Mon. Not. R. astr. Soc. (1981) **195**, 967–977

Global gravitational stability for one-dimensional polytropes

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Received 1980 October 27; in original form 1979 May 31

Summary. We investigate the stability and dynamics of an isolated cloud of gas which is contained by external pressure, and which has imposed one-dimensional symmetry, uniform density and polytropic equation of state. The energies of the cloud in its three internally conserved modes are summed to form a potential function controlling radial motions, and the associated Lagrangian is obtained. Families of stable equilibrium states are then derived – where they exist – along with conditions for marginal instability.

We show that this simple global model mimics closely the behaviour of clouds in detailed hydrostatic balance (DHB); and we use the global model to discuss the onset of gravitational instability in real clouds, and in particular, to evaluate the role of non-quasistatic compression.

1 Introduction

Detailed numerical calculations have been performed by Larson (1969, 1972) and others to follow the evolution of an isolated and initially unstable, individual protostar as it contracts towards the main sequence. By contrast, the highly chaotic, non-equilibrium thermodynamic processes which lead to the formation of such an individual protostar from the general interstellar medium have to date only been crudely represented, as for example by comparing the freefall time with the sound-crossing time (Jeans criterion), with the radiative cooling time (minimum mass; Rees 1976), or with the collision time (efficiency; Larson 1976). Such comparisons are very useful as a means of establishing general trends and dependences; but further insight can be gained only if the competing processes are more realistically represented by global time-dependent equations. In this paper, we explore a possible form for the global dynamical equations. We postpone consideration of magnetic, centrifugal and tidal stresses (by imposing spherical symmetry), and of real thermal and chemical processes (by stipulating a polytropic equation of state).

In Section 2 we develop a model for a spherically symmetric gas cloud. In Section 3 we formulate the energies of the cloud in its three internally conserved modes. These energies are summed to yield a potential function $\mathscr{U}$; and hence a Langrangian $\mathscr{L}$ controlling radial motions of the cloud is obtained. In Section 4, $\mathscr{U}$ is differentiated to give parameters of stable equilibrium and marginally unstable states. In Section 5 we show how accurately these results mimic the predictions of DHB calculations. In Section 6 we demonstrate how a cloud may become 'pre-unstable' due to non-quasistatic compression; and in Section 7 we show that highly non-quasistatic compression would seem inevitable for most protostellar clouds. In Section 8 we summarize the main results. The global treatment is extended in two appendices to the cases of cylindrical and plane-parallel symmetries.

For isothermal gas clouds, the results of this global treatment of stability are exactly the same as those derived by McCrea (1957) on the basis of the Virial Theorem. We have generalized the results to polytropic gas clouds with spherical, cylindrical and plane-parallel symmetries. We have also obtained an equation of motion which enables us to investigate non-quasistatic cloud evolution, and in particular 'pre-instability'. This equation of motion can be identified as the Time-Dependent Virial Theorem. However, we believe that the global approach used here has conceptual advantages over the Virial Theorem, which arise from its being thermodynamic rather than kinetic. We hope to develop and exploit these advantages in future work.

2 Spherical cloud model

The model cloud has constant mass M_0 and is constrained to be spherical with radius R. The gas in the cloud has uniform density ρ_{in} (subscript 'in' for internal), and obeys a polytropic equation of state with exponent η. Thus the pressure P_{in} and isothermal sound speed a_{in} of the cloud gas are given by

$$P_{\text{in}} = K\rho_{\text{in}}^{\eta} = K(3M_0/4\pi R^3)^{\eta}; \quad a_{\text{in}}^2 = kT/\bar{m} = P_{\text{in}}/\rho_{\text{in}} = K\rho_{\text{in}}^{(\eta-1)}. \tag{1}$$

K and η are constants defining a particular polytrope, k is Boltzmann's constant, T the gas kinetic temperature, and $\bar{m}$ the mean gas-particle mass. The cloud is contained by constant external pressure P_{ex} (subscript 'ex' for external).

Evidently this is a very idealized cloud model. It is spherical and has uniform density simply so that we can describe the instantaneous state with a single parameter R. Externally compressed (non-magnetic, non-rotating) clouds only have gravitationally bound, stable hydrostatic equilibrium states if $\eta > 0$. In reality such states are centrally peaked. If the external pressure containing the cloud is initially small and is increased sufficiently slowly for radial pulsations to be damped, the cloud is compressed quasistatically; it evolves through a continuous succession of stable equilibrium states with monotonically decreasing radius. If $\eta < 4/3$, the cloud eventually reaches a critical equilibrium state and becomes unstable against indefinite contraction. This quasistatic approach to instability against contraction has been mapped for an isothermal gas ($\eta = 1$, $K = a_{\text{in}}^2$) by Ebert (1955) using the DHB solutions of Emden (1907). Ebert's conclusions are supported by an approximate treatment due to McCrea (1957) based on the Virial Theorem. McCrea confirms the critical equilibrium state as the real onset of instability against contraction for an isothermal gas cloud which is compressed quasistatically. We are here able (i) to extend this conclusion to general polytropes, and (ii) to evaluate the modifications inherent in a non-quasistatic approach to instability.

Numerical calculations of the contraction of initially static but gravitationally unstable, spherically symmetric clouds indicate that, whether or not the initial state is centrally peaked, the cloud becomes increasingly centrally peaked as the contraction proceeds (e.g. Larson 1969, 1972). However, we argue – as did McCrea – that in nature a protostellar cloud is probably not afforded sufficient time to evolve quasistatically, and so at the onset of instability is unlikely to be very centrally peaked, if at all; indeed, we are already investing the cloud with an unrealistically high degree of coordination by giving it spherical symmetry and isotropic P_{ex}. For example, consider a cloud whose instability against contraction is triggered by a sudden increase in P_{ex}. This may happen as the cloud is overrun by a shock (e.g. Öpik 1953); or as the surroundings of the cloud are ionized and heated by a nearby, newly born O star (Ebert 1955). Under these circumstances the cloud may actually start its contraction with a central rarefaction. Alternatively, consider a subcloud condensing gravitationally out of a parent cloud which is itself contracting. Disney (1976) has argued that the boundary of such a subcloud will from the outset have supersonic speed relative to its centre; and that this must inhibit the development of a central density peak during the isothermal stages of the contraction. Much later on, as the protostar approaches the main sequence, a pronounced central peak is likely to develop; but hopefully we can avoid these complications and derive useful results for the earlier formative stages by assuming a uniform density within the cloud. In these earlier stages we expect an effective polytropic exponent $d\ln(P)/d\ln(\rho)\lesssim 1$ (e.g. Hoyle 1953; Shu *et al.* 1972), in which case the asymptotic similarity solutions of Larson (1969) suggest that the tendency toward central peaking is relatively weak.

The remaining assumptions underlying this model are only justifiable on grounds of expediency. (i) The epoch- and scale-dependent thermal and chemical processes defining the real equation of state are likely to have important consequences for the formation of stars and galaxies; in particular they introduce characteristic mass-scales into the problem (e.g. Hoyle 1953; Low & Lynden-Bell 1976; Rees 1976; Silk 1977), and they are only ignored here in favour of a polytropic equation of state in order to focus attention on the dynamics. (ii) Likewise, departures from spherical symmetry must be of fundamental importance; Lin, Mestel & Shu (1965) have shown that initial departures from spherical symmetry are amplified during freefall collapse, and the inclusion of magnetic, centrifugal and/or tidal stresses would introduce asphericity by anisotropically inhibiting contraction. (iii) Finally, constant P_{ex} must be supplied by an infinitely hot, infinitely rarefied gas: $a_{ex}\to\infty$ and $\rho_{ex}\to 0$, so that $P_{ex}=\rho_{ex}a_{ex}^2$ remains finite. Such a gas has zero density/inertia, so it does not feel the gravitational attraction of the underlying cloud, and it can adjust instantaneously to keep in touch with a moving cloud boundary; at the same time, if the gas particles accrete on to the cloud, its mass does not change, $dM_0/dt\propto R^2\rho_{ex}a_{ex}\to 0$.

3 'Internally conserved' energy modes

In this section we enumerate the 'internally conserved' energy modes of the cloud, and formulate how the energy of each mode changes when the cloud makes a radial excursion from its initial state with radius R_0 to a displaced state with radius R. An energy mode is 'internally conserved' if (i) its energy is a function of R only (i.e. a function of state) and (ii) it exchanges energy exclusively with other internally conserved modes and/or with the bulk kinetic energy associated with radial motion. Such a mode will contribute to the potential function $\mathscr{U}$ controlling radial motions.

The gravitational energy is $\mathscr{G}=-3GM_0^2/5R=\mathscr{G}_0(R/R_0)^{-1}$, where $\mathscr{G}_0=-3GM_0^2/5R_0$. We must also consider the work done by/against pressure on/by the cloud as it contracts/expands. This work is supplied to a compressional energy $\mathscr{B}$ defined by $d\mathscr{B}/dV=P_{ex}-P_{in}$, where V is the cloud volume. Substituting for P_{in} from equation (1) and integrating, we obtain

$$\mathscr{B}=\mathscr{B}_{ex}+\mathscr{B}_{in}=P_{ex}V_0(R/R_0)^3 + (1-\delta_{1\eta})(\eta-1)^{-1}KM_0^{\eta}V_0^{(1-\eta)}(R/R_0)^{3(1-\eta)}-\delta_{1\eta}3KM_0\ln(R/R_0), \tag{2}$$

where $\delta_{1\eta}$ is the Kronecker delta. (The constant of integration is arbitrary.)

We note that the internal energy of the cloud $\mathscr{I}$ is not necessarily an internally conserved mode. The polytropic equation of state (1) prescribes the variation of P_{in} and $a_{\text{in}} = (kT/\bar{m})^{1/2}$ with ρ_{in}. Hence, in principle, equation (1) admits arbitrary variations in T or $\bar{m}$; i.e. no unique dependence of $\mathscr{I}$, or of chemical composition, on ρ_{in} is required. Here we simply presume that the (unspecified) microscopic thermal and chemical processes (which in reality determine the variations of T and $\bar{m}$) (i) are instantaneous, and (ii) contrive to reproduce equation (1). Some internal energy is supplied/removed by compression/expansion (specifically, by $\mathscr{B}_{\text{in}}$), but the balance must be extracted from or dumped into an external reservoir, such as the radiation field. *If* the gas has constant chemical composition, *and* a constant ratio of specific heats γ (i.e. no degrees of freedom 'freezing out' or 'melting in' at quantum thresholds) we can write $\mathscr{I} = (\gamma-1)^{-1} K M_0^{\eta} V_0^{(1-\eta)} (R/R_0)^{3(1-\eta)} + \text{constant}$. In this particular circumstance, the cloud's luminosity (which can be negative) is $d(\mathscr{B}_{\text{in}} - \mathscr{I})/dt = 3(\eta-\gamma)(\gamma-1)^{-1} K M_0^{\eta} V_0^{(1-\eta)} (R/R_0)^{3(1-\eta)}\, d\ln(R)/dt$. The special case $\eta = \gamma$ then yields zero luminosity and hence represents adiabatic excursions.

The internally conserved modes can be summed to form a global potential function: $\mathscr{U} = \mathscr{G} + \mathscr{B}_{\text{in}} + \mathscr{B}_{\text{ex}}$. The kinetic energy associated with homologous radial motions of a uniform-density cloud is $\mathscr{V} = 3M_0(dR/dt)^2/10$. Hence we can construct a global Lagrangian $\mathscr{L} = \mathscr{V} - \mathscr{U}$, and deduce the equation of motion:

$$\frac{d^2R}{dt^2} = -\frac{5}{3M_0}\frac{d\mathscr{U}}{dR} = \frac{5}{3M_0R_0}\left[\mathscr{G}_0(R/R_0)^{-2} + \mathscr{K}_\eta(R/R_0)^{(2-3\eta)} - 3P_{\text{ex}}V_0(R/R_0)^2\right]. \tag{3}$$

where $\mathscr{K}_\eta = 3KM_0^{\eta}V_0^{(1-\eta)}$.

4 Stable and neutral equilibria; quasistatic compression

If $(d\mathscr{U}/dR)_{R_0} = 0$, R_0 is an equilibrium state, and the equilibrium is – respectively – stable, neutral or unstable, according as $R_0^2(d^2\mathscr{U}/dR^2)_{R_0} = 12P_{\text{ex}}V_0 + (3\eta-4)\mathscr{K}_\eta >, =,$ or < 0. Since $P_{\text{ex}}V_0$ and $\mathscr{K}_\eta$ cannot be negative, unstable equilibrium states are only available when $\eta < 4/3$.

If $(d\mathscr{U}/dR)_{R_0} = (d^2\mathscr{U}/dR^2)_{R_0} = 0$, R_0 is a neutral equilibrium state with $R_0^3(d^3\mathscr{U}/dR^3)_{R_0} = (4-3\eta)3\eta\mathscr{K}_\eta$, and

$$\begin{aligned} R_0 &\to R_{\text{nu}} = [(3/4)^{\eta}5\pi^{(1-\eta)}\eta K G^{-1} M_0^{(\eta-2)}]^{1/(3\eta-4)}, \\ P_{\text{ex}} &\to P_{\text{nu}} = (1-3\eta/4)K^{-4/(3\eta-4)}[2^8 3^{-4} 5^{-3}\pi\eta^{-3}G^3M_0^2]^{\eta/(3\eta-4)} \end{aligned} \tag{4}$$

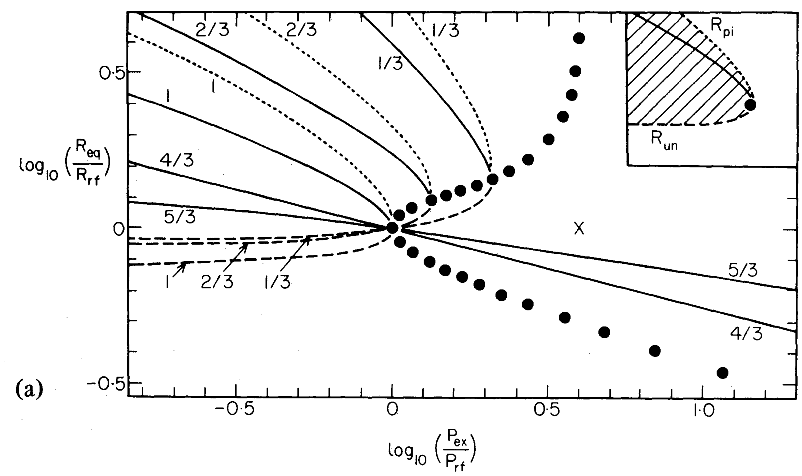
(subscript 'nu' for neutral). For $0 < \eta < 4/3$, the neutral equilibrium state is real and has $(d^3\mathscr{U}/dR^3)_{R_{\text{nu}}} > 0$, so that it represents the verge of instability against contraction (for a cloud which has been compressed quasistatically). For $\eta \leqslant 0$ or $\eta \geqslant 4/3$, the neutral equilibrium state is unreal.

The general equilibrium condition, $(d\mathscr{U}/dR)_{R_0} = 0$, reduces to

$$R_0 \to R_{\text{eq}}, \quad P_{\text{ex}} = K(3M_0/4\pi R_{\text{eq}}^3)^{\eta} - 3GM_0^2/20\pi R_{\text{eq}}^4 \tag{5}$$

(subscript 'eq' for equilibrium). The stability condition, $(d^2\mathscr{U}/dR^2)_{R_{\text{eq}}} > 0$, can then be written as $R_{\text{eq}} < R_{\text{nu}}$ for $\eta > 4/3$, $K > K_{\text{cr}} = (4\pi M_0^2/3)^{1/3}(G/5)$ for $\eta = 4/3$ (subscript 'cr' for critical) and $R_{\text{eq}} > R_{\text{nu}}$ for $0 < \eta < 4/3$. There are no stable equilibria for $\eta \leqslant 0$.

For $\eta > 0$, the pattern of stable equilibrium states, and hence the path for quasistatic evolution of a cloud subjected to monotonically increasing P_{ex}, are fundamentally different according as $\eta >, =,$ or $< 4/3$; and if $\eta = 4/3$, according as $K >, =,$ or $< K_{\text{cr}}$. This is illustrated on the (P, R)-plane in Fig. 1(a). (i) For $\eta > 4/3$, the cloud always has a stable equilibrium state and can be compressed indefinitely from $(P_{ex}, R_{eq}) = (0, R_{mx})$ to $(\infty, 0)$, where $R_{mx} = [(3/4\pi)^{(\eta-1)} 5KG^{-1}M_0^{(\eta-2)}]^{1/(3\eta-4)}$, (subscript 'mx' for maximum). (ii) For $\eta = 4/3$, equation (5) reduces to $P_{ex} = (K - K_{cr})(3M_0/4\pi)^{4/3} R_{eq}^{-4}$, so that provided $K > K_{cr}$ (as is the case for the $\eta = 4/3$ cloud represented in Fig. 1a), the cloud always has a stable equilibrium state and can be compressed indefinitely from $(P_{ex}, R_{eq}) = (0, \infty)$ to $(\infty, 0)$. (iii) For $0 < \eta < 4/3$, the cloud can be compressed quasistatically only from $(P_{ex}, R_{eq}) = (0, \infty)$ to (P_{nu}, R_{nu}), where it then becomes unstable against indefinite contraction. (iv) For $\eta = 4/3$ with $K \leqslant K_{cr}$, and for $\eta = 0$ with $P_{ex} \geqslant K$, the cloud has no equilibrium and contracts indefinitely. (v) For $\eta = 0$ with $P_{ex} < K$, and for $\eta < 0$, the cloud has an unstable equilibrium and therefore contracts/expands indefinitely according as it is released from a sufficiently small/large initial radius.



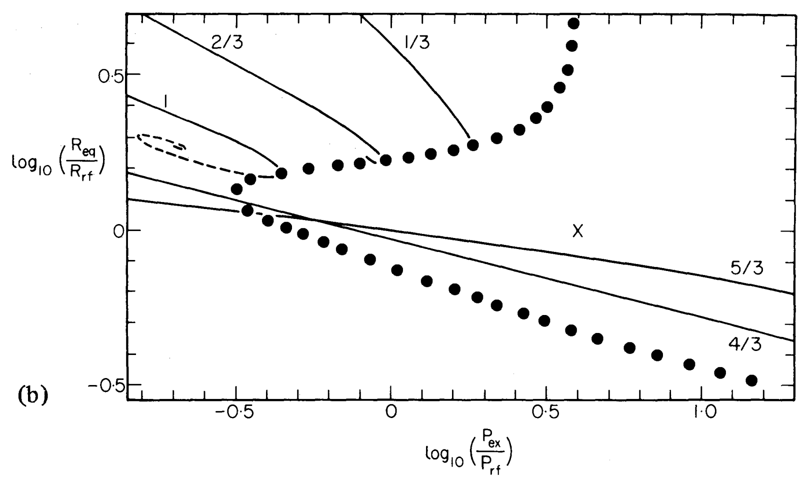


Figure 1. (a) The (P, R)-plane for uniform-density spherical clouds with standardized polytropic equation of state $P_{in} = K_\eta \rho_{in}^\eta$. The continuous lines (——) mark stable equilibrium states; the dashed lines (– – –), unstable equilibrium states, $R_{un}(P_{ex})$; and the dotted lines (·····), marginally pre-unstable states, $R_{pi}(P_{ex})$. The inset schematically divides the (P, R)-plane into two domains: clouds released from rest in the shaded domain between $R_{un}(P_{ex})$ and $R_{pi}(P_{ex})$ pulsate indefinitely; whilst clouds released in the unshaded domain contract indefinitely. The filled circles mark the limiting neutral equilibrium states, and the cross marks the standardizing state $(P_{in} = 4P_{rf}, R = R_{rf})$ to which all polytropes subscribe. The numbers (1/3, 2/3, etc.) by the curves give the corresponding values of η. (b) As (a), but for clouds in DHB; pre-instability is not defined here.

We shall henceforth refer cloud parameters to those obtaining in the neutral equilibrium state for an isothermal cloud,

$$R_{\rm nu}(\eta=1, K=K_1) = 4GM_0/15K_1 \rightarrow R_{\rm rf},$$
$$P_{\rm nu}(\eta=1, K=K_1) = 2^{-10}3^4 5^3 \pi^{-1} G^{-3} M_0^{-2} K_1^4 \rightarrow P_{\rm rf}, \tag{6}$$

(subscript 'rf' for reference), with internal pressure $P_{\rm in} = 4P_{\rm rf}$. We can then – without significant loss of generality – standardize the polytropic coefficients K be requiring that all polytropes subscribe to this same state. In other words, we restrict our consideration to a one-parameter (parameter η) family of pre-existing clouds which all have the same mass M_0, and in the particular state $R = R_{\rm rf}$ also have the same internal pressure $P_{\rm in} = 4P_{\rm rf}$. This requires that we put $K \rightarrow K_\eta = (4P_{\rm rf})^{(1-\eta)} K_1^\eta$. For these standardized polytropes we then have $K_{4/3} = 4K_{\rm cr}/3$, i.e. $K_{4/3} > K_{\rm cr}$,

$$(R_{\rm nu}/R_{\rm rf}) = \eta^{1/(3\eta-4)}, \quad (P_{\rm nu}/P_{\rm rf}) = (4-3\eta)\eta^{3\eta/(4-3\eta)}, \tag{7}$$

$$(P_{\rm ex}/P_{\rm rf}) = 4(R_{\rm eq}/R_{\rm rf})^{-3\eta} - 3(R_{\rm eq}/R_{\rm rf})^{-4}, \tag{8}$$

and $(R_{\rm mx}/R_{\rm rf}) = (4/3)^{1/(3\eta-4)}$. The loci of equilibrium states predicted by equation (8) are plotted on the (P, R)-plane in Fig. 1(a) for several representative values of η.

5 Comparison with detailed hydrostatic balance (DHB) solutions

It is appropriate to compare these results with those based on (analytic or accurate numerical) solutions of the DHB equation. Some solutions are tabulated by Airey, Miller & Sadler (1932), Chandrasekhar & Wares (1949) and Shu *et al.* (1972); additional solutions are readily computed. Here we adopt the notation of Chandrasekhar (1939) and put $K \rightarrow K_\eta$ as above. For a particular M_0 and η, the variation of equilibrium radius $R_{\rm eq}$ with external pressure $P_{\rm ex}$ is obtained in parametric form (parameter ξ) from the following two requirements. If the DHB solution is truncated at a particular ξ, corresponding to the cloud boundary, then (i) the central density ρ_c must be scaled so that the radius $r(\xi)$ contains mass $M(\xi) = M_0$, and (ii) the internal pressure at the boundary, $P(\xi)$, must exactly match the external pressure, $P_{\rm ex}$. Thus we can eliminate ρ_c in favour of M_0, and identify Chandrasekhar's $[r(\xi); P(\xi)]$ with our $(R_{\rm eq}, P_{\rm ex})$ to obtain:

for $\eta \neq 1, 4/3$

$$(R_{\rm eq}/R_{\rm rf})_{\rm DHB} = [(4\eta/5\,|\eta-1|)(\xi/3)^\eta\,|d\theta_n/d\xi\,|^{(2-\eta)}]^{1/(3\eta-4)},$$

$$(P_{\rm ex}/P_{\rm rf})_{\rm DHB} = [2^{-8/\eta}(5\,|\eta-1\,|/\eta)^3(3/\xi)^4(d\theta_n/d\xi)^{-2}]^{\eta/(3\eta-4)}\theta_n^{\eta/(\eta-1)}; \tag{9a}$$

for $\eta = 1$,

$$(R_{\rm eq}/R_{\rm rf})_{\rm DHB} = 15/4\xi(d\psi/d\xi),$$

$$(P_{\rm ex}/P_{\rm rf})_{\rm DHB} = 2^8 3^{-4} 5^{-3} \xi^4 (d\psi/d\xi)^2 \exp(-\psi), \tag{9b}$$

(apart from normalization, equation (9b) is identical to the solution obtained by Ebert 1955);

for $\eta = 4/3$, the cloud evolves homologously, i.e.

$$(P_{\rm ex}/P_{\rm rf})_{\rm DHB} = 2^{10} 3^{-4} 5^{-2} [\xi_0 \vartheta_3(\xi_0)]^4 (R_{\rm eq}/R_{\rm rf})_{\rm DHB}^{-4}, \tag{9c}$$

where $\xi_0 \simeq 2.814$ is the root of $\xi^2(d\theta_3/d\xi) = -2^{-6}3^2 5^{3/2} \simeq -1.572$, and so $[\xi_0\theta_3(\xi_0)] \simeq 1.111$. The DHB equilibrium loci defined by equations (9a–c) are plotted on the (P, R)-plane in Fig. 1(b). Qualitatively (and neglecting the unstable equilibrium states since they are not physically significant), the global uniform-density solutions (Fig. 1a) closely mimic the DHB solutions (Fig. 1b). However, the uniform-density spheres resist compression more effectively in the sense that (i) at given external pressure $P_{ex} < P_{nu,DHB}$, the equilibrium state of the uniform-density sphere is more extended; and (ii) the critical external pressure P_{nu} (representing the verge of instability against contraction for $0 < \eta < 4/3$) is greater for the uniform-density spheres.

6 'Pre-instability'

The global potential function controlling a uniform density cloud is

$$\frac{2\mathscr{U}}{3M_0K_1} = -\frac{3}{2}\left(\frac{R}{R_{rf}}\right)^{-1} + \frac{1}{6}\left(\frac{P_{ex}}{P_{rf}}\right)\left(\frac{R}{R_{rf}}\right)^{3} + (1-\delta_{1\eta})\frac{2}{3(\eta-1)}\left(\frac{R}{R_{rf}}\right)^{3(1-\eta)} - \delta_{1\eta}2\ln\left(\frac{R}{R_{rf}}\right); \tag{10}$$

$\mathscr{U}$ is here referred to the thermal translational kinetic energy $3M_0K_1/2$ in the reference state ($R = R_{rf}$, $P_{in} = 4P_{rf}$). Equation (10) is plotted in Fig. 2 for a cloud of fixed mass M_0 with isothermal equation of state ($\eta = 1$, $K = K_1$) and several representative values of the external pressure P_{ex}. In these plots, the (otherwise arbitrary) zero of potential is allocated to the 'no-contrast' state, $(R_{nc}/R_{rf}) = (P_{ex}/4P_{rf})^{-1/3\eta}$ (subscript 'nc' for no contrast), in which there is no pressure discontinuity across the cloud boundary. The pattern of potential curves is essentially the same for other η-values in (0, 4/3); but for larger η, the stable equilibrium potentials become deeper as P_{ex} approaches P_{nu}.

From the potential curves in Fig. 2 we see that, even if $P_{ex} < P_{nu}$, the cloud will contract indefinitely provided it is released from rest in a sufficiently extended state and/or is given sufficient initial radial kinetic energy $\mathscr{V}$. Specifically, if the cloud's total energy $\mathscr{U} + \mathscr{V}$

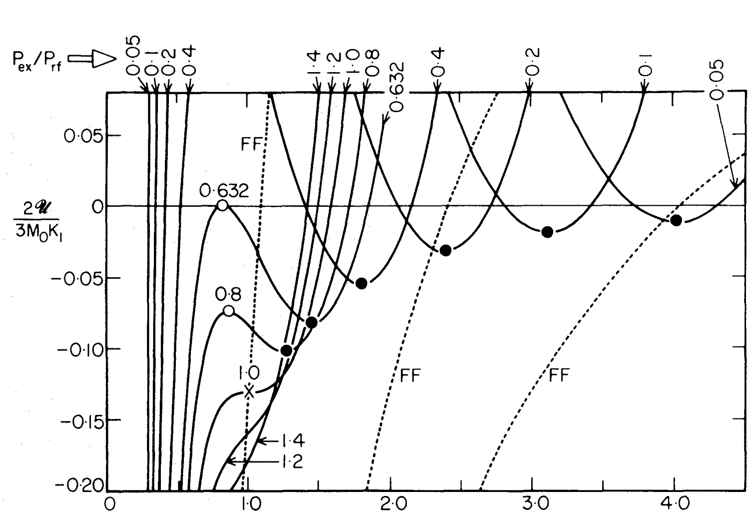


Figure 2. The continuous lines are potentials controlling radial motions of a uniform-density spherical cloud with isothermal equation of state (i.e. $\eta = 1$), for several different values of the external pressure P_{ex} (as labelled). The filled circles mark stable equilibria; the open circles mark unstable equilibria; and the cross marks the critical neutral equilibrium state. The dotted lines are freefall collapse potentials for comparison.

(which is an invariant, there being no dissipation) exceeds the potential energy of the unstable equilibrium state, then the contracting cloud gathers enough implosive momentum to overshoot both the stable and the unstable equilibrium states, and so keeps on contracting. We shall call this circumstance 'pre-instability'. For an initially stationary cloud, the minimum radius defining marginal pre-instability R_{pi} (subscript 'pi' for pre-instability) is defined relative to the radius of the unstable equilibrium state R_{un} (subscript 'un' for unstable) by $R_{pi} > R_{un}$ and $\mathscr{U}(R_{pi}) = \mathscr{U}(R_{un})$. The variation of R_{pi} with P_{ex} is shown in Fig. 1(a). In the absence of dissipation, clouds released from rest in the top left corner in Fig. 1(a) between $R_{un}(P_{ex})$ and $R_{pi}(P_{ex})$ will pulsate indefinitely; whilst clouds released anywhere else in the (P, R)-plane will contract indefinitely. Evidently pre-instability can be very important for clouds subjected to a sudden (i.e. non-quasistatic) increase in external pressure.

7 Conditions for pre-instability

From equation (3) we obtain the freefall equation $d^2R/dt^2 = -GM_0/R^2$, with initial conditions $R(t=0) = R_0$ and $dR/dt(t=0) = 0$; and well-known solution

$$\left(\frac{t}{t_{\mathrm{FF}}}\right) = \frac{2}{\pi}\left\{\cos^{-1}\left[\left(\frac{R}{R_0}\right)^{1/2}\right] + \left(\frac{R}{R_0}\right)^{1/2}\left[1-\left(\frac{R}{R_0}\right)\right]^{1/2}\right\},$$

$$t_{\mathrm{FF}} = \frac{\pi}{2}\left(\frac{R_0^3}{2GM_0}\right)^{1/2} = \left[\frac{3\pi}{32G\rho(t=0)}\right]^{1/2}, \tag{11}$$

(subscript 'FF' for freefall). We can also deduce the dynamical relaxation time for infinitesimal displacement from a stable equilibrium state, $t_{\mathrm{rl}} = (\pi/2)[3M_0/5(d^2\mathscr{U}/dR^2)_{R_{\mathrm{eq}}}]^{1/2}$ (subscript 'rl' for relaxation). If we normalize to the freefall time from the reference state, $t_{\mathrm{rf}} = \pi GM_0(2/15K_1)^{3/2}$, we obtain

$$(t_{\mathrm{rl}}/t_{\mathrm{rf}}) = 2^{-1/2}[\eta(R_{\mathrm{eq}}/R_{\mathrm{rf}})^{(1-3\eta)} - (R_{\mathrm{eq}}/R_{\mathrm{rf}})^{-3}]^{-1/2}. \tag{12}$$

A cloud will only evolve quasistatically if $t_{\mathrm{ex}} \gg (t_{\mathrm{rl}} + t_{\mathrm{dm}})$, where t_{ex} and t_{dm} are the time-scales on which, respectively, P_{ex} changes, and radial pulsations are damped (subscript 'dm' for damping). By imposing a polytropic equation of state we have rendered the compressional energy quasi-conservative and hence implicitly suppressed damping from the equation of motion: $t_{\mathrm{dm}} = \infty$. An evaluation of t_{dm} for a gas with real equation of state is outside the scope of this paper, but t_{dm} is unlikely to be less than the sound-crossing time t_{sc} (subscript 'sc' for sound-crossing), so we put $(t_{\mathrm{dm}}/t_{\mathrm{rf}}) \gtrsim$

$$(t_{\mathrm{sc}}/t_{\mathrm{rf}}) = \pi^{-1}(30/\eta)^{1/2}(R_{\mathrm{eq}}/R_{\mathrm{rf}})^{(3\eta-1)/2}. \tag{13}$$

equations (12) and (13) have been evaluated for representative stable equilibrium states. For $0 < \eta \lesssim 1.2$, all stable equilibrium states have $t_{\mathrm{rl}} \gtrsim 2t_{\mathrm{rf}}$ and $t_{\mathrm{sc}} \gtrsim 2t_{\mathrm{rf}}$, and so $(t_{\mathrm{rl}} + t_{\mathrm{dm}}) \gtrsim (t_{\mathrm{rl}} + t_{\mathrm{sc}}) \gtrsim 4t_{\mathrm{rf}}$.

If P_{ex} increases because the cloud is overrun by an ionization front, t_{ex} can be almost arbitrarily short, particularly if the front is young and consequently weak R-type. If P_{ex} increases because the cloud is overrun by any type of front (shock or ionization), $t_{\mathrm{ex}} < t_{\mathrm{sc}}$; otherwise the cloud is not 'overrun', we cannot envisage an isotropic P_{ex}, and the cloud may actually be disrupted. Finally, if P_{ex} increases because the background medium is part of a larger cloud which is itself contracting – as in the hierarchical fragmentation scheme of Hoyle (1953) – then $t_{ex} \sim 2t_{rf}$. We conclude from the calculated values of t_{rl} and t_{sc} that under realistic circumstances we should expect $t_{ex} < (t_{rl} + t_{dm})$. Therefore the evolution will be highly non-quasistatic and pre-instability must occur.

(We note that the model cloud behaves, unrealistically, as if permeated by a weightless rigid membrane which somehow transmits the external pressure so that changes therein are felt instantaneously throughout the cloud volume and the cloud responds homologously. External pressure changes are felt at the centre of a real cloud only after a time delay $\sim t_{sc}$. In a time-dependent treatment we might include this effect by requiring that the boundary transmit to the underlying cloud a retarded external pressure P_{rt} (subscript 'rt' for retarded) satisfying $dP_{rt}/dt = w(P_{ex} - P_{rt})/t_{sc}$, where w is a weighting factor which allows for the radial distribution of force required to drive homologous radial motions. However, since we estimate $w \sim 20/3$ and $t_{rl} \gtrsim t_{sc}$, the condition for pre-instability formulated above will not be significantly changed by this modification.)

8 Conclusion

(i) We have derived a simple global hydrodynamic equation (3) to describe the radial motions of a uniform, polytropic, spherical cloud contained by external pressure. (ii) We have used the associated potential function to investigate the cloud's stability; and (iii) we have shown that the results mimic closely the behaviour of clouds in DHB. (iv) We have evaluated the effect of a non-quasistatic increase in external pressure giving rise to pre-instability. (v) We have evaluated the time-scale for dynamic relaxation of a cloud to its stable equilibrium state, and the sound-crossing time in this state. Hence we obtain a quantitative condition for the quasistatic evolution of a cloud toward instability: we show that this condition is very unlikely to be fulfilled. (vi) Finally, we have repeated the basic analysis for cylindrical and plane-parallel symmetries in the two following appendices.

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Appendix A: Cylindrical filament

The axially infinite, cylindrically symmetric filament has constant mass per unit length M_0', radius R, cross-sectional area $A = \pi R^2$, uniform density $\rho_{in} = M_0'/A$, and polytropic equation of state $P_{in} = K\rho_{in}^{\eta}$. It is contained by constant external pressure P_{ex}. The potential function per unit length is

$$\mathscr{U}'(R) = (\mathscr{G}_0' - \delta_{1\eta}\mathscr{K}_1')\ln(R/R_0) + (1-\delta_{1\eta})\mathscr{K}_\eta'(R/R_0)^{2(1-\eta)}/2(\eta-1) + P_{ex}A_0(R/R_0)^2,$$

where $\mathscr{G}_0' = GM_0'^2$, and $\mathscr{K}_\eta' = 2KM_0'^{\eta}A_0^{(1-\eta)}$. The kinetic energy per unit length is $\mathscr{V}' = M_0'(dR/dt)^2/4$, and so the equation of motion is

$$\frac{d^2R}{dt^2} = -\frac{2}{M_0'}\frac{d\mathscr{U}'}{dR} = -\frac{2}{M_0'R_0}\left[\mathscr{G}_0'\left(\frac{R}{R_0}\right)^{-1} - \mathscr{K}_\eta'\left(\frac{R}{R_0}\right)^{(1-2\eta)} + 2P_{ex}A_0\left(\frac{R}{R_0}\right)\right]. \quad \text{(A3)*}$$

R_0 is an equilibrium if $\mathscr{G}_0' + 2P_{ex}A_0 - \mathscr{K}_\eta' = 0$. The equilibrium is stable, neutral , or unstable, respectively, according to whether $2P_{ex}A_0 + (\eta-1)\mathscr{K}_\eta' >$, $=$, or < 0. Thus there can only be unstable equilibria when $\eta < 1$. The neutral equilibrium represents the verge of instability against contraction, (higher order) stability, or the verge of instability against expansion, respectively, according to whether $\eta(1-\eta)\mathscr{K}_\eta' >$, $=$, or < 0. Thus instability against contraction can only be approached quasistatically for $0 < \eta < 1$; and the limiting neutral equilibrium state has

$$R_0 \to R_{nu} = \pi^{-1/2}(GM_0'^{(2-\eta)}/2\eta K)^{1/2(1-\eta)}; \quad \text{(A4)}$$

$$P_{ex} \to P_{nu} = (1-\eta)K^{1/(1-\eta)}(2\eta/GM_0')^{\eta/(1-\eta)}.$$

The general equilibrium condition is

$$R_0 \to R_{eq}, \quad P_{ex} = K(M_0'/\pi R_{eq}^2)^{\eta} - GM_0'^2/2\pi R_{eq}^2. \quad \text{(A5)}$$

(i) For $\eta > 1$, the filament always has a stable equilibrium state and can be compressed indefinitely from $(P_{ex}, R_{eq}) = (0, R_{mx})$ to $(\infty, 0)$, where $R_{mx} = \pi^{-1/2}(2KG^{-1}M_0'^{(\eta-2)})^{1/2(\eta-1)}$. (ii) For $\eta = 1$, equation (A5) reduces to $P_{ex} = (K - K_{cr})M_0'/\pi R_{eq}^2$, where $K = a_{in}^2$ and $K_{cr} = GM_0'/2$; so when $K > K_{cr}$, i.e. when $M_0' < M_{cr}' = 2a_{in}^2/G$, the filament always has a stable equilibrium state and can be compressed indefinitely from $(P_{ex}, R_{eq}) = (0, \infty)$ to $(\infty, 0)$. (iii) For $0 < \eta < 1$, stability requires $P_{ex} < P_{nu}$ and $R_{eq} > R_{nu}$, so the filament can be compressed quasistatically only from $(P_{ex}, R_{eq}) = (0, \infty)$ to (P_{nu}, R_{nu}), where it then becomes unstable against indefinite contraction. (iv) For $\eta = 1$ with $K \leqslant K_{cr}$, and for $\eta = 0$ with $P_{ex} \geqslant K$, the filament has no equilibrium and contracts indefinitely. (v) For $\eta = 0$ with $P_{ex} < K$, and for $\eta < 0$, the filament has an unstable equilibrium and therefore contracts/expands indefinitely according as it is released from a sufficiently small/large initial radius.

The pattern of equilibria $R_{eq}(P_{ex})$ and the resulting quasistatic evolution closely mimic the predictions of DHB calculations (e.g. Ostriker 1964). In fact, for the isothermal case ($\eta = 1$), there is exact agreement.

The equation of freefall is $d^2R/dt^2 = -2GM_0'/R$, with initial conditions $R(t=0) = R_0$ and $dR/dt(t=0) = 0$; and solution

$$(t/t_{FF}) = \mathrm{erf}\{[-\ln(R/R_0)]^{1/2}\}, \quad t_{FF} = (\pi/GM_0')^{1/2}(R_0/2) = 1/2[G\rho(t=0)]^{1/2}, \quad \text{(A11)}$$

where 'erf' is the error function (e.g. Abromowitz & Stegun 1964).

*Equations in the appendices for the cylindrical and plane-parallel cases are given the same numbers as the corresponding equations in the main text for the spherical case.

Appendix B: Plane-parallel sheet

The two-dimensionally infinite, plane-parallel symmetric sheet has constant mass per unit area M_0'', half-thickness R, thickness $z = 2R$, uniform density $\rho_{in} = M_0''/Z$, and polytropic equation of state $P_{in} = K\rho_{in}^{\eta}$. It is contained by external pressure P_{ex}. The potential function per unit area is

$$\mathscr{U}''(R) = (\mathscr{G}_0'' + P_{ex}Z_0)(R/R_0) - \delta_{1\eta}\mathscr{K}_1'' \ln(R/R_0) + (1-\delta_{1\eta})\mathscr{K}_\eta''(R/R_0)^{(1-\eta)}/(\eta-1),$$

where $\mathscr{G}_0'' = 4\pi GM_0''^2 R_0/3$, and $\mathscr{K}_\eta'' = KM_0''^{\eta}Z_0^{(1-\eta)}$. The kinetic energy per unit area is $\mathscr{T}'' = M_0''(dR/dt)^2/6$, and so the equation of motion is

$$\frac{d^2R}{dt^2} = -\frac{3}{M_0''}\frac{d\mathscr{U}''}{dR} = -\frac{3}{M_0''R_0}\left[\mathscr{G}_0'' - \mathscr{K}_\eta''(R/R_0)^{-\eta} + P_{ex}Z_0\right]. \tag{B3}$$

R_0 is an equilibrium if $\mathscr{G}_0'' + P_{ex}Z_0 - \mathscr{K}_\eta'' = 0$, which reduces to

$$R_0 \to R_{eq}, \quad P_{ex} = K(M_0''/2R_{eq})^{\eta} - 2\pi GM_0''^2/3. \tag{B5}$$

(i) For $\eta > 0$, the sheet always has a stable equilibrium state and can be compressed indefinitely from $(P_{ex}, R_{eq}) = (0, R_{mx})$ to $(\infty, 0)$, where $R_{mx} = (3KM_0''^{(\eta-2)}/2^{(\eta+1)}\pi G)^{1/\eta}$. (ii) For $\eta = 0$, the sheet contracts indefinitely, is always in neutral equilibrium, or expands indefinitely, respectively, according to whether $P_{ex} >, =$ or $< (K - K_{cr})$, where $K_{cr} = 2\pi GM_0''^2/3$. (iii) For $\eta < 0$, the sheet has an unstable equilibrium and therefore contracts/expands indefinitely according as it is released from sufficiently small/large initial radius.

The pattern of equilibria $R_{eq}(P_{ex})$ and the resulting quasistatic evolution closely mimic the predictions of DHB calculations. Instability against contraction cannot be approached quasistatically for any value of η.

The equation of freefall is $d^2R/dt^2 = -4\pi GM_0''$, with initial conditions $R(t=0) = R_0$ and $dR/dt(t=0) = 0$; and solution

$$(R/R_0) = 1 - (t/t_{FF})^2, \quad t_{FF} = (R_0/2\pi GM_0'')^{1/2} = 1/2[\pi G\rho(t=0)]^{1/2}. \tag{B11}$$