Global holomorphic functions in several non-commuting variables

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Abstract: We define a free holomorphic function to be a function that is locally, with respect to the free topologay, a bounded nc-function. We prove that free holomorphic functions are the functions that are locally uniformly approximable by free polynomials. We prove a realization formula and an Oka-Weil theorem for free analytic functions.

1 Introduction

1.1 Nc-functions and Free holomorphic functions

A non-commutative polynomial, also called a free polynomial, in d variables $x^1, \ldots, x^d$, is a finite linear combination of words in the variables, letting the empty word denote the constant 1. For example

$$p(x^1, x^2) \;=\; 2 + x^1 - x^1x^2x^1 + 3x^1x^1x^2$$

is a free polynomial of degree 3 in 2 variables. A free poynomial is a natural example of a graded function, which means if one evaluates it on a d-tuple of n-by-n matrices, one gets an n-by-n matrix.

Let $\mathbb{M}_n$ denote the n-by-n matrices over $\mathbb{C}$, and let $\mathbb{M}^{[d]}$ denote $\cup_{n=1}^{\infty} \mathbb{M}_n^d$. A graded function is then a map from $\mathbb{M}^{[d]}$ to $\mathbb{M} := \mathbb{M}^{[1]}$ that maps each element in $\mathbb{M}_n^d$ to an element in $\mathbb{M}_n$.

Free polynomials have two further important properties, in addition to being graded: $p(x \oplus y) = p(x) \oplus p(y)$ and $p(s^{-1}xs) = s^{-1}p(x)s$. The basic idea of non-commutative function theory is to define a class of graded functions that should bear the same relationship to free polynomials as holomorphic functions of d variables do to commutative polynomials.

This has been done in a variety of ways: by Taylor [23], in the context of the functional calculus for non-commuting operators; Voiculescu [24, 25], in the context of free probability;

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Popescu [18, 19, 20, 21], in the context of extending classical function theory to d-tuples of bounded operators; Ball, Groenewald and Malakorn [10], in the context of extending realization formulas from functions of commuting operators to functions of non-commuting operators; Alpay and Kalyuzhnyi-Verbovetzkii [6] in the context of realization formulas for rational functions that are J-unitary on the boundary of the domain; and Helton, Klep and McCullough [13, 14] and Helton and McCullough [16] in the context of developing a descriptive theory of the domains on which LMI and semi-definite programming apply.

Very recently, Kaliuzhnyi-Verbovetskyi and Vinnikov have written a monograph [17] that gives a panoramic view of the developments in the field to date. In their work, functions are defined on nc-domains. Before we say what these are, let us establish some notation. We let

$$\mathcal{I}_n := \{M \in \mathbb{M}_n \mid M \text{ is invertible}\} \tag{1.1}$$

$$\mathcal{U}_n := \{M \in \mathbb{M}_n \mid M \text{ is unitary}\}. \tag{1.2}$$

For $M_1 = (M_1^1, \ldots, M_1^d) \in \mathbb{M}_{n_1}^d$ and $M_2 = (M_2^1, \ldots, M_2^d) \in \mathbb{M}_{n_2}^d$, we define $M_1 \oplus M_2 \in \mathbb{M}_{n_1+n_2}^d$ by identifying $\mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}$ with $\mathbb{C}^{n_1+n_2}$ and direct summing M_1 and M_2 componentwise, i.e.,

$$M_1 \oplus M_2 = \Big(M_1^1 \oplus M_2^1, \ldots, M_1^d \oplus M_2^d \Big).$$

Likewise, if $M = (M^1, \ldots, M^d) \in \mathbb{M}_n^d$ and $S \in \mathcal{I}_n$, we define $S^{-1}MS \in \mathbb{M}_n^d$ by

$$S^{-1}MS = (S^{-1}M^1S, \ldots, S^{-1}M^dS).$$

Definition 1.3. If $\mathcal{D} \subseteq \mathbb{M}^d$ we say that $\mathcal{D}$ is an *nc-set* if $\mathcal{D}$ is closed with respect to the formation of direct sums and unitary conjugations, i.e.

$$\forall_{n_1,n_2}\ \forall_{M_1 \in \mathcal{D} \cap \mathbb{M}_{n_1}^d}\ \forall_{M_2 \in \mathcal{D} \cap \mathbb{M}_{n_2}^d}\ M_1 \oplus M_2 \in \mathcal{D} \cap \mathbb{M}_{n_1+n_2}^d$$

and

$$\forall_n\ \forall_{M \in \mathcal{D} \cap \mathbb{M}_n^d}\ \forall_{U \in \mathcal{U}_n}\ U^*MU \in \mathcal{D} \cap \mathbb{M}_n^d.$$

The disjoint union topology (hereinafter abbreviated as du) is the topology on $\mathbb{M}^{[d]}$ given by the sets $\mathcal{D}$ satisfying $\mathcal{D} \cap M_n^d$ is open for every $n \geq 1$. We say that a set $\mathcal{D} \subseteq \mathbb{M}^d$ is *du-open* (resp. *du-closed, du-bounded)* if $\mathcal{D} \cap M_n^d$ is open (resp. closed, bounded) for all $n \geq 1$. An *nc-domain* is an nc-set that is du-open. (We differ from the usage of nc-domain in [15] by not requiring that $\mathcal{D} \cap \mathbb{M}_n^d$ is connected and non-empty for every n).

Definition 1.4. An *nc-function* is a graded function ϕ defined on an nc-domain $\mathcal{D}$ such that

i) If $x, y \in \mathcal{D}$, then $\phi(x \oplus y) = \phi(x) \oplus \phi(y)$.

ii) If $s \in \mathcal{I}_n$ and $x, s^{-1}xs \in \mathcal{D} \cap \mathbb{M}_n^d$ then $\phi(s^{-1}xs) = s^{-1}\phi(x)s$.

We let nc($\mathcal{D}$) denote the set of all nc-functions on $\mathcal{D}$.

In this paper, we shall develop a global theory of holomorphic functions in non-commuting variables, by piecing together functions on a nice class of nc-domains, the basic free open sets.

Definition 1.5. if δ is a matrix of free polynomials in d variables, we define

$$G_\delta \ = \ \{M \in \mathbb{M}^{[d]} : \|\delta(M)\| < 1\}. \tag{1.6}$$

A set of the form (1.6) is called a *basic free open set.* The *free topology* on $\mathbb{M}^{[d]}$ is the topology that has as a basis the basic free open sets. A *free domain* is a subset of $\mathbb{M}^{[d]}$ that is open in the free topology.

Notice that the intersection of two basic free open sets is another basic free open set, because $G_{\delta_1} \cap G_{\delta_2} = G_{\delta_1 \oplus \delta_2}$. Notice also that if $\alpha \in \mathbb{C}^d$, and we define

$$\delta(x) \ = \ \begin{pmatrix} \frac{1}{\varepsilon}(x^1 - \alpha^1 \mathrm{id}) \\ \vdots \\ \frac{1}{\varepsilon}(x^d - \alpha^d \mathrm{id}) \end{pmatrix},$$

then $G_\delta \cap \mathbb{M}_1^d$ is the Euclidean ball centered at α of radius ε, so the free topology agrees with the usual topology on the scalars.

Definition 1.7. A free holomorphic function on a free domain $\mathcal{D}$ is a function ϕ such that every point M in $\mathcal{D}$ is contained in a basic free open set $G_\delta \subseteq \mathcal{D}$ on which ϕ is a bounded nc-function.

Whereas a basic free open set is an nc-domain, a general free open set may not be, since it need not be closed under direct sums.

The locally bounded condition, which one gets automatically in the scalar case, seems to play an essential rôle in developing an analytic, rather than an algebraic, theory. For example, it allows us to give a characterization of free holomorphic functions as functions that are locally limits of free polynomials.

Theorem 9.8. Let $\mathcal{D}$ be a free domain and let ϕ be a graded function defined on $\mathcal{D}$. Then ϕ is a free holomorphic function if and only if ϕ is locally approximable by polynomials.

A non-commutative power series with scalar coefficients makes sense, but only when the center is a point in $\mathbb{M}_1^d$. Given a point M, let us say in $\mathbb{M}_4^d$ for definiteness, a series

$$\sum_w a_w (x - M)^w,$$

where one is summing over non-commutative words w in d variables, can not be evaluated for $x \in \mathbb{M}_3^d$ unless there is some way of interpreting M as corresponding to an element of $\mathbb{M}_3^d$. Being locally approximable by polynomials seems therefore a natural substitute for analyticity. Rational functions (or, more generally, meromorphic functions built up from free holomorphic functions) are also free holomorphic, provided one stays away from the poles (Theorem 10.1).

The classical Oka-Weil theorem states that a holomorphic function on a neighborhood of a compact, polynomially convex set, can be uniformly approximated by polynomials. See *e.g.* [5, Chap. 7]. We derive Theorem 9.8 as a special case of a free Oka-Weil theorem.

Theorem 9.7. Let $E \subseteq \mathbb{M}^{[d]}$ be a compact set (in the free topology) that is polynomially convex. Assume that ϕ is a free holomorphic function defined on a neighborhood of E. Then ϕ can be uniformly approximated by free polynomials on E.

The corona theorem of Carleson [12] says that an N-tuple of bounded holomorphic functions on the unit disk is not contained in a proper ideal if and only if the functions are jointly bounded below by a positive constant. We obtain a free version.

Theorem 8.17. Let $\{\psi_i\}_{i=1}^N$ be bounded free holomorphic functions on G_δ. Assume for some $\varepsilon > 0$, we have

$$\sum_{i=1}^{N} \psi_i(x)^* \psi_i(x) \ \geq \ \varepsilon^2 \, \mathrm{id}.$$

Then there are bounded free holomorphic functions ϕ_i on G_δ such that

$$\sum_{i=1}^{N} \psi_i(x) \phi_i(x) \ = \ \mathrm{id}.$$

Moreover, one can choose the functions so that

$$\|(\phi_1, \dots, \phi_N)\| \ \leq \ \frac{1}{\varepsilon}.$$

Our realization formula Theorem 8.1 can be used to show that every scalar-valued function on G_δ that is bounded on commuting matrices (using the Taylor functional calculus) can be extended to a free analytic function with the same norm.

Definition 1.8. Let $\|f\|_{\delta,\mathrm{com}} = \sup\{\|f(T)\|\}$, where T ranges over *commuting* elements T in $\mathbb{M}_n^d$ that satisfy $\|\delta(T)\| \leq 1$ and $\sigma(T) \subset G_\delta$. Let $H^\infty_{\delta,\mathrm{com}}$ be the Banach algebra of holomorphic functions on G_δ with this norm.

Theorem 8.19. Let

$$I \ = \ \{\phi \in H^\infty(G_\delta) \,|\, \phi|_{\mathbb{M}_1^d} = 0\}.$$

Then $H^\infty(G_\delta)/I$ is isometrically isomorphic to $H^\infty_{\delta,\mathrm{com}}$.

1.2 The structure of free holomorphic functions

The engine that drives our results is a model and realization formula for free holomorphic functions on basic free open sets. To describe these, we must expand the notion of nc-function to consider '$\mathcal{K}$-valued' nc-functions on $\mathcal{D}$ where $\mathcal{K}$ is a separable Hilbert space. One way to model such objects would be to view them as concrete column vectors with entries in $\mathrm{nc}(\mathcal{D})$. However, we shall adopt an approach which uses tensor products. If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, we let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ denote the bounded linear transformations from $\mathcal{H}$ to $\mathcal{K}$. We identify $(\mathbb{C}^{n_1} \otimes \mathcal{K}) \oplus (\mathbb{C}^{n_2} \otimes \mathcal{K})$ and $\mathbb{C}^{n_1+n_2} \otimes \mathcal{K}$ in the obvious way. If $T_1 \in \mathcal{L}(\mathbb{C}^{n_1}, \mathbb{C}^{n_1} \otimes \mathcal{K})$ and $T_2 \in \mathcal{L}(\mathbb{C}^{n_2}, \mathbb{C}^{n_2} \otimes \mathcal{K})$, we define $T_1 \oplus T_2 \in \mathcal{L}(\mathbb{C}^{n_1+n_2}, \mathbb{C}^{n_1+n_2} \otimes \mathcal{K})$ by requiring that

$$(T_1 \oplus T_2)(v_1 \oplus v_2) = T_1(v_1) \oplus T_2(v_2)$$

for all $v_1 \in \mathbb{C}^{n_1}$, $v_2 \in \mathbb{C}^{n_2}$, and $k \in \mathcal{K}$.

Definition 1.9. We say a function f is a $\mathcal{K}$-valued nc-function if the domain of f is some nc-domain, $\mathcal{D}$,

$$\forall_n \ \forall_{x\in\mathcal{D}\cap\mathbb{M}_n^d} \ f(x) \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{K}), \tag{1.10}$$

$$\forall_{x,y\in\mathcal{D}} \ f(x \oplus y) = f(x) \oplus f(y), \text{ and} \tag{1.11}$$

$$\forall_n \ \forall_{x\in\mathcal{D}\cap\mathbb{M}_n^d} \ \forall_{s\in\mathcal{I}_n} \ s^{-1}xs \in \mathcal{D} \implies f(s^{-1}xs) = (s^{-1} \otimes \mathrm{id}_{\mathcal{K}})f(x)s. \tag{1.12}$$

If $\mathcal{D}$ is an nc-domain, we let $\mathrm{nc}_{\mathcal{K}}(\mathcal{D})$ denote the collection of $\mathcal{K}$-valued nc-functions on $\mathcal{D}$.

Let p be a free polynomial, and f be in $\mathrm{nc}_{\mathcal{K}}(\mathcal{D})$. Then we define $pf \in \mathrm{nc}_{\mathcal{K}}(\mathcal{D})$ by

$$pf(x) \ = \ [p(x) \otimes \mathrm{id}_{\mathcal{K}}]f(x).$$

Now let δ be an I-by-J matrix of free polynomials, and let u be in $\mathrm{nc}_{\ell^2(J)}(\mathcal{D})$. We define $\delta u \in \mathrm{nc}_{\ell^2(I)}(\mathcal{D})$ by matrix multiplication. Let $u = (u_1, \dots, u_J)^t$; then define δu by the formula

$$(\delta u)(x) \ = \ \begin{pmatrix} \sum_{j=1}^J [\delta_{1j}(x) \otimes \mathrm{id}_{\ell^2}]u_j(x) \\ \vdots \\ \sum_{j=1}^J [\delta_{Ij}(x) \otimes \mathrm{id}_{\ell^2}]u_j(x) \end{pmatrix} \qquad x \in \mathcal{D}.$$

Definition 1.13. Let ϕ be a graded function on G_δ. A δ *nc-model for* ϕ is a formula of the form

$$1 - \phi(y)^*\phi(x) = u(y)^*[1 - \delta(y)^*\delta(x)]u(x), \qquad x, y \in G_\delta \tag{1.14}$$

where u is in $\mathrm{nc}_{\ell^2(J)}\, G_\delta$.

Definition 1.15. Let ϕ be a graded function on G_δ. A *free δ-realization for* ϕ is an isometry

$$\mathcal{J} \ = \ \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

such that for each $n \in \mathbb{N}$ and each $x \in G_\delta \cap \mathbb{M}_n^d$

$$\phi(x) = (\mathrm{id}_{\mathbb{C}^n} \otimes A) + (\mathrm{id}_{\mathbb{C}^n} \otimes B)\delta(x)[\mathrm{id} - (\mathrm{id}_{\mathbb{C}^n} \otimes D)\delta(x)]^{-1}(\mathrm{id}_{\mathbb{C}^n} \otimes C). \tag{1.16}$$

We prove in Theorem 8.1 that every free holomorphic function that is bounded in norm by 1 on G_δ has a δ-model and a free δ-realization.

In the commutative case, and when $\mathcal{D}$ is the polydisk, the result was first proved in [1]. The extension to G_δ for scalar valued functions was first done by Ambrozie and Timotin [7]; Ball and Bolotnikov extended this result to functions of commuting operators in [9]. In the non-commutative case, the first version of this result was proved by Ball, Groenewald and Malakorn [10]. They proved a realization formula for non-commutative power series on domains that could be described in terms of certain bipartite graphs; these include the most important examples, the non-commutative polydisk and the non-commutative ball.

The statement of the theorem is as follows (we omit Statement (2) for now). We extend the notion of nc function to an $\mathcal{L}(\mathcal{H}, \mathcal{K})$-valued function in the natural way (see Definition 3.6).

Theorem 8.1 Let $\mathcal{H}, \mathcal{K}_1, \mathcal{K}_2$ be finite dimensional Hilbert spaces. Let δ be an $I \times J$ matrix whose entries are free polynomials. Let Ψ be a graded $\mathcal{L}(\mathcal{H}, \mathcal{K}_1)$-valued function on G_δ, and let Φ be a graded $\mathcal{L}(\mathcal{H}, \mathcal{K}_2)$-valued function on G_δ. The following are equivalent.
(1) $\Psi(x)^*\Psi(x) - \Phi(x)^*\Phi(x) \geq 0$ on G_δ.
(3) There exists an nc $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$-valued function Ω satsifying $\Omega\Psi = \Phi$ and such that Ω has a free δ-realization.

In the special case that Ψ is the identity, this says that every bounded free analytic function has a free δ-realization as in (1.16). One consequence is that every bounded function defined on commuting tuples in any G_δ extends to non-commuting tuples with the same norm. Another consequence, which the authors shall explore in a subsequent paper [2], is a Nevanlinna-Pick theorem for bounded nc-functions on G_δ.

2 Structure of the Paper

In Section 3 we discuss basic notions of nc domains and nc functions. We prove that every nc function on a domain $\mathcal{D}$ extends to an nc function on its envelope $\mathcal{D}^\sim$, the similarity closed set generated by $\mathcal{D}$ (Proposition 3.10).

In Section 4, we prove that locally bounded nc functions are holomorphic (Theorem 4.10). We define a free holomorphic function to be a locally bounded nc function, and prove that Montel's theorem holds for these functions (Proposition 4.14).

To prove that bounded free holomorphic functions have realizations, we use a Hahn-Banach argument. To make this work, we need to know that the set of all functions of the form

$$u(y)^*[1 - \delta(y)^*\delta(x)]u(x), \qquad u \in \mathrm{nc}_{\ell^2(J)}\, G_\delta$$

is a closed cone. Proving it is closed is delicate, so we rely on finite dimensional approximations. In Section 5 we develop the theory of partial nc-sets and partial nc-functions, which are restrictions to finite sets of nc-functions. To allow us to piece these together into an nc-function, we introduce the notion of a well-organized pair $(E, \mathcal{S})$ (Definition 5.4), which is a finite set E and a finite number of similarities with certain nice properties.

In Section 6, we show how to get δ-models and δ-realizations on well-organized pairs. In Section 7, we piece these together to get a δ nc-model on the whole set G_δ. The main theorem here is Theorem 7.10. We improve this theorem in Section 8 to get Theorem 8.1, which says one can find a free δ-realization for the multiplier Ω .

In Section 9 we use this structure theorem to derive our major consequences: the free Oka-Weil Theorem 9.7, which in particular gives a proof that a function is free holomorphic if and only if it is locally approximable by free polynomials (Theorem 9.8).

In Section 10, we prove that free meromorphic functions are free holomorphic off their singular sets. We give an index to notation and definitions in Section 11.

3 Basic Notions

3.1 du-bounded

We define the nc-norm $\|\cdot\|$ on each set $\mathbb{M}_n^d$ by the formula

$$\|M\| = \max_{1\le r\le d} \|M^r\|$$

and when metric calculations are required, we shall use the metric d, defined on each set $\mathbb{M}_n^d$ by the formula

$$d(M,N) = \max_{1\le r\le d} \|M^r - N^r\|.$$

If $M \in \mathcal{D}\cap\mathbb{M}_n^d$ and $r > 0$, we let

$$\mathrm{B}(M,r) = \{N \in \mathbb{M}_n^d \,|\, d(M,N) < r\}.$$

Evidently, a set $\mathcal{D} \subseteq \mathbb{M}^{[d]}$ is du-bounded when

$$\sup_{M\in\mathcal{D}\cap\mathbb{M}_n^d} \|M\| < \infty$$

for each $n \ge 1$. We say that a set $\mathcal{D} \subseteq \mathbb{M}^{[d]}$ is *bounded* if

$$\sup_{M\in\mathcal{D}} \|M\| < \infty.$$

Clearly, boundedness implies du-boundedness but not conversely.

3.2 Envelopes of nc-Domains

If $A \subseteq \mathbb{M}^{[d]}$, let us agree to say that A is *similarity invariant* if for each $n \ge 1$ and each $S \in \mathcal{I}_n$,

$$S^{-1}(A\cap\mathbb{M}_n^d)S \subseteq A\cap\mathbb{M}_n^d.$$

As the intersection of similarity invariant nc-sets is a similarity invariant nc-set, it is clear that if $A \subseteq \mathbb{M}^{[d]}$, then there exists a *smallest* similarity invariant nc-set containing A. We formalize this fact in the following definition.

Definition 3.1. If $A \subseteq \mathbb{M}^{[d]}$, then $A^\sim$, the envelope of A, is the unique similarity invariant nc-set satisfying $A \subseteq A^\sim$ and $A^\sim \subseteq B$ whenever B is a similarity invariant nc-set containing A.

Proposition 3.2. Let $A \subseteq \mathbb{M}^{[d]}$ and let $M \in \mathbb{M}_n^d$. $M \in A^\sim$ if and only if there exist an integer $m \ge 1$, integers $n_1,\ n_2,\ \ldots,\ n_m \ge 1$ satisfying $n = n_1+n_2+\ldots+n_m$, matrix tuples $M_1 \in A\cap\mathbb{M}_{n_1}^d, M_2 \in A\cap\mathbb{M}_{n_2}^d,\ \ldots,\ M_m \in A\cap\mathbb{M}_{n_m}^d$, and $S \in \mathcal{I}_n$ such that

$$M = S^{-1}\Big(\bigoplus_{k=1}^{m} M_k\Big)S \tag{3.3}$$

Proof. Let B denote the collection of matrix tuples M that have the form as presented in (3.3). Then B is a similarity invariant nc-set. Also, $B \subseteq C$ if C is a similarity invariant nc-set that contains A. Therefore, $B = A^{\sim}$. □

As corollaries to Proposition 3.2 we obtain the following two facts which will prove useful in the sequel. They are not new — see *e.g.* [17].

Proposition 3.4. If A is an nc-set and $N \in \mathbb{M}_n^d$, then $N \in A^{\sim}$ if and only if there exists $M \in A \cap \mathbb{M}_n^d$ and $S \in \mathcal{I}_n$ such that $N = S^{-1}MS$.

Proposition 3.5. If $\mathcal{D}$ is an nc-domain, then $\mathcal{D}^{\sim}$ is an nc-domain.

Proof. Let $\mathcal{D} \subseteq \mathbb{M}^{[d]}$ be an nc domain. Fix $n \geq 1$ and $N \in \mathcal{D}^{\sim} \cap \mathbb{M}_n^d$. By Proposition 3.4 there exist $M \in \mathcal{D} \cap \mathbb{M}_n^d$ and $S \in \mathcal{I}_n$ such that $N = S^{-1}MS$. As $\mathcal{D} \cap \mathbb{M}_n^d$ is open, there exists $\delta > 0$ such that

$$M + S\Delta S^{-1} \in D \cap \mathbb{M}_n^d$$

whenever $\Delta \in \mathbb{M}_n^d$ and $\|\Delta\| < \delta$. Consequently, if $\Delta \in \mathbb{M}_n^d$ and $\|\Delta\| < \delta$, then

$$N + \Delta = S^{-1}MS + \Delta = S^{-1}(M + S\Delta S^{-1})S \in \mathcal{D}^{\sim}.$$

□

3.3 nc-Functions

We defined nc-functions and $\mathcal{K}$-valued nc-functions in Definitions 1.4 and 1.9. We extend this to '$\mathcal{L}(\mathcal{H}, \mathcal{K})$-valued' nc-functions on $\mathcal{D}$ where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. If $T_1 \in \mathcal{L}(\mathbb{C}^{n_1} \otimes \mathcal{H}, \mathbb{C}^{n_1} \otimes \mathcal{K})$ and $T_2 \in \mathcal{L}(\mathbb{C}^{n_2} \otimes \mathcal{H}, \mathbb{C}^{n_2} \otimes \mathcal{K})$ we define $T_1 \oplus T_2 \in \mathcal{L}(\mathbb{C}^{n_1+n_2} \otimes \mathcal{H}, \mathbb{C}^{n_1+n_2} \otimes \mathcal{K})$ by requiring that

$$(T_1 \oplus T_2)((v_1 \oplus v_2) \otimes h) = T_1(v_1 \otimes h) \oplus T_2(v_2 \otimes h)$$

for all $v_1 \in \mathbb{C}^{n_1}$, $v_2 \in \mathbb{C}^{n_2}$, and $h \in \mathcal{H}$.

Definition 3.6. We say a function f is an $\mathcal{L}(\mathcal{H}, \mathcal{K})$-valued nc-function (and write $f \in \mathrm{nc}_{\mathcal{L}(\mathcal{H},\mathcal{K})}$) if the domain of f is some nc-domain, $\mathcal{D}$,

$$\forall_n \ \forall_{x \in \mathcal{D} \cap \mathbb{M}_n^d} \ f(x) \in \mathcal{L}(\mathbb{C}^n \otimes \mathcal{H}, \mathbb{C}^n \otimes \mathcal{K}), \tag{3.7}$$

$$\forall_{x,y \in \mathcal{D}} \ f(x \oplus y) = f(x) \oplus f(y), \text{ and} \tag{3.8}$$

$$\forall_n \ \forall_{x \in \mathcal{D} \cap \mathbb{M}_n^d} \ \forall_{s \in \mathcal{I}_n} \ s^{-1}xs \in \mathcal{D} \implies f(s^{-1}xs) = (s^{-1} \otimes \mathrm{id}_{\mathcal{K}}) f(x)(s \otimes \mathrm{id}_{\mathcal{H}}). \tag{3.9}$$

A simple yet important point is that if $\dim(\mathcal{H}) = \dim(\mathcal{K}) = 1$, then we can identify $\mathcal{L}(\mathbb{C}^n \otimes \mathcal{H}, \mathbb{C}^n \otimes \mathcal{K})$ with $\mathbb{M}_n$ and with this identification it is easy to verify that (3.7), (3.8), and (3.9) imply that Definition 1.4 is satisfied. Thus, theorems proved for $\mathcal{L}(\mathcal{H}, \mathcal{K})$-valued nc-functions hold for nc-functions. Likewise, theorems proved for $\mathcal{L}(\mathcal{H}, \mathcal{K})$-valued nc-functions hold for $\mathcal{K}$-valued nc-functions.

The following Proposition is also proved in [11].

Proposition 3.10. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. If $\mathcal{D}$ is an nc-domain and f is an $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued nc-function on $\mathcal{D}$, then there exists a unique nc-function $f^\sim$ on $\mathcal{D}^\sim$ such that $f^\sim|\mathcal{D} = f$.

Proof. Fix $N \in \mathcal{D}^\sim \cap \mathbb{M}_n^d$. By Proposition 3.4 there exists $M \in \mathcal{D} \cap \mathbb{M}_n^d$ and invertible $s \in \mathbb{M}_n$ such that $N = s^{-1}Ms$. We define

$$f^\sim(N) = (s^{-1} \otimes \mathrm{id}_{\mathcal{K}})f(M)(s \otimes \mathrm{id}_{\mathcal{H}}) \tag{3.11}$$

We need to prove two things: that $f^\sim$ is well defined, and that $f^\sim$ is an $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued nc-function.

To see that $f^\sim$ is well defined, fix $N \in \mathcal{D}^\sim \cap \mathbb{M}_n^d$ and then choose $M_1, M_2 \in \mathcal{D} \cap \mathbb{M}_n^d$ and invertible $s_1, s_2 \in \mathbb{M}_n$ with $s_1^{-1}M_1s_1 = N$ and $s_2^{-1}M_2s_2 = N$. If we set $s = s_1s_2^{-1}$, then as $s^{-1}M_1s = M_2 \in \mathcal{D}$, it follows from (3.9), that

$$f(s^{-1}M_1s) = (s^{-1} \otimes \mathrm{id}_{\mathcal{K}})f(M_1)(s \otimes \mathrm{id}_{\mathcal{H}}).$$

Hence,

$$\begin{aligned}(s_1^{-1} \otimes \mathrm{id}_{\mathcal{K}})f(M_1)(s_1 \otimes \mathrm{id}_{\mathcal{H}}) &= (s_2^{-1} \otimes \mathrm{id}_{\mathcal{K}})(s^{-1} \otimes \mathrm{id}_{\mathcal{K}})f(M_1)(s \otimes \mathrm{id}_{\mathcal{H}})(s_2 \otimes \mathrm{id}_{\mathcal{H}})\\ &= (s_2^{-1} \otimes \mathrm{id}_{\mathcal{K}})f(M_2)(s_2 \otimes \mathrm{id}_{\mathcal{H}}).\end{aligned}$$

This proves that $f^\sim$ is well defined.

To see that $f^\sim$ is an $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued nc-function on $\mathcal{D}^\sim$, note first that (3.7) follows immediately from (3.11). To prove (3.8) fix $N_1 \in \mathcal{D}^\sim \cap \mathbb{M}_{n_1}^d$ and $N_2 \in \mathcal{D}^\sim \cap \mathbb{M}_{n_2}^d$. Choose $M_1 \in \mathcal{D} \cap \mathbb{M}_{n_1}^d$, $N_2 \in \mathcal{D}^\sim \cap \mathbb{M}_{n_2}^d$, $s_1 \in \mathcal{I}_{n_1}$, and $s_2 \in \mathcal{I}_{n_2}$ such that $N_1 = s_1^{-1}M_1s_1$ and $N_2 = s_2^{-1}M_2s_2$. Then, as

$$N_1 \oplus N_2 = (s_1 \oplus s_2)^{-1}(M_1 \oplus M_2)(s_1 \oplus s_2),$$

and $M_1 \oplus M_2 \in \mathcal{D}$, we have using (3.11) that

$$\begin{aligned}f^\sim(N_1 \oplus N_2) &= \big((s_1 \oplus s_2)^{-1} \otimes \mathrm{id}_{\mathcal{K}}\big)f(M_1 \oplus M_2)\big((s_1 \oplus s_2) \otimes \mathrm{id}_{\mathcal{H}}\big)\\ &= \big((s_1^{-1} \otimes \mathrm{id}_{\mathcal{K}}) \oplus (s_2^{-1} \otimes \mathrm{id}_{\mathcal{K}})\big)\big(f(M_1) \oplus f(M_2)\big)\big((s_1 \otimes \mathrm{id}_{\mathcal{H}}) \oplus (s_2 \otimes \mathrm{id}_{\mathcal{H}})\big)\\ &= \big((s_1^{-1} \otimes \mathrm{id}_{\mathcal{K}})f(M_1)(s_1 \otimes \mathrm{id}_{\mathcal{H}})\big) \oplus \big((s_2^{-1} \otimes \mathrm{id}_{\mathcal{K}})f(M_2)(s_2 \otimes \mathrm{id}_{\mathcal{H}})\big)\\ &= f^\sim(N_1) \oplus f^\sim(N_2).\end{aligned}$$

This proves (3.8).

Finally, to prove (3.9), fix $N \in \mathcal{D}^\sim \cap \mathbb{M}_n^d$ and $s \in \mathcal{I}_n$. Choose $M \in \mathcal{D} \cap \mathbb{M}_n^d$ and $t \in \mathcal{I}_n$ such that $N = t^{-1}Mt$. Then, as $s^{-1}Ns = (ts)^{-1}M(ts)$,

$$\begin{aligned}f^\sim(s^{-1}Ns) &= ((ts)^{-1} \otimes \mathrm{id}_{\mathcal{K}})f(M)((ts) \otimes \mathrm{id}_{\mathcal{H}})\\ &= (s^{-1} \otimes \mathrm{id}_{\mathcal{K}})\big((t^{-1} \otimes \mathrm{id}_{\mathcal{K}})f(M)(t \otimes \mathrm{id}_{\mathcal{H}})\big)(s \otimes \mathrm{id}_{\mathcal{H}})\\ &= (s^{-1} \otimes \mathrm{id}_{\mathcal{K}})f^\sim(N)(s \otimes \mathrm{id}_{\mathcal{H}}).\end{aligned}$$

This proves (3.9). ∎

More generally, when $\mathcal{D} \subseteq \mathbb{M}^{[d]}$ is an nc-domain and $f \in \mathrm{nc}_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathcal{D})$, it is possible to extend f in the following way. If $\mathcal{V}$ is an n-dimensional vector space, T is a d-tuple of linear transformations on $\mathcal{V}$ and there exists an invertible linear map $S : \mathcal{V} \to \mathbb{C}^n$ such that

$$STS^{-1} = (ST^1S^{-1}, \dots, ST^dS^{-1}) \in \mathcal{D} \cap \mathbb{M}_n^d,$$

then define $f^{\approx} : \mathcal{V} \otimes \mathcal{H} \to \mathcal{V} \otimes \mathcal{K}$ by the formula,

$$f^{\approx}(T) = (S^{-1} \otimes \mathrm{id}_{\mathcal{K}}) f(STS^{-1})(S \otimes \mathrm{id}_{\mathcal{H}}). \tag{3.12}$$

It is straightforward to check that with this definition $f^{\approx}$ is well defined on $\mathcal{D}^{\approx}$, the set of all linear transformations on finite dimensional vector spaces that are similar to an element of $\mathcal{D}$, and that the appropriate analogs of (3.7), (3.8), and (3.9) hold.

Note to the reader: if $f \in \mathrm{nc}_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathcal{D})$, we can apply f to d-tuples of matrices on $\mathbb{C}^n$; we can apply $f^{\approx}$ to d-tuples of linear transformations on any finite dimensional vector space.

We close this section with the following useful lemmas. Both are simple modifications of results from [14].

Lemma 3.13. (cf. Lemma 2.6 in [14]). Let $\mathcal{D}$ be an nc-domain in $\mathbb{M}^{[d]}$, let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let f be an $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued nc-function on $\mathcal{D}$. Fix $n \geq 1$ and $C \in \mathbb{M}_n$. If $M, N \in \mathcal{D} \cap \mathbb{M}_n^d$ and

$$\begin{bmatrix} N & NC - CM \\ 0 & M \end{bmatrix} \in \mathcal{D} \cap \mathbb{M}_{2n}^d, \tag{3.14}$$

then

$$f(\begin{bmatrix} N & NC - CM \\ 0 & M \end{bmatrix}) = \begin{bmatrix} f(N) & f(N)C - Cf(M) \\ 0 & f(M) \end{bmatrix}. \tag{3.15}$$

Proof. Let

$$s = \begin{bmatrix} \mathrm{id}_{\mathbb{C}^n} & C \\ 0 & \mathrm{id}_{\mathbb{C}^n} \end{bmatrix}$$

so that

$$\begin{bmatrix} N & NC - CM \\ 0 & M \end{bmatrix} = s^{-1} \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} s.$$

Using (3.8) and (3.9),

$$f(\begin{bmatrix} N & NC - CM \\ 0 & M \end{bmatrix}) = f(s^{-1}(N \oplus M)s)$$

$$= (s^{-1} \otimes \mathrm{id}_{\mathcal{K}})(f(N) \oplus f(M))(s \otimes \mathrm{id}_{\mathcal{H}})$$

$$= \begin{bmatrix} \mathrm{id}_{\mathbb{C}^n} \otimes \mathrm{id}_{\mathcal{K}} & -C \otimes \mathrm{id}_{\mathcal{K}} \\ 0 & \mathrm{id}_{\mathbb{C}^n} \otimes \mathrm{id}_{\mathcal{K}} \end{bmatrix} \begin{bmatrix} f(N) & 0 \\ 0 & f(M) \end{bmatrix} \begin{bmatrix} \mathrm{id}_{\mathbb{C}^n} \otimes \mathrm{id}_{\mathcal{H}} & C \otimes \mathrm{id}_{\mathcal{H}} \\ 0 & \mathrm{id}_{\mathbb{C}^n} \otimes \mathrm{id}_{\mathcal{H}} \end{bmatrix}$$

$$= \begin{bmatrix} f(N) & f(N)C - Cf(M) \\ 0 & f(M) \end{bmatrix}.$$

□

Lemma 3.16. (cf. Proposition 2.2 in [14]). Let $\mathcal{D}$ be an nc-domain, let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let f be an $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued nc-function on $\mathcal{D}$. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and let $R:\mathcal{V}\to\mathcal{V}$, $T:\mathcal{W}\to\mathcal{W}$, and $L:\mathcal{V}\to\mathcal{W}$ be linear transformations. If $R,T\in\mathcal{D}^{\approx}$ and

$$TL = LR,$$

then

$$f^{\approx}(T)(L\otimes \mathrm{id}_{\mathcal{H}}) \;=\; (L\otimes \mathrm{id}_{\mathcal{K}})f^{\approx}(R).$$

Proof. Let $s = \begin{bmatrix} \mathrm{id}_{\mathcal{W}} & L \\ 0 & \mathrm{id}_{\mathcal{V}} \end{bmatrix}$ and use

$$f^{\approx}(s^{-1}\begin{bmatrix} T & 0 \\ 0 & R \end{bmatrix} s) \;=\; s^{-1} f^{\approx}(\begin{bmatrix} T & 0 \\ 0 & R \end{bmatrix})s.$$

□

4 Local Boundedness and Holomorphicity

In this section we shall prove that locally bounded nc-functions are automatically holomorphic. In addition we shall lay out various tools involving locally bounded and holomorphic graded functions (not necessarily assumed to be nc-functions) that will be heavily used in the sequel. Most of the content of this section also appears in [17, Chapter 7].

If $\mathcal{D}$ is an nc-domain in $\mathbb{M}^{[d]}$ and $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then we say a function f defined on $\mathcal{D}$ is a *graded* $\mathcal{L}(\mathcal{H},\mathcal{K})$*-valued function on* $\mathcal{D}$ if

$$\forall_n \; \forall_{x\in\mathcal{D}\cap\mathbb{M}_n^d} \; f(x)\in\mathcal{L}(\mathbb{C}^n\otimes\mathcal{H},\mathbb{C}^n\otimes\mathcal{K}). \tag{4.1}$$

Definition 4.2. Let $\mathcal{D}$ be an nc-domain in $\mathbb{M}^{[d]}$ and let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. We say that a graded $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued function on $\mathcal{D}$ is locally bounded if for each $n\ge 1$ and each $x\in\mathcal{D}\cap\mathbb{M}_n^d$, there exists $r>0$ such that $\mathrm{B}(x,r)\subseteq\mathcal{D}$ and

$$\sup_{y\in\mathrm{B}(x,r)} \|f(x)\| < \infty.$$

If $\mathcal{F}$ is a collection of graded $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued functions on $\mathcal{D}$, we say that $\mathcal{F}$ is locally uniformly bounded if for each $n\ge 1$ and each $x\in\mathcal{D}\cap\mathbb{M}_n^d$, there exists $r>0$ such that $\mathrm{B}(x,r)\subseteq\mathcal{D}$ and

$$\sup_{y\in\mathrm{B}(x,r)}\sup_{f\in\mathcal{F}}\|f(x)\| < \infty.$$

Proposition 4.3. Let $\mathcal{D}$ be an nc-domain in $\mathbb{M}^{[d]}$, let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let f be an $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued nc-function on $\mathcal{D}$. If f is locally bounded on $\mathcal{D}$, then $f^{\sim}$ is locally bounded on $\mathcal{D}^{\sim}$. If $\mathcal{F}$ is a locally uniformly bounded collection of graded $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued functions on $\mathcal{D}$, then $\mathcal{F}^{\sim}$ is a locally uniformly bounded collection of graded $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued functions on $\mathcal{D}^{\sim}$.

We view $\mathbb{M}^{[d]} = \cup_n \mathbb{M}_n^d$ as being endowed with the disjoint union topology, i.e., $G \subseteq \mathbb{M}^{[d]}$ is open if and only if $G \cap \mathbb{M}_n^d$ is open for each $n \geq 1$. If $K \subseteq \mathbb{M}^{[d]}$ is a compact set in this topology, then as $\mathbb{M}_n^d$ is open for each $n \geq 1$ and $K \subseteq \cup_n \mathbb{M}_n^d$, it follows that there exists $n \geq 1$ such that

$$K \subseteq \bigcup_{m=1}^{n} \mathbb{M}_m^d.$$

Fix an nc-domain $\mathcal{D} \subseteq \mathbb{M}^{[d]}$. By a *compact-open exhaustion of* $\mathcal{D}$ we mean a sequence of compact subsets of $\mathcal{D}$, $\langle K_m \rangle$, satisfying $K_m \subseteq K_{m+1}^\circ$ for all $m \geq 1$ and such that

$$\mathcal{D} = \bigcup_{m=1}^{\infty} K_m.$$

A particularly simple way to construct a compact-open exhaustion of $\mathcal{D}$ is to note that as $\mathcal{D} \cap \mathbb{M}_n^d$ is an open subset of $\mathbb{M}_n^d$ for each $n \geq 1$, for each n there exists a compact-open exhaustion, $\langle K_{n\,m} \rangle$, of $\mathcal{D} \cap \mathbb{M}_n^d$. It follows that if K_m is defined by

$$K_m = \bigcup_{n=1}^{m} K_{n\,m},$$

then $\langle K_m \rangle$ is a compact-open exhaustion of $\mathcal{D}$. In the sequel notions introduced using a compact-open exhaustion of $\mathcal{D}$ can in each case shown to be independent of the particular choice of exhaustion. Also, for convenience we assume that the exhaustion has been chosen to satisfy the property that

$$\forall_m \; K_m \subseteq \bigcup_{n=1}^{m} \mathcal{D} \cap \mathbb{M}_n^d.$$

Now let $\mathcal{D} \subseteq \mathbb{M}^{[d]}$ be an nc-domain and let $\langle K_m \rangle$ be a compact-open exhaustion of $\mathcal{D}$. If $\mathrm{lb}_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathcal{D})$ denotes the space of locally bounded graded $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued functions on $\mathcal{D}$, then for $f \in \mathrm{lb}_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathcal{D})$,

$$\rho_m(f) \stackrel{\text{def}}{=} \sup_{x \in K_m} \|f(x)\| < \infty$$

for each $m \geq 1$. It follows that $d : \mathrm{lb}_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathcal{D}) \times \mathrm{lb}_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathcal{D}) \to \mathbb{R}$ defined by

$$d(f,g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\rho_m(f-g)}{1+\rho_m(f-g)} \tag{4.4}$$

is a translation invariant metric on $\mathrm{lb}_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathcal{D})$. Clearly, d depends on the choice of exhaustion, but it is straightforward to show that the topology it generates does not. If $f \in \mathrm{lb}_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathcal{D})$ and $\langle f^{(k)} \rangle$ is a sequence in $\mathrm{lb}_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathcal{D})$ we shall write $f^{(k)} \to f$ if $d(f, f^{(k)}) \to 0$.

Definition 4.5. Let $\mathcal{D}$ be an nc-domain in $\mathbb{M}^{[d]}$ and let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Let f be a graded $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued function on $\mathcal{D}$. We say that f is holomorphic on $\mathcal{D}$ if for each $n \geq 1$, the map defined on $\mathcal{D} \cap \mathbb{M}_n^d$ by $x \to f(x)$ is a holomorphic $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued function in the entries of x.

An important tool (Proposition 4.6 below) that we shall use frequently in the sequel is based on the application of Montel's Theorem to uniformly locally bounded sequences of graded holomorphic $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued nc-functions. Unfortunately, in the cases when either $\mathcal{H}$ or $\mathcal{K}$ is infinite dimensional, the topology induced by the metric defined in (4.4) is too strong for this purpose. Accordingly, we define the following notion of *weak convergence*.

Let $\langle K_m \rangle$ be a compact-open exhaustion of an nc-domain $\mathcal{D}$ as above. If $\langle f^{(k)} \rangle$ is a sequence of graded $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued functions on $\mathcal{D}$ and f is a graded $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued function on $\mathcal{D}$, we say that $f^{(k)} \overset{\text{wk}}{\to} f$ if for each $m, n \geq 1$ such that $K_m \cap \mathbb{M}_n^d \neq \emptyset$, for each $c, d \in \mathbb{C}^n$, and for each $h \in \mathcal{H}$ and $k \in \mathcal{K}$ we have that

$$\lim_{k\to\infty} \sup_{x \in K_m \cap \mathbb{M}_n^d} \langle (f^{(k)}(x) - f(x))c \otimes h, d \otimes k \rangle = 0.$$

Proposition 4.6. Let $\mathcal{D}$ be an nc-domain and let $\langle f^{(k)} \rangle$ be a uniformly locally bounded sequence of graded holomorphic $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued functions on $\mathcal{D}$. Then there exists a subsequence $\langle f^{(k_j)} \rangle$ and a graded holomorphic $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued function f on $\mathcal{D}$ such that $f^{(k_j)} \overset{\text{wk}}{\to} f$.

Proof. The proof will proceed by doing a diagonal subsequence argument twice. First fix m and n such that $K_m \cap \mathbb{M}_n^d \neq \emptyset$. Let $\{e_i\}$ denote the standard orthonormal basis for $\mathbb{C}^n$ and fix orthonormal bases $\{\xi_l\}$ and $\{\eta_l\}$ for $\mathcal{H}$ and $\mathcal{K}$. For each $i_1, i_2 \leq n$ and each l_1 and l_2,

$$\langle f^{(k)}(x)\ c_{i_1} \otimes \xi_{l_1}, c_{i_2} \otimes \eta_{l_2} \rangle$$

is uniformly bounded on a neighborhood of $K_m \cap \mathbb{M}_n^d$. Therefore using Montel's Theorem and mathematical induction, for each $N \geq 1$, there exist an increasing sequence of integers $\langle k_{N,j} \rangle$ and holomorphic functions $g^N_{i_1,l_1,i_2,l_2}$ defined on a neighborhood of $K_m \cap \mathbb{M}_n^d$ such that

$$\langle f^{(k_{N,j})}(x)\ c_{i_1} \otimes \xi_{l_1}, c_{i_2} \otimes \eta_{l_2} \rangle \to g^N_{i_1,l_1,i_2,l_2}(x)$$

uniformly on a neighborhood of $K_m \cap \mathbb{M}_n^d$ for all $i_1, i_2 \leq n$ and $l_1, l_2 \leq N$ and with the additional property that $\langle k_{N+1,j} \rangle$ is a subsequence of $\langle k_{N,j} \rangle$ for each N. Hence, there exist holomorphic functions g_{i_1,l_1,i_2,l_2} defined on a neighborhood of $K_m \cap \mathbb{M}_n^d$ such that

$$\langle f^{(k_{N,N})}(x)\ c_{i_1} \otimes \xi_{l_1}, c_{i_2} \otimes \eta_{l_2} \rangle \to g_{i_1,l_1,i_2,l_2}(x)$$

uniformly on a neighborhood of $K_m \cap \mathbb{M}_n^d$ for all $i_1, i_2 \leq n$ and l_1, l_2.

Summarizing, we have proved the following fact.

Fact 4.7. If $\langle f^{(k)} \rangle$ is a uniformly bounded sequence of graded holomorphic $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued functions on $\mathcal{D}$, then for each m and n such that $K_m \cap \mathbb{M}_n^d \neq \emptyset$, there exist a strictly increasing sequence $\langle k_N \rangle$ and holomorphic functions g_{i_1,l_1,i_2,l_2} defined on a neighborhood of $K_m \cap \mathbb{M}_n^d$ such that

$$\langle f^{(k_N)}(x)\ c_{i_1} \otimes \xi_{l_1}, c_{i_2} \otimes \eta_{l_2} \rangle \to g_{i_1,l_1,i_2,l_2}(x)$$

uniformly on a neighborhood of $K_m \cap \mathbb{M}_n^d$ for all $i_1, i_2 \leq n$ and l_1, l_2.

Now fix n. For $m \geq n$ we use Fact 4.7 to inductively construct an increasing sequence $\langle k_{m,N} \rangle$ and holomorphic functions $g^m_{i_1,l_1,i_2,l_2}$ defined on a neighborhood of $K_m \cap \mathbb{M}^d_n$ satisfying

$$\langle f^{(k_{m,N})}(x)\ c_{i_1} \otimes \xi_{l_1}, c_{i_2} \otimes \eta_{l_2} \rangle \to g^m_{i_1,l_1,i_2,l_2}(x) \tag{4.8}$$

uniformly on a neighborhood of $K_m \cap \mathbb{M}^d_n$ for all $i_1, i_2 \leq n$ and l_1, l_2 and with $\langle k_{m,N} \rangle$ a subsequence of $\langle k_{m+1,N} \rangle$ for each m. As $K_m \subseteq K_{m+1}$, it follows from (4.8) that if $m_1 \leq m_2$, then

$$g^{m_1}_{i_1,l_1,i_2,l_2}(x) = g^{m_2}_{i_1,l_1,i_2,l_2}(x)$$

on a neighborhood of $K_{m_1} \cap \mathbb{M}^d_n$. Therefore, as $\{K_m\}$ is an exhaustion of $\mathcal{D}$, we can define an holomorphic function $g^n_{i_1,l_1,i_2,l_2} : \mathcal{D} \cap \mathbb{M}^d_n \to \mathbb{M}_n$ by the formula

$$g^n_{i_1,l_1,i_2,l_2}(x) = g^m_{i_1,l_1,i_2,l_2}(x) \text{ if } m \geq n \text{ and } x \in \mathcal{D} \cap \mathbb{M}^d_n. \tag{4.9}$$

Now define a graded holomorphic $\mathcal{L}(\mathcal{H}, \mathcal{K})$-valued function f on $\mathcal{D}$ by requiring that

$$\langle f(x)\ c_{i_1} \otimes \xi_{l_1}, c_{i_2} \otimes \eta_{l_2} \rangle = g^n_{i_1,l_1,i_2,l_2}(x)$$

whenever $n \geq 1$, $x \in \mathcal{D} \cap \mathbb{M}^d_n$, $i_1, i_2 \leq n$, $l_1, l_2 \geq 1$. By (4.8) and (4.9) it follows that

$$\langle f^{(k_{m,m})}(x)\ c_{i_1} \otimes \xi_{l_1}, c_{i_2} \otimes \eta_{l_2} \rangle \to \langle f(x)\ c_{i_1} \otimes \xi_{l_1}, c_{i_2} \otimes \eta_{l_2} \rangle$$

whenever $n \geq 1$, $x \in \mathcal{D} \cap \mathbb{M}^d_n$, $i_1, i_2 \leq n$, $l_1, l_2 \geq 1$. Hence, since $\langle f^{(k)} \rangle$ is assumed locally bounded it follows that $f(x) \in \mathcal{L}(\mathbb{C}^n \otimes \mathcal{H}, \mathbb{C}^n \otimes \mathcal{K})$ for all $x \in \mathcal{D} \cap \mathbb{M}^d_n$ and $f^{k_{m,m}} \overset{\text{wk}}{\to} f$. $\square$

Theorem 4.10. Let $\mathcal{D}$ be an nc-domain in $\mathbb{M}^{[d]}$, let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let f be an $\mathcal{L}(\mathcal{H}, \mathcal{K})$-valued nc-function on $\mathcal{D}$. If f is locally bounded on $\mathcal{D}$, then f is holomorphic on $\mathcal{D}$.

Proof. The proof will proceed in two steps. We first show that if f is locally bounded, then f is continuous. That f is holomorphic will then follow by a straightforward modification of Proposition 2.5 in [14].

Fix $M \in \mathcal{D} \cap \mathbb{M}^d_n$ and let $\epsilon > 0$. Choose $r > 0$ so that

$$\mathrm{B}(\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, r) \subseteq \mathcal{D} \cap \mathbb{M}^d_{2n}.$$

If s is chosen with $0 < s < r$ then as $\mathrm{B}(M \oplus M, s)^-$ is a compact subset of $\mathcal{D} \cap \mathbb{M}^d_{2n}$ and f is assumed locally bounded, there exists a constant B such that

$$x \in \mathrm{B}(\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, s) \implies \|f(x)\| < B. \tag{4.11}$$

Choose δ sufficiently small so that $\delta < \min\{s\epsilon/2B, s/2\}$ and $\mathrm{B}(M, \delta) \subseteq \mathcal{D}$. That f is continuous at M follows from the following claim.

$$N \in \mathrm{B}(M, \delta) \implies f(N) \in \mathrm{B}(f(M), \epsilon). \tag{4.12}$$

To prove the claim fix $N \in \mathbb{M}_n^d$ with $\|N - M\| < \delta$. Then $\|N - M\| < s/2$ and $\|(B/\epsilon)(N - M)\| < s/2$. Hence by the triangle inequality,

$$\| \begin{bmatrix} N & c(N-M) \\ 0 & M \end{bmatrix} - \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \| < s.$$

Hence, by (4.11),

$$\|f(\begin{bmatrix} N & (B/\epsilon)(N-M) \\ 0 & M \end{bmatrix})\| < B.$$

But M, N, and $\begin{bmatrix} N & c(N-M) \\ 0 & M \end{bmatrix}$ are in $\mathcal{D}$, so by Lemma 3.13,

$$f(\begin{bmatrix} N & (B/\epsilon)(N-M) \\ 0 & M \end{bmatrix}) = \begin{bmatrix} f(N) & (B/\epsilon)(f(N) - f(M)) \\ 0 & f(M) \end{bmatrix}$$

In particular, we see that $\|(B/\epsilon)(f(N) - f(M))\| < B$, or equivalently, $f(N) \in \mathrm{B}(f(M), \epsilon)$. This proves (4.12).

To see that f is holomorphic, fix $M \in \mathcal{D} \cap \mathbb{M}_n^d$. If $E \in \mathbb{M}_n^d$ is selected sufficiently small, then

$$\begin{bmatrix} M + \lambda E & E \\ 0 & M \end{bmatrix} \in \mathcal{D} \cap \mathbb{M}_{2n}^d$$

for all sufficiently small $\lambda \in \mathbb{C}$. But

$$\begin{bmatrix} M + \lambda E & E \\ 0 & M \end{bmatrix} = \begin{bmatrix} M + \lambda E & (1/\lambda)\big((M + \lambda E) - M\big) \\ 0 & M \end{bmatrix}.$$

Hence, Lemma 1.4 implies that

$$f(\begin{bmatrix} M + \lambda E & E \\ 0 & M \end{bmatrix}) = \begin{bmatrix} f(M + \lambda E) & (1/\lambda)\big(f(M + \lambda E) - f(M)\big) \\ 0 & f(M) \end{bmatrix}. \tag{4.13}$$

As the left hand side of (4.13) is continuous at $\lambda = 0$, it follows that the 1-2 entry of the right hand side of (4.13) must converge. As E is arbitrary, this implies that f is holomorphic. □

If $\mathcal{D}$ is an nc-domain, we let $H(\mathcal{D})$ denote the collection of locally bounded nc-functions on $\mathcal{D}$. In light of Theorem 4.10 we refer to the elements of $H(\mathcal{D})$ as *free holomorphic functions*. Likewise, if $\mathcal{H}$ and $\mathcal{K}$ are Hilbert space, we let $H_{\mathcal{K}}(\mathcal{D})$ (resp $H_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathcal{D})$) denote the collection of locally bounded $\mathcal{K}$-valued (resp. $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued) nc-functions on $\mathcal{D}$.

Proposition 4.14. Let $\mathcal{D}$ be an nc-domain. $H(\mathcal{D})$ equipped with the metric defined in (4.4) is complete. Furthermore, Montel's Theorem is true, i.e., if $\mathcal{F} \subseteq H(\mathcal{D})$, then $\mathcal{F}$ has compact closure if and only if $\mathcal{F}$ is locally uniformly bounded.

As mentioned above, Montel's Theorem is not true for $H_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathcal{D})$ when either $\mathcal{H}$ or $\mathcal{K}$ is infinite dimensional. However the following useful fact in many applications can take its place.

Proposition 4.15. Let $\mathcal{D}$ be an nc-domain and let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. $H_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathcal{D})$ equipped with the metric defined in (4.4) is complete. Furthermore, if $\langle f^{(k)} \rangle$ is a locally bounded sequence in $H_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathcal{D})$, then there exist an increasing sequence $\langle k_j \rangle$ and $f \in H_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathcal{D})$ such that $f^{(k_j)} \overset{\mathrm{wk}}{\to} f$.

5 Partial nc-Sets and Functions

Let $\mathbb{N}$ denote the set of positive integers. We say that E is a *partial nc-set of size* n if

$$E \subseteq \bigcup_{m=1}^{n} \mathbb{M}_m^d,$$

$E \cap \mathbb{M}_n^d \neq \emptyset$, and $M_1 \oplus M_2 \in E$ whenever $M_1 \in E \cap \mathbb{M}_{m_1}^d$, $M_2 \in E \cap \mathbb{M}_{m_2}^d$, and $m_1 + m_2 \leq n$. We do not require that partial nc-sets are closed with respect to unitary conjugations.

If E is a partial nc-set, then we say that a function $u : E \to \mathbb{M}^1$ is a *partial nc-function* if

$$\forall_{m \in \mathbb{N}}\ M \in E \cap \mathbb{M}_m^d \implies u(M) \in \mathbb{M}_m, \qquad \text{and} \tag{5.1}$$

$$\forall_{M_1, M_2 \in E}\ M_1 \oplus M_2 \in E \implies u(M_1 \oplus M_2) = u(M_1) \oplus u(M_2). \tag{5.2}$$

In a similar fashion we may define $\mathcal{K}$-valued and $\mathcal{L}(\mathcal{H}, \mathcal{K})$-valued partial nc-functions.

If E is a partial nc-set and $\mathcal{S} \subseteq \cup_n \mathcal{I}_n$, then we say that a function $u : E \to \mathbb{M}^1$ is $\mathcal{S}$*-invariant* if

$$\forall_{M \in E}\ \forall_{S \in \mathcal{S}}\ S^{-1}MS \in E \implies u(S^{-1}MS) = S^{-1}u(M)S.$$

In a similar fashion we may define $\mathcal{K}$-valued and $\mathcal{L}(\mathcal{H}, \mathcal{K})$-valued $\mathcal{S}$-invariant functions. Note that the definitions of partial nc-function and $\mathcal{S}$-invariant function are rigged in such a way that $\phi|E$ is an $\mathcal{S}$-invariant partial nc-function whenever $\mathcal{D}$ is an nc-domain, ϕ is an nc-function on $\mathcal{D}$, $\mathcal{S} \subseteq \cup_n \mathcal{I}_n$, and $E \subseteq \mathcal{D}$ is a partial nc-set.

We say that $M \in \mathbb{M}_n^d$ is *generic* if there do not exist $M_1, M_2 \in \mathbb{M}^{[d]}$ and $S \in \mathcal{I}_n$ such that $M = S^{-1}(M_1 \oplus M_2)S$. If E is a partial nc-set, we say that E *is complete* if $M_1 \oplus M_2 \in E$ implies that $M_1, M_2 \in E$. If $M \in E$, we say that M *is E-reducible* if there exist $M_1, M_2 \in E$ such that $M = M_1 \oplus M_2$. Finally, we shall let Σ_n^d denote the d-tuples of scalar matrices:

$$\Sigma_n^d \ = \ \{(\alpha^1 \mathrm{id}_{\mathbb{C}^n}, \ldots, \alpha^d \mathrm{id}_{\mathbb{C}^n}) \ : \ \alpha^r \in \mathbb{C}, 1 \leq r \leq d\}. \tag{5.3}$$

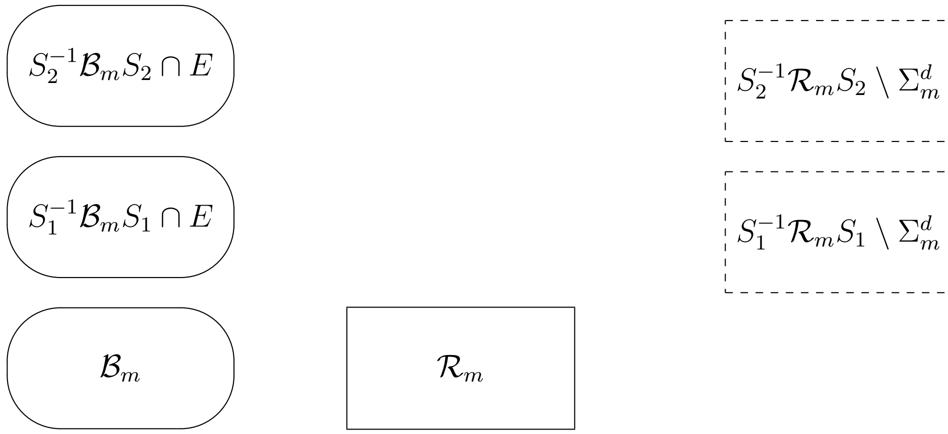
Definition 5.4. Let E be a partial nc-set of size n and $\mathcal{S} \subseteq \cup_n \mathcal{I}_n$. For each $m \leq n$ let $\mathcal{G}_m$ denote the generic elements of $E \cap \mathbb{M}_m^d$ and let $\mathcal{R}_m$ denote the E-reducible elements of $E \cap \mathbb{M}_m^d$. We say the pair $(E, \mathcal{S})$ is *well organized*, if E is finite and complete,

$$\forall_{m \leq n}\ E \cap \mathbb{M}_m^d = \mathcal{R}_m \cup \mathcal{G}_m, \tag{5.5}$$

and finally, for each $m \leq n$ there exists a set $\mathcal{B}_m \subseteq \mathcal{G}_m$ such that

$$\{\mathcal{B}_m\} \cup \{S^{-1}\mathcal{B}_m S \cap E \,|\, S \in \mathcal{S} \cap \mathcal{I}_m\} \text{ is a partition of } \mathcal{G}_m \text{ and} \tag{5.6}$$

$$\forall_{M \in E \cap \mathbb{M}_m^d}\ \forall_{S \in \mathcal{S} \cap \mathcal{I}_m}\ S^{-1}MS \in E \implies M \in \mathcal{B}_m \cup \Sigma_m^d \tag{5.7}$$



A cartoon picture: the solid sets constitute E_m. The ovals are $\mathcal{G}_m$.

We note in this definition that necessarily, as the elements of $\mathcal{R}_m$ are E-reducible and the elements of $\mathcal{G}_m$ are generic, $\mathcal{R}_m \cap \mathcal{G}_m = \emptyset$. When $m = 1$, $\mathcal{R}_m = \emptyset$, and also, (5.6) implies that $\mathcal{B}_m = \mathcal{G}_m$ and $\mathcal{S} \cap \mathcal{I}_m = \emptyset$. Note that for each $m \leq n$ and for each $S \in \mathcal{S} \cap \mathcal{I}_m$, (5.6) and (5.7) imply that

$$\mathcal{B}_m \;=\; \cup_{S\in\mathcal{S}\cap\mathcal{I}_m}(S\mathcal{G}_m S^{-1} \cap \mathcal{G}_m) \cup \{M \in \mathcal{G}_m \,:\, \nexists S \in \mathcal{S} \cap \mathcal{I}_m \text{ s.t. } S^{-1}MS \in E\}$$

(so that $\mathcal{B}_m$ is uniquely determined by $(E, \mathcal{S})$). When $(E, \mathcal{S})$ is a well organized pair of size n we set $\mathcal{B} = \cup_{m\leq n}\mathcal{B}_m$ and refer to $\mathcal{B}$ as the *base of* $(E, \mathcal{S})$. Similarly, we set $\mathcal{G} = \cup_{m\leq n}\mathcal{G}_m$ and $\mathcal{R} = \cup_{m\leq n}\mathcal{R}_m$.

If $n \in \mathbb{N}$, we say that π is an *ordered partition of* n if there exists a $\sigma \in \mathbb{N}$ such that

$$\pi : \{1, \dots, \sigma\} \to \mathbb{N} \quad \text{and} \quad \sum_{i=1}^{\sigma} \pi(i) = n. \tag{5.8}$$

We let Π_n denote the set of ordered partitions of n. If $\pi \in \Pi_n$ and is as in (5.8), we set $|\pi| = \sigma$. Finally, we let $[n]$ denote the *trivial* partition, defined by

$$[n] : \{1\} \to \mathbb{N} \quad \text{and} \quad [n](1) = n.$$

If $\pi \in \Pi_n$, we let $\mathbb{M}^d_{n,\pi}$ denote the set of $M \in \mathbb{M}^d_n$ that have the form

$$M = \bigoplus_{i=1}^{|\pi|} M_i$$

where $M_i \in \mathbb{M}^d_{\pi(i)}$ for each $i = 1, \dots, |\pi|$.

Lemma 5.9. If E is a partial nc-set, $\mathcal{S} \subseteq \cup_m \mathcal{I}_m$, and $(E, \mathcal{S})$ is well organized of size n, then for each $m \leq n$, $M \in E \cap \mathbb{M}^d_m$ if and only if there exists a partition $\pi \in \Pi_m$ and matrices $M_1, \dots, M_{|\pi|}$ such that

$$M = \oplus_{i=1}^{|\pi|} M_i \text{ and } M_i \in \mathcal{G}_{\pi(i)} \text{ for } i = 1, \dots, |\pi|. \tag{5.10}$$

Furthermore, π and $M_1, \dots, M_{|\pi|}$ satisfying (5.10) are uniquely determined by M.

Proof. Let $m \leq n$ and fix $M \in E \cap \mathbb{M}^d_m$. As E is assumed to be complete, an inductive argument implies that there exist $\pi \in \Pi_m$ and $M_1, \ldots, M_{|\pi|} \in E$ such that

$$M = \oplus_{i=1}^{|\pi|} M_i, \qquad \forall_i \ M_i \in E \cap \mathbb{M}_{\pi(i)},$$

where M_i is not E-reducible for each $i = 1, \ldots, |\pi|$. In particular, (5.5) implies that $M_i \in \mathcal{G}_{\pi(i)}$ for each i. That the decomposition is unique, follows from the fact that each of the summands, M_i is generic, and hence, irreducible. □

If $(E, \mathcal{S})$ is a well-organized pair of size n, we define $\mathcal{V}(E, \mathcal{S})$ to be the vector space consisting of the $\mathcal{S}$-invariant partial nc-functions on E, and define Grade($\mathcal{B}$) to be the vector space of graded matrix valued functions on $\mathcal{B}$, i.e., the collection of functions $\omega : \mathcal{B} \to \mathbb{M}^1$ such that

$$\forall_{m \leq n} \ \forall_{M \in \mathcal{B} \cap \mathbb{M}^d_m} \ \omega(M) \in \mathbb{M}_m.$$

Proposition 5.11. The map $\rho : \mathcal{V}(E, \mathcal{S}) \to \text{Grade}(\mathcal{B})$ defined by $\rho(\phi) = \phi \mid \mathcal{B}$ is a vector space isomorphism.

Proof. (5.1) guarantees that ρ maps into Grade($\mathcal{B}$) and clearly, ρ is linear. To see that ρ is onto, fix $\omega \in \text{Grade}(\mathcal{B})$. Define u on $\mathcal{B}$ by setting

$$u(x) = \omega(x), \qquad x \in \mathcal{B}. \tag{5.12}$$

Then use (5.6) to extend u to $\cup_{m \leq n} \mathcal{G}_m$ by the formulas

$$u(x) = S^{-1} u(SxS^{-1}) S, \qquad m \leq n, x, SxS^{-1} \in \mathcal{G}_m \tag{5.13}$$

where S is the unique element in $\mathcal{S} \cap \mathcal{I}_m$ such that $x \in S^{-1} \mathcal{B}_m S$. Finally, we extend u to $\cup_{m \leq n} \mathcal{R}_m$ by setting

$$u(x) = \oplus_{i=1}^{|\pi|} u(M_i) \qquad m \leq n, \ x \in \mathcal{R}_m \tag{5.14}$$

where $x = \oplus_{i=1}^{|\pi|} M_i$ is the unique representation of x given by Lemma 5.9.

To see that u as just defined is a a partial nc-function first fix $M_1 \in E \cap \mathbb{M}^d_{m_1}$ and $M_2 \in E \cap \mathbb{M}^d_{m_2}$ where $m_1 + m_2 \leq n$. By Lemma 5.1, there exist partitions $\pi_1 \in \Pi_{m_1}$ and $\pi_2 \in \Pi_{m_2}$ such that

$$M_1 = \oplus_{i=1}^{|\pi_1|} M_i, \qquad M_i \in \mathcal{G}_{\pi_1(i)} \text{ for } i = 1, \ldots, |\pi_1|$$

and

$$M_2 = \oplus_{i=1}^{|\pi_2|} N_i, \qquad N_i \in \mathcal{G}_{\pi_2(i)} \text{ for } i = 1, \ldots, |\pi_2|.$$

If we define $\pi \in \Pi_{m_1 + m_2}$ by

$$\pi(l) = \begin{cases} \pi_1(l) & \text{if } 1 \leq l \leq |\pi_1|, \\ \pi_2(l - |\pi_1|) & \text{if } |\pi_1| + 1 \leq l \leq |\pi_1| + |\pi_2| \end{cases}$$

and let

$$x_l = \begin{cases} M_l & \text{if } 1 \leq l \leq |\pi_1|, \\ N_{l - |\pi_1|} & \text{if } |\pi_1| + 1 \leq l \leq |\pi_1| + |\pi_2|, \end{cases}$$

then $M_1 \oplus M_2 = \oplus_l x_l$ is the unique decomposition of $M_1 \oplus M_2$ given in Lemma 5.9. Hence, using (5.14),

$$\begin{aligned} u(M_1 \oplus M_2) &= u(\oplus_l x_l) \\ &= \oplus_l u(x_l) \\ &= \oplus_{l=1}^{|\pi_1|} x_l \oplus \oplus_{l=|\pi_2|+1}^{|\pi_1|+|\pi_2|} x_l \\ &= u(M_1) \oplus u(M_2). \end{aligned}$$

To see that u is $\mathcal{S}$-invariant, fix $M \in E \cap \mathbb{M}_m^d$ and $S \in \mathcal{S} \cap \mathcal{I}_m$ satisfying $S^{-1}MS \in E$. Then (5.7) guarantees that $M \in \mathcal{B}_m \cup \Sigma_m^d$. If $M \in \Sigma_m^d$, then so is $u(M)$ by (5.14), and both M and $u(M)$ are left invariant by conjugation with S. If $M \in \mathcal{B}_m$, then using (5.13),

$$\begin{aligned} u(S^{-1}MS) &= S^{-1}u(S(S^{-1}MS)S^{-1})S \\ &= S^{-1}u(M)S. \end{aligned}$$

Summarizing, we have shown that u, as defined above, is a partial nc-function that is $\mathcal{S}$-invariant. Hence, $u \in \mathcal{V}(E, \mathcal{S})$. That $\rho(u) = \omega$ follows from (5.12). This completes the proof that ρ is onto.

To see that ρ is 1-1, notice that if $v \in \mathcal{V}(E, \mathcal{S})$ and $\rho(v) = \omega$, then as $\rho(v) = \omega$, necessarily (5.12) holds with u replaced with v. As v is $\mathcal{S}$-invariant, (5.13) also holds with u replaced with v. Finally, as v is a partial nc-function, (5.14) as well holds with u replaced with v. These facts imply that $v = u$. □

For the remainder of the section $(E, \mathcal{S})$ is a well organized pair of size n and we set $\mathcal{V} = \mathcal{V}(E, \mathcal{S})$. We define a d-tuple of linear transformations,

$$X_{\mathcal{V}} = (X_{\mathcal{V}}^1, \ldots, X_{\mathcal{V}}^d), \tag{5.15}$$

on $\mathcal{V}(E, \mathcal{S})$ by setting

$$(X_{\mathcal{V}}^r u)(x) = x^r u(x), \qquad x \in E. \tag{5.16}$$

Likewise, we define a d-tuple of linear transformations $X_{\mathcal{B}} = (X_{\mathcal{B}}^1, \ldots, X_{\mathcal{B}}^d)$ on Grade$(\mathcal{B})$ by setting

$$(X_{\mathcal{B}}^r \omega)(x) = x^r \omega(x), \qquad x \in \mathcal{B}.$$

When $\mathcal{V}_1$ and $\mathcal{V}_2$ are vector spaces and $L : \mathcal{V}_1 \to \mathcal{V}_2$ is a vector space isomorphism, we shall write $\mathcal{V}_1 \overset{L}{\sim} \mathcal{V}_2$. If in addition, T_1 is a d-tuple of linear transformations of $\mathcal{V}_1$, T_2 is a d-tuple of linear transformations of $\mathcal{V}_2$, and $T_1 = L^{-1}T_2L$ we write $T_1 \overset{L}{\sim} T_2$. Observe that with these notations, that if ρ is the isomorphism of Proposition 5.11, then

$$X_{\mathcal{V}} \overset{\rho}{\sim} X_{\mathcal{B}}. \tag{5.17}$$

For a vector space V we let $V^{(m)} = \oplus_{i=1}^m V$ and if T is a linear transformation of V we set $T^{(m)} = \oplus_{i=1}^m T$. We define $\gamma : \mathbb{M}_m \to (\mathbb{C}^m)^{(m)}$ by $\gamma(M) = \oplus_{j=1}^m M_j$ where M_j is the j^{th} column of M. If we let M_x denote the operator on $\mathbb{M}_n$ defined by $M_x(M) = xM$, then $M_x \overset{\gamma}{\sim} x^{(m)}$. It follows that if we define

$$\beta : \text{Grade}(\mathcal{B}) \to \bigoplus_{m=1}^n \bigoplus_{B \in \mathcal{B}_m} (\mathbb{C}^m)^{(m)}$$

by the formula

$$\beta(\omega) = \bigoplus_{m=1}^{n} \bigoplus_{B \in \mathcal{B}_m} \gamma(u(B)),$$

then β is an isomorphism and

$$X_{\mathcal{B}} \overset{\beta}{\sim} \bigoplus_{m=1}^{n} \bigoplus_{B \in \mathcal{B}_m} B^{(m)}. \tag{5.18}$$

Now assume that $\mathcal{D}$ is an nc-domain, $E \subseteq \mathcal{D}$, and ϕ is an nc-function defined on $\mathcal{D}$. We may define a linear transformation M_ϕ on $\mathcal{V}$ by the formula,

$$(M_\phi u)(x) = \phi(x)u(x), \qquad x \in E. \tag{5.19}$$

Noting that

$$\mathcal{V} \overset{\beta\circ\rho}{\sim} \bigoplus_{m=1}^{n} \bigoplus_{B \in \mathcal{B}_m} (\mathbb{C}^m)^{(m)},$$

$$M_\phi \overset{\beta\circ\rho}{\sim} \bigoplus_{m=1}^{n} \bigoplus_{B \in \mathcal{B}_m} M_{\phi(B)}^{(m)},$$

and

$$\bigoplus_{m=1}^{n} \bigoplus_{B \in \mathcal{B}_m} M_{\phi(B)}^{(m)} = \phi\Big(\bigoplus_{m=1}^{n} \bigoplus_{B \in \mathcal{B}_m} M_B^{(m)}\Big),$$

we have that

$$M_\phi \overset{\beta\circ\rho}{\sim} = \phi\Big(\bigoplus_{m=1}^{n} \bigoplus_{B \in \mathcal{B}_m} M_B^{(m)}\Big).$$

But

$$X_{\mathcal{V}} \overset{\beta\circ\rho}{\sim} \bigoplus_{m=1}^{n} \bigoplus_{B \in \mathcal{B}_m} B^{(m)} \in \mathcal{D}.$$

Hence, $X_{\mathcal{V}} \in \mathcal{D}^{\approx}$ and $M_\phi = \phi^{\approx}(X_{\mathcal{V}})$.

We summarize what has just been proven in the following proposition.

Proposition 5.20. Let $\mathcal{D}$ be an nc domain and assume that $E \subseteq \mathcal{D}$ and ϕ is an nc-function on $\mathcal{D}$. Also assume that $\mathcal{S} \subset \cup_{m=1}^{n} \mathcal{I}_m$, that $(E, \mathcal{S})$ is a well organized pair, and let $\mathcal{V}$ denote the vector space of $\mathcal{S}$-invariant nc-functions on E. If the d-tuple of linear transformations on $\mathcal{V}$, $X_{\mathcal{V}}$, is defined by (5.15) and (5.16) and the linear transformation on $\mathcal{V}$, M_ϕ, is defined by (5.19), then

$$M_\phi = \phi^{\approx}(X_{\mathcal{V}}).$$

6 Well Organized Models and Realizations

For $\mathcal{D}$ an nc-domain, we let $H^\infty(\mathcal{D})$ denote the bounded nc-functions on $\mathcal{D}$. As the elements of $H^\infty(\mathcal{D})$ are locally bounded, it follows from Theorem 4.10 that $H^\infty(\mathcal{D}) \subseteq H(\mathcal{D})$.

Similarly, $H^\infty_{\mathcal{L}(\mathcal{H},\mathcal{K})}$ denotes the bounded $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued nc-functions on $\mathcal{D}$, so functions Ψ for which there is a constant C such that

$$\Psi(x)^*\Psi(x) \ \leq \ C \text{ id} \quad \forall x \in \mathcal{D}.$$

We fix for the remainder of this section a matrix δ of free polynomials. By adding rows or columns of zeroes, if necessary, we can assume that δ is actually a square J-by-J matrix. Define G_δ

$$G_\delta \ = \ \{M \in \mathbb{M}^{[d]} \mid \|\delta(M)\| < 1\},$$

and assume that G_δ is non-empty.

Let us note that it is possible for G_δ to be empty at lower levels, and non-empty at higher ones. For example, if δ is the single polynomial

$$\delta(x^1, x^2) \ = \ 1 - (x^1x^2 - x^2x^1)(x^1x^2 - x^2x^1),$$

then $G_\delta \cap \mathcal{M}_1^2$ is empty, but $G_\delta \cap \mathcal{M}_2^2$ is not. If this occurs, we just start our constructions at the first m for which $G_\delta \cap \mathcal{M}_m^d$ is non-empty.

6.1 $\mathrm{H}(\mathcal{V}), \mathrm{R}(\mathcal{V}), \mathrm{P}(\mathcal{V})$ and $\mathrm{C}(\mathcal{V})$

We fix for the remainder of the sub-section a well organized pair $(E, \mathcal{S})$ of size n, with $E \subset G_\delta$. We fix Hilbert spaces $\mathcal{H}, \mathcal{K}_1$ and $\mathcal{K}_2$, with $\mathcal{H}$ finite dimensional; we shall let $\mathcal{M}$ denote an arbitrary auxiliary Hilbert space. We also fix a pair of functions Ψ in $H^\infty_{\mathcal{L}(\mathcal{H},\mathcal{K}_1)}(G_\delta)$ and Φ in $H^\infty_{\mathcal{L}(\mathcal{H},\mathcal{K}_2)}(G_\delta)$ satisfying

$$\Psi(x)^*\Psi(x) - \Phi(x)^*\Phi(x) \ \geq \ 0 \qquad \forall\, x \in G_\delta. \tag{6.1}$$

We define $\Theta(y, x)$ by

$$\Theta(y, x) \ = \ \Psi(y)^*\Psi(x) - \Phi(y)^*\Phi(x). \tag{6.2}$$

We adopt the following notations of the previous section: $\mathcal{B}$, $\mathcal{R}$, and $\mathcal{G}$ for the basic, reducible, and generic elements of $(E, \mathcal{S})$; and $\mathcal{V}$ for the vector space of $\mathcal{S}$-invariant partial nc-functions on E. We let $\mathcal{V}_{\mathcal{L}(\mathcal{H})}$ (resp. $\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathcal{M})}$) denote the vector space of $\mathcal{L}(\mathcal{H})$-valued (resp. $\mathcal{L}(\mathcal{H},\mathcal{M})$-valued) partial nc-functions on E. When $\psi \in \mathcal{V}_{\mathcal{L}(\mathcal{M})}$, M_ψ denotes the operator defined on $\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathcal{M})}$ by

$$(M_\psi u)(x) = \psi(x)u(x), \qquad x \in E. \tag{6.3}$$

We let $\mathrm{H}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ denote the set of $\mathcal{L}(\mathcal{H})$-valued graded functions h on

$$E^{[2]} = \bigcup_{m=1}^{n} E_m \times E_m \tag{6.4}$$

that have the form

$$h(y,x)=\sum_{i=1}^{\sigma} g_i(y)^* f_i(x), \qquad 1\le m\le n,\ x,y\in E\cap \mathbb{M}_m^d$$

where $\sigma\in\mathbb{N}$ and $f_i, g_i\in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})}$ for $i=1,\ldots,\sigma$. $\mathrm{H}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ is a finite dimensional vector space and is a Banach space as well, when equipped with the norm

$$\|h\|=\sup_{(y,x)\in E^{[2]}}\|h(y,x)\|.$$

We set

$$\mathrm{R}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})=\{h\in \mathrm{H}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})\,|\, h(x,y)=h(y,x)^*\} \tag{6.5}$$

and define $\mathrm{P}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ to consist of the elements $h\in \mathrm{R}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ that have the special form

$$h(y,x)=\sum_{i=1}^{\sigma} f_i(y)^* f_i(x), \qquad 1\le m\le n,\ x,y\in E\cap \mathbb{M}_m^d \tag{6.6}$$

where $\sigma\in\mathbb{N}$ and $f_i\in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})}$ for $i=1,\ldots,\sigma$. Evidently, $\mathrm{R}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ is a real subspace of $\mathrm{H}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ and $\mathrm{P}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ is a cone[1] in $\mathrm{R}(\mathcal{V})$.

Lemma 6.7. Let $\mathcal{M}$ be a finite dimensional Hilbert space, and let $F(y,x)$ be an arbitrary graded $\mathcal{L}(\mathcal{M})$-valued function on $E^{[2]}$. Let $N=\dim(\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathcal{M})})$. Then if G can be represented in the form

$$G(y,x)=\sum_{i=1}^{\sigma} g_i(y)^* F(y,x) g_i(x), \qquad 1\le m\le n,\ x,y\in E\cap \mathbb{M}_m^d,$$

where $\sigma\in\mathbb{N}$ and $g_i\in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathcal{M})}$ for $i=1,\ldots,\sigma$, then G can be represented in the form

$$G(y,x)=\sum_{i=1}^{N} f_i(y)^* F(y,x) f_i(x), \qquad 1\le m\le n,\ x,y\in E\cap \mathbb{M}_m^d, \tag{6.8}$$

where $f_i\in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathcal{M})}$ for $i=1,\ldots,N$.

Proof. Let $\langle e_l(x)\rangle_{l=1}^N$ be a basis of $\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathcal{M})}$. For each $i=1,\ldots,\sigma$, let

$$g_i(x)=\sum_{l=1}^{N} c_{il}\ e_l(x).$$

Form the $\sigma\times N$ matrix $C=[c_{il}]$. As C^*C is an $N\times N$ positive semidefinite matrix, there exists an $N\times N$ matrix $A=[a_{kl}]$ such that $C^*C=A^*A$. This leads to the formula,

$$\sum_{i=1}^{\sigma}\overline{c}_{il_1}c_{il_2}=\sum_{k=1}^{N}\overline{a}_{kl_1}a_{kl_2},$$

[1]By a cone, we mean a convex set closed under multiplication by non-negative real numbers. This is sometimes called a *wedge*.

valid for all $l_1, l_2 = 1, \ldots, N$. If $1 \le m \le n$ and $x, y \in E \cap \mathbb{M}^d_m$, then

$$\begin{aligned}
G(y,x) &= \sum_{i=1}^{\sigma} g_i(y)^* F(y,x) g_i(x) \\
&= \sum_{i=1}^{\sigma} \big(\sum_{l=1}^{N} c_{il}\ e_l(y)\big)^* F(y,x) \big(\sum_{l=1}^{N} c_{il}\ e_l(x)\big) \\
&= \sum_{l_1,l_2=1}^{N} (\sum_{i=1}^{\sigma} \overline{c}_{il_1} c_{il_2}) e_{l_1}(y)^* F(y,x) e_{l_2}(x) \\
&= \sum_{l_1,l_2=1}^{N} (\sum_{k=1}^{N} \overline{a}_{kl_1} a_{kl_2}) e_{l_1}(y)^* F(y,x) e_{l_2}(x) \\
&= \sum_{k=1}^{N} \big(\sum_{l=1}^{N} a_{kl}\ e_l(y)\big)^* F(y,x) \big(\sum_{l=1}^{N} a_{kl}\ e_l(x)\big).
\end{aligned}$$

This proves that (6.8) holds with $f_i = \sum_{l=1}^{N} a_{il}\ e_l$. □

Lemma 6.9. If $h \in \mathrm{P}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$, $x, y \in E \cap \mathbb{M}^d_m$, and $c, d \in \mathbb{C}^m \otimes \mathcal{H}$, then

$$|\langle h(y,x)c, d\rangle|^2 \le \langle h(x,x)c, c\rangle \langle h(y,y)d, d\rangle.$$

Proof. Assume that (6.6) holds.

$$\begin{aligned}
|\langle h(y,x)c, d\rangle|^2 &= \Big|\langle \sum_{i=1}^{\sigma} f_i(y)^* f_i(x) c, d\rangle\Big|^2 \\
&= \Big|\sum_{i=1}^{\sigma} \langle f_i(x)c, f_i(y)d\rangle\Big|^2 \\
&\le \Big(\sum_{i=1}^{\sigma} \ \|f_i(x)c\|\ \|f_i(y)d\| \ \Big)^2 \\
&\le \big(\sum_{i=1}^{\sigma} \ \|f_i(x)c\|^2\big)\big(\sum_{i=1}^{\sigma} \ \|f_i(y)d\|^2\big) \\
&= \big(\langle \sum_{i=1}^{\sigma} f_i(x)^* f_i(x)c, c\rangle\big)\big(\langle \sum_{i=1}^{\sigma} f_i(y)^* f_i(y)d, d\rangle\big) \\
&= \langle h(x,x)c, c\rangle \langle h(y,y)d, d\rangle.
\end{aligned}$$

□

If $u \in \mathrm{nc}_{\mathcal{L}(\mathcal{H}, \mathcal{M} \otimes \mathbb{C}^J)}(G_\delta)$, then we may define $\delta u \in \mathrm{nc}_{\mathcal{L}(\mathcal{H}, \mathcal{M} \otimes \mathbb{C}^J)}(G_\delta)$ by the formula,

$$(\delta u)(x) \ = \ (\delta(x) \otimes \mathrm{id}_{\mathcal{M}}) u(x) \qquad x \in G_\delta. \tag{6.10}$$

Definition 6.11. We let $\mathrm{C}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ and $\mathrm{C}^{\tau}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ be the cones generated in $\mathrm{R}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ by the elements in $\mathrm{R}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ of the form

$$u(y)^*[\mathrm{id} - \delta(y)^*\delta(x)]u(x), \qquad \text{and} \qquad u(y)^*[\tau^2\,\mathrm{id} - \delta(y)^*\delta(x)]u(x),$$

respectively, where $u \in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C}^J)}$, and τ is such that

$$\rho \;:=\; \max\{\|\delta(x)\| \,:\, x \in E\} \;<\; \tau \;<\; 1. \tag{6.12}$$

Proposition 6.13. $\mathrm{C}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ and $\mathrm{C}^{\tau}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ are closed cones.

Proof. By Lemma 6.7, any element of $\mathrm{C}^{\tau}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ can be represented as a sum

$$\sum_{i=1}^{N} u_i(y)^*[\tau^2\,\mathrm{id} - \delta(y)^*\delta(x)]u_i(x). \tag{6.14}$$

Suppose a sequence of sums of the form (6.14) converges to some element $h(y,x)$ in $\mathrm{R}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$. Since $\rho < \tau$, we know that each of the individual functions u_i must eventually satisfy

$$\|u_i(x)\|^2 \;\leq\; 2\,\frac{1}{\tau^2 - \rho^2} h(x,x).$$

So by compactness, a subsequence of the sequence will converge to another sum of the form (6.14). Letting $\tau = 1$ gives the proof for $\mathrm{C}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$. □

Proposition 6.15. $\mathrm{P}_{\mathcal{L}(\mathcal{H})}(\mathcal{V}) \subseteq \mathrm{C}_{\mathcal{L}(\mathcal{H})}(\mathcal{V}) \subseteq \mathrm{C}^{\tau}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$

Proof. To prove $\mathrm{P}_{\mathcal{L}(\mathcal{H})}(\mathcal{V}) \subseteq \mathrm{C}^{\tau}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$, we must show that for any $f \in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})}$, the function $f(y)^*f(x)$ is in $\mathrm{C}^{\tau}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$. Let $g(x)$ be $\frac{1}{\tau}f(x)$. Let $h_\sigma \in \mathrm{C}^{\tau}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ be

$$\begin{aligned} h_\sigma(y,x) \quad &= \quad \sum_{j=0}^{\sigma} g(y)^*(\delta(y)^*/\tau)^j[\tau^2\,\mathrm{id}_{\mathbb{C}^J} - \delta(y)^*\delta(x)](\delta(x)/\tau)^j g(x) \\ &= \quad f(y)^*f(x) - g(y)^*(\delta(y)^*/\tau)^{\sigma+1}(\delta(x)/\tau)^{\sigma+1}g(x). \end{aligned}$$

As δ/τ is a strict contraction on E, $h_\sigma(y,x)$ converges to $f(y)^*f(x)$. By Proposition 6.13, we are done. Letting $\tau = 1$, we get $\mathrm{P}_{\mathcal{L}(\mathcal{H})}(\mathcal{V}) \subseteq \mathrm{C}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$.

To show $\mathrm{C}_{\mathcal{L}(\mathcal{H})}(\mathcal{V}) \subseteq \mathrm{C}^{\tau}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$, observe that

$$u(y)^*[\mathrm{id} - \delta(y)^*\delta(x)]u(x) \;=\; u(y)^*(\tau^2\,\mathrm{id} - \delta(y)^*\delta(x))u(x) + (1-\tau^2)u(y)^*u(x). \tag{6.16}$$

The first term on the right in (6.16) is in $\mathrm{C}^{\tau}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ by definition, and the second since $\mathrm{P}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ is. □

Lemma 6.17. For each τ in $(\rho, 1)$, the function $\Theta(y,x)$ is in $\mathrm{C}^{\tau}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$.

Proof. By Proposition 6.13, $\mathrm{C}^\tau_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ is a closed cone in $\mathrm{R}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$. Therefore, by the Hahn-Banach Theorem the lemma will follow if we can show that

$$L(\Theta(y,x)) \geq 0 \tag{6.18}$$

whenever

$$L \in \mathrm{R}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})^* \text{ and } L(h) \geq 0 \text{ for all } h \in \mathrm{C}^\tau_{\mathcal{L}(\mathcal{H})}(\mathcal{V}). \tag{6.19}$$

Accordingly assume that (6.19) holds. Define $L^\sharp \in \mathrm{H}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})^*$ by the formula

$$L^\sharp(h(y,x)) = L(\frac{h(y,x)+h(x,y)^*}{2}) + iL(\frac{h(y,x)-h(x,y)^*}{2i}),$$

and then define a sesquilinear form on $\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})}$ by the formula,

$$\langle f,g\rangle_L = L^\sharp(g(y)^*f(x)), \qquad f,g \in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})}. \tag{6.20}$$

Observe that Proposition 6.15 implies that $f(y)^*f(x) \in \mathrm{C}^\tau_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ whenever $f \in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})}$. Hence, (6.19) implies that

$$\langle f,f\rangle_L \geq 0$$

for all $f \in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})}$, i.e., $\langle\cdot,\cdot\rangle_L$ is a pre-inner product on $\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})}$. It follows that $\langle\cdot,\cdot\rangle_L$ induces an inner product on the quotient, $\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})}/\mathcal{N}_L$, where

$$\mathcal{N}_L = \{f \in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})} \mid \langle f,f\rangle_L = 0\}. \tag{6.21}$$

We let H^2_L denote the Hilbert space $\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})}/\mathcal{N}_L$ equipped with this induced inner product.

Now observe that for each $r = 1,\ldots,d$, $x^r \in \mathcal{V}$, so $X^r_{\mathcal{V}}$ is a well defined operator on $\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})}$, where $X_{\mathcal{V}}$ is the linear transformation defined in (5.15) and (5.16). (6.10) implies that δ is a well-defined operator from $\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})} \otimes \mathbb{C}^J$ to $\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})} \otimes \mathbb{C}^J$.

If $f \in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C}^J)}$, it follows using (6.20) and the fact that $f(y)^*(\tau^2\mathrm{id} - \delta(y)^*\delta(x))f(x) \in \mathrm{C}^\tau_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ that

$$\begin{aligned}
\|\tau f + \mathcal{N}_L\|^2_{H^{2(J)}_L} - \|\delta f + \mathcal{N}_L\|^2_{H^{2(J)}_L} &= \langle \tau f,\tau f\rangle_{H^{2(J)}_L} - \langle \delta f,\delta f\rangle_{H^{2(J)}_L} \\
&= \tau^2 L^\sharp(f(y)^*f(x)) - L^\sharp((\delta(y)f(y))^*\delta(x)f(x)) \\
&= L^\sharp(f(y)^*(\tau^2\mathrm{id} - \delta(y)^*\delta(x))f(x)) \\
&\geq 0.
\end{aligned}$$

Hence, the formula,

$$M_\delta(f+\mathcal{N}_L) = \delta f + \mathcal{N}_L, \qquad f \in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C}^J)},$$

defines a strict contraction from $\mathrm{H}^{2(J)}_L$ to $\mathrm{H}^{2(J)}_L$. Let $M_x = (M_{x^1},\ldots,M_{x^d})$ act on H^2_L, and let $\pi : \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})} \to \mathrm{H}^2_L$ denote the canonical quotient map, $\pi(f) = f + \mathcal{N}_L$. Then

$$M_x\,\pi \;=\; \pi\,X_{\mathcal{V}}.$$

Hence by Lemma 3.16, for every ϕ that is nc on an nc-domain containing E,

$$\phi^{\approx}(M_x)\pi = \pi\phi^{\approx}(X_{\mathcal{V}}).$$

Since H^2_L is finite dimensional, we have that M_x is unitarily equivalent to some point N in $\mathbb{M}^{[d]}$. Since δ is nc, we have that $\delta^{\approx}(M_x) = M_\delta$ is unitarily equivalent to $\delta(N)$. As M_δ is a strict contraction, it follows that $N \in G_\delta$.

It follows that $\Psi^{\approx}(M_x)$ is unitarily equivalent to $\Psi(N)$, and $\Phi^{\approx}(M_x)$ is unitarily equivalent to $\Phi(N)$, so by (6.1), multiplication by $\Phi^{\approx}(M_x)$ applied to any vector yields something of smaller norm than multiplication by $\Psi^{\approx}(M_x)$. Both of these matrices are in $\mathcal{L}(H^2_L \otimes \mathcal{H}, H^2_L \otimes \mathcal{K})$.

Let $\mu = \dim(\mathcal{H})$, and let $f = (f_1, \ldots, f_\mu)^t \in H^2_L \otimes \mathcal{H}$ be the vector where f_j is the representative in $\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathbb{C})}/\mathcal{N}_L$ of the row with the constant function 1 in the j^{th} slot and zero elsewhere. We get

$$
\begin{aligned}
&\langle \Phi^{\approx}(M_x) f, \Phi^{\approx}(M_x) f \rangle_{H^2_L \otimes \mathcal{K}_2} \\
&= \langle \begin{pmatrix} (\Phi_{11}(x) & \ldots & \Phi_{1\mu}(x)) \\ (\Phi_{21}(x) & \ldots & \Phi_{2\mu}(x)) \\ & \vdots & \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} , \begin{pmatrix} (\Phi_{11}(x) & \ldots & \Phi_{1\mu}(x)) \\ (\Phi_{21}(x) & \ldots & \Phi_{2\mu}(x)) \\ & \vdots & \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} \rangle_{H^2_L \otimes \mathcal{K}_2} \\
&= \langle \begin{pmatrix} (\Phi_{11}(x), & \ldots, & \Phi_{1\mu}(x)) \\ (\Phi_{21}(x), & \ldots, & \Phi_{2\mu}(x)) \\ & \vdots & \end{pmatrix} , \begin{pmatrix} (\Phi_{11}(x), & \ldots, & \Phi_{1\mu}(x)) \\ (\Phi_{21}(x), & \ldots, & \Phi_{2\mu}(x)) \\ & \vdots & \end{pmatrix} \rangle_{H^2_L \otimes \mathcal{K}_2} \\
&= \sum_k \langle (\Phi_{k1}(x), \ldots, \Phi_{k\mu}(x)) \ , \ (\Phi_{k1}(x), \ldots, \Phi_{k\mu}(x)) \rangle_{\mathrm{H}^2_L} \\
&= \sum_k L^\sharp (\sum_{j=1}^{\mu} \Phi_{kj}(y)^* \Phi_{kj}(x)) \\
&= L^\sharp(\Phi(y)^* \Phi(x))
\end{aligned}
$$

is smaller than the same expression with Ψ in lieu of Φ. Therefore

$$L(\Theta(y,x)) \geq 0,$$

as desired. ☐

Proposition 6.22.

$$\Theta(y,x) \ = \ \Psi(y)^*\Psi(x) - \Phi(y)^*\Phi(x) \in \mathrm{C}_{\mathcal{L}(\mathcal{H})}(\mathcal{V}).$$

Proof: By Lemma 6.17, we have $\Theta \in \mathrm{C}^\tau_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ for all τ between ρ and 1. By Lemma 6.7, for each τ we have

$$\Theta(y,x) \ = \ \sum_{i=1}^{N} u_i^{(\tau)}(y)^* [\tau^2 \,\mathrm{id} - \delta(y)^*\delta(x)] u_i^{(\tau)}(x). \tag{6.23}$$

As in the proof of Proposition 6.13, we can use compactness to extract a sequence τ_i so that the right-hand side of (6.23) converges to an element in $\mathrm{C}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$. ☐

6.2 Partial nc-Models

Definition 6.24. Let $(E, \mathcal{S})$ be a well organized pair and let

$$h(y,x) = \sum_{i=1}^{\sigma} f_i(y)^* g_i(x), \tag{6.25}$$

where each f_i and g_i are graded $\mathcal{L}(\mathcal{H}, \mathcal{K}_i)$-valued functions on E. Assume $E \subset G_\delta$. A *δ-model for h* is a graded $\mathcal{L}(\mathcal{H}, \mathcal{M} \otimes \mathbb{C}^J)$-valued function u on E such that

$$h(y,x) \;=\; u(y)^*[1 - \delta(y)^*\delta(x)]u(x), \qquad x, y \in E \cap G_\delta \tag{6.26}$$

for all $x, y \in E$. If in addition, u is a partial nc-function, we say the model is *partial nc* and if the model is $\mathcal{S}$-invariant, we say the model is *$\mathcal{S}$-invariant.* If $\mathcal{M}$ is finite dimensional, we say the model is finite dimensional.

If v is a graded $\mathcal{L}(\mathcal{H}, \mathcal{K})$-valued function on E, we say v has a δ-model if $[\mathrm{id} - v(y)^*v(x)]$ does.

Before continuing we make a few clarifying remarks about the meaning of the formula in (6.26). To say that u is an $\mathcal{S}$-invariant partial $\mathcal{L}(\mathcal{H}, \mathcal{M} \otimes \mathbb{C}^J)$-valued nc-function means that

$$\forall_{m \le n}\ \forall_{x \in E \cap \mathbb{M}_m^d}\ u(x) \in \mathcal{L}(\mathbb{C}^m \otimes \mathcal{H}, \mathbb{C}^m \otimes \mathcal{M} \otimes \mathbb{C}^J), \tag{6.27}$$

$$\forall_{x,y \in E}\ x \oplus y \in E \implies u(x \oplus y) = u(x) \oplus u(y), \qquad \text{and} \tag{6.28}$$

$$\forall_{m \le n}\ \forall_{x \in E \cap \mathbb{M}_m^d}\ \forall_{S \in \mathcal{S} \cap \mathcal{I}_m}\ S^{-1}xS \in E \implies u(S^{-1}xS) = (S^{-1} \otimes \mathrm{id}_{\mathcal{M} \otimes \mathbb{C}^J})u(x)(S \otimes \mathrm{id}_{\mathcal{H}}). \tag{6.29}$$

We denote the collection of functions u satisfying these axioms by $\mathcal{V}_{\mathcal{L}(\mathcal{H}, \mathcal{M} \otimes \mathbb{C}^J)}$. In the special case when $\mathcal{M} = \ell^2$ or ${\ell^2}_N$, we say the model is *special.* Clearly, as E is finite, if a graded function v has a partial nc-model, then v has a special partial nc-model.

Proposition 6.30. Let $(E, \mathcal{S})$ be a well organized pair, with $E \subseteq G_\delta$. If $\Theta(y, x)$ is as in (6.2) and is non-negative on G_δ (*i.e.* it satisfies (6.1)), then $\Theta|_{E^{[2]}}$ has an $\mathcal{S}$-invariant finite dimensional partial nc-model.

Proof. Let $(E, \mathcal{S})$ be a well organized pair with $E \subseteq G_\delta$. By Proposition 6.22, $\Theta(y, x) \in \mathrm{C}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$. Hence, by the definition of $\mathrm{C}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ and Lemma 6.7,

$$\Theta(y,x) \;=\; u(y)^*[\mathrm{id} - \delta(y)^*\delta(x)]u(x),$$

where $u \in \mathcal{V}_{\mathcal{L}(\mathcal{H}, {\ell^2}_N^{(J)})}$. $\square$

6.3 Partial nc-Realizations

Definition 6.31. Let $(E, \mathcal{S})$ be a well organized pair of size n and let Ω be a graded $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$-valued function defined on E. A *δ-realization for* Ω is a pair $(\delta, \mathcal{J})$, where $\mathcal{J}$ is a finite sequence of operators

$$\mathcal{J} = \langle J_m \rangle_{m=1}^n = \langle \begin{bmatrix} A_m & B_m \\ C_m & D_m \end{bmatrix} \rangle_{m=1}^n$$

with J_m acting isometrically from $\mathbb{C}^m \otimes \mathcal{K}_1 \oplus (\mathbb{C}^m \otimes \ell^{2(J)})$ to $\mathbb{C}^m \otimes \mathcal{K}_2 \oplus (\mathbb{C}^m \otimes \ell^{2(I)})$ for each $m \leq n$, and such that

$$\Omega(x) = A_m + B_m \delta(x)(\mathrm{id} - D_m \delta(x))^{-1} C_m \tag{6.32}$$

for each $m \leq n$ and $x \in E \cap \mathbb{M}_m^d$. If in addition,

$$v(x) \ := \ (\mathrm{id} - D_m \delta(x))^{-1} C_m \tag{6.33}$$

is an $\ell^{2(I)}$-valued partial nc-function on E (resp. $(\mathcal{S} \cap \mathcal{I}_m)$-invariant for each $m \leq n$), we say that $(\delta, \mathcal{J})$ is *partial nc* (resp. $\mathcal{S}$-*invariant*).

Theorem 6.34. Let $(E, \mathcal{S})$ be a well organized pair, and let $\Psi \in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathcal{K}_1)}$ and Φ be in $\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathcal{K}_2)}$. If there exists a function Ω in the closed unit ball of $\mathcal{V}_{\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)}$ that has an $\mathcal{S}$-invariant partial nc δ-realization and satisfies $\Omega\Psi = \Phi$, then $[\Psi^*(y)\Psi(x) - \Phi(y)^*\Phi(x)]$ has an $\mathcal{S}$-invariant partial nc-model. The converse holds if Ψ is bounded below on E. If Ψ is not bounded below on E, then there exists a function Ω in the closed unit ball of $\mathcal{V}_{\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)}$ that has a δ-realization and satisfies $\Omega\Psi = \Phi$,

Proof. Suppose $[\Psi^*(y)\Psi(x) - \Phi(y)^*\Phi(x)]$ has an $\mathcal{S}$-invariant partial nc-model, so there exists $u \in \mathcal{V}_{\mathcal{L}(\mathcal{H},\ell^2{}_N^{(J)})}$ satisfying

$$\Psi^*(y)\Psi(x) - \Phi(y)^*\Phi(x) \ = \ u(y)^*[\mathrm{id} - \delta(y)^*\delta(x)]u(x). \tag{6.35}$$

We can rewrite (6.35) to say that for each $1 \leq m \leq n$, the map

$$J_m \ = \ \begin{bmatrix} A_m & B_m \\ C_m & D_m \end{bmatrix} \ : \ \begin{pmatrix} \Psi(x) \\ \delta(x)u(x) \end{pmatrix} \ \mapsto \ \begin{pmatrix} \Phi(x) \\ u(x) \end{pmatrix} \tag{6.36}$$

is an isometry from the span in $(\mathcal{K}_1 \oplus \ell^2{}_N^{(J)}) \otimes \mathbb{C}^m$ of

$$\left\{ \mathrm{ran} \begin{pmatrix} \Psi(x) \\ \delta(x)u(x) \end{pmatrix} \ : \ x \in E \cap \mathbb{M}_m^d \right\}$$

to the span in $(\mathcal{K}_2 \oplus \ell^2{}_N^{(J)}) \otimes \mathbb{C}^m$ of

$$\left\{ \mathrm{ran} \begin{pmatrix} \Phi(x) \\ u(x) \end{pmatrix} \ : \ x \in E \cap \mathbb{M}_m^d \right\}.$$

Replacing $\ell^2{}_N$ by ℓ^2 if necessary, we can extend J_m to an isometry from all of $(\mathcal{K}_1 \oplus \ell^{2(J)}) \otimes \mathbb{C}^m$ to $(\mathcal{K}_2 \oplus \ell^{2(J)}) \otimes \mathbb{C}^m$.

Define Ω and v on $E \cap \mathbb{M}_m^d$ by (6.32) and (6.33) respectively. Then (6.36) and the fact that J_m is an isometry yield

$$\Omega(x)\Psi(x) \ = \ \Phi(x) \quad \forall x \in E \tag{6.37}$$

$$u(x) \ = \ v(x)\Psi(x) \quad \forall x \in E \tag{6.38}$$

$$\mathrm{id} - \Omega(y)^*\Omega(x) \ = \ v(y)^*[\mathrm{id} - \delta(y)^*\delta(x)]v(x) \quad \forall (x,y) \in E^{[2]}. \tag{6.39}$$

Since u is partial nc on E, and Ψ is nc, it would follow from (6.38) that v is also partial nc on E if $\Psi(x)$ were bounded below.

Conversely, suppose Ω existed as in the statement of the theorem. Then (6.37) and (6.39) would hold, and defining $u(x) := v(x)\Psi(x)$ gives (6.35). $\square$

Remark 6.40. If Ψ is not bounded below, but each C_m and D_m in (6.36) satisfy $C_m = \mathrm{id}_{\mathbb{C}^m} \otimes C_1$ and $D_m = \mathrm{id}_{\mathbb{C}^m} \otimes D_1$, then the converse still holds. Indeed, follow the above proof through (6.39). Then define a new v by leaving $v(x)$ unchanged on $\mathcal{B}_m$, and extending it by Proposition 5.11 to be $\mathcal{S}$-invariant partial nc on E. Define $\Omega(x) := A_m + B_m\delta(x)v(x)$. Since Ψ is nc, (6.38) will still hold, and so will (6.37) and (6.39). To check (6.33), we wish to know whether

$$\mathrm{id}_{\mathbb{C}^m} \otimes C_1 \ = \ (\mathrm{id} - \mathrm{id}_{\mathbb{C}^m} \otimes D_1\delta(x))^{-1}v(x).$$

Both sides are equal on $\mathcal{B}_m$, and both sides are $\mathcal{S}$-invariant partial nc on E, therefore they agree on all of E. In Theorem 8.1, we show that C_m and D_m can be chosen with this special form.

7 Full Models and Realizations

Fix again a matrix δ of nc-polynomials, and assume that δ is J-by-J and that G_δ is non-empty. Let $\mathcal{H}, \mathcal{K}_1, \mathcal{K}_2$, and $\mathcal{M}$ be Hilbert spaces, with $\mathcal{H}, \mathcal{K}_1$ and $\mathcal{K}_2$ finite dimensional. For the rest of this section, define

$$\Theta(y,x) \ = \ \Psi(y)^*\Psi(x) - \Phi(y)^*\Phi(x), \tag{7.1}$$

where

$$\Psi \in H^\infty_{\mathcal{L}(\mathcal{H},\mathcal{K}_1)}(G_\delta), \quad \Phi \ \in H^\infty_{\mathcal{L}(\mathcal{H},\mathcal{K}_2)}(G_\delta), \tag{7.2}$$

and

$$\Theta(x,x) \ \geq \ 0, \qquad \forall x \in G_\delta. \tag{7.3}$$

We want to conclude that there exists a function Ω in the ball of $H^\infty_{\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)}$ such that

$$\Omega(x)\Psi(x) \ = \ \Phi(x), \qquad \forall x \in G_\delta. \tag{7.4}$$

Definition 7.5. Let $h(y,x)$ be an $\mathcal{L}(\mathcal{H})$-valued graded function on $G_\delta^{[2]}$. A *δ-model for h* is a graded $\mathcal{L}(\mathcal{H}, \mathcal{M} \otimes \mathbb{C}^J)$-valued function u on G_δ, such that

$$h(y,x) \ = \ u(y)^*[\mathrm{id} - \delta(y)^*\delta(x)]u(x) \tag{7.6}$$

for all $x, y \in G_\delta$. We say the model is *nc (resp. locally bounded, holomorphic)* if u is nc (resp. locally bounded, holomorphic).

Definition 7.7. Let Ω be a graded $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ valued function on G_δ. A *δ-realization for Ω* is a pair $(\delta, \mathcal{J})$, where $\mathcal{J}$ is a sequence of operators

$$\mathcal{J} = \langle J_m \rangle_{m=1}^\infty = \langle \begin{bmatrix} A_m & B_m \\ C_m & D_m \end{bmatrix} \rangle_{m=1}^\infty$$

such that J_m acts isometrically from $\mathbb{C}^m \otimes \mathcal{K}_1 \oplus (\mathbb{C}^m \otimes \ell^{2(J)})$ to $\mathbb{C}^m \otimes \mathcal{K}_2 \oplus (\mathbb{C}^m \otimes \ell^{2(J)})$ for each m, and such that

$$\Omega_m(x) \ := \ A_m + B_m\delta(x)(\mathrm{id} - D_m\delta(x))^{-1}C_m.$$

If, in addition,

$$v(x) = (\mathrm{id} - D_m\delta(x))^{-1}C_m$$

is an nc-function on G_δ, we say that $(\delta, \mathcal{J})$ is an *nc-realization.*

If, for each m, we have

$$\begin{bmatrix} A_m & B_m \\ C_m & D_m \end{bmatrix} = \begin{bmatrix} \mathrm{id}_{\mathbb{C}^m} \otimes A_1 & \mathrm{id}_{\mathbb{C}^m} \otimes B_1 \\ \mathrm{id}_{\mathbb{C}^m} \otimes C_1 & \mathrm{id}_{\mathbb{C}^m} \otimes D_1 \end{bmatrix}, \tag{7.8}$$

we say that $(\delta, \mathcal{J})$ is a *free realization.*

Note that a free realization is automatically an nc-realization. The following proposition follows by the same lurking isometry argument that proved Theorem 6.34.

Proposition 7.9. Let Ω be a graded $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ valued function on G_δ. Then Ω has a δ-realization if and only if $[\mathrm{id} - \Omega(y)^*\Omega(x)]$ has a δ-model; and Ω has a δ nc-realization if and only if $[\mathrm{id} - \Omega(y)^*\Omega(x)]$ has a δ nc-model. If Ω has a δ-realization, then automatically, the model is both locally bounded and holomorphic.

Theorem 7.10. Let Θ be as in (7.1) and satisfy (7.3). Then Θ has a δ nc-model.

The remainder of the section will be devoted to the proof of Theorem 7.10. This theorem is strengthened in Theorem 8.1, where it is shown that one can choose Ω satsifying (7.4) so that it has a free realization.

When $d = 1$, Theorem 7.10 is well-known; see *e.g.* [3] for a treatment in the case of the unit disk. In one variable generalizing to G_δ presents few difficulties.

When $d > 1$, in the commutative case, the theorem was first proved by Ambrozie and Timotin in the scalar case in [7]; it was extended to the operator valued case by Ball and Bolotnikov in [9]. See also [4] for an alternative treatment. In the non-commutative case, Ball, Groenewald and Malakorn [10] proved this theorem for G_δ's that come from certain bipartite graphs; this includes the most important examples, the non-commutative ball and the non-commutative polydisk.

We shall assume for the rest of this section that $d \geq 2$, as the $d = 1$ case can be immediately deduced from the $d = 2$ case.

7.1 Step 1

In this subsection, for each fixed $n \geq 2$, we shall construct a sequence $(E_\tau, \mathcal{S}_\tau)$ of well-organized partial nc-sets of size n. This will give rise in the next subsection, after taking a cluster point of the sequence, in an holomorphic realization of Ω on G_δ that is 'nc up to order n'.

Fix n. Many of the objects in this step of the proof (and steps 2 and 3 as well) will depend on n, though our notation will not reflect this fact. For $M \in \mathbb{M}_m^d$ we define

$$\mathrm{Comm}(M) = \{A \in \mathbb{M}_m \mid A\ M^r = M^r A,\ \text{for } r = 1, \ldots, d\}.$$

Lemma 7.11. For each $m = 1, \dots, n$, there exists a sequence, $\langle B_{m,t}\rangle_{t=1}^{\infty}$ in $G_\delta \cap \mathbb{M}_m^d$ such that

$$\forall_{t_1,t_2}\ t_1 \neq t_2 \implies B_{m,t_1} \neq B_{m,t_2}, \tag{7.12}$$

$$\forall_t\ B_{m,t} \text{ is generic}, \tag{7.13}$$

$$\forall_t\ \operatorname{Comm}(B_{m,t}) = \mathbb{C}\ \mathrm{id}_{\mathbb{C}^m}, \qquad \text{and} \tag{7.14}$$

$$\{B_{m,t}\,|\,t \in \mathbb{N}\} \text{ is dense in } G_\delta \cap \mathbb{M}_m^d. \tag{7.15}$$

Proof. This is easy to verify, because (7.13) and (7.14) only fail on sets of lower dimension than $\mathbb{M}_m^d$. □

Fix sequences $\langle B_{m,t}\rangle_{t=1}^{\infty}$ satisfying the properties of Lemma 7.11. For each $m = 1, \dots, n$ and $\tau \in \mathbb{N}$, we define

$$\mathcal{B}_{m,\tau} = \{B_{m,t}\,|\,1 \leq t \leq \tau\}. \tag{7.16}$$

We are going to inductively choose elements $\langle S_{k,m}\rangle_{k=1}^{\infty}$ in $\mathcal{I}_m$, for $1 \leq m \leq n$. Once they are chosen, we define

$$\mathcal{S}_\tau \ = \ \{S_{k,m}:\ 1 \leq k \leq \tau,\ 2 \leq m \leq n\},$$

and we define $\mathcal{R}_{m,\tau}$ to consist of all $R \in \mathbb{M}_m^d$ that have the form,

$$R = \bigoplus_{i=1}^{|\pi|} M_i$$

where π is a nontrivial partition of m and

$$M_i \ \in \ \mathcal{B}_{\pi(i),\tau} \cup \bigcup_{1\leq k\leq\tau} \left(S_{k,\pi(i)}^{-1}\mathcal{B}_{\pi(i),\tau}S_{k,\pi(i)} \cap G_\delta\right), \qquad i = 1, \dots, |\pi|.$$

Note that with this definition, as π is required to be nontrivial, $\mathcal{R}_{1,\tau} = \emptyset$. We define $E_{m,\tau} \subseteq G_\delta \cap \mathbb{M}_m^d$ by

$$E_{m,\tau} = \bigcup_{k=1}^{\tau} \left(G_\delta \cap S_{k,m}^{-1}\mathcal{B}_{m,\tau}S_{k,m}\right) \ \cup\ \mathcal{B}_{m,\tau}\ \cup\ \mathcal{R}_{m,\tau}, \qquad 2 \leq m \leq n. \tag{7.17}$$

We let $E_{1,\tau} = \mathcal{B}_{1,\tau}$. Finally, define E_τ by

$$E_\tau \ = \ \bigcup_{m=1}^{\tau} E_{m,\tau}.$$

Lemma 7.18. The set $\mathcal{S}_\tau$ can be chosen so that for each $2 \leq m \leq n$ the set $\{S_{k,m} : k \in \mathbb{N}\}$ is dense in $\mathcal{I}_m$, and

(i) $S^{-1}\mathcal{B}_{m,\tau}S \subset \mathcal{G}_m\ \forall\ S \in \mathcal{S}_\tau \cap \mathcal{I}_m$.

(ii) $\forall S_{k_1,m}, S_{k_2,m} \in \mathcal{S}_\tau \cap \mathcal{I}_m$, the set $S_{k_1,m}^{-1}S_{k_2,m}^{-1}\mathcal{B}_{m,\tau}S_{k_2,m}S_{k_1,m}$ is disjoint from $E_{m,\tau}$.

(iii) $\forall k_1 \neq k_2$ in $\{1, 2, \dots, \tau\}$, the set $S_{k_1,m}^{-1}\mathcal{B}_{m,\tau}S_{k_1,m}$ is disjoint from $S_{k_2,m}^{-1}\mathcal{B}_{m,\tau}S_{k_2,m}$ and from $\mathcal{B}_{m,\tau}$.

(iv) If $R \in R_{m,\tau}$ and for some $1 \leq k \leq \tau$ we have $S_{k,m}^{-1}RS_{k,m} \in E_\tau$, then $R \in \Sigma_m^d$.

Proof. This can be done inductively, because each of the conditions holds except on a set in $\mathcal{I}_m$ of lower dimension than the whole space. □

Lemma 7.19. For each $\tau \in \mathbb{N}$, $(E_\tau, \mathcal{S}_\tau)$ is a well organized pair of size n.

Proof. The necessary conditions follow from Lemma 7.18. □

7.2 Step 2

In this step we shall construct an $\mathcal{S}_\tau$-invariant partial nc-model for $\Theta|E_\tau^{[2]}$ (where $(E_\tau, \mathcal{S}_\tau)$ is the sequence of well organized pairs constructed in step one) that is suitable for forming a cluster point. For each $\tau \in \mathbb{N}$, let $\mathcal{V}^\tau$ denote the vector space of $\mathcal{S}_\tau$-invariant partial nc-functions on E_τ.

First observe by Proposition 6.30 that for each $\tau \in \mathbb{N}$, $\Theta|E_\tau^{[2]}$ has a special finite dimensional model, so there exist $u_\tau \in \mathcal{V}^\tau_{\mathcal{L}(\mathcal{H},\ell^{2(J)})}$ such that

$$\Theta(y,x) \;=\; u_\tau(y)^*\big((\mathrm{id} - \delta(y)^*\delta(x)) \otimes \mathrm{id}_{\ell^{2(J)}}\big)u_\tau(x) \tag{7.20}$$

for all $x, y \in E_\tau$.

If $\tau \in \mathbb{N}$, $u \in \mathcal{V}^\tau_{\mathcal{L}(\mathcal{H},\ell^{2(J)})}$, and V is a unitary operator acting on $\ell^{2(J)}$, we define $V * u \in \mathcal{V}^\tau_{\mathcal{L}(\mathcal{H},\ell^{2(J)})}$ by the formula,

$$V * u(x) = (id_{\mathbb{C}^m} \otimes V)u(x), \qquad 1 \le m \le n,\ x \in E_\tau \cap \mathbb{M}^d_m.$$

Observe that with this definition, if V is a unitary acting on $\ell^{2(J)}$, then (7.20) holds with u_τ replaced with $V_\tau * u_\tau$.

Let $\{\xi_1, \ldots, \xi_\mu\}$ be a basis for $\mathcal{H}$.

Lemma 7.21. Let $\langle M_s\rangle_{s=1}^\sigma$ be a finite sequence in $\mathbb{M}^{[d]}$ with $M_s \in \mathbb{M}_{n_s}$ for each s. Let u be a graded $\mathcal{L}(\mathcal{H}, \ell^{2(J)})$ valued function on $\{M_s \,|\, 1 \le s \le \sigma\}$. There exists a unitary operator V acting on $\ell^{2(J)}$ such that for each $s \le \sigma$,

$$\mathrm{ran}((V * u)(M_s)) \subseteq \mathbb{C}^{n_s} \otimes \ell^{2(J)}_{\mu(n_1^2+\ldots+n_s^2)}. \tag{7.22}$$

Proof. For each $1 \le r \le J$, let u^r be the r^{th} component of u. For each s, each $i, j \le n_s$, and each $1 \le \alpha \le \mu$, define μn_s^2 elements $w^r_{s,i,j,\alpha} \in \ell^2$ by

$$w^r_{s,i,j,\alpha} = \sum_{l=1}^{\infty} \langle u^r(M_s)e_j \otimes \xi_\alpha \,,\, e_i \otimes \vec{e_l}\rangle \vec{e_l} \tag{7.23}$$

In (7.23), $\{e_i\}$ is the standard basis for $\mathbb{C}^n$ and $\{\vec{e_l}\}$ denotes the standard basis for ℓ^2. For each $s \le \sigma$ define a subspace $\mathcal{W}_s$ of ℓ^2 by

$$\mathcal{W}^r_s = \mathrm{span}\{w^r_{s,i,j,\alpha} \,|\, 1 \le i, j \le n_s\}, \qquad \mathcal{W}_s = \oplus \mathcal{W}^r_s$$

and set

$$\mathcal{X}^r_s = \mathcal{W}^r_1 + \ldots \mathcal{W}^r_s.$$

If we set $\nu_s = \max_r \dim \mathcal{X}_s^r$, then there exists a unitary operator acting on ℓ^2 satisfying $V(\mathcal{X}_1^r) = {\ell^2}_{\nu_1}$, $V(\mathcal{X}_s^r \ominus \mathcal{X}_{s-1}^r) = {\ell^2}_{\nu_s} \ominus {\ell^2}_{\nu_{s-1}}$ for $s = 2, \ldots, \sigma$, and $V({\mathcal{X}_\sigma^r}^\perp) = {\ell^2_{\nu_\sigma}}^\perp$. For such a V we have that

$$V(\mathcal{X}_s^r) = {\ell^2}_{\nu_s} \subseteq {\ell^2}_{\mu(n_1^2+\ldots+n_s^2)} \tag{7.24}$$

for each $s \le \sigma$.

Now fix $s \le \sigma$ and $j \le n_s$. Using (7.23) and (7.24), we see that

$$\begin{aligned}
(V * u)(M_s)(e_j \otimes \xi_\alpha) &= (\mathrm{id}_{\mathbb{C}^{n_s}} \otimes V) u(M_s) e_j \otimes \xi_\alpha \\
&= (\mathrm{id}_{\mathbb{C}^{n_s}} \otimes V) \bigoplus_r \sum_{i,l} \langle u^r(M_s) e_j \otimes \xi_\alpha, e_i \otimes \vec{e_l} \rangle e_i \otimes \vec{e_l} \\
&= \bigoplus_r \sum_i \Big(e_i \otimes V\big(\sum_l \langle u^r(M_s) e_j \otimes \xi_\alpha, e_i \otimes \vec{e_l} \rangle \vec{e_l} \big) \Big) \\
&= \bigoplus_r \sum_i \Big(e_i \otimes V(w^r_{s,i,j,\alpha}) \Big) \\
&\in \mathbb{C}^{n_s} \otimes V(\mathcal{W}_s) \\
&\subseteq \mathbb{C}^{n_s} \otimes V(\oplus_{r=1}^J \mathcal{X}_s^r) \\
&\subseteq \mathbb{C}^{n_s} \otimes {\ell^2}^{(J)}_{\mu(n_1^2+\ldots+n_s^2)}.
\end{aligned}$$

As $e_1, \ldots e_{n_s}$ span $\mathbb{C}^{n_s}$, this proves that (7.22) holds for each $s \le \sigma$. $\square$

Fix τ and let u_τ be as in (7.20). We successively enumerate the elements of $E_1, E_2 \setminus E_1, E_3 \setminus E_2, \ldots, E_\tau \setminus E_{\tau-1}$ and apply Lemma 7.21 to obtain a unitary V_τ and integers N_t (that do not depend on τ) such that for each $t \le \tau$,

$$\mathrm{ran}((V_\tau * u)(x)) \subseteq \mathbb{C}^m \otimes {\ell^2}^{(J)}_{N_t} \qquad 1 \le m \le N_t,\ x \in E_t \cap \mathbb{M}_m^d. \tag{7.25}$$

Replacing u_τ in (7.20) with $V_\tau * u_\tau$ we thereby obtain the following improvement on (7.20).

Lemma 7.26. There exists a sequence $\langle N_t \rangle_{t=1}^\infty$ such that for each $\tau \in \mathbb{N}$, there exist

$$u_\tau \in \mathcal{V}^\tau_{\mathcal{L}(\mathcal{H}, \ell^{2(J)})} \tag{7.27}$$

such that

$$\Theta(y, x) \ = \ u_\tau(y)^* \big([\mathrm{id} - \delta(y)^* \delta(x)] \otimes \mathrm{id}_{\ell^2} \big) u_\tau(x) \tag{7.28}$$

for all $x, y \in E_\tau$ and such that for each $t \le \tau$,

$$\mathrm{ran}(u_\tau(x)) \subseteq \mathbb{C}^m \otimes {\ell^2}^{(J)}_{N_t} \qquad 1 \le m \le n,\ x \in E_t \cap \mathbb{M}_m^d. \tag{7.29}$$

7.3 Step 3

In this step we shall form a cluster point of the model described in Lemma 7.26. This will result in a model for Θ on G_δ that is 'nc to order n' as described in Lemma 7.51 below.

Fix τ and let u_τ be as in Lemma 7.26. Note that (7.27) implies that

$$u_\tau(M_1 \oplus M_2) = u_\tau(M_1) \oplus u_\tau(M_2) \tag{7.30}$$

whenever $M_1 \in \mathcal{B}_{m_1,\tau}$, $M_2 \in \mathcal{B}_{m_2,\tau}$ and $m_1 + m_2 \leq n$. Also, (7.27) implies that

$$u_\tau(S^{-1}MS) = (S^{-1} \otimes \mathrm{id}_{\ell^2(J)})u_\tau(M)(S \otimes \mathrm{id}_{\mathcal{H}}) \tag{7.31}$$

whenever $M \in \mathcal{B}_{n,\tau}$, $S \in \mathcal{S}_\tau$, and $S^{-1}MS \in G_\delta$. By Lemma 7.26 and Theorem 6.34, there exist for $m = 1, \ldots, n$ isometries (which depend on τ, though we suppress this in the notation)

$$\begin{bmatrix} A_m & B_m \\ C_m & D_m \end{bmatrix} : \mathbb{C}^m \otimes \mathcal{K}_1 \oplus (\mathbb{C}^m \otimes \ell^{2(J)}) \to \mathbb{C}^m \otimes \mathcal{K}_2 \oplus (\mathbb{C}^m \otimes \ell^{2(J)})$$

such that for each $m = 1, \ldots, n$

$$\Omega_\tau(x) := A_m + B_m\delta(x)(\mathrm{id} - D_m\delta(x))^{-1}C_m, \qquad x \in E_\tau \cap \mathbb{M}_m^d \tag{7.32}$$

satisfies

$$\Omega_\tau(x)\Psi(x) \ = \ \Phi(x), \tag{7.33}$$

and

$$v_\tau(x) \ := \ (\mathrm{id} - D_m\delta(x))^{-1}C_m, \tag{7.34}$$

satisfies

$$v_\tau(x)\Psi(x) \ = \ u_\tau(x) \qquad x \in E_\tau \cap \mathbb{M}_m^d. \tag{7.35}$$

For $m > n$, choose

$$\begin{bmatrix} A_m & B_m \\ C_m & D_m \end{bmatrix} : \mathbb{C}^m \otimes \mathcal{K}_1 \oplus (\mathbb{C}^m \otimes \ell^{2(J)}) \to \mathbb{C}^m \otimes \mathcal{K}_2 \oplus (\mathbb{C}^m \otimes \ell^{2(J)})$$

to be an arbitrary isometry.

Define an $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$-valued graded functions Ω_τ, an $\mathcal{L}(\mathcal{K}_1, \ell^{2(J)})$-valued graded function V_τ, and an $\mathcal{L}(\mathcal{H}, \ell^{2(J)})$-valued graded function U_τ, on G_δ by the formulas

$$\Omega_\tau(x) = A_m + B_m\delta(x)(\mathrm{id} - D_m\delta(x))^{-1}C_m, \qquad m \in \mathbb{N}, \ x \in G_\delta \cap \mathbb{M}_m^d \tag{7.36}$$

$$V_\tau(x) = (\mathrm{id} - D_m\delta(x))^{-1}C_m, \qquad m \in \mathbb{N}, \ x \in G_\delta \cap \mathbb{M}_m^d \tag{7.37}$$

$$U_\tau(x) = V_\tau(x)\Psi(x), \qquad m \in \mathbb{N}, \ x \in G_\delta \cap \mathbb{M}_m^d \tag{7.38}$$

Note that with these definitions that

$$\mathrm{id} - \Omega_\tau(y)^*\Omega_\tau(x) = V_\tau(y)^*(\mathrm{id} - \delta(y)^*\delta(x))V_\tau(x) \tag{7.39}$$

whenever $m \in \mathbb{N}$ and $x \in G_\delta \cap \mathbb{M}_m^d$.

It follows easily from (7.36) and (7.37) that $\langle \Omega_\tau \rangle_{\tau=1}^\infty$ and $\langle V_\tau \rangle_{\tau=1}^\infty$ are uniformly locally bounded sequences of holomorphic functions on G_δ. Hence, by Proposition 4.6 there exist a subsequence τ_j and holomorphic functions Ω and U such that

$$\Omega_{\tau_j} \to \Omega \tag{7.40}$$

and

$$V_{\tau_j} \overset{\text{wk}}{\to} V. \tag{7.41}$$

Let $U = V\Psi$. Now notice that (7.32) and (7.36) imply that

$$\Omega_\tau | E_\tau = \Omega | E_\tau \tag{7.42}$$

for each τ. Hence, as both $\Omega\Psi$ and Φ are holomorphic, (7.15) and (7.40) imply that

$$\Omega(x)\Psi(x) \;=\; \Phi(x) \tag{7.43}$$

for each $m \le n$ and $x \in G_\delta \cap \mathbb{M}_m^d$. Also notice that (7.34), (7.35) and (7.37) imply that

$$U_\tau | E_\tau \;=\; u_\tau | E_\tau \;=\; U | E_\tau \tag{7.44}$$

for each τ. Hence, it follows from (7.29) that if $m \le n$, $t \le \tau_j$ and $x \in E_t \cap \mathbb{M}_m^d$, then

$$\operatorname{ran}(U_{\tau_j}(x)) \subseteq \mathbb{C}^m \otimes \ell^2{}_{N_t}^{(J)}$$

Therefore, by (7.38) and (7.41),

$$U_{\tau_j}(x) \to U(x) \tag{7.45}$$

whenever $t \in \mathbb{N}$, $m \le n$ and $x \in E_t \cap \mathbb{M}_m^d$. Combining (7.39), (7.43), and (7.45) gives that

$$\Psi(y)^*\Psi(x) - \Phi(y)^*\Phi(x) \;=\; U(y)^*[\mathrm{id} - \delta(y)^*\delta(x)]U(x) \tag{7.46}$$

whenever $x, y \in \cup_{\tau=1}^{\infty} E_\tau$. As both the right and left hand sides of (7.46) are holomorphic in x and coholomorphic in y, it follows that

$$\forall_{m\le n}\ \forall_{x\in G_\delta\cap\mathbb{M}_m^d}\ \Psi(y)^*\Psi(x) - \Phi(y)^*\Phi(x) \;=\; U(y)^*[\mathrm{id} - \delta(y)^*\delta(x)]U(x) \tag{7.47}$$

Two additional properties of U, as constructed above, are described in the following definition.

Definition 7.48. Let $\mathcal{D}$ be an nc-domain. We say that U *is an $\mathcal{L}(\mathcal{H}, \ell^{2(J)})$-valued nc-function to order n on $\mathcal{D}$* if U is a graded $\mathcal{L}(\mathcal{H}, \ell^{2(J)})$-valued function defined on $\mathcal{D}\cap\cup_{m\le n}\mathbb{M}_m^d$, U is holomorphic,

$$x_1 \in \mathcal{D}\cap\mathbb{M}_{m_1}^d,\ x_2 \in \mathcal{D}\cap\mathbb{M}_{m_2}^d,\ m_1 + m_2 \le n \implies U(x_1 \oplus x_2) = U(x_1) \oplus U(x_2), \tag{7.49}$$

and

$$m \le n,\ x \in \mathcal{D}\cap\mathbb{M}_m^d, S \in \mathcal{I}_m,\ S^{-1}xS \in \mathcal{D}\cap\mathbb{M}_m^d \implies U(S^{-1}xS) = (S^{-1}\otimes\mathrm{id}_{\ell^{2(J)}})\, U(x)(S\otimes\mathrm{id}_{\mathcal{H}}). \tag{7.50}$$

The definition is made for a general nc-domain $\mathcal{D}$. We wish to show that (7.49) and (7.50) hold when $\mathcal{D} = G_\delta$ and U is as constructed above.

To prove (7.49) assume that $M_1 \in \mathcal{B}_{m_1,t}$ and $M_2 \in \mathcal{B}_{m_2,t}$ where $m_1 + m_2 \leq n$. Then

$$\begin{aligned}
& U(M_1 \oplus M_2) \\
(7.45) \qquad & = \lim_{j\to\infty} U_{\tau_j}(M_1 \oplus M_2) \\
(7.44) \qquad & = \lim_{j\to\infty} u_{\tau_j}(M_1 \oplus M_2) \\
(7.30) \qquad & = \lim_{j\to\infty} u_{\tau_j}(M_1) \oplus u_{\tau_j}(M_2) \\
(7.44) \qquad & = \lim_{j\to\infty} U_{\tau_j}(M_1) \oplus U_{\tau_j}(M_2) \\
(7.45) \qquad & = U(M_1) \oplus U(M_2).
\end{aligned}$$

Hence, as U is holomorphic, (7.15) implies that (7.49) holds.

To prove (7.50) assume that $M \in \mathcal{B}_{m,t}$, $S \in \mathcal{S}_t$ and $S^{-1}MS \in G_\delta$ (so that by (7.17), $S^{-1}MS \in E_{m,t}$). Then

$$\begin{aligned}
& U(S^{-1}MS) \\
(7.45) \qquad & = \lim_{j\to\infty} U_{\tau_j}(S^{-1}MS) \\
(7.44) \qquad & = \lim_{j\to\infty} u_{\tau_j}(S^{-1}MS) \\
(7.31) \qquad & = \lim_{j\to\infty} (S^{-1} \otimes \mathrm{id}_{\ell^{2(J)}})\ u_{\tau_j}(M)(S \otimes \mathrm{id}_{\mathcal{H}}) \\
(7.44) \qquad & = \lim_{j\to\infty} (S^{-1} \otimes \mathrm{id}_{\ell^{2(J)}})\ U_{\tau_j}(M)(S \otimes \mathrm{id}_{\mathcal{H}}) \\
(7.45) \qquad & = S^{-1} \otimes \mathrm{id}_{\ell^{2(J)}}\ U(M) S \otimes \mathrm{id}_{\mathcal{H}}.
\end{aligned}$$

The following lemma summarizes what has been proved. The lemma is expressed in a notation that reflects the dependence of U on n.

Lemma 7.51. Suppose Ψ is an $\mathcal{L}(\mathcal{H}, \mathcal{K}_1)$ valued nc-function on G_δ, Φ is an $\mathcal{L}(\mathcal{H}, \mathcal{K}_2)$-valued nc-function on G_δ, and suppose that $\Theta(x,x) = \Psi(x)^*\Psi(x) - \Phi(x)^*\Phi(x) \geq 0$ on G_δ. For each $n \in \mathbb{N}$ there exists U_n, such that U_n is an $\ell^{2(J)}$-valued nc-function to order n on G_δ, and such that

$$\Psi(y)^*\Psi(x) - \Phi(y)^*\Phi(x) \;=\; U_n(y)^*[\mathrm{id} - \delta(y)^*\delta(x)]U_n(x) \tag{7.52}$$

7.4 Step 4

In this step we complete the proof that Θ has a δ-model by taking a cluster point of the 'order n' models described in Lemma 7.51.

Let $\langle U_n \rangle_{n=1}^\infty$ be a sequence with U_n as in Lemma 7.51 for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, choose a dense sequence $\langle M_{n,\tau} \rangle_{\tau=1}^\infty$ in $G_\delta \cap \mathbb{M}_n^d$ and a dense sequence $\langle S_{n,\tau} \rangle$ in $\mathcal{I}_n$. As in the proof of Lemma 7.26 we may employ Lemma 7.21 to obtain a sequence of unitaries $\langle V_n \rangle_{n=1}^\infty$ acting on $\ell^{2(J)}$ such that if we define $W_n = V_n * U_n$, then W_n satisfies the conditions of Lemma 7.51 and in addition satisfies

$$\forall_{n\in\mathbb{N}}\ \exists_N\ \forall_{m\leq n}\ \forall_{s,t\leq m}\ \ \mathrm{ran}\, W_m(M_{s,t}) \subseteq \mathbb{C}^s \otimes \ell^{2(J)}_N. \tag{7.53}$$

Hence, if we use Proposition 4.6 to obtain an $\mathcal{L}(\mathcal{H}, \ell^{2(J)})$-valued holomorphic graded function W on G_δ and a subsequence $\langle n_j \rangle$ such that

$$W_{n_j} \stackrel{\text{wk}}{\to} W,$$

then

$$\forall_{n,\tau \in \mathbb{N}}\ W_{n_j}(M_{n,\tau}) \to W(M_{n,\tau}). \tag{7.54}$$

(Note that (7.54) is in finite dimensions, so weak convergence gives norm convergence). To see that W gives rise to an nc-model for Θ we need to prove the following three assertions:

$$\Psi(y)^*\Psi(x) - \Phi(y)^*\Phi(x) = W(y)^*[\text{id} - \delta(y)^*\delta(x)]W(x) \tag{7.55}$$

whenever $n \in \mathbb{N}$ and $x, y \in G_\delta \cap \mathbb{M}_n^d$,

$$W(x_1 \oplus x_2) = W(x_1) \oplus W(x_2) \tag{7.56}$$

whenever $n_1, n_2 \in \mathbb{N}$, $x_1 \in G_\delta \cap \mathbb{M}_{n_1}^d$, and $x_2 \in G_\delta \cap \mathbb{M}_{n_2}^d$, and

$$W(S^{-1}xS) = (S^{-1} \otimes \text{id}_{\ell^{2(J)}})\ W(x)S \tag{7.57}$$

whenever $n \in \mathbb{N}$, $x \in G_\delta \cap \mathbb{M}_n^d$, $S \in \mathcal{I}_n$, and $S^{-1}xS \in G_\delta$.

To see that (7.55) holds observe that (7.52) and (7.54) imply that (7.55) holds for each n whenever $x, y \in \{M_{n,\tau} \mid \tau \in \mathbb{N}\}$. Hence, as $x, y \in \{M_{n,\tau} \mid \tau \in \mathbb{N}\}$ is dense in G_δ and both sides of (7.55) are holomorphic in x and coholomorphic in y, in fact, (7.55) holds for all $x, y \in G_\delta \cap \mathbb{M}_n^d$.

(7.56) follows by noting that (7.49) and (7.54) imply that (7.56) holds whenever $x_1 \in \{M_{n_1,\tau} \mid \tau \in \mathbb{N}\}$ and $x_2 \in \{M_{n_2,\tau} \mid \tau \in \mathbb{N}\}$. Hence, by density and continuity, (7.56) holds for all $x_1 \in G_\delta \cap \mathbb{M}_{n_1}^d$ and $x_2 \in G_\delta \cap \mathbb{M}_{n_2}^d$. Likewise, (7.57) follows from (7.50) and (7.54).

This proves Theorem 7.10. ☐

8 δ nc-models and nc-realizations

Theorem 8.1. Let $\mathcal{H}, \mathcal{K}_1, \mathcal{K}_2$ be finite dimensional Hilbert spaces. Let δ be an $I \times J$ matrix with entries in $\mathbb{P}^d$, let Ψ be a graded $\mathcal{L}(\mathcal{H}, \mathcal{K}_1)$-valued function on G_δ, and let Φ be graded $\mathcal{L}(\mathcal{H}, \mathcal{K}_2)$-valued function on G_δ. Let $\Theta(y, x) = \Psi(y)^*\Psi(x) - \Phi(y)^*\Phi(x)$. The following are equivalent.
(1) $\Theta(x, x) \geq 0$ on G_δ.
(2) Θ has a δ nc-model.
(3) There exists an nc $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$-valued function Ω satsifying $\Omega\Psi = \Phi$ and such that Ω has a free δ-realization.

Proof. (1) implies (2) by Theorem 7.10. (3) implies (1) because by Proposition 7.9, we have

$$\text{id} - \Omega(y)^*\Omega(x) \ = \ v(y)^*[1 - \delta(y)^*\delta(x)]v(x).$$

Multiply by $\Psi(y)^*$ on the left and $\Psi(x)$ on the right, then restrict to the diagonal, to get $\Theta(x,x) \geq 0$.

Assume that (2) holds, i.e.,

$$\Psi(y)^*\Psi(x) - \Phi(y)^*\Phi(x) \;=\; u(y)^*[1 - \delta(y)^*\delta(x)]u(x) \tag{8.2}$$

holds, where u is an $\mathcal{L}(\mathcal{H}, \ell^{2(J)})$-valued nc-function on G_δ. Observe that if $n \in \mathbb{N}$, $S \in \mathcal{I}_n$, and we replace x with $S^{-1}xS$ in (8.2), then

$$\Psi(y)^*(S^{-1}\otimes \mathrm{id}_{\mathcal{K}_1})\Psi(x) - \Phi(y)^*(S^{-1}\otimes \mathrm{id}_{\mathcal{K}_2})\Phi(x) = u(y)^*\Big(S^{-1}\otimes \mathrm{id}_{\ell^{2(J)}} - \delta(y)^*(S^{-1}\otimes \mathrm{id}_{\ell^{2(I)}})\delta(x)\Big)u(x).$$

Hence, as $\mathcal{I}_n$ is dense in $\mathbb{M}_n$, we obtain that in fact,

$$\Psi(y)^*(C\otimes \mathrm{id}_{\mathcal{K}_1})\Psi(x) - \Phi(y)^*(C\otimes \mathrm{id}_{\mathcal{K}_2})\Phi(x) \;=\; u(y)^*\Big(C\otimes \mathrm{id}_{\ell^{2(J)}} - \delta(y)^*(C\otimes \mathrm{id}_{\ell^{2(I)}})\delta(x)\Big)u(x), \tag{8.3}$$

for all $C \in \mathbb{M}_n$. For $k = 1, \ldots, n$, define $\pi_k : \mathbb{C}^n \to \mathbb{C}$ by the formula

$$\pi_k(v) = v_k, \qquad v = (v_1, \ldots, v_n) \in \mathbb{C}^n.$$

Letting $C = \pi_l^*\pi_k$ in (8.3) and applying to $v \otimes \eta$ and $w \otimes \xi$, with v, w in $\mathbb{C}^n$ and η, ξ in $\mathcal{H}$, leads to

$$\begin{aligned} &\langle \pi_k \otimes \mathrm{id}_{\mathcal{K}_1}\Psi(x)v \otimes \eta, \pi_l \otimes \mathrm{id}_{\mathcal{K}_1}\Psi(y)w \otimes \xi\rangle - \langle \pi_k \otimes \mathrm{id}_{\mathcal{K}_2}\Phi(x)v \otimes \eta, \pi_l \otimes \mathrm{id}_{\mathcal{K}_2}\Phi(y)w \otimes \xi\rangle \\ =\;& \langle(\pi_k \otimes \mathrm{id}_{\ell^{2(J)}})u(x)v \otimes \eta, (\pi_l \otimes \mathrm{id}_{\ell^{2(J)}})u(y)w \otimes \xi\rangle \\ &- \langle(\pi_k \otimes \mathrm{id}_{\ell^{2(I)}})\delta(x)u(x)v \otimes \eta, (\pi_l \otimes \mathrm{id}_{\ell^{2(I)}})\delta(y)u(y)w \otimes \xi\rangle. \end{aligned} \tag{8.4}$$

For each $k = 1, \ldots, n$, each $v \in \mathbb{C}^n$, each $\eta \in \mathcal{H}$, and each $x \in G_\delta \cap \mathbb{M}_n^d$ define a vector $p_{k,v,\eta,x} \in \mathcal{K}_1 \oplus \ell^{2(I)}$ by

$$p_{k,v,\eta,x} = (\pi_k \otimes \mathrm{id}_{\mathcal{K}_1})\Psi(x)(v \otimes \eta) \oplus (\pi_k \otimes \mathrm{id}_{\ell^{2(I)}})\delta(x)u(x)(v \otimes \eta).$$

Also, define $q_{k,v,\eta,x} \in \mathcal{K}_2 \oplus \ell^{2(J)}$

$$q_{k,v,\eta,x} = (\pi_k \otimes \mathrm{id}_{\mathcal{K}_2})\Phi(x)(v \otimes \eta) \oplus (\pi_k \otimes \mathrm{id}_{\ell^{2(J)}})u(x)(v \otimes \eta).$$

In terms of the vectors, $p_{k,v,\eta,x}$ and $q_{k,v,\eta,x}$, (8.4) can be rewritten in the form,

$$\langle p_{k,v,\eta,x}, p_{l,w,\xi,y}\rangle = \langle q_{k,v,\eta,x}, q_{l,w,\xi,y}\rangle. \tag{8.5}$$

Hence, if we let

$$\mathcal{P}_n = \mathrm{span}\{p_{k,v,\eta,x} \,|\, k \leq n, v \in \mathbb{C}^n, \eta \in \mathcal{H}, x \in G_\delta \cap \mathbb{M}_n^d\}$$

and let

$$\mathcal{Q}_n = \mathrm{span}\{q_{k,v,\eta,x} \,|\, k \leq n, v \in \mathbb{C}^n, \eta \in \mathcal{H}, x \in G_\delta \cap \mathbb{M}_n^d\},$$

then there exists an isometry $L_n : \mathcal{P}_n \to \mathcal{Q}_n$ satisfying

$$L_n\, p_{k,v,\eta,x} \;=\; q_{k,v,\eta,x} \tag{8.6}$$

for all k, v, and x.

Now let $n \leq m$. Fix $k \leq n$, $v \in \mathbb{C}^n$, $\eta \in \mathcal{H}$, and $x \in G_\delta \cap \mathbb{M}_n^d$. Choose $v_0 \in \mathbb{C}^{m-n}$ and $x_0 \in G_\delta \cap \mathbb{M}_{m-n}^d$ and then define $v_1 = v \oplus v_0$ and $x_1 = x \oplus x_0$. We have that

$$\begin{aligned}
p_{k,v_1,\eta,x_1} &= (\pi_k \otimes \mathrm{id}_{\mathcal{K}_1})\Psi(x)(v_1 \otimes \eta) \oplus (\pi_k \otimes \mathrm{id}_{\ell^{2(I)}})\delta(x_1)u(x_1)(v_1 \otimes \eta) \\
&= (\pi_k \otimes \mathrm{id}_{\mathcal{K}_1})(\Psi(x) \oplus \Psi(x_0))(v \otimes \eta \oplus v_0 \otimes \eta) \\
&\quad \oplus (\pi_k \otimes \mathrm{id}_{\ell^{2(I)}})\delta(x \oplus x_0)u(x \oplus x_0)(v \otimes \eta \oplus v_0 \otimes \eta) \\
&= (\pi_k \otimes \mathrm{id}_{\mathcal{K}_1})(\Psi(x)v \otimes \eta \oplus \Psi(x_0)v_0 \otimes \eta) \\
&\quad \oplus (\pi_k \otimes \mathrm{id}_{\ell^{2(I)}})(\delta(x)u(x)v \otimes \eta \oplus \delta(x_0)u(x_0)v_0 \otimes \eta) \\
&= (\pi_k \otimes \mathrm{id}_{\mathcal{K}_1})(\Psi(x)v \otimes \eta) \oplus (\pi_k \otimes \mathrm{id}_{\ell^{2(I)}})(\delta(x)u(x)v \otimes \eta) \\
&= p_{k,v,\eta,x}
\end{aligned}$$

This shows that if $n \leq m$, $k \leq n$, $v \in \mathbb{C}^n$, $\eta \in \mathcal{H}$ and $x \in G_\delta \cap \mathbb{M}_n^d$, then $p_{k,v,\eta,x} = p_{k,v_1,\eta,x_1} \in \mathcal{P}_m$. Therefore,

$$\mathcal{P}_n \subseteq \mathcal{P}_m \tag{8.7}$$

whenever $n \leq m$. In like fashion, if $k \leq n$, $v \in \mathbb{C}^n$, $\eta \in \mathcal{H}$ and $x \in G_\delta \cap \mathbb{M}_n^d$, then $q_{k,v,\eta,x} = q_{k,v_1,\eta,x_1}$ so that

$$\mathcal{Q}_n \subseteq \mathcal{Q}_m. \tag{8.8}$$

Finally, observe that when $k \leq n$, $v \in \mathbb{C}^n$ and $x \in G_\delta \cap \mathbb{M}_n^d$ and v_1 and x_1 are as defined above,

$$\begin{aligned}
L_n\ p_{k,v,\eta,x} &= q_{k,v,\eta,x} \\
&= q_{k,v_1,\eta,x_1} \\
&= L_m\ p_{k,v_1,\eta,x_1} \\
&= L_m\ p_{k,v,\eta,x}.
\end{aligned}$$

Therefore, when $n \leq m$,

$$L_n = L_m|\mathcal{P}_n. \tag{8.9}$$

Let $\mathcal{P} = (\cup_{n=1}^\infty \mathcal{P}_n)^- \subseteq \mathcal{K}_1 \oplus \ell^{2(I)}$ and $\mathcal{Q} = (\cup_{n=1}^\infty \mathcal{Q}_n)^- \subseteq \mathcal{K}_2 \oplus \ell^{2(J)}$. (8.7), (8.8), and (8.9) together imply that there exists an isometry $L : \mathcal{P} \to \mathcal{Q}$ such that

$$Lp_{k,v,\eta,x} = q_{k,v,\eta,x} \tag{8.10}$$

whenever $n \in \mathbb{N}$, $k \leq n$, $v \in \mathbb{C}^n$, $\eta \in \mathcal{H}$, and $x \in G_\delta \cap \mathbb{M}_n^d$. By replacing u in (8.2) with $(\mathrm{id}_{\mathbb{C}^n} \otimes \tau^{(J)})u$ where $\tau : \ell^2 \to \ell^2$ is an isometry with $\mathrm{ran}(\tau)$ having infinite codimension in ℓ^2 we may ensure that $\mathcal{P}$ has infinite codimension in $\mathcal{K}_1 \oplus \ell^{2(I)}$ and $\mathcal{Q}$ has infinite codimension in $\mathcal{K}_2 \oplus \ell^{2(J)}$. Hence, there exists an isometry (or even a Hilbert space isomorphism) $J_1 : \mathcal{K}_1 \oplus \ell^{2(I)} \to \mathcal{K}_2 \oplus \ell^{2(J)}$ such that

$$J_1 p_{k,v,\eta,x} = q_{k,v,\eta,x} \tag{8.11}$$

whenever $n \in \mathbb{N}$, $k \leq n$, $v \in \mathbb{C}^n$ and $x \in G_\delta \cap \mathbb{M}_n^d$.

There remains to show that $J_n = \mathrm{id}_{\mathbb{C}^n} \otimes J_1$ defines an nc-realization of Ω. First, let us show that

$$(\mathrm{id}_{\mathbb{C}^n} \otimes J_1) \begin{pmatrix} \Psi(x) \\ \delta(x)u(x) \end{pmatrix} = \begin{pmatrix} \Phi(x) \\ u(x) \end{pmatrix}. \tag{8.12}$$

Fix $n \in \mathbb{N}$, $v \in \mathbb{C}^n$ $\eta \in \mathcal{H}$, and $x \in G_\delta \cap \mathbb{M}_n^d$.

$$\begin{aligned}
&(\mathrm{id}_{\mathbb{C}^n} \otimes J_1)\big(\Psi(x)v \otimes \eta \oplus (\delta(x)u(x)v \otimes \eta)\big) \\
=&(\mathrm{id}_{\mathbb{C}^n} \otimes J_1)\big(\oplus_{k=1}^n \pi_k \Psi(x)v \otimes \eta \oplus (\oplus_{k=1}^n (\pi_k \otimes \mathrm{id}_{\ell^{2(I)}})\delta(x)u(x)v \otimes \eta)\big) \\
=&(\mathrm{id}_{\mathbb{C}^n} \otimes J_1)\big(\oplus_{k=1}^n (\pi_k \Psi(x)v \otimes \eta \oplus (\pi_k \otimes \mathrm{id}_{\ell^{2(I)}})\delta(x)u(x)v \otimes \eta)\big) \\
=&\bigoplus_{k=1}^n J_1(\pi_k \Psi(x)v \otimes \eta \oplus (\pi_k \otimes \mathrm{id}_{\ell^{2(I)}})\delta(x)u(x)v \otimes \eta) \\
=&\bigoplus_{k=1}^n J_1 p_{k,v,\eta,x} = \bigoplus_{k=1}^n q_{k,v,\eta,x} \\
=&\bigoplus_{k=1}^n \pi_k \Phi(x)v \otimes \eta \oplus (\pi_k \otimes \mathrm{id}_{\ell^{2(J)}})u(x)v \otimes \eta \\
=& \oplus_{k=1}^n (\pi_k \Phi(x)v \otimes \eta \oplus (\pi_k \otimes \mathrm{id}_{\ell^{2(J)}})u(x)v \otimes \eta) \\
=& \oplus_{k=1}^n \pi_k \Phi(x)v \otimes \eta \oplus (\oplus_{k=1}^n (\pi_k \otimes \mathrm{id}_{\ell^{2(J)}})u(x)v \otimes \eta) \\
=&\Phi(x)v \otimes \eta \oplus u(x)v \otimes \eta.
\end{aligned}$$

Now, define

$$\begin{aligned}
v(x) &= (\mathrm{id} - (\mathrm{id}_{\mathbb{C}^n} \otimes D_1)\delta(x))^{-1}(\mathrm{id}_{\mathbb{C}^n} \otimes C_1), \\
\Omega(x) &= (\mathrm{id}_{\mathbb{C}^n} \otimes A_1) + (\mathrm{id}_{\mathbb{C}^n} \otimes B_1)\delta(x)v(x), \qquad \forall x \in G_\delta \cap \mathbb{M}_n^d.
\end{aligned}$$

Then Ω has a free δ-realization, because

$$\begin{bmatrix} \mathrm{id}_{\mathbb{C}^n} \otimes A_1 & \mathrm{id}_{\mathbb{C}^n} \otimes B_1 \\ \mathrm{id}_{\mathbb{C}^n} \otimes C_1 & \mathrm{id}_{\mathbb{C}^n} \otimes D_1 \end{bmatrix} \begin{pmatrix} \mathrm{id}_{\mathbb{C}^n} \otimes \mathrm{id}_{\mathcal{K}_1} \\ \delta(x)v(x) \end{pmatrix} = \begin{pmatrix} \Omega(x) \\ v(x) \end{pmatrix}, \qquad \forall x \in G_\delta \cap \mathbb{M}_n^d.$$

It follows from (8.12) that $\Omega\Psi = \Phi$ on G_δ. □

Corollary 8.13. If $\mathcal{H}$ and $\mathcal{K}_1$ are finite dimensional Hilbert spaces and if $\Phi \in \mathrm{ball}(H^\infty_{\mathcal{L}(\mathcal{H},\mathcal{K}_1)}(G_\delta))$ then there exists an isometry

$$J_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{H} \oplus \ell^{2(I)} \to \mathcal{K}_1 \oplus \ell^{2(J)}$$

so that for $x \in G_\delta \cap \mathbb{M}_n^d$,

$$\Phi(x) = \mathrm{id}_{\mathbb{C}^n} \otimes A + (\mathrm{id}_{\mathbb{C}^n} \otimes B)\delta(x)[\mathrm{id}_{\mathbb{C}^n} \otimes \mathrm{id}_{\ell^{2(J)}} - (\mathrm{id}_{\mathbb{C}^n} \otimes D)\delta(x)]^{-1}\mathrm{id}_{\mathbb{C}^n} \otimes C. \tag{8.14}$$

Consequently, Φ has the power series expansion

$$\Phi(x) = \mathrm{id}_{\mathbb{C}^n} \otimes A + \sum_{k=0}^{\infty} (\mathrm{id}_{\mathbb{C}^n} \otimes B)\delta(x)[(\mathrm{id}_{\mathbb{C}^n} \otimes D)\delta(x)]^k(\mathrm{id}_{\mathbb{C}^n} \otimes C), \tag{8.15}$$

which is absolutely convergent on G_δ.

Remark 8.16. If $\mathcal{H}$ and $\mathcal{K}_1$ are both $\mathbb{C}$, then each term

$$(\mathrm{id}_{\mathbb{C}^n} \otimes B)\delta(x)[(\mathrm{id}_{\mathbb{C}^n} \otimes D)\delta(x)]^k(\mathrm{id}_{\mathbb{C}^n} \otimes C)$$

is a non-commutative polynomial, whose terms are linear combinations of products of $k+1$ terms in the entries $\delta_{ij}(x)$. If one groups the terms by this homogeneity, then the sum of these terms has norm at most $\|\delta(x)\|^{k+1}$.

Corollary 8.13, in the case that $\delta(x) = (x^1, \dots, x^d)$, was proved by Helton, Klep and McCullough [15, Prop. 7].

A special case of Theorem 8.1 is the non-commutative corona theorem. Take $\mathcal{H}$ and $\mathcal{K}_2$ to be $\mathbb{C}$, and choose $\Phi(x) = \varepsilon$. Then we conclude:

Theorem 8.17. Let $\psi_1, \dots, \psi_k$ be in $H^\infty(G_\delta)$ and satisfy

$$\sum_{j=1}^{k} \psi_j(x)^* \psi_j(x) \;\geq\; \varepsilon^2 \,\mathrm{id}_{\mathbb{C}^n} \qquad \forall x \in G_\delta \cap \mathbb{M}_n^d.$$

Then there exist functions $\omega_1, \dots, \omega_k$ in $H^\infty(G_\delta)$ and satisfying $\|(\omega_1, \dots, \omega_k)\| \leq \frac{1}{\varepsilon}$ in $H^\infty_{\mathcal{L}(\mathbb{C}^k, \mathbb{C})}$ such that

$$\sum_{j=1}^{k} \omega_j(x)\psi_j(x) = \mathrm{id}_{\mathbb{C}^n} \qquad \forall x \in G_\delta \cap \mathbb{M}_n^d.$$

In the case $d = 1$ and G_δ is the unit disk, Theorem 8.17 is called the Toeplitz-corona theorem. It was first proved by Arveson [8]; Rosenblum showed how to deduce Carleson's corona theorem from the Toeplitz corona theorem in [22].

Another consequence of Theorem 8.1 is the following observation. Let $\mathcal{F}_\delta$ be the set of d-tuples T of commuting operators satisfying $\|\delta(T)\| \leq 1$. Recall from Definition 1.8 that

$$\|f\|_{\delta,\mathrm{com}} = \sup_{\substack{T \in \mathcal{F}_\delta \\ \sigma(T) \subseteq G_\delta}} \|f(T)\|, \tag{8.18}$$

and $H^\infty_{\delta,\mathrm{com}}$ is the set of analytic functions f on G_δ for which $\|f\|_{\delta,\mathrm{com}} < \infty$. (It follows from [4] and [**?**] that the supremum in (8.18) is the same whether T runs over commuting operators with Taylor spectrum in G_δ or commuting matrices with a spanning set of joint eigenvectors, and joint eigenvalues that lie in G_δ).

Then every free analytic function in $H^\infty(G_\delta)$ has a free δ-realization, and this gives a δ-realization for a function in $H^\infty_{\delta,\mathrm{com}}$. Conversely, every function in $H^\infty_{\delta,\mathrm{com}}$ has a δ-realization by [7], and this extends to a free δ-realization for some function ϕ in $H^\infty(G_\delta)$. So we have:

Theorem 8.19. Let

$$I \;=\; \{\phi \in H^\infty(G_\delta) \,|\, \phi|_{\mathbb{M}_1^d} = 0\}.$$

Then $H^\infty(G_\delta)/I$ is isometrically isomorphic to $H^\infty_{\delta,\mathrm{com}}$.

9 Oka Representation

Definition 9.1. The *free topology* on $\mathbb{M}^{[d]}$ is the topology that has as a basis the sets of the form G_δ where δ is a matrix of free polynomials in d variables.

That the definition actually defines a topology follows from the observation that if δ_1 and δ_2 are matrices of polynomials, then

$$G_{\delta_1} \cap G_{\delta_2} = G_{\delta_1 \oplus \delta_2}.$$

An basic property of compact polynomially convex sets in $\mathbb{C}^d$ is that they can be approximated from above by p-polyhedrons (cf. [5] Lemma 7.4). The following simple proposition asserts that compact sets in the free topology can be approximated from above by polyhedrons as well.

Proposition 9.2. Let $E \subseteq \mathbb{M}^{[d]}$ be a compact set in the free topology that is closed under (finite) direct sums. If U is a neighborhood of E, and

$$E \ \subset \ \cup_{\alpha \in A} G_{\delta_\alpha} \subseteq U,$$

then there exists $\delta \in \{\delta_\alpha : \alpha \in A\}$, a single matrix of free polynomials in d variables, and a positive number $t > 1$, such that

$$E \subseteq G_{t\delta} \subseteq G_\delta \subseteq U.$$

Proof. Since E is compact and U is open, there are $\delta_1, \dots, \delta_N$ so that

$$E \subseteq \cup_{j=1}^N G_{\delta_j} \subseteq U.$$

Claim:

$$\min_{1 \leq j \leq N} \max_{M \in E} \|\delta_j(M)\| \ < 1.$$

Indeed, otherwise there would for each j be an $M_j \in E$ such that $\|\delta_j(M_j)\| \geq 1$. Then $\oplus_{j=1}^N M_j$ would be in E, but not in any G_{δ_j}.

Choose j such that $\max_{M \in E} \|\delta_j(M)\| \ = \ r \ < 1$. Let $\delta = \delta_j$ and choose t between 1 and $1/r$. ☐

Definition 9.3. By an $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued *free holomorphic function* is meant a graded function $\phi : D \to \mathcal{L}(\mathcal{H},\mathcal{K})$ such that D is a free open set, ϕ is an $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued graded function on D, and for every $M \in D$, there exists a basic free neighborhood G_δ of M in D such that ϕ is bounded and nc on G_δ.

If δ is a matrix of polynomials, we shall let

$$K_\delta \ := \ \{M \in \mathbb{M}^{[d]} : \|\delta(M)\| \leq 1\}. \tag{9.4}$$

Definition 9.5. Let $E \subset \mathbb{M}^{[d]}$. The polynomial hull of E is defined to be

$$\hat{E} \ := \ \bigcap \{K_\delta \ : \ E \subseteq K_\delta\}.$$

(If E is not contained in any K_δ, we declare $\hat{E}$ to be $\mathbb{M}^{[d]}$.) We say a compact set is polynomially convex if it equals its polynomial hull. We say an open set D is polynomially convex if for any compact set $E \subset D$, the polynomial hull of E is a compact subset of D.

Note that $\hat{E}$ is always an nc set, so if $\hat{E}$ is compact and contained in some open set U, then by Proposition 9.2 it is contained in a single basic free open set in U.

Example 9.6. Consider the free annulus A

$$A \ := \bigcup_{0 \leq \theta \leq 2\pi} \{x \in \mathbb{M} \ : \ \|x - \frac{3}{4} e^{i\theta} \mathrm{id}\| < \frac{1}{4}\}.$$

Suppose D is a polynomially convex free open set containing A. Letting E range over the compact subsets of $\{z \in \mathbb{C} \ : \frac{1}{2} < |z| < 1\}$, and using that $\hat{E} \subset D$, we conclude that $D \cap \mathbb{M}_1 \supseteq \mathbb{D}$, so D contains all normal matrices with spectrum in $\mathbb{D}$.

For each $r < 1$, we let $E = r\overline{\mathbb{D}}$. By Proposition 9.2, we conclude that there exists δ such that $\hat{E} \subset G_\delta \subset D$. As δ is a contractive matrix-valued function on $r\overline{\mathbb{D}}$, it has a realization formula,- and so is contractive on all matrices M with $\|M\| \leq r$. (Note that in one variable, a polynomial is uniquely defined on $\mathbb{M}$ by its action on $\mathbb{M}_1 = \mathbb{C}$). We conclude therefore that D must contain the open unit matrix ball:

$$\{M : \|M\| < 1\} \ \subseteq \ D.$$

So polynomial convexity has filled in the holes at all levels.

The following theorem is the free analogue of the Oka-Weil theorem.

Theorem 9.7. Let $\mathcal{H}$ and $\mathcal{K}$ be finite dimensional Hilbert spaces. Let $E \subseteq \mathbb{M}^{[d]}$ be a compact set in the free topology, and assume that E is polynomially convex. Let U be a free open set containing E, and let ϕ be a free holomorphic $\mathcal{L}(\mathcal{H}, \mathcal{K})$-valued function defined on U. Then ϕ can be uniformly approximated on E by $\mathcal{L}(\mathcal{H}, \mathcal{K})$-valued free polynomials.

Proof. For each point M in E, there is a matrix δ_M of free polynomials such that $M \in G_{\delta_M} \subseteq U$ and ϕ is bounded on G_{δ_M}. By Proposition 9.2, we can find a single matrix δ of free polynomials, and $t > 1$, such that $E \subseteq G_{t\delta} \subseteq G_\delta$ and such that ϕ is bounded on G_δ. Hence, by Theorem 8.1, ϕ has a δ free realization. Using the resulting Neumann series for ϕ (which converges uniformly on $G_{t\delta}$) yields that ϕ can be uniformly approximated by polynomials on E. □

As an application of Theorem 9.7 the following result gives a purely holomorphic characterization of free holomorphic functions. If ϕ is a graded function defined on a free open set $\mathcal{D}$, let us agree to say that ϕ *is locally approximable by polynomials* if for each $M \in \mathcal{D}$ and $\epsilon > 0$, there exists a free neighborhood U of M and a free polynomial p such that

$$\sup_{x \in U \cap \mathcal{D}} \|\phi(x) - p(x)\| < \epsilon.$$

Theorem 9.8. Let $\mathcal{D}$ be a free open set and let ϕ be a graded function defined on $\mathcal{D}$. Then ϕ is a free holomorphic function if and only if ϕ is locally approximable by polynomials.

Proof. Sufficiency follows because the uniform limit of free polynomials is nc and bounded. For necessity, let M be in $\mathcal{D}$. Then since $\mathcal{D}$ is open, there exists a matrix δ of free polynomials, and $t > 1$, such that $M \in G_{t\delta} \subseteq G_\delta$. Now apply Theorem 9.7 with $E = K_{t\delta}$. □

10 Free Meromorphic Functions

It is a natural question to ask whether rational functions are free holomorphic away from their poles. A rational function means any function that can be built up from free polynomials by finitely many arithmetic operations. We shall say the polar set of a rational function ϕ is the set of $x \in \mathbb{M}_n^d$ at which, at some stage in the evaluation of the function $\phi(x)$, one has to divide by a matrix that is not invertible. This depends on the presentation of the function, so to be careful one should define rational functions as an equivalence class of such expressions; see *e.g.* [17, p.7] for a discussion. We can extend this notion to meromorphic functions on a free open set $\mathcal{D}$, defining them to be any function that can be built up from free holomorphic functions on $\mathcal{D}$ by finitely many arithmetic operations.

Theorem 10.1. Let ϕ and ψ be free holomorphic functions on a free open set $\mathcal{D}$. Then $\psi[\phi]^{-1}$ and $[\phi]^{-1}\psi$ are free holomorphic off the zero set of ϕ.

Proof. It is sufficient to prove that if ϕ is free holomorphic on $\mathcal{D}$ and $\phi(M)$ is invertible for some point M in $\mathcal{D}$, then there is a free open neighborhood of M in $\mathcal{D}$ on which $\phi(x)^{-1}$ is bounded.

Since $\mathcal{D}$ is open, there exists δ such that $M \in G_\delta \subseteq \mathcal{D}$, and such that ϕ is bounded by B on G_δ. Let $T = \phi(M)$, and let p be a polynomial in one variable satisfying $p(T) = 0, p(0) = 1$. Let $\delta' = \delta \oplus (2p \circ \phi)$. If $N \in G_{\delta'}$, then $\|p \circ \phi(N)\| \leq \frac{1}{2}$, so

$$\|[\mathrm{id} - p \circ \phi(N)]^{-1}\| \ \leq \ 2.$$

Let $\phi(N) = S$, and let $c, \beta_j \in \mathbb{C}$ satisfy

$$1 - p(z) \ = \ cz \prod (z - \beta_j).$$

Then

$$[\mathrm{id} - p \circ \phi(N)]^{-1} \ = \ \frac{1}{c} S^{-1} \prod (S - \beta_j)^{-1}.$$

Therefore

$$\|S^{-1}\| \ \leq \ 2|c| \prod (B + |\beta_j|),$$

so $\phi(x)^{-1}$ is bounded on $G_{\delta'}$, as required. ☐

11 Index

11.1 Notation

$\mathbb{M}_n$ The n-by-n matrices

$\mathbb{M}^{[d]} = \cup_{n=1}^\infty \mathbb{M}_n^d$ Second paragraph, Subsection 1.1

$\mathcal{I}_n$ Invertible n-by-n matrices (1.1)

$\mathcal{U}_n$ Unitary n-by-n matrices (1.2)

nc($\mathcal{D}$) Definition 1.4

$G_\delta = \{x \in \mathbb{M}^{[d]} : \|\delta(x)\| < 1\}$ (1.6)

$\|f\|_{\delta,\text{com}}, H^\infty_{\delta,\text{com}}$ Definition 1.8
$\mathcal{L}(\mathcal{H},\mathcal{K})$ Bounded linear transformations between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ Subsection 1.2
$nc_{\mathcal{K}}(\mathcal{D})$ Line after (1.12)
$A^\sim$ Envelope of A, Definition 3.1
$\text{nc}_{\mathcal{L}(\mathcal{H},\mathcal{K})}$ The $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued nc functions on D, Definition 3.6
$f^\sim$ (3.11)
$f^\approx$ (3.12)
$d(f,g)$ (4.4)
$\mathcal{R}_m, \mathcal{B}_m, \mathcal{G}_m$ Definition 5.4
$\mathcal{B}, \mathcal{G}, \mathcal{R}$ Paragraph following Definition 5.4
$\mathcal{V}(E,\mathcal{S}), \text{Grade}(\mathcal{B})$ Paragraph before Proposition 5.11
$X_{\mathcal{V}}, X^r_{\mathcal{V}}$ (5.15) and (5.16)
$X_{\mathcal{V}} \overset{\rho}{\sim} X_{\mathcal{B}}$ (5.17)
M_ϕ (5.19), (6.3)
$H^\infty(\mathcal{D}), H^\infty_{\mathcal{L}(\mathcal{H},\mathcal{K})}$ First two paragraphs of Section 6
$E^{[2]}$ (6.4)
$\mathcal{V}_{\mathcal{L}(\mathcal{H})}, \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathcal{M})}$ Second paragraph, Subsection 6.1
$\text{H}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ Third paragraph, Subsection 6.1
$\text{R}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ (6.5)
$\text{P}_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ (6.6)
$\text{C}_{\mathcal{L}(\mathcal{H})}(\mathcal{V}), \text{C}^\tau_{\mathcal{L}(\mathcal{H})}(\mathcal{V})$ Definition 6.11
$\mathcal{N}_L$ (6.21)
H^2_L line after (6.21)
$K_\delta = \{x \in \mathbb{M}^{[d]} : \|\delta(x)\| \leq 1\}$ (9.4)
$\hat{E}$ Definition 9.5

11.2 Definitions

nc-set, du-open, du-closed, du-bounded, nc-domain: Definition 1.3
nc-function: Definition 1.4
basic free open set, free domain, free topology: Definition 1.5
$\mathcal{K}$-valued nc-function: Definition 1.9
δ nc-model: Definition 1.13
free δ-realization: Definition 1.15
envelope of A: Definition 3.1
$\mathcal{L}(\mathcal{H},\mathcal{K})$-valued nc functions on $\mathcal{D}$, Definition 3.6
locally bounded, locally uniformly bounded: Definition 4.2
partial nc-set: First paragraph, Section 5
partial nc-function: Second paragraph, Section 5
$\mathcal{S}$-invariant function: Third paragraph, Section 5
generic, complete, E-reducible: Fourth paragraph, Section 5
well-organized pair: Definition 5.4
base $\mathcal{B}$ of well-organized pair: Paragraph following Definition 5.4
ordered partition: (5.8)

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