Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 87 – 109
© TÜBİTAK
doi:10.3906/mat-1212-15

Global existence, uniform decay, and exponential growth of solutions for a system of viscoelastic Petrovsky equations

Faramarz TAHAMTANI*, **Amir PEYRAVI**
Department of Mathematics, College of Sciences, Shiraz University, Shiraz, 71454, Iran

Received: 12.12.2012 • **Accepted:** 19.06.2013 • **Published Online:** 09.12.2013 • **Printed:** 20.01.2014

Abstract: In this paper, we study the initial-boundary value problem for a system of nonlinear viscoelastic Petrovsky equations. Introducing suitable perturbed energy functionals and using the potential well method we prove uniform decay of solution energy under some restrictions on the initial data and the relaxation functions. Moreover, we establish a growth result for certain solutions with positive initial energy.

Key words: Global existence, uniform decay, exponential growth, viscoelastic Petrovsky equation

1. Introduction

In this paper, we investigate the following initial-boundary value problem:

$$\begin{cases} |u_t|^\rho u_{tt} + \Delta^2 u - \Delta u_{tt} - (g_1 * \Delta^2 u)(t) - \Delta u_t + |u_t|^{p-1}u_t = f_1(u,v), & (x,t) \in \Omega \times [0,T), \\ |v_t|^\rho v_{tt} + \Delta^2 v - \Delta v_{tt} - (g_2 * \Delta^2 v)(t) - \Delta v_t + |v_t|^{q-1}v_t = f_2(u,v), & (x,t), \in \Omega \times [0,T), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega, \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \qquad x \in \Omega, \\ u(x,t) = \partial_\nu u(x,t) = 0, \qquad v(x,t) = \partial_\nu v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T), \end{cases} \tag{1.1}$$

where $\rho > 0$, $p,q \geq 1$, $T > 0$, Ω is a bounded domain of $\mathrm{R}^n (n = 1,2,3)$ with a smooth boundary $\partial\Omega$ so that the divergence theorem can be applied, ν denotes the outward normal derivative, g_1 and g_2 are positive functions satisfying some conditions to be specified later, and

$$(g_i * \phi)(t) = \int_0^t g_i(t-\tau)\phi(\tau)d\tau, \quad i = 1,2.$$

By taking

$$\begin{aligned} f_1(u,v) &= (r+1)\left[a|u+v|^{r-1}(u+v) + b|u|^{\frac{r-3}{2}}|v|^{\frac{r+1}{2}}u\right], \\ f_2(u,v) &= (r+1)\left[a|u+v|^{r-1}(u+v) + b|v|^{\frac{r-3}{2}}|u|^{\frac{r+1}{2}}v\right], \end{aligned} \tag{1.2}$$

where $a > 1, b > 0$, and $r \geq 3$, one can easily verify

$$uf_1(u,v) + vf_2(u,v) = (r+1)F(u,v), \qquad \forall (u,v) \in R^2,$$

*Correspondence: tahamtani@shirazu.ac.ir
2010 *AMS Mathematics Subject Classification:* 35L05, 35L55, 35L70, 35B40.

where

$$F(u,v) = a|u+v|^{r+1} + 2b|uv|^{\frac{r+1}{2}}. \tag{1.3}$$

In [6] Cavalcanti et al. studied the following nonlinear viscoelastic problem:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma\Delta u_t = 0, \quad x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0, \ x \in \partial\Omega, \quad t > 0, \end{cases} \tag{1.4}$$

where Ω is open bounded in $R^n, n \geq 1$. Under the assumptions $0 < \rho \leq \frac{2}{n-2}$ if $n \geq 3$ or $\rho > 0$ if $n = 1, 2$ and $g(t)$ decays exponentially, they obtained the global existence of weak solutions for $\gamma \geq 0$ and the uniform exponential decay rates of the energy for $\gamma > 0$. In the presence of a nonlinear source term, the decay result has been extended by [23]. In the case of $\gamma = 0$ when a source term competes with the dissipation induced by the viscoelastic term, Messaoudi and Tatar [24] studied the equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = b|u|^{p-2}u, \quad x \in \Omega, \quad t > 0,$$

with the initial and boundary conditions $(1.4)_2$ and $(1.4)_3$. They used the potential well method to show that the damping induced by the viscoelastic term is enough to ensure global existence and uniform decay of solutions provided that the initial data are in some stable set. Han and Wang [12], investigated a related problem with linear damping

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t = 0, \quad x \in \Omega, \quad t > 0.$$

Using the Faedo–Galerkin method, they showed the global existence of weak solutions and obtained uniform exponential decay of solutions by introducing a perturbed energy functional. Recently, these results have been extended by Wu [34] to a more general case where a source term and a nonlinear damping term are present

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + |u_t|^p u_t = |u|^r u, \quad x \in \Omega, \quad t > 0, \tag{1.5}$$

where the initial and boundary conditions are as $(1.4)_2$ and $(1.4)_3$. In the case of $\rho = 0$, and in the absence of a dispersive term, there is a substantial number of papers dealing with equation (1.5). For example, we may recall the work by Cavalcanti et al. [7] in which the following equation:

$$u_{tt} - k_0\Delta u + \int_0^t \mathrm{div}[a(x)g(t-\tau)\nabla u(\tau)]d\tau + b(x)h(u_t) + f(u) = 0, \quad x \in \Omega, \quad t > 0,$$

has been considered. Under some conditions on the relaxation function g and for $a(x) + b(x) \geq \rho > 0$, they improved the results of [8] by establishing stability for exponential decay function g and linear function h, and polynomial stability for polynomial decay function g and nonlinear function h. For some other related papers in connection with the existence, finite time blow-up, and asymptotic properties of solutions of nonlinear wave equations, we refer the reader to [4, 5, 9, 10, 13, 18, 19, 20, 35, 37, 38] and references therein.

The following initial boundary value problem:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + (g_1 * \Delta u)(t) - \gamma_1 \Delta u_t + a_1 |u_t|^{p-1} u_t = f_1(u,v), & x \in \Omega, \quad t > 0, \\ |v_t|^\rho v_{tt} - \Delta v - \Delta v_{tt} + (g_2 * \Delta v)(t) - \gamma_2 \Delta v_t + a_2 |v_t|^{q-1} v_t = f_2(u,v), & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega, \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \qquad x \in \Omega, \\ u(x,t) = v(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0, \end{cases} \tag{1.6}$$

has been investigated by many people. When $\rho = 0$ and there are no dispersion terms, in the absence of strong damping ($\gamma_i = 0$), the system has been investigated by several authors and results concerning existence, decay, and blow-up were obtained [1, 2, 15, 22, 25, 26, 28, 29, 32, 36]. In the case of $g_i = 0$, Agre and Rammaha [1] proved several results on local and global existence of weak solutions with the nonlinear functions $f_1(u,v)$ and $f_2(u,v)$ as given in (1.2). The strong nonlinearities on f_1 and f_2 allowed them to prove the local existence result only for $n \leq 3$. Involving the Nehari Manifold and under some conditions on the parameters in the same system, the authors in [2] obtained several results on the global existence, uniform decay rates, and blow-up of solutions in finite time when the initial energy is nonnegative. Recently, based on the potential well method and a lemma by Nakao [28], Said-Houari [29] established global existence, and polynomial and exponential decay rates for the energy of the same problem with different nonlinear source terms. In [36], Wu studied the following nonlinear wave equations:

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{p-1} u_t + m_1^2 u = f_1(u,v), \\ v_{tt} - \Delta v + |v_t|^{q-1} v_t + m_2^2 v = f_2(u,v), \end{cases} \tag{1.7}$$

with $r = 3$, in (1.2), and initial-boundary values $(1.6)_3 - (1.6)_5$. Wu discussed the blow-up properties of (1.7) in 2 cases. In the first case, $p = q = 1$, the main result contains the estimates of the upper bound of the blow-up time. In the second case, $1 < p, q < 3$, the nonexistence of global solutions is proved and estimates for the blow-up time are also given. This work improved the work [15], in which similar results have been established for (1.7) in the absence of damping terms.

In the presence of the viscoelastic terms ($g_i \neq 0$), Said-Houari et al. [32] obtained global existence and a uniform decay rate result under some restrictions on initial data and for some classes of kernels g_i. They showed that the decay rate of the energy depends on those of the relaxation functions. Their result improved the one in [25] in which only the exponential and polynomial decay rates are obtained. In another work [26], they obtained a global nonexistence result for the same system when the initial energy is considered to be positive. In the weak damping case ($p = 1, q = 1$), Ma et al. [22] showed the solutions with arbitrarily positive initial energy blow-up in finite time with the following nonlinear functions:

$$\begin{cases} f_1(u,v) = a_1 |v|^{s+1} |u|^{r-1} v, \\ f_2(u,v) = a_1 |u|^{r+1} |v|^{s-1} v, \end{cases}$$

where $r, s > 1$. They used the concavity method, which is based on defining a positive function $\eta(t)$ and showing that $\eta(t)^{-\alpha}$ is a concave function for some $\alpha > 0$. In the presence of strong damping terms ($\gamma_i \neq 0$) as well as absence of nonlinear damping terms ($a_i = 0$), another coupled system was investigated in [14] . The authors proved that, under suitable assumptions on the functions g_i, f_i and certain initial data in the stable set, the decay rate of the solution energy is exponential. They also showed that, for certain initial data in the unstable set, there are solutions with positive initial energy that blow-up in finite time.

In [21], Liu studied the following system:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + (g_1 * \Delta u)(t) + f_1(u,v) = 0, \\ |v_t|^\rho v_{tt} - \Delta v - \Delta v_{tt} + (g_2 * \Delta v)(t) + f_2(u,v) = 0, \end{cases} \tag{1.8}$$

where the functions f_1 and f_2 satisfy

$$\begin{cases} |f_1(u,v)| \le d \min\{(|u|^{\beta_1} + |v|^{\beta_2}), |u|^{\beta-1}|v|^\beta\}, \\ |f_2(u,v)| \le d \min\{(|u|^{\beta_3} + |v|^{\beta_4}), |u|^{\beta}|v|^{\beta-1}\}, \end{cases}$$

for some constant $d > 0$ and

$$\beta_i \ge 1, \qquad (n-2)\beta_i \le n, \qquad i = 1,2,3,4.$$

$$\beta > 1 \quad \text{if } n = 1,2; \quad 1 < \beta \le \frac{n-1}{n-2} \quad \text{if } n \ge 3.$$

The author used the perturbed energy method to prove an exponential decay result if both g_1 and g_2 are decaying exponentially and a polynomial decay result if both g_1 and g_2 are decaying polynomially. This is an extension of the result obtained by Messaoudi and Tatar [25] for the system (1.8) with $\rho = 0$ and in the absence of dispersive terms. Recently, motivated by the works [30, 33], Said-Houari [31] studied (1.6) with $\gamma_i = 0$. He proved that the energy associated with the system grows as an exponential function as time goes to infinity, provided that the initial data are large enough.

In recent years, these results have been extended to Petrovsky equations. The single Petrovsky wave equation of the form

$$u_{tt} + \Delta^2 u + h(u_t) = f(u), \qquad x \in \Omega, \quad t > 0, \tag{1.9}$$

with the boundary and initial conditions $(1.1)_3 - (1.1)_5$, has been widely investigated. For $f = -q(x)u(x,t)$, equation (1.9) has been considered by Guesmia [11], where q is a positive function in $L^\infty(\Omega)$ and h is a continuous and increasing function that satisfies $h(0) = 0$. When $h(u_t) = au_t|u_t|^{p-2}$ and $f(u) = bu|u|^{r-2}$ where $a, b > 0$ and $r, p > 2$, Messaoudi [27] established an existence result when $p \ge r$ with arbitrary initial data. He also showed that the solution blows up if $p < r$ and the initial energy is negative. In [3], Amroun and Benaissa obtained the global solvability of (1.9) subject to the same boundary and initial conditions as $(1.1)_3 - (1.1)_5$, where $f(u) = bu|u|^{r-2}$ and h satisfies

$$c_1|s| \le |h(s)| \le c_2|s|^m, \quad |s| \ge 1, \quad c_1, c_2 > 0,$$

where

$$1 \le m \le \infty \quad \text{if} \quad n = 1,2,3,4 \quad \text{or} \quad 1 \le m \le \frac{n+4}{n-4} \quad \text{if} \quad n \ge 5.$$

The key point to their proof is the use of the stable set method combined with the Faedo–Galerkin procedure. In the presence of strong damping, we mention also the work by Li et al. [16] in which they considered the following Petrovsky equation:

$$u_{tt} + \Delta^2 u - \Delta u_t + u_t|u_t|^{p-1} = u|u|^{r-1}, \qquad x \in \Omega, \quad t \ge 0,$$

with the boundary and initial conditions $(1.1)_3 - (1.1)_5$. The authors obtained the global existence and uniform decay of solutions if the initial data are in some stable set without any interaction between the damping mechanism $u_t|u_t|^{p-1}$ and the source term $u|u|^{r-1}$. Moreover, they established the blow-up properties of the local solution if $r > p$ and the initial energy is less than the potential well depth. In [17], inspiring the work [36], they also considered the following Petrovsky equations:

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{p-1} u_t = f_1(u, v), \\ v_{tt} + \Delta^2 v + |v_t|^{q-1} v_t = f_2(u, v), \end{cases}$$

in a bounded domain $\Omega \subseteq R^n, (n = 1, 2, 3)$ with nonlinear functions $f_1(u, v)$ and $f_2(u, v)$ given in (1.2). They proved the global existence of solutions and established the uniform decay rates by means of Nakao's inequality. Improving on the method of [36], they showed the blow-up of solutions and the lifespan estimates when $p = 1, q = 1$. In the case $1 < p, q < 3$, they also obtained a blow-up result when the initial energy is negative or nonnegative at less than the mountain pass level value.

Motivated by the above studies, we consider the decay and growth propositions of the solution for problem (1.1). We prove the global existence and uniform decay of solutions by using the potential well method and introducing a perturbed energy function. We also prove that the energy will grow provided that the initial data are large enough.

The outline of our paper is as follows. In section 2 we present some notations, assumptions, and lemmas needed later and state the main results of this article. Section 3 is devoted to proving the global existence and uniform decay of solutions: Theorem 2.2. The growth result of solutions, Theorem 2.3, is proved in Section 4.

2. Preliminaries and main results

In this section, we introduce some notations and lemmas needed in the proof of our main results. Throughout this paper, we use the standard Lebesgue space $L^p(\Omega)$, the Sobolev spaces $H_0^2(\Omega)$ and $H_0^1(\Omega)$ with their usual scalar products and norms. First, we give the following Sobolev–Poincaré inequality, which will be used frequently in this paper.

Lemma 2.1 (*Sobolev–Poincaré inequality*). *Let q be a number with $2 \leq q < +\infty$; $n \leq 3$; then for $u \in H_0^2(\Omega)$ there is a constant $C_* = C(\Omega, q)$ such that*

$$\|u\|_q \leq C_* \|\Delta u\|_2. \tag{2.1}$$

In the sequel by c_i or C_i, we denote various positive constants, which may be different at different occurrences. For nonlinear terms we assume

(G1) $p, q \geq 1$ and

$$\begin{aligned} &r \geq 3 \ \text{ if } \ n = 1, 2; \quad r = 3 \ \text{ if } \ n = 3, \\ &\rho > 0 \ \text{ if } \ n = 1, 2; \quad \rho = 2 \ \text{ if } \ n = 3. \end{aligned} \tag{2.2}$$

For the relaxation functions we present the following assumptions

(G2) $g_i : R^+ \to R^+$, $i = 1, 2$, are nonincreasing bounded C^1-functions such that

$$g_i(0) > 0, \qquad 1 - \int_0^\infty g_i(\tau) d\tau = l_i > 0, \quad i = 1, 2.$$

(G3) There exist 2 positive nonincreasing differentiable functions ξ_1, ξ_2 such that

$$g_i'(t) \leq -\xi_i(t)g_i(t), \quad i=1,2, \quad \forall t \geq 0,$$

where

$$\int_0^{+\infty} \xi_i(t)dt = +\infty, \quad i=1,2.$$

Similar to in [2, 17], we consider that

(G4) There exist constants $c_0, c_1 > 0$ such that

$$c_0\left(|u|^{r+1} + |v|^{r+1} \right) \leq F(u,v) \leq c_1\left(|u|^{r+1} + |v|^{r+1} \right).$$

Remark 2.1 The assumption $(G1)$ is needed to guarantee the local existence of weak solutions for the initial-boundary value problem (1.1). The condition $(G2)$ guarantees the hyperbolicity of the system (1.1). There is a wide class of functions satisfying $(G2)$ and $(G3)$. Examples can be found in [32].

Remark 2.2 It is easy to see that $F(u,v) \leq c_1\left(|u|^{r+1} + |v|^{r+1} \right)$, for all $(u,v) \in R^2$, where $c_1 = 2^r a + b$. Moreover, for a fixed $a, r > 1$, there exists a constant $c_0 > 0$ such that $F(u,v) \geq c_0\left(|u|^{r+1} + |v|^{r+1} \right)$ provided b is chosen large enough.

Next, for the problem (1.1), we consider the following functionals:

$$\begin{aligned} I(t) = I(u,v) &= \left(1 - \int_0^t g_1(\tau)d\tau\right) \|\Delta u\|_2^2 + \left(1 - \int_0^t g_2(\tau)d\tau\right) \|\Delta v\|_2^2 \\ &\quad + (g_1 \circ \Delta u)(t) + (g_2 \circ \Delta v)(t) - (r+1)\int_\Omega F(u,v)dx, \end{aligned} \tag{2.3}$$

$$\begin{aligned} J(t) = J(u,v) &= \frac{1}{2}\left(1 - \int_0^t g_1(\tau)d\tau\right) \|\Delta u\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g_2(\tau)d\tau\right) \|\Delta v\|_2^2 \\ &\quad + \frac{1}{2}(g_1 \circ \Delta u)(t) + \frac{1}{2}(g_2 \circ \Delta v)(t) - \int_\Omega F(u,v)dx. \end{aligned} \tag{2.4}$$

$$E(t) = \frac{1}{\rho+2}\left(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}\right) + \frac{1}{2}\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 \right) + J(t), \tag{2.5}$$

defined on $H_0^2(\Omega)$, where

$$(g_i \circ \phi)(t) = \int_0^t g_i(t-\tau)\|\phi(t) - \phi(\tau)\|_2^2 d\tau, \quad i=1,2.$$

Lemma 2.2 *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$\frac{d}{dt}E(t) = \frac{1}{2}\left[(g_1' \circ \Delta u)(t) + (g_2' \circ \Delta v)(t) \right] - \frac{1}{2}\left[g_1(t)\|\Delta u\|_2^2 + g_2(t)\|\Delta v\|_2^2 \right],$$

$$-\|\nabla u_t\|_2^2 - \|\nabla v_t\|_2^2 - \|u_t\|_{p+1}^{p+1} - \|v_t\|_{q+1}^{q+1} \leq 0. \tag{2.6}$$

Proof. Multiplying the first equation in (1.1) by u_t and the second by v_t, integrating over Ω, using the boundary conditions and (2.5), we obtain (2.6).

The following lemmas play critical roles in the proof of our main results.

Lemma 2.3 *Assume that* $(2.2)_1$ *holds. Then there exists* $\eta > 0$ *such that for any* $(u,v) \in H_0^2(\Omega) \times H_0^2(\Omega)$ *one has*

$$\int_\Omega F(u,v)dx \leq \eta \left(l_1\|\Delta u\|_2^2 + l_2\|\Delta v\|_2^2 \right)^{\frac{r+1}{2}}. \tag{2.7}$$

Proof. Using Minkowski's inequality, we get

$$\|u+v\|_{r+1}^2 \leq 2\left(\|u\|_{r+1}^2 + \|v\|_{r+1}^2 \right). \tag{2.8}$$

Moreover, by Hölder's inequality, (2.1) and Young's inequality we get

$$\|uv\|_{\frac{r+1}{2}} \leq \|u\|_{r+1}\|v\|_{r+1} \leq C_*^2\|\Delta u\|_2\|\Delta v\|_2$$

$$\leq C_*^2\left(\frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|\Delta v\|_2^2\right) \leq c\left(l_1\|\Delta u\|_2^2 + l_2\|\Delta v\|_2^2 \right), \tag{2.9}$$

for some positive constant c. Combining (1.3), (2.8), and (2.9) and using the embedding $H_0^2(\Omega) \hookrightarrow L^{r+1}(\Omega)$, we obtain (2.7).

Lemma 2.4 *There exist 2 positive constants* μ_1 *and* μ_2 *such that*

$$\int_\Omega |f_i(u,v)|^2 dx \leq \mu_i \left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 \right)^r, \qquad i = 1, 2.$$

Proof. Clearly we have

$$|f_1(u,v)| \leq C_1\left(|u+v|^r + |u|^{\frac{r-1}{2}}|v|^{\frac{r+1}{2}}\right) \leq C_2\left(|u|^r + |v|^r + |u|^{\frac{r-1}{2}}|v|^{\frac{r+1}{2}}\right).$$

Using Young's inequality we obtain

$$|u|^{\frac{r-1}{2}}|v|^{\frac{r+1}{2}} \leq (C_3|u|^r + C_4|v|^r).$$

Therefore,

$$|f_1(u,v)| \leq C_5(|u|^r + |v|^r).$$

Thus, by lemma 2.1 we get

$$\int_\Omega |f_1(u,v)|^2 dx \leq C_6(\|u\|_{2r}^{2r} + \|v\|_{2r}^{2r}) \leq \mu_1\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right)^r.$$

The same way can be followed to obtain a similar inequality for f_2.

We define

$$d(t) = \inf_{(u,v)\in H_0^2(\Omega)\times H_0^2(\Omega),(u,v)\neq(0,0)} \sup_{\lambda\geq 0} J(\lambda u,\lambda v), \qquad t\geq 0.$$

Then we obtain the following result.

Lemma 2.5 *For $t \geq 0$ we have*

$$0 < d_1 \leq d(t) \leq d_2(u,v) = \sup_{\lambda\geq 0} J(\lambda u,\lambda v),$$

where

$$d_1 := \frac{r-1}{2(r+1)}\left(\frac{1}{\eta(r+1)}\right)^{\frac{2}{r-1}},$$

and

$$d_2(u,v) := \frac{r-1}{2(r+1)}\left[\frac{(\Gamma(t))^{\frac{r+1}{r-1}}}{\left(\,(r+1)\int_\Omega F(u,v)dx\,\right)^{\frac{2}{r-1}}}\right],$$

where

$$\Gamma(t) := \left(1-\int_0^t g_1(\tau)d\tau\right)\|\Delta u\|_2^2 + (g_1\circ\Delta u)(t) + \left(1-\int_0^t g_2(\tau)d\tau\right)\|\Delta v\|_2^2 + (g_2\circ\Delta v)(t).$$

Proof For fixed $(u,v)\in H_0^2(\Omega)\times H_0^2(\Omega)\backslash(0,0)$, we define

$$\mathcal{G}(\lambda) = J(\lambda u,\lambda v) = \frac{1}{2}\lambda^2\left\{\left(1-\int_0^t g_1(\tau)d\tau\right)\|\Delta u\|_2^2 + (g_1\circ\Delta u)(t)\right.$$
$$\left.+\left(1-\int_0^t g_2(\tau)d\tau\right)\|\Delta v\|_2^2 + (g_2\circ\Delta v)(t) - 2\lambda^{r-1}\int_\Omega F(u,v)dx\right\}.$$

A direct calculation shows that

$$\mathcal{G}'(\lambda_1) = 0 \ \text{ where } \ \lambda_1 = \left(\frac{\Gamma(t)}{(r+1)\int_\Omega F(u,v)dx}\right)^{\frac{1}{r-1}}.$$

Therefore,

$$\sup_{\lambda\geq 0}\mathcal{G}(\lambda) = \mathcal{G}(\lambda_1) = d_2(u,v). \tag{2.10}$$

Then from (2.10) the desired result can be obtained by using the inequality (2.7) and the fact that $1 - \int_0^t g_i(\tau)d\tau > l_i,\ i=1,2$. □

Now we state a local existence theorem for solutions of the system (1.1) that can be established by combining the arguments in [1, 6, 27].

Theorem 2.1 *Suppose that (G1) holds. Let $(u_0,u_1),(v_0,v_1)\in H_0^2(\Omega)\times H_0^1(\Omega)$ be given. Then there exists a unique weak solution (u,v) of (1.1) such that*

$$u,v\in C([0,T],H_0^2(\Omega)),$$

$$u_t \in C([0,T); H_0^1(\Omega)) \cap L^{p+1}(\Omega \times (0,T)), \quad v_t \in C([0,T); H_0^1(\Omega)) \cap L^{q+1}(\Omega \times (0,T)),$$

for some $T > 0$.

We state our results as follows.

Theorem 2.2 *Suppose that* $(G1) - (G3)$ *hold. Assume that* $(u_0, u_1), (v_0, v_1) \in H_0^2(\Omega) \times H_0^1(\Omega)$ *and satisfy*

$$I(0) > 0; \qquad E(0) < d_1. \tag{2.11}$$

Then, for each $t_0 > 0$, *there exist 2 positive constants* K *and* κ *such that the solution of* (1.1) *satisfies*

$$E(t) \le K e^{-\kappa \int_{t_0}^{t} \xi(s) ds}, \qquad \forall t \ge t_0. \tag{2.12}$$

Theorem 2.3 *Suppose that* $(G1)$, $(G2)$, *and* $(G4)$ *hold and* $r+1 > \max(\rho+2, p+1, q+1)$. *For any fixed number* $0 < \delta < 1$, *assume that* $(u_0, u_1), (v_0, v_1) \in H_0^2(\Omega) \times H_0^1(\Omega)$ *and satisfy*

$$I(0) < 0, \qquad E(0) < \delta d_1. \tag{2.13}$$

Assume further that there exists a fixed number

$$2 < \theta < \frac{r+1}{1 + (r-1)(\delta/2)},$$

such that

$$\max\left(\int_0^\infty g_1(\tau) d\tau, \int_0^\infty g_2(\tau) d\tau\right) < \frac{(\theta/2) - 1}{(\theta/2) - 1 + 1/(2\theta)}. \tag{2.14}$$

Then the norm $\|(u,v)\|_{r+1}$ *of solutions grows exponentially where*

$$\|(u,v)\|_{r+1} := \|u\|_{r+1} + \|v\|_{r+1}.$$

3. Global existence and energy decay

Lemma 3.1 *Suppose that* (2.2) *holds. Let* $(u_0, u_1), (v_0, v_1) \in H_0^2(\Omega) \times H_0^1(\Omega)$ *and satisfy* (2.11). *Then* $I(t) > 0$ *for all* $t \ge 0$.

Proof. Since $I(0) > 0$, then by continuity, there exists $T_* \le T$ such that $I(t) \ge 0$ for all $t \in [0, T_*)$. Using the fact that $1 - \int_0^t g_i(\tau) d\tau > l_i$, for any $t \in [0, T_*)$ we have

$$\begin{aligned} J(t) &= \frac{r-1}{2(r+1)} \left(1 - \int_0^t g_1(\tau) d\tau\right) \|\Delta u\|_2^2 + \frac{r-1}{2(r+1)} \left(1 - \int_0^t g_2(\tau) d\tau\right) \|\Delta v\|_2^2 \\ &\quad + \frac{r-1}{2(r+1)} \Big((g_1 \circ \Delta u)(t) + (g_2 \circ \Delta v)(t) \Big) + \frac{1}{r+1} I(u,v) \\ &\ge \frac{r-1}{2(r+1)} \Big(l_1 \|\Delta u\|_2^2 + l_2 \|\Delta v\|_2^2 \Big). \end{aligned} \tag{3.1}$$

Therefore, from (2.5), (2.6), and (3.1) we have

$$l_1\|\Delta u\|_2^2 + l_2\|\Delta v\|_2^2 \le \frac{2(r+1)}{r-1}J(t) \le \frac{2(r+1)}{r-1}E(t) \le \frac{2(r+1)}{r-1}E(0). \tag{3.2}$$

By (2.11), (3.2), and lemma 2.3 we obtain

$$\begin{aligned}(r+1)\int_\Omega F(u,v)dx &\le \eta(r+1)\left(l_1\|\Delta u\|_2^2 + l_2\|\Delta v\|_2^2 \right)^{\frac{r+1}{2}} \\ &\le \eta(r+1)\left(\frac{2(r+1)}{r-1}E(0)\right)^{\frac{r-1}{2}} \left(l_1\|\Delta u\|_2^2 + l_2\|\Delta v\|_2^2 \right) \\ &< \left(1-\int_0^t g_1(\tau)d\tau\right)\|\Delta u\|_2^2 + \left(1-\int_0^t g_2(\tau)d\tau\right)\|\Delta v\|_2^2.\end{aligned}$$

This shows that $I(t) > 0$ for all $t \in [0, T_*)$. By repeating this procedure, T_* can be extended to T.

Lemma 3.2 *Suppose that* (2.2) *and* (2.11) *hold. If* $(u_0,u_1),(v_0,v_1) \in H_0^2(\Omega)\times H_0^1(\Omega)$, *then the solution of* (1.1) *is global and bounded.*

Proof. Using (2.6) and lemma 3.1 we have

$$\begin{aligned}E(0) \ge E(t) &= \frac{1}{\rho+2}\left(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}\right) + \frac{1}{2}\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 \right) + J(t) \\ &\ge \frac{1}{\rho+2}\left(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}\right) + \frac{1}{2}\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 \right) + \frac{r-1}{2(r+1)}\left(l_1\|\Delta u\|_2^2 + l_2\|\Delta v\|_2^2 \right).\end{aligned}$$

Therefore,

$$\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 + \|\Delta u\|_2^2 + \|\Delta v\|_2^2 \le CE(0), \tag{3.3}$$

where C is a positive constant that depends only on ρ, l_1, l_2, and r.

Remark 3.1 When, in (1.2), $a \le 0$ and $b \le 0$, then any solution of (1.1) with $(u_0,u_1),(v_0,v_1) \in H_0^2(\Omega)\times H_0^1(\Omega)$ is global in time and lemma 3.2 and Theorem 2.2 hold without condition (2.11).

To prove Theorem 2.2 we need to define the following perturbed energy functional:

$$G(t) = ME(t) + \varepsilon\psi(t) + \chi(t),$$

where M and ε are positive constants that will be specified later and

$$\begin{aligned}\psi(t) &= \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx + \frac{1}{\rho+1}\int_\Omega |v_t|^\rho v_t v dx \\ &\quad + \frac{1}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \int_\Omega (\nabla u.\nabla u_t + \nabla v.\nabla v_t)\, dx,\end{aligned} \tag{3.4}$$

$$\chi(t) = \int_\Omega \left(\Delta u + \Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1}\right)\int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau dx$$

$$+\int_\Omega\left(\Delta v+\Delta v_t-\frac{|v_t|^\rho v_t}{\rho+1}\right)\int_0^t g_2(t-\tau)(v(t)-v(\tau))d\tau dx. \tag{3.5}$$

It is straightforward to see that $G(t)$ and $E(t)$ are equivalent in the sense that there exist 2 positive constants β_1 and β_2, depending on ε and M, such that for $t\geq 0$

$$\beta_1 E(t)\leq G(t)\leq \beta_2 E(t). \tag{3.6}$$

Lemma 3.3 *Suppose that* $(G1)$ *and* $(G2)$ *hold. Let* $(u_0,u_1),(v_0,v_1)\in H_0^2(\Omega)\times H_0^1(\Omega)$ *and satisfy* (2.11). *Then any solution of* (1.1) *satisfies*

$$\psi'(t)\leq\frac{1}{\rho+1}\left(\|u_t\|_{\rho+2}^{\rho+2}+\|v_t\|_{\rho+2}^{\rho+2}\right)-\frac{1}{3}\left(\,l_1\|\Delta u\|_2^2+l_2\|\Delta v\|_2^2\,\right)$$

$$+\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2+k_1(g_1\circ\Delta u)(t)+k_2(g_2\circ\Delta v)(t)$$

$$+k_3\|u_t\|_{p+1}^{p+1}+k_4\|v_t\|_{q+1}^{q+1}+(r+1)\int_\Omega F(u,v)dx, \tag{3.7}$$

for some positive constants k_1,k_2,k_3, *and* k_4.

Proof By taking the time derivative of (3.4) and using problem (1.1), we get

$$\psi'(t)=\frac{1}{\rho+1}\left(\|u_t\|_{\rho+2}^{\rho+2}+\|v_t\|_{\rho+2}^{\rho+2}\right)-\|\Delta u\|_2^2-\|\Delta v\|_2^2+\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2$$

$$-\int_\Omega uu_t|u_t|^{p-1}dx-\int_\Omega vv_t|v_t|^{q-1}dx+(r+1)\int_\Omega F(u,v)dx$$

$$+\int_\Omega\int_0^t g_1(t-\tau)\Delta u(t)\Delta u(\tau)d\tau dx+\int_\Omega\int_0^t g_2(t-\tau)\Delta v(t)\Delta v(\tau)d\tau dx. \tag{3.8}$$

Using Young's inequality, (2.1), and (3.3), for $\gamma_1,\gamma_2>0$, we have

$$\left|-\int_\Omega uu_t|u_t|^{p-1}dx\right|\leq\gamma_1\|u\|_{p+1}^{p+1}+c(\gamma_1)\|u_t\|_{p+1}^{p+1}$$

$$\leq\gamma_1 c_1\|\Delta u\|_2^2+c(\gamma_1)\|u_t\|_{p+1}^{p+1}, \tag{3.9}$$

where $c_1=C_*^{p+1}(CE(0))^{\frac{p-1}{2}}$. Similarly

$$\left|-\int_\Omega vv_t|v_t|^{q-1}dx\right|\leq\gamma_2\|v\|_{q+1}^{q+1}+c(\gamma_2)\|v_t\|_{q+1}^{q+1}$$

$$\leq\gamma_2 c_2\|\Delta v\|_2^2+c(\gamma_2)\|v_t\|_{q+1}^{q+1}, \tag{3.10}$$

where $c_2=C_*^{q+1}(CE(0))^{\frac{q-1}{2}}$. For the last 2 terms in the right-hand side of (3.8) we have

$$\left|\int_\Omega\int_0^t g_1(t-\tau)\Delta u(t)\Delta u(\tau)d\tau dx\right|$$

$$\leq \int_\Omega \left(\int_0^t g_1(t-\tau)|\Delta u(\tau) - \Delta u(t)||\Delta u(t)|d\tau \right) dx + \int_0^t g_1(\tau)d\tau \|\Delta u\|_2^2$$

$$\leq (1+\gamma_1)(1-l_1)\|\Delta u\|_2^2 + \frac{1}{4\gamma_1}(g_1 \circ \Delta u)(t). \tag{3.11}$$

Analogously,

$$\left| \int_\Omega \int_0^t g_2(t-\tau)\Delta v(t)\Delta v(\tau)d\tau dx \right| \leq (1+\gamma_2)(1-l_2)\|\Delta v\|_2^2 + \frac{1}{4\gamma_2}(g_2 \circ \Delta v)(t). \tag{3.12}$$

Inserting (3.9)–(3.12) into (3.8) we obtain

$$\psi'(t) \leq \frac{1}{\rho+1}\left(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2} \right) + (r+1)\int_\Omega F(u,v)dx + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2$$

$$+ \Big(\gamma_1 c_1 + (1+\gamma_1)(1-l_1) - 1 \Big) \|\Delta u\|_2^2 + \Big(\gamma_2 c_2 + (1+\gamma_2)(1-l_2) - 1 \Big) \|\Delta v\|_2^2$$

$$+ \frac{1}{4\gamma_1}(g_1 \circ \Delta u)(t) + \frac{1}{4\gamma_2}(g_2 \circ \Delta v)(t) + c(\gamma_1)\|u_t\|_{p+1}^{p+1} + c(\gamma_2)\|v_t\|_{q+1}^{q+1}.$$

Letting $\gamma_1 = \frac{2l_1}{3(c_1+1-l_1)}$ and $\gamma_2 = \frac{2l_2}{3(c_2+1-l_2)}$, the estimate (3.7) follows. □

Lemma 3.4 *Suppose that* (G1) *and* (G2) *hold. Let* $(u_0,u_1),(v_0,v_1) \in H_0^2(\Omega) \times H_0^1(\Omega)$ *and satisfying* (2.11)*. Then there exist positive constants* k_5*,* k_6*,* k_7*, and* k_8 *such that the functional*

$$\chi_1(t) = \int_\Omega \left(\Delta u + \Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1} \right) \int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau dx,$$

satisfies, for all $\gamma > 0$*,*

$$\chi_1'(t) \leq \gamma \Big(k_5\|\Delta u\|_2^2 + k_6\|\Delta v\|_2^2 \Big) + \varphi_1(\gamma)(g_1 \circ \Delta u)(t)$$

$$+\gamma(1-l_1)\|u_t\|_{p+1}^{p+1} - \left(\frac{1}{\rho+1}\int_0^t g_1(\tau)d\tau \right) \|u_t\|_{\rho+2}^{\rho+2}$$

$$-\frac{k_7}{4\gamma}(g_1' \circ \Delta u)(t) + \left[k_8\left(\gamma + \frac{1}{\gamma} \right) - \int_0^t g_1(\tau)d\tau \right] \|\nabla u_t\|_2^2. \tag{3.13}$$

where $\varphi_1(\gamma)$ *is a positive function of* γ*, which will be given in the proof.*

Proof. By the first equation in (1.1), we get

$$\chi_1'(t) = \int_\Omega \Delta u \int_0^t g_1(t-\tau)(\Delta u(t) - \Delta u(\tau))d\tau dx$$

$$- \int_\Omega \left(\int_0^t g_1(t-\tau)\Delta u(\tau)d\tau \right) \left(\int_0^t g_1(t-\tau)(\Delta u(t) - \Delta u(\tau)d\tau) \right) dx$$

$$+ \int_\Omega |u_t|^{p-1}u_t \left(\int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau \right) dx$$

$$
\begin{aligned}
&+\int_\Omega \Delta u \int_0^t g_1'(t-\tau)(u(t)-u(\tau))d\tau dx \\
&-\int_\Omega \nabla u_t . \int_0^t g_1'(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau dx \\
&-\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g_1'(t-\tau)(u(t)-u(\tau))d\tau dx \\
&-\int_\Omega f_1(u,v)\left(\int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau\right)dx \\
&-\left(\|\nabla u_t\|_2^2+\frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2}\right)\int_0^t g_1(\tau)d\tau+\int_\Omega u_t\ \Delta u\ dx\int_0^t g_1(\tau)d\tau. \qquad (3.14)
\end{aligned}
$$

We estimate the terms in the right-hand side of (3.14). First, using Young's inequality for $\gamma > 0$ we have

$$
\begin{aligned}
&\int_\Omega \Delta u \int_0^t g_1(t-\tau)(\Delta u(t)-\Delta u(\tau))d\tau dx \\
&\quad\le \gamma\|\Delta u\|_2^2+\frac{1}{4\gamma}\int_\Omega\left(\int_0^t g_1(t-\tau)(\Delta u(t)-\Delta u(\tau))d\tau\right)^2 dx \\
&\quad\le \gamma\|\Delta u\|_2^2+\frac{1}{4\gamma}(1-l_1)(g_1\circ\Delta u)(t). \qquad (3.15)
\end{aligned}
$$

For the second term we obtain

$$
\begin{aligned}
&\left|-\int_\Omega\left(\int_0^t g_1(t-\tau)\Delta u(\tau)d\tau\right)\left(\int_0^t g_1(t-\tau)(\Delta u(t)-\Delta u(\tau))\right)dx\right| \\
&\quad\le \gamma\int_\Omega\left(\int_0^t g_1(t-\tau)\Delta u(\tau)d\tau\right)^2 dx+\frac{1}{4\gamma}\int_\Omega\left(\int_0^t g_1(t-\tau)(\Delta u(t)-\Delta u(\tau))\right)^2 dx \\
&\quad\le \gamma\int_\Omega\left(\int_0^t g_1(t-\tau)\Delta u(\tau)d\tau\right)^2 dx+\frac{1}{4\gamma}(1-l_1)(g_1\circ\Delta u)(t). \qquad (3.16)
\end{aligned}
$$

The first integral in the right-hand side of (3.16) can be estimated in the form

$$
\begin{aligned}
&\int_\Omega\left(\int_0^t g_1(t-\tau)\Delta u(\tau)d\tau\right)^2 dx \\
&\quad\le \int_\Omega\left(\int_0^t g_1(t-\tau)(|\Delta u(\tau)-\Delta u(t)|+|\Delta u(t)|)d\tau\right)^2 dx \\
&\quad\le 2\int_\Omega\left(\int_0^t g_1(t-\tau)|\Delta u(\tau)-\Delta u(t)|d\tau\right)^2+2\int_\Omega\left(\int_0^t g_1(t-\tau)|\Delta u(t)|d\tau\right)^2 dx \\
&\quad\le 2(1-l_1)(g_1\circ\Delta u)(t)+2(1-l_2)^2\|\Delta u\|_2^2. \qquad (3.17)
\end{aligned}
$$

Using (3.17), for the inequality (3.16) we have

$$\left| -\int_\Omega \left(\int_0^t g_1(t-\tau)\Delta u(\tau) d\tau \right) \left(\int_0^t g_1(t-\tau)(\Delta u(t) - \Delta u(\tau)) \right) dx \right|$$

$$\leq (2\gamma + \frac{1}{4\gamma})(1-l_1)(g_1 \circ \Delta u)(t) + 2\gamma(1-l_1)^2 \|\Delta u\|_2^2. \tag{3.18}$$

We use Young's inequality, (2.1), and (3.3) to estimate the third term as

$$\int_\Omega u_t |u_t|^{p-1} \int_0^t g_1(t-\tau)(u(t)-u(\tau)) d\tau dx$$

$$\leq \int_0^t g_1(t-\tau) \left(\gamma \|u_t\|_{p+1}^{p+1} + c(\gamma) \|u(t)-u(\tau)\|_{p+1}^{p+1} \right) d\tau$$

$$\leq \gamma(1-l_1)\|u_t\|_{p+1}^{p+1} + c(\gamma) C_*^{p+1} \int_0^t g_1(t-\tau) \|\Delta u(t) - \Delta u(\tau)\|_2^{p+1} d\tau$$

$$\leq \gamma(1-l_1)\|u_t\|_{p+1}^{p+1} + c(\gamma) c_3 (g_1 \circ \Delta u)(t), \tag{3.19}$$

where $c_3 = C_*^{p+1} \left(2CE(0)\right)^{\frac{p-1}{2}}$. Concerning the fourth term we have

$$\int_\Omega \Delta u \int_0^t g_1'(t-\tau)(u(t)-u(\tau)) d\tau dx \leq \gamma \|\Delta u\|_2^2 - \frac{1}{4\gamma} g_1(0) C_*^2 (g_1' \circ \Delta u)(t). \tag{3.20}$$

By the Young and Poincaré inequalities we get

$$\left| -\int_\Omega \nabla u_t \int_0^t g_1'(t-\tau)(\nabla u(t) - \nabla u(\tau)) d\tau dx \right|$$

$$\leq \gamma \|\nabla u_t\|_2^2 + \frac{1}{4\gamma} \int_\Omega \left(\int_0^t g_1'(t-\tau)(\nabla u(t) - \nabla u(\tau)) d\tau \right)^2 dx$$

$$\leq \gamma \|\nabla u_t\|_2^2 - \frac{g_1(0)}{4\gamma} \int_\Omega \int_0^t g_1'(t-\tau) |\nabla u(t) - \nabla u(\tau)|^2 d\tau dx$$

$$\leq \gamma \|\nabla u_t\|_2^2 - \frac{g_1(0)}{4\gamma} \lambda^{-1} (g_1' \circ \Delta u)(t), \tag{3.21}$$

where λ denotes the Poincaré constant. To estimate the sixth term, we use Young's inequality, the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$ with ρ satisfying $(2.2)_2$, and the inequality (3.3) to obtain

$$-\frac{1}{\rho+1} \int_\Omega |u_t|^\rho u_t \int_0^t g_1'(t-\tau)(u(t)-u(\tau)) d\tau dx$$

$$\leq \frac{\gamma}{\rho+1} \|u_t\|_{2(\rho+1)}^{2(\rho+1)} + \frac{1}{4\gamma(\rho+1)} \int_\Omega \left(\int_0^t g_1'(t-\tau)(u(t)-u(\tau)) ds \right)^2 dx$$

$$\leq \gamma c_4 \|\nabla u_t\|_2^2 - \frac{c_5}{4\gamma} (g_1' \circ \Delta u)(t), \tag{3.22}$$

where $c_4=\frac{c_s(CE(0))^\rho}{\rho+1}$ and $c_5=\frac{g_1(0)C_*^2}{\rho+1}$, where c_s is the embedding constant. Using Young's inequality again and the lemma 2.4 and (3.3), for the seventh term in the right-hand side of (3.14), we have

$$\begin{aligned}
&\left|-\int_\Omega f_1(u,v)\left(\int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau\right)dx\right|\\
&\quad\le \gamma\mu_1\Big(\ \|\Delta u\|_2^2+\|\Delta v\|_2^2\ \Big)^r+\frac{1}{4\gamma}(1-l_1)C_*^2(g_1\circ\Delta u)(t)\\
&\quad\le \gamma c_6\Big(\ \|\Delta u\|_2^2+\|\Delta v\|_2^2\ \Big)+\frac{1}{4\gamma}(1-l_1)C_*^2(g_1\circ\Delta u)(t), \qquad (3.23)
\end{aligned}$$

where $c_6=\mu_1(CE(0))^{r-1}$. Finally, exploiting the Poincaré inequality, we have

$$\int_\Omega u_t\ \Delta u\ dx\int_0^t g_1(\tau)d\tau\le\gamma(1-l_1)\|\Delta u\|_2^2+\frac{\lambda^{-1}}{4\gamma}(1-l_1)\|\nabla u_t\|_2^2. \qquad (3.24)$$

where λ is the Poincaré constant. Combining (3.14),(3.15), (3.18)-(3.24), the estimate (3.13) follows with $k_5=2(1+(1-l_1)^2)+c_6+1-l_1$, $k_6=c_6$, $k_7=g_1(0)(C_*^2+\lambda^{-1})+c_5$, $k_8=\frac{\lambda^{-1}}{4}(1-l_1)+c_4+1$ and

$$\varphi_1(\gamma)=\frac{1}{4\gamma}\Big(\ (1-l_1)(2+C_*^2+8\gamma^2)+4\gamma c(\gamma)c_3\ \Big).$$

Repeating the same discussion in lemma 3.4, we have the following result.

Lemma 3.5 *Suppose that* $(G1)$ *and* $(G2)$ *hold. Let* $(u_0,u_1),(v_0,v_1)\in H_0^2(\Omega)\times H_0^1(\Omega)$ *and satisfying* (2.11). *Then there exist positive constants* k_9, k_{10}, k_{11}, *and* k_{12} *such that the functional*

$$\chi_2(t)=\int_\Omega\left(\Delta v+\Delta v_t-\frac{|v_t|^\rho v_t}{\rho+1}\right)\int_0^t g_2(t-\tau)(v(t)-v(\tau))d\tau dx,$$

satisfies, for all $\gamma>0$,

$$\begin{aligned}
\chi_2'(t)&\le\gamma\Big(\ k_9\|\Delta u\|_2^2+k_{10}\|\Delta v\|_2^2\ \Big)+\varphi_2(\gamma)(g_2\circ\Delta v)(t)\\
&\quad+\gamma(1-l_2)\|v_t\|_{q+1}^{q+1}-\left(\frac{1}{\rho+1}\int_0^t g_2(\tau)d\tau\right)\|v_t\|_{\rho+2}^{\rho+2}\\
&\quad-\frac{k_{11}}{4\gamma}(g_2'\circ\Delta v)(t)+\left[k_{12}\left(\gamma+\frac{1}{\gamma}\right)-\int_0^t g_2(\tau)d\tau\right]\|\nabla v_t\|_2^2.
\end{aligned}$$

where

$$\varphi_2(\gamma)=\frac{1}{4\gamma}\Big(\ (1-l_2)(2+C_*^2+8\gamma^2)+4\gamma c(\gamma)c_7\ \Big).$$

in which $c_7=C_*^{q+1}(2CE(0))^{\frac{q-1}{2}}$.

Now, we are in a position to prove Theorem 2.2.

Proof of Theorem 2.2. The assumption $(G2)$ guarantees that for any $t_0 > 0$ we have

$$\int_0^t g_i(\tau)d\tau \geq \int_0^{t_0} g_i(\tau)d\tau =: \hat{g}_i, \quad i = 1,2, \qquad \forall t \geq t_0.$$

By using the definition of the function $G(t)$ and lemmas 3.3–3.5 we deduce

$$\begin{aligned}
G'(t) \leq & -\left(\frac{\varepsilon l_1}{3} - \gamma(k_5 + k_9)\right)\|\Delta u\|_2^2 - \left(\frac{\varepsilon l_2}{3} - \gamma(k_6 + k_{10})\right)\|\Delta v\|_2^2 \\
& + \left(\frac{M}{2} - \frac{k_7}{4\gamma}\right)(g_1' \circ \Delta u)(t) + \left(\frac{M}{2} - \frac{k_{11}}{4\gamma}\right)(g_2' \circ \Delta v)(t) \\
& - \left(M - k_8(\gamma + \frac{1}{\gamma}) + \hat{g}_1 - \varepsilon\right)\|\nabla u_t\|_2^2 - \left(M - k_{12}(\gamma + \frac{1}{\gamma}) + \hat{g}_2 - \varepsilon\right)\|\nabla v_t\|_2^2 \\
& - \Big(M - \varepsilon k_3 - \gamma(1 - l_1)\Big)\|u_t\|_{p+1}^{p+1} - \Big(M - \varepsilon k_4 - \gamma(1 - l_2)\Big)\|v_t\|_{q+1}^{q+1} \\
& - \left(\frac{\hat{g}_1 - \varepsilon}{\rho + 1}\right)\|u_t\|_{\rho+2}^{\rho+2} - \left(\frac{\hat{g}_2 - \varepsilon}{\rho + 1}\right)\|v_t\|_{\rho+2}^{\rho+2} + \varepsilon(r + 1)\int_\Omega F(u,v)dx \\
& + (\varepsilon k_1 + \varphi_1(\gamma))(g_1 \circ \Delta u)(t) + (\varepsilon k_2 + \varphi_2(\gamma))(g_2 \circ \Delta v)(t).
\end{aligned} \tag{3.25}$$

We choose ε and γ small enough such that $\varepsilon < \min(\hat{g}_1, \hat{g}_2)$ and

$$\gamma < \min\left\{\frac{\varepsilon l_1}{3(k_5 + k_9)}, \frac{\varepsilon l_2}{3(k_6 + k_{10})}\right\}.$$

With ε and γ fixed, we take M sufficiently large such that

$$M > \max\left\{\frac{k_7}{2\gamma}, \frac{k_{11}}{2\gamma}, (k_8 + k_{12})(\gamma + \frac{1}{\gamma}), \varepsilon k_3 + \gamma(1 - l_1), \varepsilon k_4 + \gamma(1 - l_2)\right\}.$$

Therefore, there exist positive constants κ_1 and κ_2 such that for all $t \geq t_0$ we have

$$G'(t) \leq -\kappa_1 E(t) + \kappa_2 \Big((g_1 \circ \Delta u)(t) + (g_2 \circ \Delta v)(t)\Big). \tag{3.26}$$

Multiplying (3.26) by $\xi(t) = \min(\xi_1(t), \xi_2(t))$, using the condition $(G3)$ and (2.6), we get

$$\begin{aligned}
\xi(t)G'(t) & \leq -\kappa_1\xi(t)E(t) + \kappa_2\xi(t)((g_1 \circ \Delta u)(t) + (g_2 \circ \Delta v)(t)) \\
& \leq -\kappa_1\xi(t)E(t) - \kappa_2((g_1' \circ \Delta u)(t) + (g_2' \circ \Delta v))(t) \\
& \leq -\kappa_1\xi(t)E(t) - 2\kappa_2 E'(t).
\end{aligned}$$

In other words, for all $t \geq t_0$ we have

$$(\xi(t)G(t) + 2\kappa_2 E(t))' \leq \xi'(t)G(t) - \kappa_1\xi(t)E(t). \tag{3.27}$$

Let us define

$$\mathcal{E}(t) = \xi(t)G(t) + 2\kappa_2 E(t). \tag{3.28}$$

Using the fact that $\xi(t)$ is a positive nonincreasing function, we have $\xi(t) < \xi(0)$ for all $t \geq t_0$. Then, by (3.6), it is not difficult to see that $\mathcal{E}(t)$ is equivalent to $E(t)$. Therefore, by (3.27) and (3.28), we find

$$\mathcal{E}'(t) \leq \beta_2 \xi'(t)E(t) - \kappa_1 \xi(t)E(t) \leq -\kappa_1 \xi(t)E(t) \leq -\kappa \xi(t)\mathcal{E}(t), \tag{3.29}$$

for some positive constant κ. Integrating (3.30) over (t_0, t) gives the estimate

$$\mathcal{E}(t) \leq \mathcal{E}(t_0)e^{-\kappa \int_{t_0}^{t} \xi(s)ds}, \qquad \forall t \geq t_0. \tag{3.30}$$

Consequently, by using (3.6), (3.28), and (3.30), the estimate (2.12) follows.

4. Exponential growth

In this section we prove an unboundedness result, Theorem 2.3, for certain solutions of (1.1) with positive initial energy. For this purpose, we first give a lemma that will be used later.

Lemma 4.1 *Suppose that* $(u_0, u_1), (v_0, v_1) \in H_0^2(\Omega) \times H_0^1(\Omega)$ *and satisfy* (2.13)*. Then* $I(t) < 0$ *for* $0 \leq t < T$ *and*

$$d_1 < \frac{r-1}{2(r+1)}\Gamma(t) < \frac{r-1}{2}\int_\Omega F(u,v)dx, \qquad \forall t, \quad 0 \leq t < T. \tag{4.1}$$

Proof Let $I(0) < 0$; we have to prove that $I(t) < 0$ for all $t \in [0, T)$. This can be shown by contradiction. Suppose that there exists $t^* > 0$ such that $I(t^*) = 0$ and $I(t) < 0$ for $t \in [0, t^*)$. Therefore,

$$\Gamma(t) < (r+1)\int_\Omega F(u,v)dx, \qquad \forall t \in [0, t^*).$$

Then, by the use of lemma 2.5, we obtain

$$d_1 < \frac{r-1}{2(r+1)}\Gamma(t), \qquad \forall t \in [0, t^*).$$

Therefore,

$$d_1 < \frac{r-1}{2}\int_\Omega F(u,v)dx, \qquad \forall t \in [0, t^*).$$

Since $t \mapsto \int_\Omega F(u,v)dx$ is continuous, we have $\int_\Omega F(u(t^*), v(t^*))dx \neq 0$. In view of lemma 2.5 and (2.4), we have

$$d_1 \leq \frac{r-1}{2}\int_\Omega F(u(t^*), v(t^*))dx = \frac{r-1}{2(r+1)}\Gamma(t^*) = J(u(t^*), v(t^*)),$$

which is impossible, since $J(u(t^*), v(t^*)) \leq E(t^*) < d_1$. The inequality (4.1) can be obtained by using lemma 2.5 again. This completes the proof. □

Proof of Theorem 2.3. Since $E(0) < \delta d_1$, then $E(0) < d_1$. Let us define

$$H(t) = \delta d_1 - E(t), \tag{4.2}$$

which is an increasing function by (2.6) and

$$H'(t) \geq \|u_t\|_{p+1}^{p+1} + \|v_t\|_{q+1}^{q+1} \geq 0. \tag{4.3}$$

Also

$$H(t) \geq H(0) = \delta d_1 - E(0) > 0,$$

and

$$H(t) \leq \delta d_1 + \int_\Omega F(u,v)dx \leq \left[\delta\left(\frac{r-1}{2}\right)+1\right]\int_\Omega F(u,v)dx, \qquad \forall t \in [0,T).$$

We consider the following functional

$$\mathcal{L}(t) = H(t) + \frac{\varepsilon}{\rho+1}\int_\Omega \Big(|u_t|^\rho u_t u + |v_t|^\rho v_t v \Big)\, dx + \frac{\varepsilon}{2}\int_\Omega \Big(|\nabla u|^2 + |\nabla v|^2 \Big)\, dx + \varepsilon \int_\Omega (\nabla u . \nabla u_t + \nabla v . \nabla v_t),$$

for small $\varepsilon > 0$ to be specified later. By taking the time derivative of the function $\mathcal{L}(t)$, using problem (1.1), performing several integration by parts, and using the relation

$$\int_\Omega F(u,v)dx = H(t) - \delta d_1 + \frac{1}{\rho+2}\Big(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2} \Big) + \frac{1}{2}(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2) + \frac{1}{2}\Gamma(t),$$

we get, for $2 < \theta < r+1$,

$$\begin{aligned}
\mathcal{L}'(t) = {} & H'(t) + \varepsilon\left(\frac{1}{\rho+1} + \frac{\theta}{\rho+2}\right)\Big(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2} \Big) \\
& + \varepsilon\left(1+\frac{\theta}{2}\right)(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2) + \varepsilon(r+1-\theta)\int_\Omega F(u,v)dx \\
& - \varepsilon \int_\Omega \Big(|u_t|^{p-1}u_t u + |v_t|^{q-1} v_t v \Big)\, dx + \frac{\varepsilon\theta}{2}\Big((g_1 \circ \Delta u)(t) + (g_2 \circ \Delta v)(t) \Big) \\
& + \varepsilon\left[\left(\frac{\theta}{2}-1\right) - \left(\frac{\theta}{2}-1\right)\int_0^t g_1(\tau)d\tau\right]\|\Delta u\|_2^2 + \varepsilon\theta H(t) \\
& + \varepsilon\left[\left(\frac{\theta}{2}-1\right) - \left(\frac{\theta}{2}-1\right)\int_0^t g_2(\tau)d\tau\right]\|\Delta v\|_2^2 - \varepsilon\theta\delta d_1 \\
& + \varepsilon \int_\Omega\int_0^t g_1(t-\tau)(\Delta u(\tau) - \Delta u(t))\Delta u(t) d\tau dx \\
& + \varepsilon \int_\Omega\int_0^t g_2(t-\tau)(\Delta v(\tau) - \Delta v(t))\Delta v(t) d\tau dx.
\end{aligned} \tag{4.4}$$

By Young's inequality, from (4.4) we obtain

$$\begin{aligned}
\mathcal{L}'(t) \geq {} & H'(t) + \varepsilon\left(\frac{1}{\rho+1} + \frac{\theta}{\rho+2}\right)\Big(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2} \Big) + \varepsilon(r+1-\theta)\int_\Omega F(u,v)dx \\
& - \varepsilon \int_\Omega \Big(|u_t|^{p-1}u_t u + |v_t|^{q-1} v_t v \Big)\, dx + \varepsilon\left(\frac{\theta}{2} - \eta\right)\Big((g_1 \circ \Delta u)(t) + (g_2 \circ \Delta v)(t) \Big)
\end{aligned}$$

$$+\varepsilon\left(1+\frac{\theta}{2}\right)(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2)+\varepsilon\theta H(t)-\varepsilon\theta\delta d_1$$

$$+\varepsilon\left[\left(\frac{\theta}{2}-1\right)-\left(\frac{\theta}{2}-1+\frac{1}{4\eta}\right)\int_0^t g_1(\tau)d\tau\right]\|\Delta u\|_2^2$$

$$+\varepsilon\left[\left(\frac{\theta}{2}-1\right)-\left(\frac{\theta}{2}-1+\frac{1}{4\eta}\right)\int_0^t g_2(\tau)d\tau\right]\|\Delta v\|_2^2, \tag{4.5}$$

Taking $0<\eta<\frac{\theta}{2}$ and using (2.14), the inequality (4.5) can be rewritten as

$$\mathcal{L}'(t)\geq H'(t)+\varepsilon\left(\frac{1}{\rho+1}+\frac{\theta}{\rho+2}\right)\Big(\|u_t\|_{\rho+2}^{\rho+2}+\|v_t\|_{\rho+2}^{\rho+2}\Big)$$

$$+\varepsilon\left(1+\frac{\theta}{2}\right)(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2)+\varepsilon(r+1-\theta)\int_\Omega F(u,v)dx$$

$$+\varepsilon\alpha_1\Big((g_1\circ\Delta u)(t)+(g_2\circ\Delta v)(t)\Big)+\varepsilon\alpha_2(\|\Delta u\|_2^2+\|\Delta v\|_2^2)$$

$$+\varepsilon\theta H(t)-\varepsilon\theta\delta d_1-\varepsilon\int_\Omega\Big(|u_t|^{p-1}u_tu+|v_t|^{q-1}v_tv\Big)dx, \tag{4.6}$$

where

$$\alpha_1=\frac{\theta}{2}-\eta>0,\qquad \alpha_2=\left(\frac{\theta}{2}-1\right)-\left(\frac{\theta}{2}-1+\frac{1}{4\eta}\right)\max\left(\int_0^\infty g_1(\tau)d\tau,\int_0^\infty g_2(\tau)d\tau\right)>0.$$

Using (4.1) and taking $\sigma=(r+1)-\theta\big(1+(r-1)(\delta/2)\big)>0$, the estimate (4.6) reduces to

$$\mathcal{L}'(t)\geq H'(t)+\varepsilon\left(\frac{1}{\rho+1}+\frac{\theta}{\rho+2}\right)\Big(\|u_t\|_{\rho+2}^{\rho+2}+\|v_t\|_{\rho+2}^{\rho+2}\Big)$$

$$+\varepsilon\left(1+\frac{\theta}{2}\right)(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2)+\varepsilon\sigma\int_\Omega F(u,v)dx$$

$$+\varepsilon\alpha_1\Big((g_1\circ\Delta u)(t)+(g_2\circ\Delta v)(t)\Big)+\varepsilon\alpha_2(\|\Delta u\|_2^2+\|\Delta v\|_2^2)$$

$$+\varepsilon\theta H(t)-\varepsilon\int_\Omega\Big(|u_t|^{p-1}u_tu+|v_t|^{q-1}v_tv\Big)dx. \tag{4.7}$$

Since $2\leq p+1<r+1$, using the Hölder inequality and the standard interpolation inequality, we get

$$\left|\int_\Omega|u_t|^{p-1}u_tudx\right|\leq\|u(t)\|_{p+1}\;\|u_t(t)\|_{p+1}^p\leq\|u(t)\|_2^k\;\|u(t)\|_{r+1}^{1-k}\;\|u_t(t)\|_{p+1}^p, \tag{4.8}$$

where $\frac{k}{2}+\frac{1-k}{r+1}=\frac{1}{p+1}$, which gives $k=\frac{2(r-p)}{(p+1)(r-1)}>0$. From the condition $(G4)$, we have

$$c_0\Big(\|u\|_{r+1}^{r+1}+\|v\|_{r+1}^{r+1}\Big)\leq\mathcal{F}(t)=\int_\Omega F(u,v)dx,\qquad\forall t\in[0,+\infty). \tag{4.9}$$

Using lemma 4.1, since $I(t) < 0$, from (2.3) we obtain

$$\|\Delta u\|_2^2 + \|\Delta v\|_2^2 \leq c_1 \mathcal{F}(t), \qquad \forall t \in [0, +\infty).$$

Therefore, an application of (2.1) yields

$$\|u(t)\|_2^2 \qquad \text{and} \qquad \|v(t)\|_2^2 \quad \leq c_2 \mathcal{F}(t), \qquad \forall t \in [0, +\infty), \tag{4.10}$$

From (4.8)–(4.10) we deduce

$$\left| \int_\Omega |u_t|^{p-1} u_t u dx \right| \leq c_3 \mathcal{F}^{\frac{k}{2}}(t)\ \mathcal{F}^{\frac{1-k}{r+1}}(t)\ \|u_t(t)\|_{p+1}^p \leq c_3 \mathcal{F}^{\frac{1}{p+1}}(t)\ \|u_t(t)\|_{p+1}^p. \tag{4.11}$$

Consequently, by using Young's inequality, from (4.11) we get

$$\left| \int_\Omega |u_t|^{p-1} u_t u dx \right| \leq \frac{\delta_1^{p+1}}{p+1} \mathcal{F}(t) + \frac{p}{p+1} \delta_1^{-\frac{p+1}{p}} \|u_t\|_{p+1}^{p+1}, \tag{4.12}$$

and similarly

$$\left| \int_\Omega |v_t|^{q-1} v_t v dx \right| \leq \frac{\delta_2^{q+1}}{q+1} \mathcal{F}(t) + \frac{q}{q+1} \delta_2^{-\frac{q+1}{q}} \|v_t\|_{q+1}^{q+1}, \tag{4.13}$$

where $\delta_1, \delta_2 > 0$ will be chosen later. We use (4.7), (4.12), (4.13), and (4.3) to obtain

$$\begin{aligned}
\mathcal{L}'(t) \geq & \left[1 - \varepsilon \left(\frac{p}{p+1} \delta_1^{-\frac{p+1}{p}} + \frac{q}{q+1} \delta_2^{-\frac{q+1}{q}} \right) \right] H'(t) \\
& + \varepsilon \left(\sigma - \frac{\delta_1^{p+1}}{p+1} - \frac{\delta_2^{q+1}}{q+1} \right) \int_\Omega F(u, v) dx + \varepsilon \theta H(t) \\
& + \varepsilon \alpha_1 \Big((g_1 \circ \Delta u)(t) + (g_2 \circ \Delta v)(t) \Big) + \varepsilon \alpha_2 (\|\Delta u\|_2^2 + \|\Delta v\|_2^2) \\
& + \varepsilon \left(\frac{1}{\rho+1} + \frac{\theta}{\rho+2} \right) \Big(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2} \Big) + \varepsilon \left(1 + \frac{\theta}{2} \right) (\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2).
\end{aligned} \tag{4.14}$$

Taking δ_1, δ_2, and ε so small such that

$$\sigma - \frac{\delta_1^{p+1}}{p+1} - \frac{\delta_2^{q+1}}{q+1} > 0, \quad 1 - \varepsilon \left(\frac{p}{p+1} \delta_1^{-\frac{p+1}{p}} + \frac{q}{q+1} \delta_2^{-\frac{q+1}{q}} \right) > 0,$$

then there exists $\Lambda_1 > 0$ such that (4.14) takes the form

$$\mathcal{L}'(t) \geq \Lambda_1 \left(H(t) + \int_\Omega F(u, v) dx + \|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 \right). \tag{4.15}$$

Therefore, we have

$$\mathcal{L}(t) \geq \mathcal{L}(0) > 0, \qquad \forall t \geq 0,$$

where

$$\mathcal{L}(0) = H(0) + \frac{\varepsilon}{\rho+1}\int_\Omega \Big(|u_1|^\rho u_1 u_0 + |v_1|^\rho v_1 v_0 \Big)\, dx$$
$$+\frac{\varepsilon}{2}\int_\Omega \Big(|\nabla u_0|^2 + |\nabla v_0|^2 \Big)\, dx + \varepsilon\int_\Omega (\nabla u_0.\nabla u_1 + \nabla v_0.\nabla v_1)dx.$$

On the other hand, since $2 < \rho+2 < r+1$, we use the standard interpolation inequality again to get

$$\|u(t)\|_{\rho+2} \le \|u(t)\|_2^k\ \|u(t)\|_{r+1}^{1-k},$$

where $\frac{k}{2} + \frac{1-k}{r+1} = \frac{1}{\rho+2}$, which gives $k = \frac{2(r-(\rho+1))}{(\rho+2)(r-1)} > 0$. Thus, with the same way followed to obtain the inequalities (4.12) and (4.13), we have

$$\left|\int_\Omega |u_t|^\rho u_t u dx\right| \le \frac{\delta_3^{\rho+2}}{\rho+2}\mathcal{F}(t) + \frac{\rho+1}{\rho+2}\delta_3^{-\frac{\rho+2}{\rho+1}}\|u_t\|_{\rho+2}^{\rho+2}, \tag{4.16}$$

and

$$\left|\int_\Omega |v_t|^\rho v_t v dx\right| \le \frac{\delta_3^{\rho+2}}{\rho+2}\mathcal{F}(t) + \frac{\rho+1}{\rho+2}\delta_3^{-\frac{\rho+2}{\rho+1}}\|v_t\|_{\rho+2}^{\rho+2}, \tag{4.17}$$

where $\delta_3 > 0$ is an arbitrary constant. Using Young's inequality we obtain

$$\int_\Omega \Big(\nabla u.\nabla u_t + \nabla v.\nabla v_t \Big)\, dx \le \frac{1}{2}\int_\Omega \Big(|\nabla u|^2 + |\nabla v|^2 \Big)\, dx + \frac{1}{2}\int_\Omega \Big(|\nabla u_t|^2 + |\nabla v_t|^2 \Big)\, dx. \tag{4.18}$$

An application of the Poincaré inequality yields

$$\int_\Omega \Big(|\nabla u|^2 + |\nabla v|^2 \Big)\, dx \le \lambda^{-1}\int_\Omega \Big(|\Delta u|^2 + |\Delta v|^2 \Big)\, dx, \tag{4.19}$$

where λ is the Poincaré constant. Consequently, from (4.16)–(4.19), we get

$$\mathcal{L}(t) \le \Lambda_2 \left(H(t) + \int_\Omega F(u,v)dx + \|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 \right). \tag{4.20}$$

for some constant $\Lambda_2 > 0$. Combining (4.15) and (4.20), we arrive at

$$\mathcal{L}'(t) \ge \kappa_0\ \mathcal{L}(t), \qquad \forall t \ge 0, \tag{4.21}$$

where κ_0 is a positive constant. A simple integration of (4.21) over $(0,t)$ then gives

$$\mathcal{L}(t) \ge \mathcal{L}(0)\ e^{\kappa_0 t}, \qquad \forall t \ge 0. \tag{4.22}$$

Using (4.2), (4.16)–(4.19), and the condition $(G4)$, for sufficiently small ε, we get

$$\mathcal{L}(t) \le \widetilde{\kappa}_0 \Big(\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1} \Big), \tag{4.23}$$

for some $\widetilde{\kappa}_0 > 0$. A combination of (4.22) and (4.23) completes the proof.

References

[1] Agre, K., Rammaha, M.A.: Systems of nonlinear wave equations with damping and source terms, Differential Integral Equations. 19(11), 1235–1270 (2006).

[2] Alves, C.O., Cavalcanti, M.M., Domingos Cavalcanti, V.N., Rammaha, M. A., Toundykov, D.: On existence, uniform decay rates and blow up for solutions of systems of nonlinear wave equations with damping and source terms, Discrete Contin. Dyn. Syst. Ser. S 2(3), 583–608 (2009).

[3] Amroun, N.E., Benaissa, A.: Global existence and energy decay of solutions to a Petrovsky equation with general nonlinear dissipation and source term. Georgian. Math. J. 13, 397–410 (2006).

[4] Berrimi, S., Messaoudi, S.A.: Existence and decay of solutions of a viscoelastic equation with a nonlinear source. Nonlinear Anal. 64, 2314–2331 (2006).

[5] Berrimi, S., Messaoudi, S.A.: Exponential decay of solutions to a viscoelastic equation with nonlinear localized damping. Electron. J. Differential Equations. 88, 1–10 (2004).

[6] Cavalcanti, M.M., Domingos Cavalcanti, V.N., Ferreira, J.: Existence and uniform decay for a nonlinear viscoelastic equation with strong damping. Math. Meth. Appl. Sci. 24, 1043–1053 (2001).

[7] Cavalcanti, M.M., Oquendo, H.P.: Frictional versus viscoelastic damping in a semilinear wave equation. SIAM. J. Control. Optim. 42(4), 1310–1324 (2003).

[8] Cavalcanti, M.M., Domingos Cavalcanti, V.N., Soriano, J.A.: Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping. Electron. J. Differential Equations. 2002(44), 1–14 (2002).

[9] Cavalcanti, M.M., Domingos Cavalcanti, V.N.: Existence and asymptotic stability for evolution problems on manifolds with damping and source terms. J. Math. Anal. Appl. 291(1), 109–127 (2004).

[10] Cavalcanti, M.M., Domingos Cavalcanti, V.N., Ma, T.F., Soriano, J.A.: Global existence and asymptotic stability for viscoelastic problems. Differential Integral Equations. 15 (6), 731–748 (2002).

[11] Guesmia, A.: Existence globale et stabilisation interne non linéaire d'un système de Petrovsky. Bell. Belg. Math. Soc. 5, 583–594 (1998).

[12] Han, X.S., Wang, M.X.: Global existence and uniform decay for a nonlinear viscoelastic equation with damping. Nonlinear Anal. 70, 3090–3098 (2009).

[13] Han, X.S., Wang, M.X.: General decay of energy for a viscoelastic equation with nonlinear damping. Math. Meth. Appl. Sci. 32(3), 346–358 (2009).

[14] Liang, F., Gao, H.: Exponential energy decay and blow-up of solutions for a system of nonlinear viscoelastic wave equations with strong damping. Boundary Value Problems 2011. 22, (2011).

[15] Li, M.R., Tsai, L.Y.: On a system of nonlinear wave equations. Taiwanese J. Math. 7, 557–573 (2003).

[16] Li, G., Sun, Y., Liu, W.: Global existence and blow up of solutions for a strongly damped Petrovsky system with nonlinear damping. Appl. Anal. 91(3), 575–586 (2012).

[17] Li, G., Sun, Y., Liu, W.: Global existence, uniform decay and blow up of solutions for a system of Petrovsky equations. Nonlinear Anal. 74, 1523–1538 (2011).

[18] Liu, W.J.: General decay and blow-up of solution for a quasilinear viscoelastic problem with nonlinear source. Nonlinear Anal. 73(6), 1890–1904 (2010).

[19] Liu, W.J.: General decay of solutions to a viscoelastic wave equation with nonlinear localized damping. Annal. Academ. Scien. Fenni. Math. 34, 291–306 (2009).

[20] Liu, W.J.: Global existence, asymptotic behavior and blow-up of solutions for a viscoelastic equation with strong damping and nonlinear source. Topol. Methods Nonlinear Anal. 36(1), 153–178 (2010).

[21] Liu, W.J.: Uniform decay of solutions for a quasilinear system of viscoelastic equations. Nonlinear Anal. 71, 2257–2267 (2010).

[22] Ma, J., Mu, C., Zeng, R.: A blow up result for viscoelastic equations with arbitrary positive initial energy. Boundary Value Problems 2011. 6, (2011).

[23] Messaoudi, S.A., Tatar, N-E.: Global existence and asymptotic behavior for a nonlinear viscoelastic problem. Math. Sci. Res. J. 7(4), 136–149 (2003).

[24] Messaoudi, S.A., Tatar, N-E.: Global existence and uniform stability of solutions for a quasilinear viscoelastic problem. Math. Meth. Appl. Sci. 30, 665–680 (2007).

[25] Messaoudi, S.A., Tatar, N-E.: Uniform stabilization of solutions of a nonlinear system of viscoelastic equations. Appl. Anal. 87(3), 247–263 (2008).

[26] Messaoudi, S.A., Said-Houari, B.: Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms. J. Math. Anal. Appl. 365, 277–287 (2010).

[27] Messaoudi, S.A.: Global existence and nonexistence in a system of Petrovsky. J. Math. Anal. Appl. 265, 296–308 (2002).

[28] Nakao, M.: Asymptotic stability of the bounded or almost periodic solution of the wave equation. J. Math. Anal. Appl. 56, 336–343 (1977).

[29] Said-Houari, B.: Global existence and decay of solutions of a nonlinear system of wave equations. Appl. Anal. 91(3), 475–489 (2012).

[30] Said-Houari, B.: Global nonexistence of positive initial-energy solutions of a system of nonlinear wave equations with damping and source terms. Differential Integral Equations. 23(1–2), 79–92 (2010).

[31] Said-Houari, B.: Exponential growth of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms. Z. Angew. Math. Phys. 62, 115–133 (2011).

[32] Said-Houari, B., Messaoudi, S.A., Guesmia, A.: General decay of solutions of a nonlinear system of viscoelastic wave equations. Nonlinear. Differ. Equ. Appl. 18, 659–684 (2011).

[33] Vitillaro, E.: Global existence theorems for a class of evolution equations with dissipation. Arch. Ration. Mech. Anal. 149, 155–182 (1999).

[34] Wu, S.T.: General decay of solutions for a viscoelastic equation with nonlinear damping and source terms. Acta. Math. Scientia. 31B, 1436–1448 (2011).

[35] Wu, S.T.: General decay of energy for a viscoelastic equation with damping and source term. Taiwanese J. Math. 16(1), 113–128 (2012).

[36] Wu, S.T.: Blow-up of solutions for a system of nonlinear wave equations with nonlinear damping. Electron. J. Differential Equations. 2009(105), 1–11 (2009).

[37] Xu, R., Yang, Y., Liu, Y.: Global well-posedness for strongly damped viscoelastic wave equation. Appl. Anal. 92(1), 138–157 (2013).

[38] Yu, S.Q.: On the strongly damped wave equation with nonlinear damping and source terms. Electron. J. Qual. Theory Differ. Eqns. 39, 1–18 (2009).