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GLOBAL GORENSTEIN DIMENSIONS

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Dedicated to our advisor, Salah-Eddine Kabbaj

Abstract. In this paper, we prove that the global Gorenstein projective dimension of a ring R is equal to the global Gorenstein injective dimension of R and that the global Gorenstein flat dimension of R is smaller than the common value of the terms of this equality.

1. Introduction

Throughout this paper, R denotes a non-trivial associative ring with identity and all modules are, if not specified otherwise, left R-modules. All the results, except Proposition 2.6, are formulated for left modules, and the corresponding results for right modules hold as well. For an R-module M, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$, and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective, injective, and flat dimensions of M. We use $l.\mathrm{gldim}(R)$ and $r.\mathrm{gldim}(R)$ to denote, respectively, the classical left and right global dimensions of R, and $\mathrm{wgldim}(R)$ to denote the weak global dimension of R. Recall that the left finitistic projective dimension of R is the quantity $l.\mathrm{FPD}(R) = \sup\{\mathrm{pd}_R(M) \mid M \textit{ is an } R\textit{-module with } \mathrm{pd}_R(M) < \infty\}$.

Furthermore, we use $\mathrm{Gpd}_R(M)$, $\mathrm{Gid}_R(M)$, and $\mathrm{Gfd}_R(M)$ to denote, respectively, the Gorenstein projective, injective, and flat dimensions of M (see [3, 4, 8]).

The main result of this paper is an analog of a classical equality that is used to define the global dimension of R; see [12, Theorems 9 and 10]. For Noetherian rings the following theorem is proved in [4, Theorem 12.3.1].

Theorem 1.1. *The following equality holds:*

$$\sup\{\mathrm{Gpd}_R(M) \mid M \textit{ is an } R\textit{-module}\} = \sup\{\mathrm{Gid}_R(M) \mid M \textit{ is an } R\textit{-module}\}.$$

We call the common value of the quantities in the theorem the *left Gorenstein global dimension* of R and denote it by $l.\mathrm{Ggldim}(R)$. Similarly, we set

$$l.\mathrm{wGgldim}(R) = \sup\{\mathrm{Gfd}_R(M) \mid M \textit{ is an } R\textit{-module}\}$$

and call this quantity the *left weak Gorenstein global dimension* of R.

Corollary 1.2. *The following inequalities hold:*

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(1) $l.\mathrm{wGgldim}(R) \leq \sup\{l.\mathrm{Ggldim}(R), r.\mathrm{Ggldim}(R)\}$,
(2) $l.\mathrm{FPD}(R) \leq l.\mathrm{Ggldim}(R) \leq l.\mathrm{gldim}(R)$,
(3) $l.\mathrm{wGgldim}(R) \leq \mathrm{wgldim}(R)$.

Equalities hold in (2) *and* (3) *if* $\mathrm{wgldim}(R) < \infty$.

The theorem and its corollary are proved in Section 2.

2. Proofs of the main results

The proofs use the following results:

Lemma 2.1. *If* $\sup\{\mathrm{Gpd}_R(M) \,|\, M \text{ is an } R\text{-module}\} < \infty$*, then, for a positive integer* n*, the following are equivalent:*

(1) $\sup\{\mathrm{Gpd}_R(M) \,|\, M \text{ is an } R\text{-module}\} \leq n$,
(2) $\mathrm{id}_R(P) \leq n$ *for every* R*-module* P *with finite projective dimension.*

Proof. Use [6, Theorem 2.20] and [12, Theorem 9.8]. □

The proof of the main theorem depends on the notions of strong Gorenstein projectivity and injectivity, which were introduced in [1] as follows:

Definition 2.2 ([1, Definition 2.1])**.** An R-module M is called strongly Gorenstein projective if there exists an exact sequence of projective R-modules

$$\mathbf{P} = \ \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

such that $M \cong \operatorname{Ker} f$ and such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective R-module.

Strongly Gorenstein injective modules are defined dually.

Remark 2.3. It is easy to see that an R-module M is strongly Gorenstein projective if and only if there exists a short exact sequence of R-modules $0 \to M \to P \to M \to 0$, where P is projective, and $\mathrm{Ext}^i_R(M, Q) = 0$ for some integer $i > 0$ and for every R-module Q with finite projective dimension (or for every projective R-module Q).

Strongly Gorenstein injective modules are characterized in similar terms.

The principal role of these modules is to characterize the Gorenstein projective and injective modules, as follows:[1]

Lemma 2.4 ([1, Theorems 2.7])**.** *An* R*-module is Gorenstein projective (resp., injective) if and only if it is a direct summand of a strongly Gorenstein projective (resp., injective)* R*-module.*

Proof of Theorem 1.1. For every integer n we need to show:

$$\mathrm{Gpd}_R(M) \leq n \text{ for every } R\text{-module } M \iff \mathrm{Gid}_R(M) \leq n \text{ for every } R\text{-module } M.$$

We prove only the direct implication; the converse one has a dual proof.

[1] In [1], the base ring is assumed to be commutative. However, for the result needed here, one can show easily that this assumption is not necessary.

Assume first that M is strongly Gorenstein projective. By Remark 2.3 there is a short exact sequence $0 \to M \to P \to M \to 0$ with P is projective. The Horseshoe Lemma (see [10, Remark, page 187]) gives a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & P & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_0 & \to & I_0 \oplus I_0 & \to & I_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_n & \to & E_n & \to & I_n & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where I_i is injective for $i = 0, ..., n-1$. Since P is projective, $\mathrm{id}_R(P) \leq n$ (by Lemma 2.1); hence E_n is injective. On the other hand, from [7, Theorem 2.2], $\mathrm{pd}_R(E) \leq n$ for every injective R-module E. Then, $\mathrm{Ext}^i_R(E, I_n) = 0$ for all $i \geq n+1$. Then, from Remark 2.3, I_n is strongly Gorenstein injective, and so $\mathrm{Gid}_R(M) \leq n$. This implies, from [6, Proposition 2.19], that $\mathrm{Gid}_R(G) \leq n$ for any Gorenstein projective R-module G, since every Gorenstein projective R-module is a direct summand of a strongly Gorenstein projective R-module (Lemma 2.4).

Finally, consider an R-module M with $\mathrm{Gpd}_R(M) \leq m \leq n$. We can assume that $\mathrm{Gpd}_R(M) \neq 0$. Then, there exists a short exact sequence $0 \to K \to N \to M \to 0$ such that N is Gorenstein projective and $\mathrm{Gpd}_R(K) \leq m-1$ [6, Proposition 2.18]. By induction, $\mathrm{Gid}_R(K) \leq n$ and $\mathrm{Gid}_R(N) \leq n$. Therefore, using [6, Theorems 2.22 and 2.25] and the long exact sequence of Ext, we get that $\mathrm{Gid}_R(M) \leq n$. □

Proof of Corollary 1.2. (1) We may assume that $\sup\{l.\mathrm{Ggldim}(R), r.\mathrm{Ggldim}(R)\} < \infty$. Then, the character module, $I^* = \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$, of every injective right R-module I has finite projective dimension (by [7, Theorem 2.2] and [12, Theorem 3.52]). Then, similarly to the proof of [6, Proposition 3.4], we get that every Gorenstein projective R-module is Gorenstein flat. Therefore, $l.\mathrm{wGgldim}(R) \leq \sup\{l.\mathrm{Ggldim}(R), r.\mathrm{Ggldim}(R)\}$.

(2) and (3) The inequality $l.\mathrm{FPD}(R) \leq l.\mathrm{Ggldim}(R)$ follows from [6, Theorem 2.28].

The inequalities $l.\mathrm{Ggldim}(R) \leq l.\mathrm{gldim}(R)$ and $l.\mathrm{wGgldim}(R) \leq \mathrm{wgldim}(R)$ hold true since every projective (resp., flat) module is Gorenstein projective (resp., Gorenstein flat).

If $\mathrm{wgldim}(R) < \infty$, then, from [10, Corollary 3], $l.\mathrm{FPD}(R) = l.\mathrm{Ggldim}(R) = l.\mathrm{gldim}(R)$ and, from [1, Corollary 3.8], $l.\mathrm{wGgldim}(R) = \mathrm{wgldim}(R)$. □

Remark 2.5. It is well-known that there are examples of rings for which the left and right global dimensions differ (see [5, pages 74-75] and [9]). Then, by Corollary 1.2, the same examples show that there are also examples of rings for which the left and right Gorenstein global dimensions differ. However, as the classical case [12, Corollary 9.23], we have $l.\mathrm{Ggldim}(R) = r.\mathrm{Ggldim}(R)$ if R is Noetherian [4, Theorem 12.3.1].

For the case where $l.\mathrm{Ggldim}(R) = 0$ or $r.\mathrm{Ggldim}(R) = 0$, we have the following result, which is [2, Theorem 2.2] in a non-commutative setting. Recall that a ring is called quasi-Frobenius if it is Noetherian and both left and right self-injective (see [11]).

Proposition 2.6. *The following are equivalent:*

(1) *R is quasi-Frobenius,*
(2) $l.\mathrm{Ggldim}(R) = 0$*,*
(3) $r.\mathrm{Ggldim}(R) = 0$*.*

Proof. The implications $1 \Rightarrow 2$ and $1 \Rightarrow 3$ are well-known (see, for example, [4, Exercise 5, page 257]).

The implication $2 \Rightarrow 1$ follows from Lemma 2.1 and the Faith-Walker Theorem [11, Theorem 7.56]. The implication $3 \Rightarrow 1$ is proved similarly. □

We finish with a generalization of a result of Iwanaga; see [4, Proposition 9.1.10].

Corollary 2.7. *Assume that* $l.\mathrm{Ggldim}(R) \leq n$ *holds for some non-negative integer n. If for an R-module M one of the numbers* $\mathrm{pd}_R(M)$*,* $\mathrm{id}_R(M)$*, or* $\mathrm{fd}_R(M)$ *is finite, then all of them are less than or equal to n.*

Proof. If $\mathrm{pd}_R(M)$ is finite, then [6, Proposition 2.27] and the assumption give $\mathrm{pd}_R(M) = \mathrm{Gpd}_R(M) \leq n$. The argument for $\mathrm{id}_R(M) < \infty$ is similar. Finally, Corollary 1.2(2) and the assumption give $l.\mathrm{FPD}(R) \leq n$, and then $\mathrm{fd}_R(M) < \infty$ implies $\mathrm{pd}_R(M) < \infty$ by [10, Proposition 6]. □

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