Global Hyperbolicity of Sliced Spaces

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Abstract

We show that for generic sliced spacetimes global hyperbolicity is equivalent to space completeness under the assumption that the lapse, shift and spatial metric are uniformly bounded. This leads us to the conclusion that simple sliced spaces are timelike and null geodesically complete if and only if space is a complete Riemannian manifold.

1 Introduction

The singularity theorems of general relativity inform us that the singular behaviour met in the simplest isotropic and homogeneous but anisotropic spacetimes is a more general phenomenon: It arises in all those circumstances which, under canonical causality assumptions such as global hyperbolicity, globally share the same geodesic properties (geodesic focusing) as that found in the simplest models.

However, this by no means exhausts the available possibilities even in the simplest examples. For instance, the geometric properties of the universal covering space of AdS space are refreshingly unlike those of the usual FRW spaces. AdS space is not globally hyperbolic but it is (null and spacelike) geodesically complete.

The purpose of this paper is to discuss a connection between global hyperbolicity and space completeness in spacetimes which are very general, but are closer in a sense to AdS behaviour rather than that of the usual FRW universes and their generalizations. In a simple sub-class of such spacetimes, all three, geodesic completeness, slice completeness and global hyperbolicity, are equivalent.

2 Slice completeness and global hyperbolicity

Consider a spacetime of the form $(\mathcal{V}, g)$ with $\mathcal{V} = \mathcal{M} \times \mathcal{I}$, $\mathcal{I}$ being an interval in $\mathbb{R}$ and $\mathcal{M}$ a smooth manifold of dimension n, in which the smooth, $(n+1)$–dimensional, Lorentzian metric g splits as follows:

$$g \equiv -N^2(\theta^0)^2 + g_{ij}\,\theta^i\theta^j, \quad \theta^0 = dt, \quad \theta^i \equiv dx^i + \beta^i dt. \tag{2.1}$$

Here $N = N(t, x^i)$ is called the *lapse function*, $\beta^i(t, x^j)$ is called the *shift function* and the spatial slices $\mathcal{M}_t\,(= \mathcal{M} \times \{t\})$ are spacelike submanifolds endowed with the time-dependent spatial metric $g_t \equiv g_{ij}dx^i dx^j$. We call such a spacetime a *sliced space* [1].

Sliced spaces are time-oriented by increasing t and may be thought of as warped products $\mathcal{M} \times_N \mathbb{R}$, $N : \mathcal{M} \to \mathbb{R}$ with the lapse as their warping function (in general the lapse is defined in the extended space $\mathcal{M} \times \mathbb{R}$). Notice, however, that such a warped product is different than the usual warped product form $\mathbb{R} \times_f \mathcal{M}$, $f : \mathbb{R} \to \mathbb{R}$, which includes, for instance, the FRW spaces. The simplest example of a sliced space which cannot be written in the usual warped product form is the (universal covering space of) AdS spacetime.

For simplicity we choose $\mathcal{I} = \mathbb{R}$. The following hypothesis insures that the parameter t measures, up to a positive factor bounded above and below, the proper time along the normals to the slices $\mathcal{M}_t$. We say that a sliced space has *uniformly bounded lapse* if the lapse function N is bounded below and above by positive numbers N_m and N_M,

$$0 < N_m \leq N \leq N_M. \tag{2.2}$$

A sliced space has *uniformly bounded shift* if the g_t norm of the shift vector β, projection on the tangent space to $\mathcal{M}_t$ of the tangent to the lines $\{x\} \times \mathcal{I}$, is uniformly bounded by a number B.

A sliced space has *uniformly bounded spatial metric* if the time-dependent metric $g_t \equiv g_{ij}dx^i dx^j$ is uniformly bounded below and above for all $t \in \mathcal{I}$ by a metric $\gamma = g_0$, that is there exist numbers $A, D > 0$ such that for all tangent vectors v to $\mathcal{M}$ it holds that

$$A\gamma_{ij}v^i v^j \leq g_{ij}v^i v^j \leq D\gamma_{ij}v^i v^j. \tag{2.3}$$

We prove the following result.

Theorem 2.1 *Let $(\mathcal{V}, g)$ be a sliced space with uniformly bounded lapse N, shift β and spatial metric g_t. Then the following are equivalent:*

1. *$(\mathcal{M}_0, \gamma)$ is a complete Riemannian manifold*

2. *The spacetime $(\mathcal{V}, g)$ is globally hyperbolic*

PROOF. (1) $\Rightarrow$ (2). This was proved in [2], but we present here a somewhat different, but completely equivalent, proof which is based on Penrose's definition of global hyperbolicity [3], equivalent to Leray's original definition [4]. The strong causality of $(\mathcal{V}, g)$ follows if we prove that each slice $\mathcal{M}_t$ intersects more than once no inextendible, future-directed causal curve $C : \mathcal{I} \to \mathcal{V} : s \mapsto C(s)$. If $n = (-N, 0)$ is the timelike normal to $\mathcal{M}_t$, the tangent to this curve is such that,

$$g\left(\frac{dC}{ds}, n\right) \equiv -N\frac{dt}{ds} < 0, \tag{2.4}$$

therefore on C we have that,

$$\frac{dt}{ds} > 0, \tag{2.5}$$

and hence C can be reparametrized using t and cuts each $\mathcal{M}_t$ at most once. To prove that the sets of the form $J^+(u) \bigcap J^-(v)$ with $u, v \in \mathcal{V}$ are compact we proceed as follows. Suppose there is a pair of points $(x_1, t_1), (x_2, T)$ of $\mathcal{V}$ such that the set $J^+((x_1, t_1)) \bigcap J^-((x_2, T))$ is noncompact. This means that there exists a future-directed, causal curve C from (x_1, t_1) to (x_2, T) which is inextendible. Consider a Cauchy sequence of numbers (t_n) which converges to T and the corresponding points (c_n, t_n) of the curve C, where c_n (with components $C^i(t_n)$) are points of $\mathcal{M}$. It follows that the sequence c_n cannot converge to the point $c(T)$. But this is impossible, since the estimates of [2], p. 347, show that c_n is a Cauchy sequence in the complete Riemannian manifold $(\mathcal{M}, \gamma)$. Thus the sets $J^+(u) \bigcap J^-(v)$ are compact and hence $(\mathcal{V}, g)$ is globally hyperbolic.

(2) $\Rightarrow$ (1). Suppose that $(\mathcal{M}_0, \gamma)$ is not complete. Then from the Hopf-Rinow theorem we can find a geodesic $c : [0, \delta) \to \mathcal{M}_0$ of finite length which cannot be extended to the arclength value $s = \delta < \infty$. We take two times $t_1 < t_2$ greater than zero, such that $\delta < (t_2 - t_1)/2 \equiv \delta^*/2$. Since δ is given by the geometry of the slice, this is a hypothesis on $t_2 - t_1$, i.e., on the minimum length of the spacetime slab.

Define on $\mathcal{V}$ the future-directed causal curve $\bar{c} : [0, \delta) \to \mathcal{V}$ with

$$\bar{c} = (t + t_1 = s, c(s)),$$

and the past-directed causal curve $\tilde{c} : [0, \delta) \to \mathcal{V}$ with

$$\tilde{c} = (\delta^* - t + t_1 = s, c(s)).$$

The curve $\bar{c}$ is causal if

$$-N^2(\bar{c}) + g_{ij}(\bar{c}) \left(\frac{dc^i}{ds} + \beta^i \right) \left(\frac{dc^j}{ds} + \beta^j \right) \leq 0. \tag{2.6}$$

Since c is a geodesic on $(\mathcal{M}_0, \gamma)$ we have

$$\gamma_{ij}(c) \frac{dc^i}{ds} \frac{dc^j}{ds} = 1, \tag{2.7}$$

and therefore Condition (2.6) will hold whenever[1]

$$D + B \leq N_m^2. \tag{2.8}$$

Similar reasoning for the curve $\tilde{c}$.

The curve $\bar{c}$ starts from the point $(-t_1, c(0))$ and proceeds to the future in the past of c, while $\tilde{c}$ starts from the point $(t_2, c(0))$ and develops to the past in the future of $c \, (\equiv (t = 0, c(s)))$. Therefore for each $t \in [0, \delta)$, since $t < \delta^* - t$, we conclude that

$$(-t_1, c(0)) \prec \bar{c}(t) \ll \tilde{c}(t) \prec (t_2, c(0)), \tag{2.9}$$

where $\prec, \ll$ are the causality (J) and chronology (I) relations respectively. It follows that the diamond-shaped set $J^+(-t_1, c(0)) \bigcap J^-(t_2, c(0))$ contains the curve $\bar{c}([0, \delta))$. But since the set $c([0, \delta))$ does not have compact closure in $(\mathcal{M}_0, \gamma)$, it follows that $\bar{c}([0, \delta))$ cannot have compact closure in $J^+(-t_1, c(0)) \bigcap J^-(t_2, c(0))$. This is however impossible, for the set $J^+(-t_1, c(0)) \bigcap J^-(t_2, c(0))$ is compact because $\mathcal{V}$ is globally hyperbolic. Hence, the curve c is extendible. ■

[1]inequality (2.8) could be lifted by replacing $\bar{c}$ by a curve $(k(t + t_1) = s, c(s))$, $0 \leq s < \delta$ with k an appropriate positive number. This curve $\bar{c}$ is in the past of $c \equiv (t = 0, s = c(s))$, $0 \leq s < \delta$ if on it $t < 0$, i.e. $t_1 > k^{-1}\delta$. Analogous reasoning for $\tilde{c}$.

3 Geodesic completeness of trivially sliced spaces

In this Section we are interested in the question under what conditions is global hyperbolicity equivalent to geodesic completeness. What is the class of sliced spaces in which such an equivalence holds? In a sliced space belonging to this class, in view of the results of the previous Section, geodesic completeness of the spacetime would be guessed very simply: It would suffice to look at the completeness of a slice.

The tangent vector u to a geodesic parametrized by arc length, or by the canonical parameter in the case of a null geodesic, with components dx^α/ds in the natural frame, satisfies in an arbitrary frame the geodesic equations,

$$u^\alpha \nabla_\alpha u^\beta \equiv u^\alpha \partial_\alpha u^\beta + \omega^\beta_{\alpha\gamma} u^\alpha u^\gamma = 0. \tag{3.1}$$

In the adapted frame the components of u become,

$$u^0 = \frac{dt}{ds}, \quad u^i = \frac{dx^i}{ds} + \beta^i \frac{dt}{ds}, \tag{3.2}$$

while the Pfaff derivatives are given by,

$$\partial_0 \equiv \partial_t - \beta^i \partial_i, \quad \partial_i \equiv \frac{\partial}{\partial x^i}. \tag{3.3}$$

It holds therefore that,

$$u^\alpha \partial_\alpha u^\beta \equiv \frac{dt}{ds} \left(\partial_t - \beta^i \partial_i\right) u^\beta + \left(\frac{dx^i}{ds} + \beta^i \frac{dt}{ds}\right) \partial_i u^\beta \equiv \frac{du^\beta}{ds}. \tag{3.4}$$

Since $u^0 \equiv dt/ds$, setting

$$v^i = \frac{dx^i}{dt} + \beta^i, \tag{3.5}$$

so that

$$u^i = v^i u^0, \tag{3.6}$$

Eq. (3.1) with $\beta = 0$ gives the 0-component of the geodesic equations which can be written in the form,

$$\partial_t u^0 + u^0 \left(\omega^0_{00} + 2\omega^0_{0i} v^i + \omega^0_{ij} v^i v^j\right) = 0. \tag{3.7}$$

On the other hand, the k-component of the geodesic equations is

$$\partial_t v^k + v^i \partial_i v^k + \omega^k_{00} + 2\omega^k_{0i} v^i + \omega^k_{ij} v^i v^j = 0. \tag{3.8}$$

Using the expressions for the connection coefficients, we conclude that when the lapse and shift are constant functions and $\partial_0 g_{ij} = 0$, Eq. (3.7) gives

$$t = \text{const.} \times s, \tag{3.9}$$

while Eq. (3.8), using (3.6), becomes,

$$\frac{d^2 x^i}{ds^2} + \tilde{\Gamma}^k_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 0, \tag{3.10}$$

that is the geodesic equation in the Riemannian manifold $\mathcal{M}$. This result means that in a sliced space $(\mathcal{V}, g)$ with constant lapse and shift and time-independent spatial metric g_{ij}, called here a *trivially sliced space*, a curve $x^\alpha(s) = (t(s), x^i(s))$ is a geodesic if and only if $x^i(s)$ is a geodesic in the Riemannian manifold $\mathcal{M}$.

From the result proved above, it follows that all geodesics of trivially sliced $(\mathcal{V}, g) \equiv \mathcal{M} \times \mathbb{R}$ (with $g \equiv -dt^2 + g_{ij}(x) dx^i dx^j$) have one of the following forms: either (μ, x_0) for some $x_0 \in \mathcal{M}_0$ and μ a constant, or $(\mu, c(s))$ with $c(s)$ a geodesic of $\mathcal{M}_0$, or $(\mu s, x_0)$, or in the general case $(\mu s, c(s))$ (we have taken $t = \mu s$).

Evidently, such geodesics will be complete if and only if $c(s)$ is complete, hence $(\mathcal{V}, g)$ is timelike and null geodesically complete, if and only if $(\mathcal{M}_0, \gamma)$ is complete. We have therefore the following result.

Theorem 3.1 *Let $(\mathcal{V}, g)$ be a trivially sliced space. Then the following are equivalent:*

1. *The spacetime $(\mathcal{V}, g)$ is timelike and null geodesically complete*
2. *$(\mathcal{M}_0, \gamma)$ is a complete Riemannian manifold*
3. *The spacetime $(\mathcal{V}, g)$ is globally hyperbolic.*

This result provides a partial converse to the completeness theorem given in [2] (Thm. 3.2) under the restricted assumptions given above, and gives necessary and sufficient conditions for the nonexistence of singularities in this case.

It appears that to prove a generic singularity theorem for more general sliced spaces having non-constant lapse and shift functions and time-dependent spatial metric, one needs information about the extrinsic curvature of the slices.

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