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Global existence of strong solutions for incompressible hydrodynamic flow of liquid crystals with vacuum

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Abstract. We consider the Cauchy problem for incompressible hydrodynamic flow of nematic liquid crystals in three dimensions. We prove the global existence and uniqueness of the strong solutions with nonnegative ρ_0 and small initial data.

1. Introduction

In this paper, we study the following incompressible hydrodynamic flow of nematic liquid crystals in $\mathbb{R}^3 \times (0, +\infty)$ (See [5, 10, 22]):

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

$$(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = \mu \Delta \mathbf{u} - \lambda \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \tag{2}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{3}$$

$$\mathbf{d}_t + (\mathbf{u} \cdot \nabla)\mathbf{d} = \theta(\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}). \tag{4}$$

Here $\rho : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{R}^1$ denotes the density function of the fluid, $\mathbf{u} : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{R}^3$ denotes the velocity field of the fluid, $\mathbf{d} : \mathbb{R}^3 \times (0, +\infty) \to \mathbf{S}^2$ denotes the macroscopic average of the nematic liquid crystal orientation field, and $P(x, t)$ is a scalar function representing the pressure. $\mu > 0$, $\lambda > 0$, $\theta > 0$ are viscosity of the fluid, competition between kinetic and potential energy, and microscopic elastic relaxation time respectively. The symbol $\otimes$ is the usual Kronecker multiplication, e.g. $\mathbf{u} \otimes \mathbf{u} = (u_i u_j)_{1 \le i,j \le 3}$, and the notation $\nabla \mathbf{d} \odot \nabla \mathbf{d}$ denotes the 3×3 matrix whose (i, j)-th entry is given by $\nabla_i \mathbf{d} \cdot \nabla_j \mathbf{d}$, for $1 \le i, j \le 3$.

We consider system (1)–(4) equipped with the following initial conditions:

$$(\rho, \mathbf{u}, \mathbf{d})(x, 0) = (\rho_0, \mathbf{u}_0, \mathbf{d}_0), \text{ with } \nabla \cdot \mathbf{u}_0 = 0, \tag{5}$$

and the following boundary conditions (see also [7]):

$$\rho, \mathbf{u} \text{ vanish at infinity and } \mathbf{d} \text{ is constant at infinity (in some weak sense).} \tag{6}$$

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The hydrodynamic flow of incompressible liquid crystals was first derived by Ericksen [5] and Leslie [10] in 1960s. However, its rigorous mathematical analysis did not take place until 1990s, when Lin [12] and Lin-Liu [14–16] addressed the existence and partial regularity theory of suitable weak solutions to the incompressible hydrodynamic flow of liquid crystals of variable length. More precisely, they considered the homogeneous case $\rho \equiv 1$ and the approximate equation of incompressible hydrodynamic flow of liquid crystals $\Big($i.e., $|\nabla \mathbf{d}|^2\mathbf{d}$ is replaced by $\dfrac{(1-|\mathbf{d}|^2)\mathbf{d}}{\epsilon^2}\Big)$, and proved in [14] the local existence of classical solutions and the global existence of weak solutions in dimension two and three. For any fixed $\epsilon > 0$, they also showed the existence and uniqueness of global classical solutions either in dimension two or dimension three when the fluid viscosity μ is sufficiently large; in [15], Lin and Liu extended the classical theorem by Caffarelli-Kohn-Nirenberg [1] on the Navier-Stokes equation that asserts the one dimensional parabolic Hausdorff measure of the singular set of any *suitable* weak solution is zero. See also [17, 20] for relevant results. It is a very interesting question to ask whether there exists a global weak solution for the incompressible hydrodynamic flow equations (1)–(4) similar to the Leray-Hopf type solutions in the context of the Navier-Stokes equation. This question has been answered firmly by [13] when $N = 2$ and $\rho = 1$. When $\rho \neq$ constant, Liu and Zhang in [19] obtained the global weak solutions in dimension three with the initial density $\rho_0 \in L^2$. Jiang and Tan in [8] improved the condition of ρ_0, i.e. $\rho_0 \in L^\gamma$, $\gamma > \dfrac{3}{2}$. However, the estimates depend on ε, and therefore one cannot take the limit $\varepsilon \to 0$. Wen and Ding in [22] proved the local existence and uniqueness of the strong solutions to the model (1)–(4) for a bounded domain in $\mathbb{R}^N$ ($N = 2$ or 3), provided that the initial density $\rho_0 \geq 0$. Furthermore, they got the global existence and uniqueness of the strong solutions with small enough initial data and $\inf\limits_{x\in\Omega} \rho_0 > 0$ in 2D. Very recently, Li and Wang proved in [11] the existence and uniqueness of the local strong solutions with large initial data and the global strong solutions with small data in Besov space for the initial density away from vacuum in 3D. It leads us to focus on the global existence and uniqueness of strong solutions with small enough initial data and nonnegative ρ_0 for the model in 3D.

2. Main results

Before stating the main results, we explain the notations and conventions used throughout this paper. We denote

$$\int f = \int_{\mathbb{R}^3} f dx \text{ and } \int_0^t \int f = \int_0^t \int_{\mathbb{R}^3} f dx dt.$$

For $1 \leq r \leq \infty$, we denote the standard Sobolev spaces as follows:

$$L^r = L^r(\mathbb{R}^3),\ D^{k,r} = \left\{ v \in L^1_{loc}(\mathbb{R}^3) : |\nabla^k v|_{L^r} < \infty \right\};$$
$$W^{k,r} = L^r \cap D^{k,r},\ H^k = W^{k,2},\ D^k = D^{k,2},\ D^1 = \left\{ v \in L^6 : |\nabla v|_{L^2} < \infty \right\},$$

here $|\cdot|_{L^r}$ and $|\cdot|_{W^{k,r}}$ denote the norm in L^r and $W^{k,r}$ respectively.

Our main results are stated as follows:

Theorem 2.1. *Assume that $\rho_0 \geq 0$, $\rho_0 \in H^1 \cap L^\infty$, $\mathbf{u}_0 \in D^2 \cap D^1$, $\mathbf{d}_0 \in D^1 \cap D^3(\mathbb{R}^3, \mathbf{S}^2)$, and the following compatible conditions are valid*

$$\mu\Delta\mathbf{u}_0 - \nabla P_0 - \lambda\nabla\cdot(\nabla\mathbf{d}_0 \odot \nabla\mathbf{d}_0) = \sqrt{\rho_0}\mathbf{g}, \text{ in } \mathbb{R}^3, \tag{7}$$

for some $(P_0, \mathbf{g}) \in H^1 \times L^2$. There exists a sufficiently small positive constant ε_0 such that if

$$|\sqrt{\rho_0}\mathbf{u}_0|^2_{L^2} + |\nabla\mathbf{u}_0|^2_{H^1} + |\nabla\mathbf{d}_0|^2_{H^2} + |\mathbf{g}|^2_{L^2} < \varepsilon_0, \tag{8}$$

then for arbitrary positive T, problems (1)–(6) admit a unique strong solution $(\rho, \mathbf{u}, \mathbf{d})$ *satisfying*

$$\begin{aligned}
&\rho \in C([0,T]; H^1) \cap L^\infty(Q_T),\ \rho_t \in C([0,T]; L^2),\\
&\nabla \mathbf{u} \in C([0,T]; H^1) \cap L^2([0,T]; W^{1,6}),\ \mathbf{u}_t \in L^2([0,T]; D^1),\ \ \sqrt{\rho}\mathbf{u}_t \in L^\infty([0,T]; L^2),\\
&P \in C([0,T]; H^1) \cap L^2([0,T]; W^{1,6}),\ |\mathbf{d}| = 1,\ \text{in } \mathbb{R}^3 \times [0,T],\\
&\nabla \mathbf{d} \in C([0,T]; H^2) \cap L^2([0,T]; H^3),\ \mathbf{d}_t \in C([0,T]; H^1) \cap L^2([0,T]; H^2).
\end{aligned}$$

3. Preliminaries

In this section, we give some lemmas which will be used in the next section.

Lemma 3.1. *(Interpolation inequality) Assume* $1 \le s \le r \le t \le \infty$ *and*

$$\frac{1}{r} = \frac{\vartheta}{s} + \frac{1-\vartheta}{t}. \tag{9}$$

Suppose also $f \in L^s \cap L^t$*. Then* $f \in L^r$*, and*

$$|f|_{L^r} \le C|f|_{L^s}^{\vartheta} |f|_{L^t}^{1-\vartheta}. \tag{10}$$

Lemma 3.2. *(Gagliardo-Nirenberg inequality) For* $p \in [2,6]$*,* $q \in (1,\infty)$*, and* $r \in (3,\infty)$*, there exists some generic constant* $C > 0$ *which may depend on* q*,* r *such that for* $f \in H^1$ *and* $g \in L^q \cap D^{1,r}$*, we have*

$$|f|_{L^p}^p \le C|f|_{L^2}^{\frac{6-p}{2}} |\nabla f|_{L^2}^{\frac{3p-6}{2}}, \tag{11}$$

and

$$|g|_{C(\mathbb{R}^3)} \le C|g|_{L^q}^{\frac{q(r-3)}{3r+q(r-3)}} |\nabla g|_{L^r}^{\frac{3r}{3r+q(r-3)}}. \tag{12}$$

One interesting case is when $q = 2$ and $r = 6$, we recover the $C(\mathbb{R}^3)$ norm stated above.

4. Proof of main results

In this section we establish some a priori estimates globally in time by a modified energy method motivated by [2, 3] and then prove Theorem 2.1. The local existence and uniqueness of solution for problems (1)–(6) can be proved by a similar iteration procedure shown in [18, 22] or Galerkin's method shown in [7, 14, 21] and a standard domain expansion technique mentioned in [4, 7]. For simplicity, we omit the proof in this paper. In the following, we denote by C the generic constants dependent on μ, λ, θ and the initial data, but independent of ρ, $\mathbf{u}$, $\mathbf{d}$ and T.

Lemma 4.1. *(Basic energy law) For any* $t \ge 0$*, it holds*

$$\begin{aligned}
&\int (\rho|\mathbf{u}|^2 + \lambda|\nabla \mathbf{d}|^2) + 2\mu \int_0^t \int |\nabla \mathbf{u}|^2 + 2\lambda\theta \int_0^t \int |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2\\
= &\int (\rho_0|\mathbf{u}_0|^2 + \lambda|\nabla \mathbf{d}_0|^2).
\end{aligned} \tag{13}$$

and

$$0 \le \rho(x,t) \le \sup_{x \in \mathbb{R}^3} \rho_0(x). \tag{14}$$

Proof. Firstly, we rewrite (2) by (1) into

$$\rho\mathbf{u}_t + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \mu\Delta\mathbf{u} - \lambda\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d}). \tag{15}$$

Then multiplying (15) by $\mathbf{u}$, and then integrating over $\mathbb{R}^3$, we use (3) and integration by parts to give

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho|\mathbf{u}|^2 + \mu\int|\nabla\mathbf{u}|^2\\ = \ & -\lambda\int\left[\Delta\mathbf{d}\cdot\nabla\mathbf{d} + \nabla\left(\frac{|\nabla\mathbf{d}|^2}{2}\right)\right]\cdot\mathbf{u} = -\lambda\int(\mathbf{u}\cdot\nabla)\mathbf{d}\cdot\Delta\mathbf{d}.\end{aligned} \tag{16}$$

Here $\Delta\mathbf{d}\cdot\nabla\mathbf{d} = \sum_{i=1}^3\Delta\mathbf{d}_i\nabla\mathbf{d}_i$. Then multiplying (4) by $(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d})$, and then integrating over $\mathbb{R}^3$, one obtains

$$\int(\mathbf{d}_t + \mathbf{u}\cdot\nabla\mathbf{d})\cdot\Delta\mathbf{d} = \theta\int|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2, \tag{17}$$

where we have used the fact that $|\mathbf{d}| = 1$ to get

$$(\mathbf{d}_t + \mathbf{u}\cdot\nabla\mathbf{d})\cdot|\nabla\mathbf{d}|^2\mathbf{d} = \frac{1}{2}\left[|\nabla\mathbf{d}|^2(|\mathbf{d}|^2)_t + \mathbf{u}\cdot\nabla(|\mathbf{d}|^2)|\nabla\mathbf{d}|^2\right] = 0. \tag{18}$$

By using integration by parts and (6), we have

$$\int\mathbf{d}_t\cdot\Delta\mathbf{d} = -\frac{1}{2}\frac{d}{dt}\int|\nabla\mathbf{d}|^2. \tag{19}$$

Hence we obtain

$$\frac{\lambda}{2}\frac{d}{dt}\int|\nabla\mathbf{d}|^2 + \lambda\theta\int|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2 = \lambda\int(\mathbf{u}\cdot\nabla)\mathbf{d}\cdot\Delta\mathbf{d}. \tag{20}$$

It is easy to see that, by adding (16) and (20) and then integrating over $[0, t]$, (13) follows.
Finally, (14) follows by the characteristic method (cf. [9]). □

Lemma 4.2. *For any $t \geq 0$, it holds*

$$\begin{aligned}&\sup_{t\geq 0}\left(|\nabla\mathbf{u}|_{L^2}^2 + |\nabla\mathbf{d}|_{H^2}^2 + |\sqrt{\rho}\mathbf{u}_t|_{L^2}^2 + |\mathbf{d}_t|_{H^1}^2\right)\\ + \ & \int_0^t\left(|\nabla\mathbf{u}|_{H^1}^2 + |\Delta\mathbf{d}|_{H^2}^2 + |\sqrt{\rho}\mathbf{u}_t|_{L^2}^2 + |\nabla\mathbf{u}_t|_{L^2}^2 + |\nabla\mathbf{d}_t|_{H^1}^2\right) \leq C.\end{aligned} \tag{21}$$

Proof. First of all, taking the inner product of (4) by $\Delta\mathbf{d}$, and then by using integration by parts and the fact $|\mathbf{d}| = 1$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\nabla\mathbf{d}|^2 + \theta\int|\Delta\mathbf{d}|^2 &= \theta\int|\nabla\mathbf{d}|^4 + \int(\mathbf{u}\cdot\nabla)\mathbf{d}\cdot\Delta\mathbf{d}\\ &\leq C|\nabla\mathbf{d}|_{L^2}|\Delta\mathbf{d}|_{L^2}^3 + \int(\mathbf{u}\cdot\nabla)\mathbf{d}\cdot\Delta\mathbf{d}\\ &\leq C|\nabla\mathbf{d}|_{H^1}^2|\Delta\mathbf{d}|_{L^2}^2 + \int(\mathbf{u}\cdot\nabla)\mathbf{d}\cdot\Delta\mathbf{d}.\end{aligned} \tag{22}$$

Here we have used the following fact obtained from Lemma 3.2 and the elliptic estimate in the whole space $\mathbb{R}^3$:

$$|\nabla\mathbf{d}|_{L^4}^4 \leq C|\nabla\mathbf{d}|_{L^2}|\nabla^2\mathbf{d}|_{L^2}^3 \leq C|\nabla\mathbf{d}|_{L^2}|\Delta\mathbf{d}|_{L^2}^3, \tag{23}$$

and this term is the main difficulty for the problem considered in a bounded domain. Then applying ∇ to (4), one obtains

$$\nabla \mathbf{d}_t + \nabla \mathbf{u} \cdot \nabla \mathbf{d} + \mathbf{u} \cdot \nabla^2 \mathbf{d} = \theta \nabla \Delta \mathbf{d} + 2\theta (\nabla \mathbf{d} : \nabla^2 \mathbf{d})\mathbf{d} + \theta |\nabla \mathbf{d}|^2 \nabla \mathbf{d}, \tag{24}$$

and then multiplying (24) by $\nabla \Delta \mathbf{d}$ and using integration by parts, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int |\Delta \mathbf{d}|^2 + \theta \int |\nabla \Delta \mathbf{d}|^2 \\ = & \int \left[\nabla \mathbf{u} \cdot \nabla \mathbf{d} + \mathbf{u} \cdot \nabla^2 \mathbf{d} - 2\theta (\nabla \mathbf{d} : \nabla^2 \mathbf{d})\mathbf{d} - \theta |\nabla \mathbf{d}|^2 \nabla \mathbf{d}\right] \cdot \nabla \Delta \mathbf{d} \\ \leq & \int |\nabla \mathbf{u}||\nabla \mathbf{d}||\nabla \Delta \mathbf{d}| + \int |\mathbf{u}||\nabla^2 \mathbf{d}||\nabla \Delta \mathbf{d}| \\ & +2\theta \int |\nabla \mathbf{d}||\nabla^2 \mathbf{d}||\nabla \Delta \mathbf{d}| + \theta \int |\nabla \mathbf{d}|^3 |\nabla \Delta \mathbf{d}| \\ = & I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{25}$$

The Hölder inequality, Lemma 3.2, Lemma 4.1, the elliptic estimate in $\mathbb{R}^3$ and the Cauchy inequality imply that

$$\begin{aligned} I_1 & \leq C|\nabla \mathbf{u}|_{L^6} |\nabla \mathbf{d}|_{L^3} |\nabla \Delta \mathbf{d}|_{L^2} \\ & \leq C|\Delta \mathbf{u}|_{L^2} |\nabla \mathbf{d}|_{L^2}^{\frac{1}{2}} |\Delta \mathbf{d}|_{L^2}^{\frac{1}{2}} |\nabla \Delta \mathbf{d}|_{L^2} \leq C|\Delta \mathbf{d}|_{L^2}^{\frac{1}{2}} \left(|\Delta \mathbf{u}|_{L^2}^2 + |\nabla \Delta \mathbf{d}|_{L^2}^2\right), \\ I_2 & \leq C|\mathbf{u}|_{L^6} |\nabla^2 \mathbf{d}|_{L^3} |\nabla \Delta \mathbf{d}|_{L^2} \leq C|\nabla \mathbf{u}|_{L^2} |\Delta \mathbf{d}|_{H^1}^2, \\ I_3 & \leq C|\nabla \mathbf{d}|_{L^6} |\nabla^2 \mathbf{d}|_{L^3} |\nabla \Delta \mathbf{d}|_{L^2} \leq C|\Delta \mathbf{d}|_{L^2} |\Delta \mathbf{d}|_{H^1}^2, \\ I_4 & \leq C|\nabla \mathbf{d}|_{L^6}^3 |\nabla \Delta \mathbf{d}|_{L^2} \leq C|\Delta \mathbf{d}|_{L^2}^2 |\Delta \mathbf{d}|_{H^1}^2. \end{aligned}$$

On the other hand, multiplying (15) by $\mathbf{u}_t$, we use (3), (14) and integration by parts to give

$$\begin{aligned} & \frac{\mu}{2}\frac{d}{dt}\int |\nabla \mathbf{u}|^2 + \int \rho |\mathbf{u}_t|^2 \\ = & -\int \rho (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{u}_t - \lambda \int \Delta \mathbf{d} \cdot \nabla \mathbf{d} \cdot \mathbf{u}_t \\ \leq & C|\sqrt{\rho}|_{L^\infty} |\mathbf{u}|_{L^6} |\nabla \mathbf{u}|_{L^3} |\sqrt{\rho}\mathbf{u}_t|_{L^2} + |\Delta \mathbf{d}|_{L^2} |\nabla \mathbf{d}|_{L^3} |\mathbf{u}_t|_{L^6} \\ \leq & \frac{1}{2}|\sqrt{\rho}\mathbf{u}_t|_{L^2}^2 + C|\nabla \mathbf{u}|_{L^2}^2 |\nabla \mathbf{u}|_{H^1}^2 + C|\Delta \mathbf{d}|_{L^2}^{\frac{1}{2}} \left(|\Delta \mathbf{d}|_{L^2}^2 + |\nabla \mathbf{u}_t|_{L^2}^2\right). \end{aligned} \tag{26}$$

It follows from the estimates for the stationary Stokes equations (see [6]), (14), the Hölder inequality and Lemma 3.2 that

$$\begin{aligned} |\nabla^2 \mathbf{u}|_{L^2}^2 & \leq C|-\rho \mathbf{u}_t - \rho (\mathbf{u} \cdot \nabla)\mathbf{u} - \lambda \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d})|_{L^2}^2. \\ & \leq C|\rho|_{L^\infty} |\sqrt{\rho}\mathbf{u}_t|_{L^2}^2 + C|\mathbf{u}|_{L^6}^2 |\nabla \mathbf{u}|_{L^3}^2 + C|\nabla^2 \mathbf{d}|_{L^3}^2 |\nabla \mathbf{d}|_{L^6}^2 \\ & \leq C|\sqrt{\rho}\mathbf{u}_t|_{L^2}^2 + C|\nabla \mathbf{u}|_{L^2}^2 |\nabla \mathbf{u}|_{H^1}^2 + C|\Delta \mathbf{d}|_{L^2}^2 |\Delta \mathbf{d}|_{H^1}^2. \end{aligned} \tag{27}$$

We infer from (26) and (27) that

$$\begin{aligned} & \frac{d}{dt}\int |\nabla \mathbf{u}|^2 + \int |\nabla^2 \mathbf{u}|^2 + \int \rho |\mathbf{u}_t|^2 \\ \leq & C\left(|\nabla \mathbf{u}|_{L^2} + |\nabla \mathbf{u}|_{L^2}^2 + |\Delta \mathbf{d}|_{L^2}^{\frac{1}{2}} + |\Delta \mathbf{d}|_{L^2}^2\right)\left(|\nabla \mathbf{u}|_{H^1}^2 + |\nabla \mathbf{u}_t|_{L^2}^2 + |\Delta \mathbf{d}|_{H^1}^2\right). \end{aligned} \tag{28}$$

Secondly, applying ∂_t to (15), we have

$$\begin{aligned}
& \rho\mathbf{u}_{tt} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u}_t - \mu\Delta\mathbf{u}_t + \nabla P_t \\
= \quad & -\rho_t\left[\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] - \rho\mathbf{u}_t\cdot\nabla\mathbf{u} - \left[\Delta\mathbf{d}_t\cdot\nabla\mathbf{d} + \Delta\mathbf{d}\cdot\nabla\mathbf{d}_t + \nabla\left(\frac{|\nabla\mathbf{d}|^2}{2}\right)_t\right].
\end{aligned} \tag{29}$$

Then multiplying (29) by $\mathbf{u}_t$, and then using (1), (3) and integration by parts, one obtains

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int\rho|\mathbf{u}_t|^2 + \mu\int|\nabla\mathbf{u}_t|^2 \\
= \quad & -\int\rho_t\left[\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\right]\cdot\mathbf{u}_t - \int\rho(\mathbf{u}_t\cdot\nabla)\mathbf{u}\cdot\mathbf{u}_t - \int(\Delta\mathbf{d}_t\cdot\nabla\mathbf{d} + \Delta\mathbf{d}\cdot\nabla\mathbf{d}_t)\cdot\mathbf{u}_t \\
= \quad & -\int\rho\mathbf{u}\cdot\left(\nabla\mathbf{u}_t + \nabla\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla^2\mathbf{u}\right)\cdot\mathbf{u}_t - \int\rho\mathbf{u}\cdot\left[\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\right]\cdot\nabla\mathbf{u}_t \\
& -\int\rho(\mathbf{u}_t\cdot\nabla)\mathbf{u}\cdot\mathbf{u}_t - \int(\Delta\mathbf{d}_t\cdot\nabla\mathbf{d} + \Delta\mathbf{d}\cdot\nabla\mathbf{d}_t)\cdot\mathbf{u}_t \\
\leq \quad & \int 2\rho|\mathbf{u}||\nabla\mathbf{u}_t||\mathbf{u}_t| + \rho|\mathbf{u}||\nabla\mathbf{u}|^2|\mathbf{u}_t| + \rho|\mathbf{u}|^2|\nabla^2\mathbf{u}||\mathbf{u}_t| + \rho|\mathbf{u}|^2|\nabla\mathbf{u}||\nabla\mathbf{u}_t| \\
& +\rho|\mathbf{u}_t|^2|\nabla\mathbf{u}| + |\Delta\mathbf{d}_t||\nabla\mathbf{d}||\mathbf{u}_t| + |\Delta\mathbf{d}||\nabla\mathbf{d}_t||\mathbf{u}_t| \\
= \quad & J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7.
\end{aligned} \tag{30}$$

It yields from the Hölder inequality, Lemma 3.1, Lemma 3.2, Lemma 4.1, the elliptic estimate in $\mathbb{R}^3$ and the Cauchy inequality that

$$\begin{aligned}
J_1 &\leq C|\rho\mathbf{u}|_{L^3}|\nabla\mathbf{u}_t|_{L^2}|\mathbf{u}_t|_{L^6} \leq C|\rho\mathbf{u}|_{L^2}^{\frac{1}{2}}|\rho\mathbf{u}|_{L^6}^{\frac{1}{2}}|\nabla\mathbf{u}_t|_{L^2}^2 \leq C|\nabla\mathbf{u}|_{L^2}^{\frac{1}{2}}|\nabla\mathbf{u}_t|_{L^2}^2, \\
J_2 &\leq C|\mathbf{u}|_{L^6}|\nabla\mathbf{u}|_{L^2}|\nabla\mathbf{u}|_{L^6}|\mathbf{u}_t|_{L^6} \leq |\nabla\mathbf{u}|_{L^2}^2\left(|\Delta\mathbf{u}|_{L^2}^2 + |\nabla\mathbf{u}_t|_{L^2}^2\right), \\
J_3 &\leq C|\mathbf{u}|_{L^6}^2|\nabla^2\mathbf{u}|_{L^2}|\mathbf{u}_t|_{L^6} \leq C|\nabla\mathbf{u}|_{L^2}^2\left(|\Delta\mathbf{u}|_{L^2}^2 + |\nabla\mathbf{u}_t|_{L^2}^2\right), \\
J_4 &\leq C|\mathbf{u}|_{L^6}^2|\nabla\mathbf{u}|_{L^6}|\nabla\mathbf{u}_t|_{L^2} \leq C|\nabla\mathbf{u}|_{L^2}^2\left(|\Delta\mathbf{u}|_{L^2}^2 + |\nabla\mathbf{u}_t|_{L^2}^2\right), \\
J_5 &\leq C|\sqrt{\rho}\mathbf{u}_t|_{L^3}|\nabla\mathbf{u}|_{L^2}|\mathbf{u}_t|_{L^6} \leq C|\sqrt{\rho}\mathbf{u}_t|_{L^2}^{\frac{1}{2}}|\mathbf{u}_t|_{L^6}^{\frac{3}{2}}|\nabla\mathbf{u}|_{L^2} \\
&\leq C|\sqrt{\rho}\mathbf{u}_t|_{L^2}^{\frac{1}{2}}|\nabla\mathbf{u}_t|_{L^2}^{\frac{3}{2}}|\nabla\mathbf{u}|_{L^2} \leq |\nabla\mathbf{u}|_{L^2}\left(|\sqrt{\rho}\mathbf{u}_t|_{L^2}^2 + |\nabla\mathbf{u}_t|_{L^2}^2\right), \\
J_6 &\leq C|\Delta\mathbf{d}_t|_{L^2}|\nabla\mathbf{d}|_{L^3}|\mathbf{u}_t|_{L^6} \leq C|\Delta\mathbf{d}_t|_{L^2}|\Delta\mathbf{d}|_{L^2}^{\frac{1}{2}}|\nabla\mathbf{u}_t|_{L^2} \\
&\leq C|\Delta\mathbf{d}|_{L^2}^{\frac{1}{2}}\left(|\Delta\mathbf{d}_t|_{L^2}^2 + |\nabla\mathbf{u}_t|_{L^2}^2\right), \\
J_7 &\leq C|\Delta\mathbf{d}|_{L^2}|\nabla\mathbf{d}_t|_{L^3}|\mathbf{u}_t|_{L^6} \leq C|\Delta\mathbf{d}|_{L^2}|\nabla\mathbf{d}_t|_{L^2}^{\frac{1}{2}}|\Delta\mathbf{d}_t|_{L^2}^{\frac{1}{2}}|\nabla\mathbf{u}_t|_{L^2} \\
&\leq C|\Delta\mathbf{d}|_{L^2}\left(|\nabla\mathbf{d}_t|_{L^2}^2 + |\Delta\mathbf{d}_t|_{L^2}^2 + |\nabla\mathbf{u}_t|_{L^2}^2\right).
\end{aligned}$$

Thirdly, Applying ∂_t to (4), one obtains

$$\mathbf{d}_{tt} + (\mathbf{u}_t\cdot\nabla)\mathbf{d} + (\mathbf{u}\cdot\nabla)\mathbf{d}_t = \theta\Delta\mathbf{d}_t + 2\theta(\nabla\mathbf{d}:\nabla\mathbf{d}_t)\mathbf{d} + \theta|\nabla\mathbf{d}|^2\mathbf{d}_t. \tag{31}$$

Multiplying (31) by $\mathbf{d}_t$, and then by using integration by parts, (4) and the fact $|\mathbf{d}| = 1$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\mathbf{d}_t|^2 + \theta\int |\nabla \mathbf{d}_t|^2 \\
= \;& -\int (\mathbf{u}_t\cdot\nabla)\mathbf{d}\cdot\mathbf{d}_t + \theta\int |\nabla\mathbf{d}|^2|\mathbf{d}_t|^2 \\
= \;& \int (\mathbf{u}_t\cdot\nabla)\mathbf{d}\cdot\Big[(\mathbf{u}\cdot\nabla)\mathbf{d} - \theta(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d})\Big] + \theta\int |\nabla\mathbf{d}|^2|\mathbf{d}_t|^2 \\
\le \;& C\int |\mathbf{u}_t||\nabla\mathbf{d}|^2|\mathbf{u}| + C\int |\mathbf{u}_t||\nabla\mathbf{d}||\Delta\mathbf{d}| + C\int |\nabla\mathbf{d}|^2|\mathbf{d}_t|^2 \\
= \;& K_1 + K_2 + K_3.
\end{aligned} \tag{32}$$

We have similarly as the estimates about $I_k (k = 1, 2, 3, 4)$ and $J_k (k = 1, 2, \ldots, 7)$ that

$$\begin{aligned}
K_1 &\le C|\mathbf{u}_t|_{L^6}|\nabla\mathbf{d}|_{L^3}^2|\mathbf{u}|_{L^6} \le C|\nabla\mathbf{u}_t|_{L^2}|\nabla\mathbf{d}|_{L^2}|\Delta\mathbf{d}|_{L^2}|\nabla\mathbf{u}|_{L^2} \\
&\le C|\nabla\mathbf{u}_t|_{L^2}|\Delta\mathbf{d}|_{L^2}|\nabla\mathbf{u}|_{L^2} \le C|\nabla\mathbf{u}|_{L^2}\left(|\nabla\mathbf{u}_t|_{L^2}^2 + |\Delta\mathbf{d}|_{L^2}^2\right), \\
K_2 &\le C|\mathbf{u}_t|_{L^6}|\nabla\mathbf{d}|_{L^3}|\Delta\mathbf{d}|_{L^2} \le C|\nabla\mathbf{u}_t|_{L^2}|\nabla\mathbf{d}|_{L^2}^{\frac{1}{2}}|\Delta\mathbf{d}|_{L^2}^{\frac{3}{2}} \le C|\nabla\mathbf{u}_t|_{L^2}|\Delta\mathbf{d}|_{L^2}^{\frac{3}{2}} \\
&\le C|\Delta\mathbf{d}|_{L^2}^{\frac{1}{2}}\left(|\nabla\mathbf{u}_t|_{L^2}^2 + |\Delta\mathbf{d}|_{L^2}^2\right), \\
K_3 &\le C|\nabla\mathbf{d}|_{L^3}^2|\mathbf{d}_t|_{L^6}^2 \le C|\Delta\mathbf{d}|_{L^2}|\nabla\mathbf{d}_t|_{L^2}^2.
\end{aligned}$$

Then multiplying (31) by $\Delta\mathbf{d}_t$, and then using integration by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla\mathbf{d}_t|^2 + \theta\int |\Delta\mathbf{d}_t|^2 \\
= \;& \int \Big[(\mathbf{u}_t\cdot\nabla)\mathbf{d} + (\mathbf{u}\cdot\nabla)\mathbf{d}_t - 2\theta(\nabla\mathbf{d} : \nabla\mathbf{d}_t)\mathbf{d} - \theta|\nabla\mathbf{d}|^2\mathbf{d}_t\Big]\cdot\Delta\mathbf{d}_t \\
\le \;& \int |\mathbf{u}_t||\nabla\mathbf{d}||\Delta\mathbf{d}_t| + |\mathbf{u}||\nabla\mathbf{d}_t||\Delta\mathbf{d}_t| + 2\theta|\nabla\mathbf{d}||\nabla\mathbf{d}_t||\Delta\mathbf{d}_t| + \theta|\nabla\mathbf{d}|^2|\mathbf{d}_t||\Delta\mathbf{d}_t| \\
= \;& L_1 + L_2 + L_3 + L_4,
\end{aligned} \tag{33}$$

It yield as before that

$$\begin{aligned}
L_1 &\le C|\mathbf{u}_t|_{L^6}|\nabla\mathbf{d}|_{L^3}|\Delta\mathbf{d}_t|_{L^2} \le C|\nabla\mathbf{u}_t|_{L^2}|\Delta\mathbf{d}|_{L^2}^{\frac{1}{2}}|\Delta\mathbf{d}_t|_{L^2} \\
&\le C|\Delta\mathbf{d}|_{L^2}^{\frac{1}{2}}\left(|\nabla\mathbf{u}_t|_{L^2}^2 + |\Delta\mathbf{d}_t|_{L^2}^2\right), \\
L_2 &\le C|\mathbf{u}|_{L^6}|\nabla\mathbf{d}_t|_{L^3}|\Delta\mathbf{d}_t|_{L^2} \le C|\nabla\mathbf{u}|_{L^2}|\nabla\mathbf{d}_t|_{L^2}^{\frac{1}{2}}|\Delta\mathbf{d}_t|_{L^2}^{\frac{3}{2}} \\
&\le C|\nabla\mathbf{u}|_{L^2}\left(|\nabla\mathbf{d}_t|_{L^2}^2 + |\Delta\mathbf{d}_t|_{L^2}^2\right), \\
L_3 &\le C|\nabla\mathbf{d}|_{L^3}|\nabla\mathbf{d}_t|_{L^6}|\Delta\mathbf{d}_t|_{L^2} \le C|\Delta\mathbf{d}|_{L^2}^{\frac{1}{2}}|\Delta\mathbf{d}_t|_{L^2}^2, \\
L_4 &\le C|\nabla\mathbf{d}|_{L^6}^2|\mathbf{d}_t|_{L^6}|\Delta\mathbf{d}_t|_{L^2} \le C|\nabla\mathbf{d}|_{L^6}^2|\nabla\mathbf{d}_t|_{L^2}|\Delta\mathbf{d}_t|_{L^2} \\
&\le C|\Delta\mathbf{d}|_{L^2}^2\left(|\nabla\mathbf{d}_t|_{L^2}^2 + |\Delta\mathbf{d}_t|_{L^2}^2\right).
\end{aligned}$$

Combining (16), (22), (25), (28), (30), (32), (33) and all the estimates about I_n, J_n, K_n and L_n together, we have

$$\frac{d}{dt}F(t) + G(t) \le CH(t)G(t), \tag{34}$$

where

$$\begin{aligned} F(t) &= |\sqrt{\rho}\mathbf{u}|_{L^2}^2 + |\nabla\mathbf{u}|_{L^2}^2 + |\nabla\mathbf{d}|_{H^1}^2 + |\sqrt{\rho}\mathbf{u}_t|_{L^2}^2 + |\mathbf{d}_t|_{H^1}^2, \\ G(t) &= |\nabla\mathbf{u}|_{H^1}^2 + |\Delta\mathbf{d}|_{H^1}^2 + |\sqrt{\rho}\mathbf{u}_t|_{L^2}^2 + |\nabla\mathbf{u}_t|_{L^2}^2 + |\nabla\mathbf{d}_t|_{H^1}^2, \\ H(t) &= |\nabla\mathbf{u}|_{L^2}^{\frac{1}{2}} + |\nabla\mathbf{u}|_{L^2}^2 + |\nabla\mathbf{d}|_{H^1}^{\frac{1}{2}} + |\nabla\mathbf{d}|_{H^1}^2. \end{aligned}$$

In conclusion, we get

$$\frac{d}{dt}F(t) + (1 - CH(t))G(t) \le 0. \tag{35}$$

We set now δ_0 small enough such that $\delta_0 + 2\delta_0^{\frac{1}{4}} < \frac{1}{C}$, where C is the constant shown in (35). And then we choose ε_0 small enough such that $2c_1\left(\varepsilon_0 + \varepsilon_0^3\right) < \delta_0$, where c_1 is a Sobolev constant which will be used in the following, and suppose that (8) holds. We claim that for all $t \ge 0$,

$$|\nabla\mathbf{u}|_{L^2}^2(t) + |\nabla\mathbf{d}|_{H^1}^2(t) + |\mathbf{d}_t|_{H^1}^2(t) < \delta_0. \tag{36}$$

If it is not the case, then let t_1 be the first time $t > 0$ such that

$$|\nabla\mathbf{u}|_{L^2}^2(t) + |\nabla\mathbf{d}|_{H^1}^2(t) + |\mathbf{d}_t|_{H^1}^2(t) \ge \delta_0. \tag{37}$$

For all $t < t_1$,

$$|\nabla\mathbf{u}|_{L^2}^2(t) + |\nabla\mathbf{d}|_{H^1}^2(t) + |\mathbf{d}_t|_{H^1}^2(t) < \delta_0, \tag{38}$$

then it yields

$$H(t) \le \delta_0 + 2\delta_0^{\frac{1}{4}} < \frac{1}{C}, \text{ for } t < t_1. \tag{39}$$

So we have

$$1 - CH(t) \ge 0, \text{ for } t < t_1. \tag{40}$$

Hence, for all $t < t_1$, we have

$$\frac{d}{dt}F(t) \le 0, \tag{41}$$

it implies

$$\begin{aligned} & |\sqrt{\rho}\mathbf{u}|_{L^2}^2(t) + |\nabla\mathbf{u}|_{L^2}^2(t) + |\nabla\mathbf{d}|_{H^1}^2(t) + |\sqrt{\rho}\mathbf{u}_t|_{L^2}^2(t) + |\mathbf{d}_t|_{H^1}^2(t) \\ \le\ & |\sqrt{\rho_0}\mathbf{u}_0|_{L^2}^2 + |\nabla\mathbf{u}_0|_{L^2}^2 + |\nabla\mathbf{d}_0|_{H^1}^2 + |\sqrt{\rho}\mathbf{u}_t|_{L^2}^2(0) + |\mathbf{d}_t|_{H^1}^2(0) \\ \le\ & |\sqrt{\rho_0}\mathbf{u}_0|_{L^2}^2 + |\nabla\mathbf{u}_0|_{L^2}^2 + |\nabla\mathbf{d}_0|_{H^1}^2 + |\mathbf{g}|_{L^2}^2 + |\sqrt{\rho_0}\mathbf{u}_0\cdot\nabla\mathbf{u}_0|_{L^2}^2 + |\mathbf{u}_0\cdot\nabla\mathbf{d}_0|_{L^2}^2 \\ & +\theta|\Delta\mathbf{d}_0|_{L^2}^2 + \theta\left||\nabla\mathbf{d}_0|^2\right|_{L^2}^2 + |\nabla\mathbf{u}_0\cdot\nabla\mathbf{d}_0|_{L^2}^2 + |\mathbf{u}_0\cdot\nabla^2\mathbf{d}_0|_{L^2}^2 + \theta|\nabla\Delta\mathbf{d}|_{L^2}^2 \\ & +\theta|\nabla\mathbf{d}_0 : \nabla^2\mathbf{d}_0|_{L^2}^2 + \theta\left||\nabla\mathbf{d}_0|^3\right|_{L^2}^2 \\ \le\ & c_1\left(|\sqrt{\rho_0}\mathbf{u}_0|_{L^2}^2 + |\nabla\mathbf{u}_0|_{H^1}^2 + |\nabla\mathbf{u}_0|_{H^1}^4 + |\mathbf{g}|_{L^2}^2 + |\nabla\mathbf{d}_0|_{H^2}^6 + |\nabla\mathbf{d}_0|_{H^2}^2\right) \\ \le\ & 2c_1\left(\varepsilon_0 + \varepsilon_0^3\right) \\ <\ & \delta_0. \end{aligned} \tag{42}$$

Then we have

$$|\nabla\mathbf{u}|_{L^2}^2(t) + |\nabla\mathbf{d}|_{H^1}^2(t) + |\mathbf{d}_t|_{H^1}^2(t) \le 2c_1\left(\varepsilon_0 + \varepsilon_0^3\right) < \delta_0. \tag{43}$$

Finally, let $t \to t_1$, we have from the continuity of the local strong solution that

$$\left(|\nabla \mathbf{u}|_{L^2}^2 + |\nabla \mathbf{d}|_{H^1}^2 + |\mathbf{d}_t|_{H^1}^2\right)(t_1) \le 2c_1\left(\varepsilon_0 + \varepsilon_0^3\right) < \delta_0, \tag{44}$$

it yields a contradiction with the definition of t_1 in (37). Then we conclude (36) holds for $t \ge 0$. Furthermore, by integrating (35) over $[0, t]$, we have

$$|\sqrt{\rho}\mathbf{u}_t|_{L^2}^2(t) + \int_0^t G(t) \le C, \text{ for } t \ge 0. \tag{45}$$

Finally, multiplying (24) by $\nabla\Delta\mathbf{d}$, and then integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}\theta\int|\nabla\Delta\mathbf{d}|^2 \quad &\le \int|\nabla\mathbf{d}_t||\nabla\Delta\mathbf{d}| + \int|\nabla\mathbf{u}||\nabla\mathbf{d}||\nabla\Delta\mathbf{d}| + \int|\mathbf{u}||\nabla^2\mathbf{d}||\nabla\Delta\mathbf{d}| \\ &\quad +2\theta\int|\nabla\mathbf{d}||\nabla^2\mathbf{d}||\nabla\Delta\mathbf{d}| + \theta\int|\nabla\mathbf{d}|^3|\nabla\Delta\mathbf{d}| \\ &= R_1 + R_2 + R_3 + R_4 + R_5. \end{aligned} \tag{46}$$

The Hölder inequality, (36) and Lemma 3.2 imply that

$$\begin{aligned} R_1 &\le C|\nabla\mathbf{d}_t|_{L^2}|\nabla\Delta\mathbf{d}|_{L^2} \le C|\nabla\Delta\mathbf{d}|_{L^2} \\ R_2 &\le C|\nabla\mathbf{u}|_{L^2}|\nabla\mathbf{d}|_{L^\infty}|\nabla\Delta\mathbf{d}|_{L^2} \le C|\nabla\Delta\mathbf{d}|_{L^2}^{\frac{7}{4}}, \\ R_3 &\le C|\mathbf{u}|_{L^6}|\nabla^2\mathbf{d}|_{L^3}|\nabla\Delta\mathbf{d}|_{L^2} \le C|\nabla\mathbf{u}|_{L^2}|\Delta\mathbf{d}|_{L^2}^{\frac{1}{2}}|\nabla\Delta\mathbf{d}|_{L^2}^{\frac{3}{2}} \le C|\nabla\Delta\mathbf{d}|_{L^2}^{\frac{3}{2}}, \\ R_4 &\le C|\nabla\mathbf{d}|_{L^6}|\nabla^2\mathbf{d}|_{L^3}|\nabla\Delta\mathbf{d}|_{L^2} \le C|\Delta\mathbf{d}|_{L^2}^{\frac{3}{2}}|\nabla\Delta\mathbf{d}|_{L^2}^{\frac{3}{2}} \le C|\nabla\Delta\mathbf{d}|_{L^2}^{\frac{3}{2}}, \\ R_5 &\le C|\nabla\mathbf{d}|_{L^6}^3|\nabla\Delta\mathbf{d}|_{L^2} \le C|\Delta\mathbf{d}|_{L^2}^3|\nabla\Delta\mathbf{d}|_{L^2} \le C|\nabla\Delta\mathbf{d}|_{L^2}. \end{aligned}$$

Then it follows by the Cauchy inequality that

$$\sup_{t\ge0}|\Delta\mathbf{d}|_{D^1}^2 \le C, \text{ for } t \ge 0. \tag{47}$$

Similarly, by (36), (45) and (47), we have

$$\int_0^t |\Delta\mathbf{d}|_{D^2}^2 \le C, \text{ for } t \ge 0. \tag{48}$$

then (21) follows by (36), (45), (47) and (48). □

Lemma 4.3. *For any $t \ge 0$, it holds*

$$\sup_{t\ge0}(|\mathbf{u}|_{D^2} + |P|_{H^1}) + \int_0^t\left(|\nabla\mathbf{u}|_{W^{1,6}}^2 + |P|_{W^{1,6}}^2\right) \le C. \tag{49}$$

and

$$\sup_{t\ge0}(|\rho|_{H^1} + |\rho_t|_{L^2}) \le C\exp(1+t). \tag{50}$$

Proof. Firstly, By using Lemma 3.2, Lemma 4.2, the Hölder inequality and a similar discussion as (27), one obtains

$$\begin{aligned} |\nabla^2\mathbf{u}|_{L^2}^2 &\le C|\rho|_{L^\infty}|\sqrt{\rho}\mathbf{u}_t|_{L^2}^2 + C|\mathbf{u}|_{L^6}^2|\nabla\mathbf{u}|_{L^3}^2 + C|\nabla^2\mathbf{d}|_{L^3}^2|\nabla\mathbf{d}|_{L^6}^2 \\ &\le C + C|\nabla^2\mathbf{u}|_{L^2}. \end{aligned} \tag{51}$$

Then it follows by the Cauchy inequality that

$$|\mathbf{u}|_{D^2}(t) \leq C, \ t \geq 0. \tag{52}$$

On the other hand, using the regularity theory for the stationary Stokes equations (see [4, 6]) and Lemma 3.2 again, we have

$$\begin{aligned} & |\nabla \mathbf{u}|_{W^{1,6}} \\ \leq \ & C\left(|\rho \mathbf{u}_t|_{L^6} + |\rho \mathbf{u} \cdot \nabla \mathbf{u}|_{L^6} + |\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d})|_{L^6}\right) \\ \leq \ & C\left(|\rho|_{L^\infty} |\nabla \mathbf{u}_t|_{L^2} + |\rho|_{L^\infty}\left(|\nabla \mathbf{u}|_{L^2} + |\nabla \mathbf{u}|_{L^6}\right)|\nabla \mathbf{u}|_{L^6} + |\nabla \mathbf{d}|_{L^\infty}|\Delta \mathbf{d}|_{L^6}\right) \\ \leq \ & C\left(|\nabla \mathbf{u}_t|_{L^2} + |\nabla \mathbf{u}|_{H^1}^2 + |\nabla \mathbf{d}|_{H^2}|\nabla \Delta \mathbf{d}|_{L^2}\right) \\ \leq \ & C\left(|\nabla \mathbf{u}_t|_{L^2} + |\nabla \mathbf{u}|_{H^1}^2 + |\nabla \Delta \mathbf{d}|_{L^2}\right). \end{aligned} \tag{53}$$

Then we have

$$\begin{aligned} \int_0^t |\nabla \mathbf{u}|_{W^{1,6}}^2 \ & \leq \ C \int_0^t \left(|\nabla \mathbf{u}_t|_{L^2}^2 + |\nabla \mathbf{u}|_{H^1}^4 + |\Delta \mathbf{d}|_{H^1}^2\right) \\ & \leq \ C \int_0^t |\nabla \mathbf{u}_t|_{L^2}^2 + C \sup_{t \geq 0} |\nabla \mathbf{u}|_{H^1}^2 \int_0^t |\nabla \mathbf{u}|_{H^1}^2 + C \int_0^t |\Delta \mathbf{d}|_{H^1}^2 \\ & \leq \ C. \end{aligned} \tag{54}$$

And the estimates about the pressure P follows by (2) and the estimates about $\mathbf{u}$ and $\mathbf{d}$ immediately. It completes the proof of (49).

Secondly, we turn to give the estimates about the density. To derive these, we first observe that $\nabla \rho$ satisfies

$$(\nabla \rho)_t + \mathbf{u} \cdot \nabla^2 \rho + \nabla \mathbf{u} \cdot \nabla \rho = 0. \tag{55}$$

Then multiplying (55) by $\nabla \rho$, integrating over $\mathbb{R}^3$, and then by using integration by parts and (3), we obtain

$$\frac{d}{dt} \int |\nabla \rho|^2 \leq C \int |\nabla \mathbf{u}||\nabla \rho|^2 \leq C |\nabla \mathbf{u}|_{L^\infty} |\nabla \rho|_{L^2}^2. \tag{56}$$

Lemma 3.2 yields that

$$|\nabla \mathbf{u}|_{L^\infty} \leq C |\nabla \mathbf{u}|_{L^2}^{\frac{1}{4}} |\nabla^2 \mathbf{u}|_{L^6}^{\frac{3}{4}}. \tag{57}$$

Then it follows by (56), (57) and Lemma 4.2 that

$$\frac{d}{dt} |\nabla \rho|_{L^2}^2 \leq C\left(|\nabla \mathbf{u}|_{W^{1,6}}^2 + 1\right)|\nabla \rho|_{L^2}^2, \tag{58}$$

Then we have from the Gronwall inequality and (54) that

$$|\nabla \rho|_{L^2} \leq C \exp(1 + t). \tag{59}$$

Then (50) follows by (1). This completes the proof of Lemma 4.3. □

Proof of Theorem 2.1 The priori estimates obtained in Lemma 4.1-4.3 allow us to extend the unique local strong solution to $[0, T]$ for any fixed positive $T > 0$. Therefore, the proof of Theorem 2.1 is completed. □

Remark 4.4. *As shown in the proof of (50), we can not get a uniform estimates for $t \in [0, \infty)$. It yields that the strong solution obtained in Theorem 2.1 can not be extended to the case $t \in [0, \infty)$.*

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