, \chi(u; r_0))du. \quad (2.8)$$

Using this characteristic, we can represent $\mathcal{F}$ as the integral as follows.

$$\begin{aligned}\mathcal{F}(u_1, r_1) = h(0, r_0)\exp\left\{\int_0^{u_1}\left[\frac{g-\tilde{g}}{2r} - \frac{Q^2}{2r^3}g - iA_0\right]_\chi du'\right\} \\ + \int_0^{u_1}\exp\left\{\int_u^{u_1}\left[\frac{g-\tilde{g}}{2r} - \frac{Q^2}{2r^3}g - iA_0\right]_\chi du'\right\}[f]_\chi du, \quad (2.9)\end{aligned}$$

where we denoted

$$f = \left[\frac{Q^2}{2r^3}g - \frac{1}{2r}(g-\tilde{g}) - \frac{iQ}{2r}g\right]\bar{h} - 4\pi gr\frac{\partial V(|\bar{h}|)}{\partial \bar{h}^*}. \quad (2.10)$$

We estimate

$$|\bar{h}| \leq \frac{1}{r}\int_0^r \frac{\|h\|_X}{(1+s+u)^2}ds = \frac{x}{(1+u)(1+r+u)}. \quad (2.11)$$

And,

$$
\begin{aligned}
|h(u,r)-h(u,r')| &\leq \int_{r'}^{r}\left|\frac{\partial h}{\partial s}(u,s)\right| ds \leq x\int_{r'}^{r}\frac{ds}{(1+s+u)^3}\\
&= \frac{1}{2}\left[\frac{1}{(1+r'+u)^2}-\frac{1}{(1+r+u)^2}\right]. \qquad (2.12)
\end{aligned}
$$

From this we have

$$
\begin{aligned}
|(h-\bar{h})(u,r)| &\leq \frac{1}{r}\int_0^r |h(u,r)-h(u,r')|dr'\\
&\leq \frac{x}{2r}\int_0^r\left[\frac{1}{(1+r'+u)^2}-\frac{1}{(1+r+u)^2}\right]dr\\
&= \frac{xr}{2(1+r+u)^2(1+u)}. \qquad (2.13)
\end{aligned}
$$

Thus,

$$
\int_0^\infty \frac{|h-\bar{h}|^2}{r}dr \leq \frac{x^2}{4(1+u)^2}\int_0^\infty\frac{s}{(1+s+u)^4}ds = \frac{x^2}{24(1+u)^4}. \qquad (2.14)
$$

This, combined with (1.10), implies

$$
g(u,0) \geq \exp\left[-\frac{\pi x^2}{6(1+u)^4}\right]. \qquad (2.15)
$$

Using this inequality, we estimate

$$
\begin{aligned}
\int_{r'}^{r}\left|\frac{\partial g}{\partial s}(u,s)\right| &\leq 4\pi\int_{r'}^{r}\frac{|h-\bar{h}|^2}{s}ds\\
&\leq \frac{\pi x^2}{(1+u)^2}\int_{r'}^{r}\frac{s}{(1+s+u)^4}ds\\
&= \frac{\pi x^2}{(1+u)^2}\left\{\frac{1}{2}\left[\frac{1}{(1+r'+u)^2}-\frac{1}{(1+r+u)^2}\right]\right.\\
&\qquad \left.-\frac{1}{3}(1+u)\left[\frac{1}{(1+r'+u)^3}-\frac{1}{(1+r+u)^3}\right]\right\}, \qquad (2.16)
\end{aligned}
$$

and obtain

$$
\begin{aligned}
|(g-\bar{g})(u,r)| &\leq \frac{1}{r}\int_0^r |g(u,r)-g(u,r')|dr' \leq \frac{1}{r}\int_0^r\int_{r'}^{r}\left|\frac{\partial g}{\partial s}(u,s)\right| ds dr'\\
&\leq \frac{\pi x^2}{(1+u)^2 r}\int_0^r\left\{\frac{1}{2}\left[\frac{1}{(1+r'+u)^2}-\frac{1}{(1+r+u)^2}\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{3}(1+u)\left[\frac{1}{(1+r'+u)^3}-\frac{1}{(1+r+u)^3}\right]\Bigg\}dr' \\
&= \frac{\pi x^2 r^2}{3(1+u)^3(1+r+u)^3}. \qquad (2.17)
\end{aligned}
$$

The charge Q is estimated as

$$
\begin{aligned}
|Q(u,r)| &= \left|4\pi i\int_0^r s(\bar{h}^*h-\bar{h}h^*)dr\right| \le 8\pi\int_0^r s|\bar{h}||h|ds \\
&\le \frac{8\pi x^2}{(1+u)}\int_0^r \frac{s}{(1+s+u)^3}ds = \frac{4\pi x^2 r^2}{(1+u)^2(1+r+u)^2}. \qquad (2.18)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{1}{r}\int_0^r \frac{|Q|^2}{s^2}ds &\le \frac{16\pi^2 x^4}{(1+u)^4 r}\int_0^r \frac{s^2}{(1+s+u)^4}ds \\
&= \frac{16\pi^2 x^4 r^2}{3(1+u)^6(1+r+u)^3}. \qquad (2.19)
\end{aligned}
$$

We also estimate the potential term as

$$
\begin{aligned}
\left|\frac{1}{r}\int_0^r s^2 V(|\bar{h}|)ds\right| &\le \frac{K_0}{r}\int_0^r s^2|\bar{h}|^{p+1}ds \\
&\le \frac{K_0 x^{p+1}}{(1+u)^{p+1}r}\int_0^r \frac{s^2}{(1+s+u)^{p+1}}ds \\
&\le \frac{K_0 x^{p+1}}{(1+u)^{p+1}r}\int_0^r \frac{s^2}{(1+s+u)^4}ds \\
&= \frac{K_0 x^{p+1} r^2}{3(1+u)^{p+2}(1+r+u)^3}. \qquad (2.20)
\end{aligned}
$$

From (2.17) and (2.19) we have

$$
\begin{aligned}
|g-\tilde{g}| &\le |g-\bar{g}| + \frac{1}{r}\int_0^r \frac{|Q|^2}{s^2}ds + \frac{8\pi}{r}\int_0^r s^2 V(|\bar{h}|)ds \\
&\le \frac{C_0(x^2+x^4+x^{p+1})r^2}{(1+u)^3(1+r+u)^3}. \qquad (2.21)
\end{aligned}
$$

Combination of (2.19) and (2.20) with (2.15) yields

$$
\begin{aligned}
\tilde{g} &\ge \bar{g}(u,0) - \frac{1}{r}\int_0^r \frac{|Q|^2}{s^2}ds - \frac{8\pi}{r}\int_0^r s^2 V(|\bar{h}|)ds \\
&\ge \exp\left[-\frac{\pi x^2}{6}\right] - \frac{16}{3}\pi^2 x^4 - \frac{8\pi K_0}{3}x^{p+1}. \qquad (2.22)
\end{aligned}
$$

Let x_1 be the smallest positive root of the equation

$$\exp\left[-\frac{\pi x^2}{6}\right] - \frac{16}{3}\pi^2 x^4 - \frac{8\pi K_0}{3} x^{p+1} = 0. \tag{2.23}$$

Then, the function

$$k = k(x) = \exp\left[-\frac{\pi x^2}{6}\right] - \frac{16}{3}\pi^2 x^4 - \frac{8\pi K_0}{3} x^{p+1} \tag{2.24}$$

satisfies $0 < k \le 1$ for all $x \in [0, x_1)$. From (2.10) we estimate

$$\begin{aligned} |f| &\le \left|\frac{Q^2}{2r^3} g\bar{h}\right| + \left|\frac{1}{2r}(g-\tilde{g})\bar{h}\right| + \left|\frac{Q}{2r} g\bar{h}\right| + 4\pi \left| gr \frac{\partial V(|\bar{h}|)}{\partial \bar{h}^*}\right| \\ &= \{1\} + \{2\} + \{3\} + \{4\}. \end{aligned} \tag{2.25}$$

Combining (2.18) with (2.11), we have

$$\{1\} \le \frac{8\pi^2 x^5 r}{(1+u)^5 (1+r+u)^5} \le \frac{8\pi^2 x^5}{(1+u)^5 (1+r+u)^4}. \tag{2.26}$$

From (2.21) and (2.11) we obtain

$$\{2\} \le \frac{1}{2r} |g - \tilde{g}| |\bar{h}| \le \frac{C_1 (x^3 + x^5 + x^{p+2})}{(1+u)^4 (1+r+u)^3}. \tag{2.27}$$

Similarly

$$\{3\} \le \frac{2\pi x^3}{(1+u)^3 (1+r+u)^2}, \tag{2.28}$$

and

$$\{4\} \le 4\pi K_0 r |\bar{h}|^p \le \frac{4\pi K_0 r x^p}{(1+u)^p (1+r+u)^p} \le \frac{4\pi K_0 x^p}{(1+u)^3 (1+r+u)^2}. \tag{2.29}$$

Adding (2.26)–(2.29), we have

$$|f| \le \frac{C_2 (x^3 + x^5 + x^p + x^{p+2})}{(1+u)^3 (1+r+u)^2}. \tag{2.30}$$

For $r(u) = \chi(u; r_0)$ we have

$$r(u) = r_1 + \frac{1}{2}\int_u^{u_1} \tilde{g}(u', r(u')) du' \ge r_1 + \frac{1}{2} k (u_1 - u), \tag{2.31}$$

and, since $k \in (0, 1]$ for $x \in (0, x_1]$,

$$1 + u + r_1 + \frac{k}{2}(u_1 - u) \ge k\left(1 + \frac{u_1}{2} + r_1\right) \ge \frac{k}{2}(1 + u_1 + r_1). \tag{2.32}$$

Thus,

$$\begin{aligned}\int_0^{u_1} |[f]_\chi| du &\leq C_2(x^3+x^5+x^p+x^{p+2})\int_0^{u_1}\left[\frac{1}{(1+u)^3(1+r+u)^2}\right]_\chi du\\ &\leq \frac{4C_2(x^3+x^5+x^p+x^{p+2})}{(1+r_1+u_1)^2 k^2}\int_0^\infty \frac{du}{(1+u)^3}\\ &\leq \frac{C_3(x^3+x^5+x^p+x^{p+2})}{(1+r_1+u_1)^2 k^2}. \qquad (2.33)\end{aligned}$$

Now we estimate

$$\begin{aligned}&\left|\int_0^{u_1}\left[\frac{g-\tilde{g}}{2r}-\frac{Q^2}{2r^3}g-iA_0\right]_\chi du\right|\\ &\quad\leq \int_0^{u_1}\left|\left[\frac{g-\tilde{g}}{2r}\right]_\chi\right| du' + \int_0^{u_1}\left|\left[\frac{Q^2}{2r^3}g\right]_\chi\right| du + \int_0^{u_1}\left|[A_0]_\chi\right| du\\ &\quad\leq I_1+I_2+I_3. \qquad (2.34)\end{aligned}$$

Combining the estimate (2.32) with (2.21), we have

$$\begin{aligned}I_1 &\leq \frac{C_0(x^2+x^4+x^{p+1})}{2}\int_0^{u_1}\left[\frac{r}{(1+u)^3(1+r+u)^3}\right]_\chi du\\ &\leq \frac{C_0(x^2+x^4+x^{p+1})}{2}\int_0^\infty \frac{1}{(1+u)^5} \leq C_5(x^2+x^4+x^{p+1}). \qquad (2.35)\end{aligned}$$

From (2.18) we obtain immediately

$$\begin{aligned}I_2 &\leq \int_0^{u_1}\left[\frac{8\pi^2x^4r}{(1+u)^4(1+r+u)^4}\right]_\chi du\\ &\leq \int_0^\infty \frac{8\pi^2x^4}{(1+u)^7}du = \frac{8}{3}\pi^2x^4, \qquad (2.36)\end{aligned}$$

and from the estimate

$$\begin{aligned}|A_0| &\leq \int_0^r \frac{|Q|}{s^2}ds \leq \frac{4\pi x^2}{(1+u)^2}\int_0^r \frac{ds}{(1+s+u)^2}\\ &= \frac{4\pi x^2}{(1+u)^3(1+r+u)} \qquad (2.37)\end{aligned}$$

we have

$$I_3 \leq \int_0^\infty \frac{4\pi x^2}{(1+u)^4}du = \frac{8}{3}\pi x^2. \qquad (2.38)$$

Adding (2.3), (2.36) and (2.38), we obtain

$$\int_0^{u_1} \left| \left[\frac{g-\tilde{g}}{2r} - \frac{Q^2}{2r^3} g - iA_0 \right]_\chi \right| du \le C_6 (x^2 + x^4 + x^{p+1}). \tag{2.39}$$

From (3.26) we have

$$1 + r_0 \ge 1 + r_1 + \frac{1}{2} k u_1 \ge \frac{k}{2}(1 + r_1 + u_1), \tag{2.40}$$

and

$$|h(0, r_0)| \le \frac{d}{(1+r_0)^2} \le \frac{d}{(1 + r_1 + \frac{1}{2}ku_1)^2} \le \frac{4d}{k^2 \left(1 + r_1 + u_1\right)^2}. \tag{2.41}$$

Combining this with (2.33) and (2.39), we obtain

$$\begin{aligned} |\mathcal{F}(u_1, r_1)| &\le |h(0, r_0)| \exp \left\{ \int_0^{u_1} \left| \left[\frac{g-\tilde{g}}{2r} - \frac{Q^2}{2r^3} g - iA_0 \right]_\chi \right| du \right\} \\ &\quad + \int_0^{u_1} \exp \left\{ \int_u^{u_1} \left| \left[\frac{g-\tilde{g}}{2r} - \frac{Q^2}{2r^3} g - iA_0 \right]_\chi \right| du' \right\} |[f]_\chi| du \\ &\le \frac{4 \exp[C_6 (x^2 + x^4 + x^{p+1})]}{\left(1 + r_1 + u_1\right)^2 k^2} \left[d + C_3 (x^3 + x^5 + x^p + x^{p+1}) \right], \end{aligned} \tag{2.42}$$

from which we have

$$\sup_{r,u \ge 0} \{(1+r+u)^2 |\mathcal{F}(u,r)|\} \le \frac{C_7}{k^2}(d + x^3 + x^5 + x^p + x^{p+1}) \exp[C_6 (x^2 + x^4 + x^{p+1})], \tag{2.43}$$

where we set $C_7 = 4\max\{C_3, 1\}$. Now let us denote

$$\mathcal{G}(u,r) = \frac{\partial \mathcal{F}}{\partial r}(u,r)$$

with

$$\mathcal{G}(0, r_0) = \frac{\partial h}{\partial r}(0, r_0).$$

Differentiating (2.5) with respect to r we obtain

$$\begin{aligned} D\mathcal{G}(u_1, r_1) &= \left[\frac{1}{2} \frac{\partial \tilde{g}}{\partial r} + \frac{g - \tilde{g}}{2r} - \frac{Q^2 g}{2r^3} - iA_0 \right] \mathcal{G} \\ &\quad + \left[\frac{1}{2r} \frac{\partial}{\partial r}(g - \tilde{g}) - \frac{1}{2r^2}(g - \tilde{g}) + \left(\frac{3^2}{2r^4} - \frac{Q}{r^3} \frac{\partial Q}{\partial r} \right) g - \frac{Q^2}{2r^3} \frac{\partial g}{\partial r} - i \frac{\partial A_0}{\partial r} \right] \mathcal{F} \end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{\partial^2 V(|\bar{h}|)}{\partial \bar{h}^* \partial \bar{h}}\frac{\partial \bar{h}^*}{\partial r}+\frac{\partial^2 V(|\bar{h}|)}{\partial \overline{h^*}^2}\frac{\partial \bar{h}^*}{\partial r}\right]\\
&+\left(\frac{Q^2}{2r^3}g-\frac{g-\tilde{g}}{2r}-\frac{iQ}{2r}g\right)\frac{\partial \bar{h}}{\partial r}\\
&+\left[\frac{1}{2r}\frac{\partial}{\partial r}(g-\tilde{g})+\left(\frac{3Q^2}{2r^4}g-\frac{Q}{r^3}\frac{\partial Q}{\partial r}\right)g+\frac{i}{2r}\frac{\partial Q}{\partial r}\right.\\
&\left.+\left(\frac{Q^2}{2r^3}-\frac{iQ}{2r}\right)\frac{\partial g}{\partial r}+\frac{iQ}{2r^2}g-\frac{1}{2r^2}(g-\tilde{g})\right]\bar{h}. \qquad (2.44)
\end{aligned}
$$

As previously we can solve the linear equation (2.44) with respect to $\mathcal{G}$ by using the characteristic introduced in (2.7) and (2.8) as follows:

$$
\begin{aligned}
\mathcal{G}(u_1,r_1) &= \frac{\partial h}{\partial r}(0,r_0)\exp\left\{\int_0^{u_1}\left[\frac{1}{2}\frac{\partial \tilde{g}}{\partial r}+\frac{g-\tilde{g}}{2r}-\frac{Q^2 g}{2r^3}-iA_0\right]_\chi du\right\}\\
&+\int_0^{u_1}\exp\left\{\int_u^{u_1}\left[\frac{1}{2}\frac{\partial \tilde{g}}{\partial r}+\frac{g-\tilde{g}}{2r}-\frac{Q^2 g}{2r^3}-iA_0\right]_\chi du'\right\}[f_1]_\chi du, \qquad (2.45)
\end{aligned}
$$

where we set

$$
\begin{aligned}
f_1 \;=\; &\left[\frac{1}{2r}\frac{\partial}{\partial r}(g-\tilde{g})-\frac{1}{2r^2}(g-\tilde{g})+\left(\frac{3Q^2}{2r^4}-\frac{Q}{r^3}\frac{\partial Q}{\partial r}\right)g-\frac{Q^2}{2r^3}\frac{\partial g}{\partial r}-i\frac{\partial A_0}{\partial r}\right]\mathcal{F}\\
&+\left(\frac{Q^2}{2r^3}g-\frac{g-\tilde{g}}{2r}-\frac{iQ}{2r}g\right)\frac{\partial \bar{h}}{\partial r}+\left[\frac{\partial^2 V(|\bar{h}|)}{\partial \bar{h}^* \partial \bar{h}}\frac{\partial \bar{h}}{\partial r}+\frac{\partial^2 V(|\bar{h}|)}{\partial \overline{h^*}^2}\frac{\partial \bar{h}^*}{\partial r}\right]\\
&+\left[\frac{1}{2r}\frac{\partial}{\partial r}(g-\tilde{g})+\left(\frac{3Q^2}{2r^4}g-\frac{Q}{r^3}\frac{\partial Q}{\partial r}\right)g+\frac{i}{2r}\frac{\partial Q}{\partial r}g\right.\\
&\left.+\left(\frac{Q^2}{2r^3}-\frac{iQ}{2r}\right)\frac{\partial g}{\partial r}+\frac{iQ}{2r^2}g-\frac{1}{2r^2}(g-\tilde{g})\right]\bar{h}. \qquad (2.46)
\end{aligned}
$$

We write

$$
\begin{aligned}
|f_1| \;\le\; &\\
&\left\{\left|\frac{1}{2r}\frac{\partial}{\partial r}(g-\tilde{g})-\frac{1}{2r^2}(g-\tilde{g})\right|+\left|\left(\frac{2Q^2}{3r^4}-\frac{Q}{r^3}\frac{\partial Q}{\partial r}\right)g\right|+\left|\frac{Q^2}{2r^3}\frac{\partial g}{\partial r}\right|+\left|\frac{\partial A_0}{\partial r}\right|\right\}|\mathcal{F}|\\
&+\left\{\left|\frac{Q^2}{2r^3}g-\frac{g-\tilde{g}}{2r}-\frac{iQ}{2r}g\right|+2K_0|\bar{h}|^{p-1}\right\}\left|\frac{\partial \bar{h}}{\partial r}\right|\\
&+\left\{\left|\frac{1}{2r}\frac{\partial}{\partial r}(g-\tilde{g})+\left(\frac{2Q^2}{3r^4}g-\frac{Q}{r^3}\frac{\partial Q}{\partial r}\right)g+\frac{i}{2r}\frac{\partial Q}{\partial r}g\right|\right.\\
&\left.+\left|\left(\frac{Q^2}{2r^3}-\frac{iQ}{2r}\right)\frac{\partial g}{\partial r}+\frac{iQ}{2r^2}g-\frac{1}{2r^2}(g-\tilde{g})\right|\right\}|\bar{h}|\\
=\; &(A_1+A_2+A_3+A_4)|\mathcal{F}|+A_5+(A_6+A_7)|\bar{h}|. \qquad (2.47)
\end{aligned}
$$

From (1.11) we compute

$$\frac{\partial \tilde{g}}{\partial r} = \frac{\partial \bar{g}}{\partial r} + \frac{1}{r^2}\int_0^r \frac{Q^2}{s^2} g ds - \frac{Q^2}{r^3} g + \frac{8\pi}{r^2}\int_0^r s^2 V(|\bar{h}|) g ds - 8\pi r V(|\bar{h}|) g. \tag{2.48}$$

Thus, using the formula, $\frac{\partial \bar{g}}{\partial r} = \frac{g-\bar{g}}{r}$, and the previous estimates (2.19) and (2.20) we have

$$\begin{aligned}
\left|\frac{\partial \tilde{g}}{\partial r}\right| &\le \frac{|g-\bar{g}|}{r} + \frac{1}{r^2}\int_0^r \frac{|Q|^2}{s^2} g ds + \frac{|Q|^2}{r^3} + \frac{8\pi}{r^2}\int_0^r s^2 |V(|\phi|)| ds + 8\pi r |V(|\phi|)| \\
&\le \frac{\pi x^2 r}{3(1+u)^3(1+r+u)^3} + \frac{16\pi^2 x^4 r}{3(1+u)^6(1+r+u)^3} + \frac{16\pi^2 x^4 r}{(1+u)^4(1+r+u)^4} \\
&\quad + \frac{8\pi K_0 r x^{p+1}}{3(1+u)^{p+3}(1+r+u)^4} + \frac{8\pi K_0 r x^{p+1}}{3(1+u)^{p+1}(1+r+u)^{p+1}} \\
&\le \frac{C_8(x^2+x^4+x^{p+1})r}{(1+u)^3(1+r+u)^3}.
\end{aligned} \tag{2.49}$$

As in (2.13), we have

$$\left|\frac{\partial g}{\partial r}\right| \le \frac{4\pi|h-\bar{h}|^2}{r} \le \frac{\pi x^2 r}{(1+u)^2(1+r+u)^4}. \tag{2.50}$$

The estimate (2.21) implies immediately

$$\frac{|g-\tilde{g}|}{2r^2} \le \frac{C_0(x^2+x^4+x^{p+1})}{2(1+u)^3(1+r+u)^3}. \tag{2.51}$$

Combining (2.49), (2.50) and (2.51), we immediately have

$$A_1 \le \frac{C_9(x^2+x^4+x^{p+1})}{(1+u)^2(1+r+u)^3}. \tag{2.52}$$

We calculate and estimate

$$\begin{aligned}
\left|\frac{\partial Q}{\partial r}\right| &\le 4\pi r|\bar{h}^* h - \bar{h} h^*| \le 8\pi r|\bar{h}||h| \\
&\le \frac{8\pi r x^2}{(1+u)(1+r+u)^3}.
\end{aligned} \tag{2.53}$$

This, combined with (2.18) provides us with

$$\begin{aligned}
A_2 &\le \frac{32\pi^2 x^4}{9(1+u)^4(1+r+u)^4} + \frac{32\pi^2 x^4}{(1+u)^3(1+r+u)^5} \\
&\le \frac{C_{10} x^4}{(1+u)^3(1+r+u)^4}.
\end{aligned} \tag{2.54}$$

As previously, (2.50) and (2.18) imply

$$A_3 \le \frac{8\pi^3 r^2 x^6}{(1+u)^6 (1+r+u)^8} \le \frac{8\pi^3 x^6}{(1+u)^6 (1+r+u)^6}, \tag{2.55}$$

and

$$A_4 \le \frac{|Q| g}{r^2} \le \frac{4\pi x^2}{(1+u)^2 (1+r+u)^2}. \tag{2.56}$$

Similarly, from previous estimates, we deduce

$$\begin{aligned} A_5 \quad &\le \quad \left(\left| \frac{Q^2}{2r^3} \right| + \left| \frac{g-\tilde{g}}{2r} \right| + \frac{|Q|}{2r} + 2K_0 |\bar{h}|^{p-1} \right) \frac{|h-\bar{h}|}{r} \\ &\le \quad \frac{C_{11}(x^3+x^5+x^p+x^{p+2})}{(1+u)^3 (1+r+u)^4}. \end{aligned} \tag{2.57}$$

By (2.53)

$$\left| \frac{i}{r} \frac{\partial Q}{\partial r} \right| \le \frac{8\pi x^2}{(1+u)(1+r+u)^3}. \tag{2.58}$$

This, and the previous estimates (2.49), (2.50) and (2.54) yield

$$\begin{aligned} A_6 \quad &\le \quad \frac{C_9(x^2+x^4+x^{p+1})}{(1+u)^2 (1+r+u)^3} + \frac{C_{10} x^4}{(1+u)^3 (1+r+u)^4} + \frac{4\pi x^2}{(1+u)(1+r+u)^3} \\ &\le \quad \frac{C_{12}(x^2+x^4+x^{p+1})}{(1+u)(1+r+u)^3}. \end{aligned} \tag{2.59}$$

By estimates (2.18) and (2.50) we have

$$\begin{aligned} A_7 \quad &\le \quad \left(\frac{|Q|^2}{2r^3} + \frac{|Q|}{2r} \right) \left| \frac{\partial g}{\partial r} \right| + \frac{|Q|}{2r^2} \\ &\le \quad \frac{8\pi^3 r^2 x^6}{(1+u)^6 (1+r+u)^6} + \frac{4\pi^2 x^3}{(1+u)^3 (1+r+u)^4} + \frac{2\pi x^2}{(1+u)^2 (1+r+u)^2} \\ &\le \quad \frac{C_{13}(x^2+x^3+x^6)}{(1+u)^2 (1+r+u)^2}. \end{aligned} \tag{2.60}$$

Summing (2.52)–(2.60) up, we obtain

$$\begin{aligned} |f_1| \quad &\le \quad \frac{C_{14}(x^2+x^4+x^6+x^p)|\mathcal{F}(u,r)|}{(1+u)^2 (1+r+u)^2} + \frac{C_{10}(x^3+x^5+x^p+x^{p+2})}{(1+u)^4 (1+r+u)^4} \\ &\quad + \left[\frac{C_{11}(x^2+x^4+x^{p+1})}{(1+u)(1+r+u)^3} + \frac{C_{13}(x^2+x^3+x^6)}{(1+u)^2 (1+r+u)^2} \right] |\bar{h}| \\ &\le \quad \frac{C_{14}(x^2+x^4+x^6+x^p)}{(1+u)^2 (1+r+u)^4} \sup_{u,r \ge 0} \left\{ (1+r+u)^2 |\mathcal{F}(u,r)| \right\} \\ &\quad + \frac{C_{15}(x^3+x^4+x^5+x^6++x^7+x^p+x^{p+2})}{(1+u)^2 (1+r+u)^3}. \end{aligned} \tag{2.61}$$

Now, using the inequality along the characteristics, as in (3.26) and (3.27), we know

$$1 + r(u) + \frac{u}{2} \geq k(1 + r_1 + \frac{u_1}{2}) \geq \frac{k}{2}(1 + r_1 + u_1), \tag{2.62}$$

and thus we obtain in general

$$\begin{aligned}
\int_0^{u_1} \left[\frac{r^s}{(1+u)^p\,(1+r+u)^q}\right]_\chi du &\leq \int_0^{u_1} \left[\frac{1}{(1+u)^p\,(1+r+u)^{q-s}}\right]_\chi du \\
&\leq \frac{2^m}{k^m\,(1+r_1+u_1)^m} \int_0^\infty \frac{du}{(1+u)^{q-s+p-m}} du \\
&= \frac{2^m}{(q-s+p-m-1)k^m\,(1+r_1+u_1)^m}
\end{aligned} \tag{2.63}$$

for $q-s+p-m>1$, where q,s,p,m are positive integers. Applying this inequality, we find that

$$\begin{aligned}
\int_0^{u_1} |[f_1]_\chi| du &\leq \frac{C_{16}(x^2+x^4+x^6+x^p)}{k\,(1+r_1+u_1)} \sup_{u,r\geq 0}\{(1+r+u)^2\,|\mathcal{F}(u,r)|\} \\
&\quad + \frac{8C_{15}(x^3+x^4+x^5+x^6++x^7+x^p+x^{p+2})}{k^3\,(1+r_1+u_1)^3} \\
&\leq \frac{C_{17}}{k^3\,(1+r_1+u_1)^3}\{(x^2+x^4+x^6+x^p)(d+x^3+x^p+x^{p+1}) \\
&\quad + \exp[C_6(x^2+x^4+x^{p+1})] + x^3+x^4+x^5+x^6++x^7+x^p+x^{p+2}\} \\
&\leq \frac{C_{17}\exp[C_6(x^2+x^4+x^{p+1})]}{k^3\,(1+r_1+u_1)^3} \\
&\quad \times\{(x^2+x^4+x^6+x^p)(d+x^3+x^p+x^{p+1})+ \\
&\quad +x^3+x^4+x^5+x^6++x^7+x^p+x^{p+2}\}.
\end{aligned} \tag{2.64}$$

Using the estimate (2.49), we have

$$\begin{aligned}
\int_0^{u_1} \frac{1}{2}\left|\left[\frac{\partial\tilde{g}}{\partial r}\right]_\chi\right| du &\leq C_7(x^2+x^4+x^{p+1})\int_0^{u_1}\left[\frac{r}{(1+u)^3\,(1+r+u)^3}\right]_\chi du \\
&\leq C_7(x^2+x^4+x^{p+1})\int_0^\infty \frac{1}{(1+u)^5} du \\
&\leq C_{18}(x^2+x^4+x^{p+1}).
\end{aligned} \tag{2.65}$$

Combining this with (2.39), we obtain

$$\begin{aligned}
&\left|\int_0^{u_1}\left[\frac{1}{2}\frac{\partial\tilde{g}}{\partial r} + \frac{g-\tilde{g}}{2r} - \frac{Q^2 g}{2r^3} - iA_0\right]_\chi du\right| \\
&\qquad \leq \int_0^{u_1}\left|\left[\frac{g-\tilde{g}}{2r} - \frac{Q^2 g}{2r^3} - iA_0\right]_\chi\right| du + \int_0^{u_1}\frac{1}{2}\left|\left[\frac{\partial\tilde{g}}{\partial r}\right]_\chi\right| du \\
&\qquad \leq C_{19}(x^2+x^4+x^{p+1}).
\end{aligned} \tag{2.66}$$

Similarly to (2.41) we estimate

$$\left|\frac{\partial h}{\partial r}(0,r_0)\right| \le \frac{\|h_0\|_{X_0}}{\left(1+r_0+\frac{u_0}{2}\right)^3} \le \frac{8d}{k^3\left(1+r_1+u_1\right)^3} \tag{2.67}$$

for all $x \in [0, x_1)$. Thus from (2.67) and (2.64)

$$\begin{aligned} |\mathcal{G}(u_1,r_1)| &\le \left(\left|\frac{\partial h}{\partial r}(0,r_0)\right| + \int_0^{u_1} |[f_1]_X| du\right) \exp[C_{19}(x^2+x^4+x^{p+1})] \\ &\le \frac{C_{21}\exp[C_{20}(x^2+x^4+x^{p+1})]}{k^3\left(1+r_1+u_1\right)^3}\{d+(x^2+x^4+x^6+x^p) \\ &\quad (d+x^3+x^5+x^{p+1})+x^3+x^4+x^5+x^6+x^7+x^p+x^{p+2}\}, \end{aligned} \tag{2.68}$$

where we set $C_{20} = C_6 + C_{19}$. From this we have

$$\begin{aligned} \sup_{u,r\ge 0}\left\{(1+r+u)^3\left|\frac{\partial \mathcal{F}(u,r)}{\partial r}\right|\right\} &\le \frac{C_{21}\exp[C_{20}(x^2+x^4+x^{p+1})]}{k^3}] \\ &\times\{d+(x^2+x^4+x^6+x^p)(d+x^3+x^5+x^{p+1}) \\ &+x^3+x^4+x^5+x^6+x^7+x^p+x^{p+2}\}. \end{aligned} \tag{2.69}$$

Combining (2.69) with (2.43), we thus have

$$\begin{aligned} \|\mathcal{F}(h)\|_X &\le \frac{C_7\exp[C_6(x^2+x^4+x^{p+1})]}{k^2}(d+x^3+x^5+x^p+x^{p+1}) \\ &\quad + \frac{C_{21}\exp[C_{20}(x^2+x^4+x^{p+1})]}{k^3}\{d+(x^2+x^4+x^6+x^p) \\ &\quad (d+x^3+x^5+x^{p+1})+x^3+x^4+x^5+x^6+x^7+x^p+x^{p+2}\} \\ &\le \frac{C_{22}}{k^3}\exp[C_{20}(x^2+x^4+x^{p+1})](2+x^2+x^4+x^6+x^p) \\ &\quad \times(d+x^3+x^4+x^5+x^6+x^7+x^p+x^{p+1}+x^{p+2}). \end{aligned} \tag{2.70}$$

Let us set

$$\begin{aligned} \Lambda_1(x) &= \frac{xk^3\exp\left[-C_{20}(x^2+x^4+x^{p+1})\right]}{C_{22}(2+x^2+x^4+x^6+x^p)} \\ &\quad -(x^3+x^4+x^5+x^6+x^7+x^p+x^{p+1}+x^{p+2}). \end{aligned} \tag{2.71}$$

Then, we find that $\Lambda_1(0) = 0, \Lambda_1'(0) > 0$, and $\Lambda_1(x) \to -\infty$ as $x \to \infty$. Thus there exists $x_0 \in (0, x_1)$ such that $\Lambda_1(x)$ is monotone increasing function on $[0, x_0]$. For every $x \in (0, x_0)$ we deduce that if $d < \Lambda_1(x)$, then $\|\mathcal{F}\|_X \le x$, and $\mathcal{F} : B(0,x) \to B(0,x)$.

We now show that the mapping $h \mapsto \mathcal{F}(h)$ contracts in Y. In order to prove this we consider equations (2.1) for h_1 and h_2 in X with $h_1(0,r) = h_2(0,r)$, and take difference between them. We put $\Theta = \mathcal{F}(h_1) - \mathcal{F}(h_2)$, and use the obvious notations $g_j = g(h_j)$, $Q_j = Q(h_j)$, $\mathcal{F}_j = \mathcal{F}(h_j)$, $\mathcal{G}_j = \mathcal{G}(h_j)$, and $A_{0j} = A_0(h_j)$ for $j = 1, 2$. We assume

$$\max\{\|h_1\|_X, \|h_2\|_X\} < x.$$

Then, we have

$$\begin{aligned}
\frac{\partial\Theta}{\partial u} - \frac{1}{2}\tilde{g}_1\frac{\partial\Theta}{\partial r} \;=\; & \frac{1}{2}(g_1 - \tilde{g}_2)\mathcal{G}_2 + \frac{1}{2r}(\tilde{g}_1 - \tilde{g}_1)(\mathcal{F}_1 - \mathcal{F}_2 - \bar{h}_1 + \bar{h}_2) \\
& + \frac{1}{2r}(g_1 - g_2 - \tilde{g}_1 + \tilde{g}_2)(\mathcal{F}_2 - \bar{h}_2) - \frac{1}{2r^3}(Q_1^2 - Q_2^2)(\mathcal{F}_2 - \bar{h}_2)g_1 \\
& - \frac{1}{2r^3}Q_2^2(\mathcal{F}_1 - \mathcal{F}_2 - \bar{h}_1 + \bar{h}_2)g_1 - \frac{1}{2r^3}Q_2^2(\mathcal{F}_2 - \bar{h}_2)(g_1 - g_2) \\
& - \frac{i}{2r}(Q_1 - Q_2)g_1\bar{h}_1 - \frac{i}{2r}Q_2(g_1 - g_2)\bar{h}_1 \\
& - \frac{i}{2r}Q_2 g_2(\bar{h}_1 - \bar{h}_2) - i(\mathcal{F}_1 - \mathcal{F}_2)A_{01} - i\mathcal{F}_2(A_{01} - A_{02}) \\
& - 4\pi(g_1 - g_2)r\frac{\partial V(|\bar{h}_1|)}{\partial \bar{h}_1^*} - 4\pi g_2 r\left[\frac{\partial V(|\bar{h}_1|)}{\partial \bar{h}_1^*} - \frac{\partial V(|\bar{h}_2|)}{\partial \bar{h}_2^*}\right].
\end{aligned}$$

Now, as previously, we use the characteristic $\chi_j = \chi_j(u,r)$ defined by

$$\frac{dr}{du} = -\tilde{g}_1(\chi_j(u,r),u) \qquad ; \qquad r(0) = r_0.$$

As previously we can represent Θ as

$$\Theta(u_1, r_1) = \int_0^{u_1} \exp\left\{\int_u^{u_1}\left[\frac{g_1 - \tilde{g}_1}{2r} - \frac{Q_2^2}{2r^3}g - iA_{01}\right]_{\chi_1} du'\right\}[\varphi]_{\chi_1}du,$$

where φ is, after rearrangement of terms,

$$\begin{aligned}
\varphi \;=\; & (\bar{h}_1 - \bar{h}_2)\left[\frac{Q_2^2 g_1}{2r^3} - \frac{iQ_2 g_2}{2r} - \frac{g_1 - \tilde{g}_1}{2r}\right] \\
& + (g_1 - g_2)\left[\frac{Q_2^2(\bar{h}_2 - \mathcal{F}_2)}{2r^3} - \frac{iQ_2\bar{h}_1}{2r} - 4\pi r\frac{\partial V(|\bar{h}_1|)}{\partial \bar{h}_1^*}\right] \\
& - (Q_1 - Q_2)\left[\frac{(Q_1 + Q_2)(\mathcal{F}_1 - \bar{h}_1)g_1}{2r^3} + \frac{ig_1\bar{h}_1}{2r}\right] \\
& - i(A_{01} - A_{02})\mathcal{F}_2 + \frac{1}{2}(\tilde{g}_1 - \tilde{g}_2)\mathcal{G}_2 \\
& + \frac{1}{2r}[g_1 - \tilde{g}_1 - (g_2 - \tilde{g}_2)](\mathcal{F}_2 - \bar{h}_2) \\
& - 4\pi g_2 r\left[\frac{\partial V(|\bar{h}_1|)}{\partial \bar{h}_1^*} - \frac{\partial V(|\bar{h}_2|)}{\partial \bar{h}_2^*}\right].
\end{aligned} \tag{2.72}$$

We write

$$
\begin{aligned}
|\varphi| \quad \leq \quad & |\bar{h}_1 - \bar{h}_2| \left| \frac{Q_2^2 g_1}{2r^3} - \frac{iQ_2 g_2}{2r} - \frac{g_1 - \tilde{g}_1}{2r} \right| \\
& + |g_1 - g_2| \left| \frac{Q_2^2(\bar{h}_2 - \mathcal{F}_2)}{2r^3} - \frac{iQ_2 \bar{h}_1}{2r} - 4\pi r \frac{\partial V(|\bar{h}_1|)}{\partial \bar{h}_1^*} \right| \\
& + |Q_1 - Q_2| \left| \frac{(Q_1 + Q_2)(\mathcal{F}_1 - \bar{h}_1) g_1}{2r^3} + \frac{i g_1 \bar{h}_1}{2r} \right| \\
& + |A_{01} - A_{02}| |\mathcal{F}_2| + \frac{1}{2} |\tilde{g}_1 - \tilde{g}_2| |\mathcal{G}_2| \\
& + \frac{1}{2r} |g_1 - \tilde{g}_1 - (g_2 - \tilde{g}_2)| |\mathcal{F}_2 - \bar{h}_2| + 4\pi g_2 r \left| \frac{\partial V(|\bar{h}_1|)}{\partial \bar{h}_1^*} - \frac{\partial V(|\bar{h}_2|)}{\partial \bar{h}_2^*} \right| \\
= \quad & B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7. \qquad (2.73)
\end{aligned}
$$

By the similar computations as before we have the following estimates

$$
|\bar{h}_1 - \bar{h}_2| \leq \frac{1}{r} \int_0^r |h_1 - h_2| ds \leq \frac{y}{(1+u)(1+r+u)}, \qquad (2.74)
$$

and

$$
\begin{aligned}
|h_1 - h_2 - (\bar{h}_1 - \bar{h}_2)| \quad & \leq \quad |h_1 - h_2| + |\bar{h}_1 - \bar{h}_2| \\
& \leq \quad y \left[\frac{1}{(1+r+u)^2} + \frac{1}{r} \int_0^r \frac{ds}{(1+s+u)^2} \right] \\
& \leq \quad \frac{2y}{(1+u)(1+r+u)}. \qquad (2.75)
\end{aligned}
$$

Thus, using (2.13), we estimate

$$
\begin{aligned}
\left| |h_1 - \bar{h}_1|^2 - |h_2 - \bar{h}_2|^2 \right| \quad & \leq \quad |(h_1 - h_2) - (\bar{h}_1 - \bar{h}_2)| (|h_1 - \bar{h}_1| + |h_2 - \bar{h}_2|) \\
& \leq \quad \frac{2xyr}{(1+u)^2 (1+r+u)^3}. \qquad (2.76)
\end{aligned}
$$

Thanks to the mean value theorem we have

$$
\begin{aligned}
|g_1 - g_2| \quad & \leq \quad 4\pi \int_r^\infty \frac{1}{s} \left| |h_1 - \bar{h}_1|^2 - |h_2 - \bar{h}_2|^2 \right| \\
& \leq \quad 8\pi \int_r^\infty \frac{sxy}{(1+u)^2 (1+s+u)^3} ds \\
& \leq \quad \frac{4\pi yx}{3(1+u)^2 (1+r+u)^2}. \qquad (2.77)
\end{aligned}
$$

And

$$\begin{aligned}
|\bar{g}_1 - \bar{g}_2| &\leq \frac{1}{r}\int_0^r |g_1 - g_2| ds \leq \frac{4\pi xy}{3r\,(1+u)^2}\int_0^r \frac{ds}{(1+s+u)^2} \\
&= \frac{4\pi xy}{3\,(1+u)^2\,(1+r+u)}. \qquad (2.78)
\end{aligned}$$

By the similar computation to (2.7)

$$\begin{aligned}
|Q_1 - Q_2| &\leq 4\pi \int_0^r s|\bar{h}_1^* h_1 - \bar{h}_1 h_1^* - \bar{h}_2^* h_2 + \bar{h}_2 h_2^*| ds \\
&\leq 8\pi \int_0^r s(|\bar{h}_1 - \bar{h}_2||h_1| + |h_1 - h_2||\bar{h}_2|) ds \\
&\leq \frac{8\pi r^2 xy}{(1+u)^2\,(1+r+u)^2}. \qquad (2.79)
\end{aligned}$$

We use (2.77) and (2.79) to estimate

$$\begin{aligned}
|A_{01} - A_{02}| &\leq \int_0^r \frac{|Q_1 - Q_2| g_1 + |g_1 - g_2||Q_2|}{s^2} ds \\
&\leq \int_0^r \frac{|Q_1 - Q_2|}{s^2} ds + \int_0^r \frac{|g_1 - g_2||Q_2|}{s^2} ds \\
&\leq \frac{8\pi xy}{(1+u)^2}\int_0^r \frac{ds}{(1+s+u)^2} + \frac{16\pi^2 xy}{(1+u)^4}\int_0^r \frac{ds}{(1+s+u)^4} \\
&= \frac{8\pi rxy}{(1+u)^3\,(1+r+u)} + \frac{16\pi^2 xy\left[\frac{r^3}{3} + r^2(1+\frac{u}{2}) + r(1+\frac{u}{2})\right]}{(1+u)^7\,(1+r+u)^3} \\
&\leq \frac{20\pi^2 xy}{(1+u)^3}. \qquad (2.80)
\end{aligned}$$

From (2.79) again

$$\begin{aligned}
\frac{1}{r}\int_0^r \frac{1}{s^2}|Q_1^2 - Q_2^2| ds &\leq \frac{1}{r}\int_0^r \frac{1}{s^2}|Q_1 - Q_2|(|Q_1| + |Q_2|) ds \\
&\leq \frac{8\pi xy}{(1+u)^4\, r}\int_0^r \frac{s^2}{(1+s+u)^4} ds \\
&= \frac{32\pi^2 r^2 xy}{3\,(1+u)^6\,(1+r+u)^3}. \qquad (2.81)
\end{aligned}$$

From (2.18) and (2.77) we have

$$\begin{aligned}
\frac{1}{r}\int_0^r \frac{1}{s^2}|Q_1|^2 |g_1 - g_2| ds &\leq \frac{64\pi^3 x^5 y}{3\,(1+u)^6\, r}\int_0^r \frac{s^2}{(1+s+u)^4} ds \\
&= \frac{64\pi^3 x^5 y r^2}{9\,(1+u)^6\,(1+r+u)^3}. \qquad (2.82)
\end{aligned}$$

By the mean value theorem we have

$$\begin{aligned}
|V(|\bar{h}_1|) - V(|\bar{h}_2|)| &\leq \int_0^1 \left|\frac{\partial V}{\partial \bar{h}}(|t\bar{h}_1 + (1-t)\bar{h}_2|)\right| |\bar{h}_1 - \bar{h}_2| \\
&\leq K_0 \int_0^1 |t\bar{h}_1 + (1-t)\bar{h}_2|^p |\bar{h}_1 - \bar{h}_2| \\
&\leq K_0 2^p (|\bar{h}_1|^p + |\bar{h}_2|^p)|\bar{h}_1 - \bar{h}_2| \\
&\leq \frac{K_0 2^p x^p y}{(1+u)^{p+1}(1+r+u)^{p+1}} \\
&\leq \frac{K_0 2^p x^p y}{(1+u)^4 (1+r+u)^4}, \qquad (2.83)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{r}\int_0^r s^2 |V(|\bar{h}_1|) - V(|\bar{h}_2|)| g ds &\leq \frac{K_0 2^p x^p y}{r(1+u)^4} \int_0^r \frac{s^2}{(1+s+u)^4} ds \\
&= \frac{K_0 2^{p+2} r^2 x^p y}{3(1+u)^6 (1+r+u)^3}, \qquad (2.84)
\end{aligned}$$

and

$$\begin{aligned}
\frac{8\pi}{r}\int_0^r s^2 |V(|\bar{h}_1|)||g_1 - g_2| ds &\leq \frac{256\pi^3 K_0 x^{p+2} y}{r(1+u)^{p+3}} \int_0^r \frac{s^2}{(1+s+u)^4} ds \\
&= \frac{256\pi^3 K_0 r^2 x^{p+2}}{(1+u)^{p+3}(1+r+u)^3}. \qquad (2.85)
\end{aligned}$$

From the estimates (2.78)–(2.85), we obtain

$$\begin{aligned}
|\tilde{g}_1 - \tilde{g}_2| &\leq |\bar{g}_1 - \bar{g}_2| + \frac{1}{r}\int_0^r \frac{1}{s^2}|Q_1^2 - Q_2^2| ds + \frac{1}{r}\int_0^r \frac{1}{s^2}|Q_1|^2 |g_1 - g_2| ds \\
&\quad + \frac{8\pi}{r}\int_0^r s^2 |V(|\bar{h}_1|) - V(|\bar{h}_2|)| g ds + \frac{8\pi}{r}\int_0^r s^2 |V(|\bar{h}_1|)||g_1 - g_2| ds \\
&\leq \frac{4\pi y x}{3(1+u)^2(1+r+u)^2} + \frac{32\pi^2 r^2 x y}{3(1+u)^6(1+r+u)^3} + \frac{64\pi^3 x^5 y r^2}{9(1+u)^6(1+r+u)^3} \\
&\quad + \frac{K_0 2^{p+5}\pi r^2 x^p y}{3(1+u)^6(1+r+u)^3} + \frac{256\pi^3 K_0 r^2 x^{p+2}}{(1+u)^{p+3}(1+r+u)^3} \\
&\quad + \frac{K_0 2^{p+4}\pi r x^p y}{3(1+u)^6(1+r+u)^3} + \frac{128\pi^3 K_0 r x^{p+2}}{(1+u)^{p+3}(1+r+u)^3} \\
&\leq \frac{C_{23}(x + x^5 + x^p + x^{p+2}) y}{(1+u)^3(1+r+u)}. \qquad (2.86)
\end{aligned}$$

Similarly to (2.77) we estimate

$$|g_1 - g_2 - (\bar{g}_1 - \bar{g}_2)| \leq \frac{1}{r}\int_0^r \int_{r'}^r \left|\frac{\partial}{\partial r}(g_1 - g_2)\right| ds dr'$$

$$\begin{aligned}
&\leq \frac{4\pi}{r}\int_0^r\int_{r'}^r \frac{1}{s}\left||h_1-\bar{h}_1|^2-|h_2-\bar{h}_2|^2\right| \\
&\leq \frac{8\pi xy}{r\,(1+u)^2}\int_0^r\int_{r'}^r \frac{ds}{(1+s+u)^3}dr' \\
&= \frac{4\pi rxy}{(1+u)^3\,(1+r+u)^2}. \qquad (2.87)
\end{aligned}$$

Combining (2.87) with (2.81), we have

$$\begin{aligned}
\frac{1}{2r}|g_1-\tilde{g}_1-(g_2-\tilde{g}_2)| &\leq \frac{1}{2r}|g_1-g_2-(\bar{g}_1-\bar{g}_2)|+\frac{1}{2r^2}\int_0^r\frac{1}{s^2}|Q_1^2-Q_2^2|ds \\
&\quad +\frac{1}{2r^2}\int_0^r\frac{1}{s^2}|Q_1|^2|g_1-g_2|ds+\frac{4\pi}{r^2}\int_0^r s^2|V(|\bar{h}_1|)-V(|\bar{h}_2|)|gds \\
&\leq \frac{C_{24}(x+x^5+x^p+x^{p+2})y}{(1+u)^3\,(1+r+u)^2}. \qquad (2.88)
\end{aligned}$$

From (2.74) and (2.21) we deduce

$$\begin{aligned}
B_1 &\leq |\bar{h}_1-\bar{h}_2|\left(\frac{|Q_2|^2}{2r^3}+\frac{|Q_2|}{2r}+\frac{|g_1-\tilde{g}_1|}{2r}\right) \\
&\leq \frac{C_{25}(x^2+x^4+x^{p+1})}{(1+u)^3\,(1+r+u)^3}. \qquad (2.89)
\end{aligned}$$

Similarly we have the following estimates:

$$\begin{aligned}
B_2 &\leq |g_1-g_2|\left(\frac{|Q_2|^2(|\bar{h}_2|+|\mathcal{F}_2|)}{2r^3}+\frac{|Q_2||\bar{h}_1|}{2r}+4\pi r\left|\frac{\partial V(|\bar{h}_1|)}{\partial \bar{h}_1^*}\right|\right) \\
&\leq \frac{32\pi^3 rx^6 y}{(1+u)^7\,(1+r+u)^7}+\frac{32\pi^2 x^5 ry}{(1+u)^6\,(1+r+u)^6}|\mathcal{F}_2| \\
&\quad +\frac{16\pi K_0 rx^{p+1}y}{3\,(1+u)^{p+2}\,(1+r+u)^{p+2}}+\frac{8\pi^2 rx^4 y}{(1+u)^5\,(1+r+u)^5}, \qquad (2.90)
\end{aligned}$$

$$\begin{aligned}
B_3 &\leq |Q_1-Q_2|\left(\frac{(|Q_1|+|Q_2|)(|\mathcal{F}_1|+|\bar{h}_1|)}{2r^3}+\frac{|\bar{h}_1|}{2r}\right) \\
&\leq \frac{32\pi^2 rx^3 y}{(1+u)^4\,(1+r+u)^4}|\mathcal{F}_1|+\frac{32\pi^2 rx^4 y}{(1+u)^5\,(1+r+u)^5} \\
&\quad +\frac{8\pi rx^2 y}{(1+u)^3\,(1+r+u)^3}, \qquad (2.91)
\end{aligned}$$

$$B_4 \leq \frac{8\pi^2 xy}{(1+u)^3\,(1+r+u)^2}|\mathcal{F}_2|, \qquad (2.92)$$

$$B_5 \leq \frac{C_{23}(x + x^5 + x^p + x^{p+2})y}{2\,(1+u)^3\,(1+r+u)}|\mathcal{G}_2|, \tag{2.93}$$

$$B_6 \leq \frac{C_{24}(x + x^5 + x^p + x^{p+2})y}{(1+u)^3\,(1+r+u)^2}|\mathcal{F}_2| + \frac{C_{24}(x^2 + x^6 + x^{p+1} + x^{p+3})y}{(1+u)^4\,(1+r+u)^3}, \tag{2.94}$$

and finally

$$\begin{aligned} B_7 &\leq 4\pi g_2 \int_0^1 \left| \frac{\partial^2 V}{\partial \bar{h} \partial \bar{h}^*}(t\bar{h}_1 + (1-t)\bar{h}_2) \right| |\bar{h}_1 - \bar{h}_2| dt \\ &\leq 2^{p+2}\pi K_0 r(|\bar{h}_1|^{p-1} + |\bar{h}_2|^{p-1})|\bar{h}_1 - \bar{h}_2| \\ &\leq \frac{2^{p+3}\pi K_0 r x^p y}{(1+u)^{p+1}\,(1+r+u)^{p+1}}. \end{aligned} \tag{2.95}$$

Summarizing the above estimates, we have

$$\begin{aligned} \sum_{j=1}^{7} B_j \leq\; & \frac{C_{26} y(x^2 + x^4 + x^6 + x^p + x^{p+1} + x^{p+3})}{(1+u)^3\,(1+r+u)^2} \\ &+ \frac{C_{27} y(x + x^3 + x^5 + x^p + x^{p+2}) \max_{j=1,2}\left[\sup_{r,u\geq 0}\{(1+r+u)^2\,|\mathcal{F}_j(u,r)|\}\right]}{(1+u)^3\,(1+r+u)^4} \\ &+ \frac{C_{23} y(x + x^3 + x^5 + x^p + x^{p+2}) \sup_{r,u\geq 0}\{(1+r+u)^3\,|\mathcal{G}_2(u,r)|\}}{2\,(1+u)^3\,(1+r+u)^4} \\ &\leq \frac{C_{26} y(x^2 + x^4 + x^6 + x^p + x^{p+1} + x^{p+3})}{(1+u)^3\,(1+r+u)^2} \\ &+ \frac{C_{28} y(x + x^3 + x^5 + x^p + x^{p+2}) \max\{\|\mathcal{F}_1\|_X, \|\mathcal{F}_2\|_X\}}{(1+u)^3\,(1+r+u)^4}. \end{aligned} \tag{2.96}$$

In the previous part of the proof (see paragraphs below (2.71)) we find that for each $j = 1, 2$ $\|\mathcal{F}_j\|_X = \|\mathcal{F}(h_j)\|_X < \|h_j\|_X = x_j \leq x$ for $x \in (0, x_0)$ and $d_j < \lambda_1(x_j)$, which we are assuming to hold. Thus, for $x \in (0, x_0)$ we have the estimate

$$\begin{aligned} \sum_{j=1}^{7} B_j &\leq \frac{C_{26}(x^2 + x^4 + x^6 + x^p + x^{p+1} + x^{p+3})y}{(1+u)^3\,(1+r+u)^2} \\ &\quad + \frac{C_{28}(x^2 + x^4 + x^6 + x^{p+1} + x^{p+3})y}{(1+u)^3\,(1+r+u)^4} \\ &\leq \frac{C_{29}(x^2 + x^4 + x^6 + x^p + x^{p+1} + x^{p+3})y}{(1+u)^3\,(1+r+u)^2}. \end{aligned} \tag{2.97}$$

Now, using the inequality, $1+r(u)+\frac{u}{2}\geq k(1+r_1+\frac{u_1}{2})\geq\frac{1}{2}(1+r_1+u_1)$ along the characteristics and the formula for the integrals as in (3.56), we estimate

$$\begin{aligned}\int_0^{u_1}|[\varphi]_\chi|du &\leq \int_0^{u_1}[\sum_{j=1}^7 B_j]_\chi du\\ &\leq \frac{C_{30}(x^2+x^4+x^6+x^p+x^{p+1}+x^{p+3})y}{k^2\,(1+r_1+u_1)^2}.\end{aligned}\tag{2.98}$$

On the other hand, similarly to (2.39), we have

$$\int_u^{u_1}\left|\left[\frac{g_1-\tilde{g}_1}{2r}-\frac{Q_2^2}{2r^3}g-iA_{01}\right]_{\chi_1}\right|du'\leq C_6(x^2+x^4+x^{p+1}).\tag{2.99}$$

Thus,

$$\begin{aligned}|\Theta(u_1,r_1)| &\leq \int_0^{u_1}\exp\left\{\int_u^{u_1}\left[\frac{g_1-\tilde{g}_1}{2r}-\frac{Q_2^2}{2r^3}g-iA_{01}\right]_{\chi_1}du'\right\}|[\varphi]_{\chi_1}|du\\ &\leq \frac{C_{30}(x^2+x^4+x^6+x^p+x^{p+1}+x^{p+3})y\exp[C_6(x^2+x^4+x^{p+1})]}{k^2\,(1+r_1+u_1)^2}.\end{aligned}$$

We finally have

$$\|\Theta\|_Y\leq\Lambda_2(x)y,$$

where we set

$$\Lambda_2(x)=\frac{C_{30}(x^2+x^4+x^6+x^p+x^{p+1}+x^{p+3})\exp[C_6(x^2+x^4+x^{p+1})]}{k^2(x)}.$$

The function $\Lambda_2(x)$ is a smooth, monotone increasing function on $[0,x_0]$, and $\Lambda_2(0)=0$, and there exists $x_2\in(0,x_0]$ such that $\Lambda_2(x)<1$ for all x in $(0,x_0]$. The mapping $h\mapsto\mathcal{F}(h)$ contracts in Y for $\|h\|_X\leq x_0$. This proves the global existence and uniqueness of solution. From (2.14) and (1.10) we find that $g\to1$ as $u\to\infty$ for each $r\geq0$. This, combined with (2.21), in turn, implies $\tilde{g}\to1$ as $u\to\infty$. Thus our metric, represented in the form of (0.1), becomes the Minkowski metric written in terms of the Bondi coordinate system. This completes the proof of Theorem 2.1. □

3 Completeness along the time-like lines

In the previous section we proved that the space-time is complete along the time-like and null geodesics toward the future for small initial data. The corresponding initial Bondi mass and charge decay out to infinity, and no final mass and charge remain. In this section we study completeness properties of the space-time in the case when the final Bondi mass and charge are positive numbers. Our study in this section also closely follows [6], but extends it substantially to the case of "charged black hole". Throughout this section we assume the Higgs potential function, $V(|\phi|) \geq 0$. We introduce the local mass function defined by

$$m(u,r) = \frac{r}{2}\left(1 - \frac{\tilde{g}}{g} + \frac{Q^2}{r^2}\right), \tag{3.1}$$

which represents the total mass inside a ball of radius r at the retarded time u. By the expression in (1.10) g is a monotone nondecreasing function of r, and thus from (1.11) we have

$$0 < \tilde{g} \leq \bar{g} \leq g \leq 1.$$

The fact that $\tilde{g} \leq g$, in particular, implies $m(u,r) \geq 0$. We set

$$\sup_{0<r<\infty} m(u,r) := M_1(u), \qquad \inf_{0<r<\infty} |Q(u,r)| := Q_1(u) \tag{3.2}$$

for each $u \in [0,\infty)$. Then, we assume

$$\lim_{u\to\infty} M_1(u) = M_2 < \infty, \qquad \lim_{u\to\infty} Q_1(u) = Q_2 < \infty \tag{3.3}$$

exist. Then, there exists $u_0 > 0$ such that $M_1(u) + Q_1(u) < \infty$ for all $u \geq u_0$. Let us introduce the real-valued function $R(u) = R(M_1(u), Q_1(u))$ defined on $[u_0, \infty)$ by

$$R(u) = \begin{cases} M_1 + \sqrt{M_1^2 - Q_1^2} & \text{if } M_1 > Q_1 \\ M_1 & \text{if } M_1 = Q_1 \\ 0 & \text{if } M_1 < Q_1 \end{cases}. \tag{3.4}$$

We set $R_0 = \lim_{u\to\infty} R(u) = R(M_2, Q_2)$. The purpose of this section is the proof of the following proposition.

Proposition 3.1 *Suppose the above assumption described in* (3.3) *holds. Then, the time-like line $r = r_1$ is complete toward future if $r_1 > R_0$.*

Proof. Given $u \geq u_0$, we consider $r_1 > R(u)$. Then, since $\lim_{r\to\infty} \tilde{g}(u,r) = 1$, we have

$$\begin{aligned} -\log \tilde{g}(u,r_1) &= \int_{r_1}^{\infty} \frac{1}{\tilde{g}}\frac{\partial \tilde{g}}{\partial r}(u,r)dr \\ &= \int_{r_1}^{\infty} \left(\frac{2m}{r^2} - \frac{2Q^2}{r^3} - 8\pi r V\right)\frac{g}{\tilde{g}}dr \\ &= \int_{r_1}^{\infty} \left(\frac{2m}{r^2} - \frac{2Q^2}{r^3} - 8\pi r V\right)\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr \\ &\leq \int_{r_1}^{\infty} \frac{2M_1(u)}{r^2}\left(1 - \frac{2M_1(u)}{r} + \frac{Q_1^2(u)}{r^2}\right)^{-1} dr \\ &= -\log\{f(r_1, M_1, Q_1)\}, \end{aligned} \tag{3.5}$$

where we set

$$f(r_1, M_1, Q_1)(u) = \begin{cases} \exp\left[-\frac{M_1}{\sqrt{M_1^2-Q_1^2}} \log\left(\frac{r_1-M_1+\sqrt{M_1^2-Q_1^2}}{r_1-M_1-\sqrt{M_1^2-Q_1^2}}\right)\right] & \text{if } M_1 > Q_1 \\ \exp\left[-\frac{2M_1}{r_1-M_1}\right] & \text{if } M_1 = Q_1 \\ \exp\left[-\pi M_1 + \frac{2M_1}{\sqrt{Q_1^2-M_1^2}} \arctan\left(\frac{r_1-M_1}{\sqrt{Q_1^2-M_1^2}}\right)\right] & \text{if } M_1 < Q_1 \end{cases} .$$

Now, we fix $r_1 > R_0$. Let $u_1 > u_0$ be chosen so that $|R_0 - R(u)| < r_1 - R_0$ if $u > u_1$. Then, $r_1 > R(u)$ for all $u > u_1$. Since

$$\lim_{u\to\infty} f(r_1, M_1, Q_1)(u) = f(r_1, M_2, Q_2) > 0,$$

we can choose $u_2 > u_1$ so that

$$f(r_1, M_1, Q_1)(u) \geq \frac{1}{2} f(r_1, M_2, Q_2) \qquad \text{if } u > u_2. \tag{3.6}$$

Let $u_3 > u_2$. From the fact $g \geq \tilde{g}$, we estimate

$$\begin{aligned} \int_0^{u_3} \sqrt{g\tilde{g}}(u, r_1)du &\geq \int_0^{u_3} \tilde{g}(u, r_1)du \geq \int_{u_0}^{u_3} f(r_1, M_1, Q_1)(u)du \\ &\geq \int_{u_2}^{u_3} \frac{1}{2} f(r_1, M_2, Q_2)du = \frac{1}{2} f(r_1, M_2, Q_2)(u_3 - u_2) \to \infty \end{aligned} \tag{3.7}$$

as $u_3 \to \infty$, i.e., the proper time along the line $r = r_1$ tends to infinity as the parameter $u \to \infty$. This completes the proof of the proposition. □

Remark 3.1. If we assume only $\lim_{u\to\infty} M_1(u) < \infty$ instead of (3.2) and (3.3), then by the obvious modification of the estimate in (3.5) (just ignoring the terms with Q_1^2) we can infer that the time-like line $r = r_1$ is complete toward future if $r_1 > 2M_2$.

Remark 3.2. In the case of real scalar field $\phi = \phi^*$ we have $Q = 0$, and our R_0 becomes the Schwartzschild radius independent of the form of nonlinearity in $V(|\phi|)$.

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