Global hypoellipticity and compactness of resolvent for Fokker-Planck operator

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Abstract. In this paper we study the Fokker-Planck operator with potential $V(x)$, and analyze some kind of conditions imposed on the potential to ensure the validity of global hypoelliptic estimates (see Theorem 1.1). As a consequence, we obtain the compactness of resolvent of the Fokker-Planck operator if either the Witten Laplacian on 0-forms has a compact resolvent or some additional assumption on the behavior of the potential at infinity is fulfilled. This work improves the previous results of Hérau-Nier [5] and Helffer-Nier [3], by obtaining a better global hypoelliptic estimate under weaker assumptions on the potential.

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1. Introduction and main results

In this work we consider the Fokker-Planck operator

$$P = y \cdot \partial_x - \partial_x V(x) \cdot \partial_y - \triangle_y + \frac{|y|^2}{4} - \frac{n}{2}, \quad (x, y) \in \mathbb{R}^{2n} \tag{1.1}$$

where x denotes the space variable and y denotes the velocity variable, and $V(x)$ is a potential defined in the whole space $\mathbb{R}^n_x$. There have been extensive works concerned with the operator P, with various techniques from different fields such as partial differential equation, spectral theory and statistical physics. In this paper we will focus on analyzing some kind of conditions imposed on the potential $V(x)$, so that the Fokker-Planck operator P admits a global hypoelliptic estimate and has a compact resolvent. This problem is linked closely with the trend to equilibrium for the Fokker-Planck operator, and has been studied by Desvillettes-Villani, Helffer-Nier, Hérau-Nier and some other authors (see [2, 3, 5] and the references therein). It is believed that the global estimate and the compactness of resolvent are related to the properties of the potential $V(x)$. In the particular case of quadratic potential,

the theory is well developed. As far as general potential is concerned, different kind of assumptions on $V(x)$ had been explored firstly by Hérau-Nier [5] and then generalized by Helffer-Nier [3]. This work is motivated by the previous works of Hérau-Nier and Helffer-Nier, and can be seen as an improvement of their results. Our main result is the following.

Theorem 1.1. *Let* $V(x) \in C^2(\mathbb{R}^n)$ *be a real-valued function satisfying that*

$$\forall \, |\alpha| = 2, \; \exists \, C_\alpha > 0, \quad \left| \partial_x^\alpha V(x) \right| \leq C_\alpha \left(1 + |\partial_x V(x)|^2 \right)^{\frac{s}{2}} \quad \text{with } s < \frac{4}{3}. \quad (1.2)$$

Then there is a constant C, *such that for any* $u \in C_0^\infty(\mathbb{R}^{2n})$ *one has*

$$\left\| |\partial_x V(x)|^{\frac{2}{3}} u \right\|_{L^2} \leq C \left\{ \| Pu \|_{L^2} + \| u \|_{L^2} \right\}, \quad (1.3)$$

and

$$\left\| (1 - \triangle_x)^{\frac{\delta}{2}} u \right\|_{L^2} + \left\| \left(1 - \triangle_y + |y|^2 \right)^{\frac{1}{2}} u \right\|_{L^2} \leq C \left\{ \| Pu \|_{L^2} + \| u \|_{L^2} \right\}, \quad (1.4)$$

where δ *equals to* $\frac{2}{3}$ *if* $s \leq \frac{2}{3}$, $\frac{4}{3} - s$ *if* $\frac{2}{3} < s \leq \frac{10}{9}$, *and* $\frac{2}{3} - \frac{s}{2}$ *if* $\frac{10}{9} < s < \frac{4}{3}$. *As a result the operator* P *has a compact resolvent if the potential* $V(x)$ *satisfies additionally that*

$$\lim_{|x| \to +\infty} |\partial_x V(x)| = +\infty.$$

Here and throughout the paper we will use $\| \cdot \|_{L^2}$ *to denote the norm of the complex Hilbert space* $L^2(\mathbb{R}^{2n})$, *and denote by* $C_0^\infty(\mathbb{R}^{2n})$ *the set of smooth compactly supported functions.*

Remark 1.2. In particular, if the assumption (1.2) is fulfilled with $s = \frac{2}{3}$, then we have the following hypoelliptic estimate which seems to be optimal:

$$\forall \, u \in C_0^\infty(\mathbb{R}^{2n}), \quad \left\| |\partial_x V(x)|^{\frac{2}{3}} u \right\|_{L^2} + \left\| (1 - \triangle_x)^{\frac{1}{3}} u \right\|_{L^2} \leq C \left\{ \| Pu \|_{L^2} + \| u \|_{L^2} \right\}.$$

Moreover one can deduce from the above estimate a better regularity in the velocity variable y, that is,

$$\forall \, u \in C_0^\infty(\mathbb{R}^{2n}), \quad \left\| \left(1 - \triangle_y + |y|^2 \right) u \right\|_{L^2} \leq C \left\{ \| Pu \|_{L^2} + \| u \|_{L^2} \right\}.$$

This can be seen in Proposition 2.1 in the next section.

To analyze the compactness of resolvent of the operator P, the hypoellipticity techniques are an efficient tool, one of which is referred to Kohn's method [7] and another is based on nilpotent Lie groups (see [4, 8]). Kohn's method had been used by Hérau-Nier [5] to study such a potential $V(x)$ that behaves at infinity as a

high-degree homogeneous function. More precisely, if $V(x)$ satisfies that for some $C, M \geq 1$,

$$\frac{1}{C} \langle x \rangle^{2M-1} \leq \left(1 + |\partial_x V(x)|^2\right)^{\frac{1}{2}} \text{ and } \forall |\gamma| \geq 0, \ \left|\partial_x^\gamma V(x)\right| \leq C_\gamma \langle x \rangle^{2M-|\gamma|}, \quad (1.5)$$

where $\langle x \rangle = \left(1 + |x|^2\right)^{\frac{1}{2}}$, then Hérau-Nier established the following isotropic hypoelliptic estimate, by use of the global pseudo-differential calculus,

$$\left\|\Lambda_{x,y}^{\frac{1}{4}} u\right\|_{L^2} \leq C \left\{\|Pu\|_{L^2} + \|u\|_{L^2}\right\} \quad (1.6)$$

with $\Lambda_{x,y} = \left(1 - \triangle_x - \triangle_y + \frac{1}{2}|V(x)|^2 + \frac{1}{2}|y|^2\right)^{\frac{1}{2}}$. By developing the approach of Hérau-Nier, Helffer-Nier [3] obtained the same estimate as above for more general $V(x)$ which satisfies that, with some constants $c > 0$ and $k > 0$,

$$\frac{1}{c} \langle x \rangle^{\frac{1}{k}} \leq \left(1 + |\partial_x V(x)|^2\right)^{\frac{1}{2}} \leq c \langle x \rangle^k,$$
$$\forall |\gamma| \geq 1, \quad \left|\partial_x^\gamma V(x)\right| \leq C_\gamma \left(1 + |\partial_x V(x)|^2\right)^{\frac{1}{2}}. \quad (1.7)$$

As for the Kohn's proof for the hypoellipticity, the exponent $\frac{1}{4}$ in (1.6) is not optimal. A better exponent, which seems to be $\frac{2}{3}$ as seen in [8], can be obtained via explicit method in the particular case when $V(x)$ is a non-degenerate quadratic form. Moreover Helffer-Nier [3] studied such a $V(x)$ that satisfies

$$\forall |\alpha| = 2, \quad \left|\partial_x^\alpha V(x)\right| \leq C_\alpha \left(1 + |\partial_x V(x)|^2\right)^{\frac{1-\rho}{2}} \quad \text{with} \ \rho > \frac{1}{3}, \quad (1.8)$$

and obtained the estimate

$$\left\||\partial_x V(x)|^{\frac{2}{3}} u\right\|_{L^2} \leq C \left\{\|Pu\|_{L^2} + \|u\|_{L^2}\right\}. \quad (1.9)$$

This generalized the quadratic potential case, and their main tool is the nilpotent technique that initiated by [8] and then developed by [4]. Although the estimate (1.9) is better, the condition (1.8) is stronger than (1.7) for the second derivatives, and comparing with (1.6), we see that in (1.9) some information on the Sobolev regularity in x is missing. In (1.2) we get rid of the assumptions on the behavior of $\partial_x V(x)$ at infinity. This generalizes the conditions (1.5) and (1.7). Moreover, the exponent in (1.3) is $\frac{2}{3}$, better than $\frac{1}{4}$ established in (1.6). Besides, we have relaxed the condition (1.8) by allowing the number ρ there to take values in the interval $]-\frac{1}{3}, +\infty[$. As seen in the proof presented in Section 3, our approach is direct, which seems simpler for it doesn't touch neither complicated nilpotent group techniques nor pseudo-differential calculus.

Another direction to get the compact resolvent is to analyze the relationship between P and the Witten Laplace operator $\triangle_{V/2}^{(0)}$ defined by

$$\triangle_{V/2}^{(0)} = -\triangle_x + \frac{1}{4}\,|\partial_x V(x)|^2 - \frac{1}{2}\triangle_x V(x).$$

In [3], Helffer-Nier stated a conjecture which says that the Fokker-Planck operator P has a compact resolvent if and only if the Witten Laplacian $\triangle_{V/2}^{(0)}$ has a compact resolvent. The necessity part is well-known, and under rather weak assumptions on the potential V, saying $V \in C^\infty(\mathbb{R}^{2n})$ for instance, the Witten Laplacian $\triangle_{V/2}^{(0)}$ has a compact resolvent if the Fokker-Planck operator P has a compact resolvent. The reverse implication still remains open, and some partial answers have been obtained by [3,5]. For example, suppose $V \in C^\infty(\mathbb{R}^{2n})$ such that

$$\forall\, |\gamma| \geq 0, \ \ \forall\, x \in \mathbb{R}^{2n}, \quad \left|\partial_x^\gamma V(x)\right| \leq C_\gamma \left(1 + |\partial_x V(x)|^2\right)^{\frac{1}{2}},$$

$$\exists\, M,\ C > 1, \ \ \forall\, x \in \mathbb{R}^{2n}, \quad |\partial_x V(x)| \leq C \, \langle x \rangle^M,$$

and

$$\exists\, \kappa > 0, \ \ \forall\, |\alpha| = 2, \ \ \forall\, x \in \mathbb{R}^{2n}, \quad \left|\partial_x^\alpha V(x)\right| \leq C_\alpha \left(1 + |\partial_x V(x)|^2\right)^{\frac{1}{2}} \langle x \rangle^{-\kappa}.$$

Then the operator P has a compact resolvent if the Witten Laplace operator $\triangle_{V/2}^{(0)}$ has a compact resolvent (see [3, Corollary 5.10]). Due to Theorem 1.1, we can generalize the previous results as follows.

Corollary 1.3. *Let $V(x)$ satisfy the condition* (1.2). *Then the Fokker-Planck operator P has a compact resolvent if the Witten Laplacian $\triangle_{V/2}^{(0)}$ has a compact resolvent.*

Remark 1.4. In fact the above corollary is just a consequence of a part of Theorem 1.1. This can be seen at the end of Section 3.

The paper is organized as follow. In the next section we introduce some notations used throughout the paper, and then present some regularity results on the velocity variable y. Since the proof of Theorem 1.1 is quite lengthy, we divide it into two parts and proceed to handle them in Section 3 and Section 4. The proof of Corollary 1.3 will be presented in Section 3.

2. Notations and regularity in velocity variable

We firstly list some notations used throughout the paper in Subsection 2.1, and then establish the regularity in the velocity variable y in Subsection 2.2. This will give the desired estimate on the second term on the left of (1.4).

2.1. Notations

Throughout the paper we denote by (ξ, η) the dual variables of (x, y), and denote by $\langle \cdot, \cdot \rangle_{L^2}$ the inner product of the complex Hilbert space $L^2(\mathbb{R}^{2n})$. Set

$$D_{x_j} = -i\,\partial_{x_j}, \quad D_{y_j} = -i\,\partial_{y_j} \text{ and } D_x = (D_{x_1}, \cdots, D_{x_n}), \quad D_y = (D_{y_1}, \cdots, D_{y_n}).$$

Let Λ_y be the operator given by

$$\Lambda_y = \left(1 + \frac{1}{2}|y|^2 - \Delta_y\right)^{\frac{1}{2}}.$$

Observing $|\partial_x V(x)|$ is only continuous, we have to replace it sometimes by a equivalent C^1 function $f(x)$ given by

$$f(x) = \left(1 + |\partial_x V(x)|^2\right)^{\frac{1}{2}}.$$

Denoting $Q = y \cdot D_x - \partial_x V(x) \cdot D_y$ and $L_j = \partial_{y_j} + \frac{y_j}{2}, j = 1, \cdots n$, we can write the operator P given in (1.1) as

$$P = iQ + \sum_{j=1}^{n} L_j^* L_j. \tag{2.1}$$

2.2. Regularity in the velocity variable

In view of the expression (2.1), we see that the required estimate on the term $\|\Lambda_y u\|_{L^2}$ is easy to get, without any assumption on the potential $V(x)$. Indeed, As a result of (2.1), we have

$$\forall u \in C_0^\infty(\mathbb{R}^{2n}), \quad \sum_{j=1}^{n} \|L_j u\|_{L^2}^2 \le \operatorname{Re}\langle Pu, u \rangle_{L^2}, \tag{2.2}$$

from which one can deduce that

$$\forall u \in C_0^\infty(\mathbb{R}^{2n}), \quad \|\Lambda_y u\|_{L^2}^2 \le C\left\{|\langle Pu, u \rangle_{L^2}| + \|u\|_{L^2}^2\right\}. \tag{2.3}$$

This gives the desired estimate on the second term on the left of (1.4).

For constant potential, *i.e.*, $\partial_x V(x) = 0$, starting from the regularity in x, we can derive a better Sobolev exponent, which is known to be 2, for the regularity in y variable (see for instance [1]). When general potential is involved, we have the following estimate.

Proposition 2.1. *There exists a constant C such that for any $u \in C_0^\infty(\mathbb{R}^{2n})$,*

$$\left\| \Lambda_y^2 u \right\|_{L^2} \leq C \left\{ \left\| |\partial_x V(x)|^{\frac{2}{3}} u \right\|_{L^2} + \left\| (1 - \triangle_x)^{\frac{1}{3}} u \right\|_{L^2} + \|Pu\|_{L^2} \right\}, \qquad (2.4)$$

or equivalently,

$$\sum_{j=1}^n \left\| L_j L_j^* u \right\|_{L^2} \leq C \left\{ \left\| |\partial_x V(x)|^{\frac{2}{3}} u \right\|_{L^2} + \left\| (1 - \triangle_x)^{\frac{1}{3}} u \right\|_{L^2} + \|Pu\|_{L^2} \right\}. \quad (2.5)$$

Proof. In this proof we show (2.5). Using (2.2) gives

$$\begin{aligned}
\left\| L_j L_j^* u \right\|_{L^2}^2 &\leq \operatorname{Re} \left\langle P L_j^* u, \ L_j^* u \right\rangle_{L^2} \\
&= \operatorname{Re} \left\langle [P, \ L_j^*] u, \ L_j^* u \right\rangle_{L^2} + \operatorname{Re} \left\langle P u, \ L_j L_j^* u \right\rangle_{L^2} \\
&\leq \operatorname{Re} \left\langle [P, \ L_j^*] u, \ L_j^* u \right\rangle_{L^2} + \frac{1}{2} \left\| L_j L_j^* u \right\|_{L^2}^2 + 2 \|Pu\|_{L^2}^2 .
\end{aligned}$$

Hence

$$\left\| L_j L_j^* u \right\|_{L^2}^2 \leq 2 \left| \left\langle [P, \ L_j^*] u, \ L_j^* u \right\rangle_{L^2} \right| + 4 \|Pu\|_{L^2}^2 .$$

Now assume the following estimate holds, for any $\varepsilon > 0$,

$$\begin{aligned}
\left| \left\langle [P, L_j^*] u, L_j^* u \right\rangle_{L^2} \right| &\leq \varepsilon \left\| L_j L_j^* u \right\|_{L^2}^2 \\
&\quad + C_\varepsilon \left\{ \left\| |\partial_x V|^{\frac{2}{3}} u \right\|_{L^2}^2 + \left\| (1 - \triangle_x)^{\frac{1}{3}} u \right\|_{L^2}^2 + \|Pu\|_{L^2}^2 \right\}.
\end{aligned} \qquad (2.6)$$

Then combining the above two inequalities and then letting ε small enough, we get the desired estimate (2.5). In order to show (2.6), we make use of the following commutation relations satisfied by $iQ, L_j, L_k^*, j, k = 1, 2, \cdots, n$,

$$[iQ, \ L_j^*] = -\frac{1}{2} \partial_{x_j} V(x) + \partial_{x_j}, \quad [L_j, \ L_k] = [L_j^*, \ L_k^*] = 0, \quad [L_j, \ L_k^*] = \delta_{jk};$$

this gives

$$[P, \ L_j^*] = -\frac{1}{2} \partial_{x_j} V(x) + \partial_{x_j} + L_j^*.$$

Then

$$\begin{aligned}
\left| \left\langle [P, \ L_j^*] u, \ L_j^* u \right\rangle_{L^2} \right| &\leq \left\langle L_j^* u, \ L_j^* u \right\rangle_{L^2} + \left| \left\langle \left(-\frac{1}{2} \partial_{x_j} V(x) + \partial_{x_j} \right) u, \ L_j^* u \right\rangle_{L^2} \right| \\
&\leq C \left\{ \left\| |\partial_x V(x)|^{\frac{2}{3}} u \right\|_{L^2}^2 + \left\| (1 - \triangle_x)^{\frac{1}{3}} u \right\|_{L^2}^2 + \left\| L_j u \right\|_{L^2}^2 \right\} \\
&\quad + C \left\{ \left\| L_j |\partial_x V(x)|^{\frac{1}{3}} u \right\|_{L^2}^2 + \left\| L_j (1 - \triangle_x)^{\frac{1}{6}} u \right\|_{L^2}^2 \right\}.
\end{aligned}$$

Moreover, note that

$$
\begin{aligned}
\left\| L_j \, |\partial_x V(x)|^{\frac{1}{3}} u \right\|_{L^2}^2 &= \left\langle L_j^* L_j u, \ |\partial_x V(x)|^{\frac{2}{3}} u \right\rangle_{L^2} \\
&= \left\langle L_j L_j^* u, \ |\partial_x V(x)|^{\frac{2}{3}} u \right\rangle_{L^2} - \left\langle u, \ |\partial_x V(x)|^{\frac{2}{3}} u \right\rangle_{L^2},
\end{aligned}
$$

and hence

$$
\forall \, \varepsilon > 0, \ \left\| L_j \, |\partial_x V(x)|^{\frac{1}{3}} u \right\|_{L^2}^2 \le \varepsilon \left\| L_j L_j^* u \right\|_{L^2}^2 + C_\varepsilon \left\{ \left\| |\partial_x V(x)|^{\frac{2}{3}} u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}.
$$

Similarly,

$$
\forall \, \varepsilon > 0, \ \left\| L_j \, (1-\triangle_x)^{\frac{1}{6}} u \right\|_{L^2}^2 \le \varepsilon \left\| L_j L_j^* u \right\|_{L^2}^2 + C_\varepsilon \left\{ \left\| (1-\triangle_x)^{\frac{1}{3}} u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}.
$$

These inequalities yield (2.6). The proof of Proposition 2.1 is thus completed. □

3. Proof of Theorem 1.1: the first part

In this section we only show (1.3) and postpone (1.4) to the next section. Let $V(x)$ satisfy the assumption (1.2). Then using the notation

$$
f(x) = \left(1 + |\partial_x V(x)|^2 \right)^{\frac{1}{2}},
$$

we have

$$
\forall \, x \in \mathbb{R}^n, \quad |\partial_x f| \le C f(x)^s \quad \text{with } s < \frac{4}{3}. \tag{3.1}
$$

The following is the main result of this section.

Proposition 3.1. *Suppose f satisfies the condition* (3.1). *Then*

$$
\exists \, C > 0, \ \forall \, u \in C_0^\infty(\mathbb{R}^{2n}), \quad \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2} \le C \left\{ \| Pu \|_{L^2} + \| u \|_{L^2} \right\}. \tag{3.2}
$$

Proof. To simplify the notation, we will use the capital letter C to denote different suitable constants. Let $R \in C^1(\mathbb{R}^{2n})$ be a real-valued function given by

$$
R = R(x, y) = 2 f(x)^{-\frac{2}{3}} \partial_x V(x) \cdot y.
$$

We can verify that

$$
\forall \, u \in C_0^\infty(\mathbb{R}^{2n}), \quad \| R u \|_{L^2} \le C \left\| |y| \, f(x)^{\frac{1}{3}} u \right\|_{L^2} \le C \left\| \Lambda_y f(x)^{\frac{1}{3}} u \right\|_{L^2}.
$$

Recall $P = iQ + \sum_{j=1}^{n} L_j^* L_j$ with $Q = y \cdot D_x - \partial_x V(x) \cdot D_y$ and $L_j = \partial_{y_j} + \frac{y_j}{2}$. Then the above inequalities together with the relation

$$\operatorname{Re} \langle Pu, \ Ru \rangle_{L^2} = \operatorname{Re} \langle iQu, \ Ru \rangle_{L^2} + \operatorname{Re} \sum_{j=1}^{n} \langle L_j^* L_j u, \ Ru \rangle_{L^2}$$

yield

$$\operatorname{Re} \langle iQu, \ Ru \rangle_{L^2} \leq \|Pu\|_{L^2}^2 + \left\| \Lambda_y f(x)^{\frac{1}{3}} u \right\|_{L^2}^2 + \sum_{j=1}^{n} \left| \langle L_j^* L_j u, \ Ru \rangle_{L^2} \right|. \quad (3.3)$$

Next we will proceed to treat the terms on both sides of (3.3) by the following three steps.

Step I. Firstly we will show that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that the estimate

$$\left\| \Lambda_y f(x)^{\frac{1}{3}} u \right\|_{L^2}^2 \leq \varepsilon \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 + C_\varepsilon \left\{ \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right\} \quad (3.4)$$

holds for all $u \in C_0^\infty(\mathbb{R}^{2n})$. To confirm this, we use (2.3) to get

$$\left\| \Lambda_y f(x)^{\frac{1}{3}} u \right\|_{L^2}^2 \leq \operatorname{Re} \left\langle P f(x)^{\frac{1}{3}} u, \ f(x)^{\frac{1}{3}} u \right\rangle_{L^2}$$
$$= \operatorname{Re} \left\langle Pu, \ f(x)^{\frac{2}{3}} u \right\rangle_{L^2} + \operatorname{Re} \left\langle \left[P, \ f(x)^{\frac{1}{3}} \right] u, \ f(x)^{\frac{1}{3}} u \right\rangle_{L^2}.$$

The upper bound of the term $\operatorname{Re} \left\langle Pu, \ f(x)^{\frac{2}{3}} u \right\rangle_{L^2}$ can be obtained by Cauchy-Schwarz's inequality. Then the required estimate (3.4) will follow if the following inequality holds: for any $\varepsilon_1, \varepsilon_2 > 0$, there exists a constant $C_{\varepsilon_1, \varepsilon_2}$ such that

$$\left\langle \left[P, \ f(x)^{\frac{1}{3}} \right] u, \ f(x)^{\frac{1}{3}} u \right\rangle_{L^2}$$
$$\leq \varepsilon_1 \left\| \Lambda_y f(x)^{\frac{1}{3}} u \right\|_{L^2}^2 + \varepsilon_2 \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 + C_{\varepsilon_1, \varepsilon_2} \|u\|_{L^2}^2. \quad (3.5)$$

To prove (3.5), we use (3.1); this gives

$$\left| \left[P, \ f(x)^{\frac{1}{3}} \right] \right| \leq C |y| f(x)^{s - \frac{2}{3}},$$

and hence

$$\forall \varepsilon_1 > 0, \ \operatorname{Re} \left\langle \left[P, f(x)^{\frac{1}{3}} \right] u, \ f(x)^{\frac{1}{3}} u \right\rangle_{L^2} \leq \varepsilon_1 \left\| \Lambda_y f(x)^{\frac{1}{3}} u \right\|_{L^2}^2 + C_{\varepsilon_1} \left\| f(x)^{s - \frac{2}{3}} u \right\|_{L^2}^2.$$

Since $s - \frac{2}{3} < \frac{2}{3}$ for $s < \frac{4}{3}$ then the following interpolation inequality holds:

$$\forall \, \varepsilon_2 > 0, \quad \left\| f(x)^{s-\frac{2}{3}} u \right\|_{L^2}^2 \leq \varepsilon_2 \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 + C_{\varepsilon_2} \| u \|_{L^2}^2 .$$

Now combination of the above inequalities yields (3.5).

Step II. Next we will show that there exists a constant $C > 0$ such that

$$\left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 \leq C \left\{ \mathrm{Re} \, \langle i \, Qu, \ Ru \rangle_{L^2} + \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right\} . \qquad (3.6)$$

Since $Q = y \cdot D_x - \partial_x V(x) \cdot D_y$ and $R = 2 f(x)^{-\frac{2}{3}} \partial_x V(x) \cdot y$, then it's a straight-forward verification to see that

$$\frac{i}{2} \left[R, \ Q \right] = f(x)^{-\frac{2}{3}} |\partial_x V(x)|^2 - y \cdot \partial_x \left(f(x)^{-\frac{2}{3}} \partial_x V(x) \cdot y \right) .$$

As a result, we use the relation

$$\mathrm{Re} \, \langle i \, Qu, \ Ru \rangle_{L^2} = \frac{i}{2} \, \langle [R, \ Q] \, u, \ u \rangle_{L^2}$$

to get

$$\mathrm{Re} \, \langle i \, Qu, \ Ru \rangle_{L^2} = \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 - \left\| f(x)^{-\frac{1}{3}} u \right\|_{L^2}^2$$
$$- \left\langle \left(y \cdot \partial_x \left(f(x)^{-\frac{2}{3}} \partial_x V(x) \cdot y \right) \right) u, \ u \right\rangle_{L^2} .$$

This gives

$$\left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 \leq \mathrm{Re} \, \langle i \, Qu, Ru \rangle_{L^2} + \| u \|_{L^2}^2 + \left\langle \left| y \cdot \partial_x \left(f(x)^{-\frac{2}{3}} \partial_x V(x) \cdot y \right) \right| u, \ u \right\rangle_{L^2} .$$

Moreover, by use of (3.1), we compute

$$\left| y \cdot \partial_x \left(f(x)^{-\frac{2}{3}} \partial_x V(x) \cdot y \right) \right| \leq C f(x)^{s-\frac{2}{3}} |y|^2 \leq C f(x)^{\frac{2}{3}} |y|^2 ,$$

which implies that for any $\varepsilon > 0$,

$$\left\langle \left| y \cdot \partial_x \left(f(x)^{-\frac{2}{3}} \partial_x V(x) \cdot y \right) \right| u, \ u \right\rangle_{L^2} \leq C \left\| |y| \, f(x)^{\frac{1}{3}} u \right\|_{L^2}^2 \leq C \left\| \Lambda_y f(x)^{\frac{1}{3}} u \right\|_{L^2}^2$$
$$\leq \varepsilon \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 + C_\varepsilon \left\{ \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right\} ,$$

the last inequality using (3.4). Consequently,

$$\forall \, \varepsilon > 0, \ \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 \leq \mathrm{Re} \, \langle i \, Qu, Ru \rangle_{L^2} + \varepsilon \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 + C_\varepsilon \left\{ \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right\} .$$

Letting $\varepsilon > 0$ small enough gives (3.6).

Step III. Now we prove that for any $\varepsilon > 0$ there exists a constant C_ε such that

$$\sum_{j=1}^{n} \left| \left\langle L_j^* L_j u, \ Ru \right\rangle_{L^2} \right| \leq \varepsilon \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 + C_\varepsilon \left\{ \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}. \tag{3.7}$$

As a preliminary step, we firstly show the following estimate:

$$\forall \varepsilon > 0, \quad \left\| \langle y \rangle^2 u \right\|_{L^2}^2 \leq \varepsilon \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2} + C_\varepsilon \left\{ \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}, \tag{3.8}$$

where $\langle y \rangle = \left(1 + |y|^2\right)^{\frac{1}{2}}$. Using (2.3) gives

$$\begin{aligned}
\left\| \langle y \rangle^2 u \right\|_{L^2}^2 &\leq C \left\{ \operatorname{Re} \langle P \langle y \rangle u, \ \langle y \rangle u \rangle_{L^2} + \| \langle y \rangle u \|_{L^2}^2 \right\} \\
&= C \left\{ \operatorname{Re} \left\langle Pu, \ \langle y \rangle^2 u \right\rangle_{L^2} + \operatorname{Re} \langle [P, \langle y \rangle] u, \ \langle y \rangle u \rangle_{L^2} \right\} + C \| \langle y \rangle u \|_{L^2}^2.
\end{aligned}$$

This together with (2.3) implies that

$$\left\| \langle y \rangle^2 u \right\|_{L^2}^2 \leq C \left\{ \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right\} + C \left| \langle [P, \langle y \rangle] u, \ \langle y \rangle u \rangle_{L^2} \right|. \tag{3.9}$$

Moreover observe that

$$|[P, \langle y \rangle] u| \leq C \left\{ |\partial_x V(x)| \, |u| + |\partial_y u| + |u| \right\} \leq C \left\{ f(x) \, |u| + |\partial_y u| + |u| \right\},$$

and hence for any $\varepsilon > 0$,

$$\left| \langle [P, \langle y \rangle] u, \ \langle y \rangle u \rangle_{L^2} \right| \leq \varepsilon \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 + C_\varepsilon \left\{ \left\| \Lambda_y f(x)^{\frac{1}{3}} u \right\|_{L^2}^2 + \| \Lambda_y u \|_{L^2}^2 \right\}.$$

This along with (3.4) and (2.3) gives

$$\forall \varepsilon > 0, \quad \left| \langle [P, \langle y \rangle] u, \ \langle y \rangle u \rangle_{L^2} \right| \leq \varepsilon \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 + C_\varepsilon \left\{ \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}. \tag{3.10}$$

Now combining (3.9) and (3.10), we get (3.8). As a result of (3.8), we have

$$\forall \varepsilon > 0, \quad \left\| \Lambda_y \langle y \rangle u \right\|_{L^2}^2 \leq \varepsilon \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 + C_\varepsilon \left\{ \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}. \tag{3.11}$$

Indeed by (2.3) one has

$$\begin{aligned}
\left\| \Lambda_y \langle y \rangle u \right\|_{L^2}^2 &\leq C \left\{ \operatorname{Re} \langle P \langle y \rangle u, \ \langle y \rangle u \rangle_{L^2} + \| \langle y \rangle u \|_{L^2}^2 \right\} \\
&\leq C \left| \langle [P, \langle y \rangle] u, \ \langle y \rangle u \rangle_{L^2} \right| + C \left\{ \|Pu\|_{L^2}^2 + \left\| \langle y \rangle^2 u \right\|_{L^2} \right\}.
\end{aligned}$$

So (3.11) can be deduced from (3.8) and (3.10). Now we are ready to prove (3.7). Observe

$$\left| \left\langle L_j^* L_j u, \ Ru \right\rangle_{L^2} \right| = \left| \left\langle f(x)^{\frac{1}{3}} L_j u, \ f(x)^{-\frac{1}{3}} L_j Ru \right\rangle_{L^2} \right|$$
$$\leq \left\| \Lambda_y f(x)^{\frac{1}{3}} u \right\|_{L^2}^2 + \left\| f(x)^{-\frac{1}{3}} L_j Ru \right\|_{L^2}^2.$$

Then in view of (3.4), we see that the required inequality (3.7) will follow if the following estimate holds:

$$\forall \, \varepsilon > 0, \quad \left\| f(x)^{-\frac{1}{3}} L_j Ru \right\|_{L^2}^2 \leq \varepsilon \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 + C_\varepsilon \left\{ \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right\}. \quad (3.12)$$

Since

$$L_j Ru = 2u f(x)^{-\frac{2}{3}} \partial_{y_j} (\partial_x V(x) \cdot y) + 2 f(x)^{-\frac{2}{3}} (\partial_x V(x) \cdot y) \partial_{y_j} u$$
$$+ f(x)^{-\frac{2}{3}} y_j (\partial_x V(x) \cdot y) u,$$

then

$$\left\| f(x)^{-\frac{1}{3}} L_j Ru \right\|_{L^2}^2 \leq C \left\{ \| u \|_{L^2}^2 + \left\| \Lambda_y \langle y \rangle u \right\|_{L^2}^2 \right\}.$$

This along with (3.11) gives (3.12), completing the proof (3.7).

Now we combine the inequalities (3.3), (3.4), (3.6) and (3.7), to obtain

$$\forall \, \varepsilon > 0, \quad \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 \leq \varepsilon \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 + C_\varepsilon \left\{ \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right\}.$$

Taking $\varepsilon = \frac{1}{2}$ gives the desired estimate (3.2). This completes the proof of Proposition 3.1. $\qquad\square$

The rest of this section is occupied by the proof of Corollary 1.3 which is in fact a consequence of Proposition 3.1.

Proof of Corollary 1.3. Observe the compactness of the resolvent $(1+P)^{-1}$ can be deduced from Proposition 3.1 together with the condition that

$$\lim_{|x| \to +\infty} f(x) = +\infty. \quad (3.13)$$

Then it suffices to show (3.13) holds whenever $1 + \triangle_{V/2}^{(0)}$ has a compact resolvent. Let's suppose that $\left(1 + \triangle_{V/2}^{(0)} \right)^{-1}$ is compact and that, contrary to the condition (3.13), there exists a sequence $\left\{ x_\mu \right\}_{\mu \geq 1}$ in $\mathbb{R}^n$ and a constant M such that

$$\lim_{\mu \to +\infty} |x_\mu| = +\infty \quad \text{and} \quad \sup_{\mu \geq 1} f(x_\mu) \leq M.$$

As to be seen in Lemma 4.2, the condition (3.1) allows us to find a positive number r and a constant C, both independent of μ, such that

$$\forall x \in B(x_\mu; r) \overset{\text{def}}{=} \left\{ z \in \mathbb{R}^n; \ |z - x_\mu| \leq r \right\}, \quad C^{-1} \leq \frac{f(x)}{f(x_\mu)} \leq C, \quad (3.14)$$

due to the fact $1 \leq f(x_\mu) \leq M$. Moreover since $|x_\mu| \to +\infty$ we could choose a subsequence, still denoted by $\{x_\mu\}_{\mu \geq 1}$, such that the Euclidean balls $B(x_\mu; r)$ are mutually disjoint. Now we take

$$h_\mu(x) = \chi(x - x_\mu)$$

with χ a smooth function such that

$$\text{supp } \chi \subset \left\{ z \in \mathbb{R}^n; \ |z| < r \right\}, \quad \int_{\mathbb{R}^n} |\chi(x)|^2 \, dx = 1.$$

It then follows that $h_\mu \in C_0^\infty \left(B(x_\mu, r) \right)$ and

$$\langle h_\mu, \ h_\nu \rangle_{L^2(\mathbb{R}^n)} = \delta_{\mu, \nu}. \quad (3.15)$$

Furthermore we could find a constant $C_{M,r}$ depending only on M and r, such that

$$\forall \mu \geq 1, \ \forall x \in \mathbb{R}^n, \quad \left| (1 - \triangle_x) h_\mu(x) \right| \leq C_{M,r},$$

and that by virtue of (3.14) and (3.1),

$$\forall \mu \geq 1, \ \forall x \in \mathbb{R}^n, \quad \left| \left(\frac{1}{4} |\partial_x V(x)|^2 - \frac{1}{2} \triangle V(x) \right) h_\mu(x) \right| \leq C_{M,r}.$$

As a result there exists a constant $\tilde{C}_{M,r}$ depending only on M and r, such that

$$\sup_{\mu \geq 1} \left\| \left(1 + \triangle_{V/2}^{(0)} \right) h_\mu \right\|_{L^2(\mathbb{R}^n)} \leq \tilde{C}_{M,r}.$$

Since $\left(1 + \triangle_{V/2}^{(0)} \right)^{-1}$ is compact then $\{h_\mu\}_{\mu \geq 1}$ admits a strongly convergent subsequence in $L^2(\mathbb{R}^n)$. This contradicts (3.15). The proof of Corollary 1.3 is hence completed. $\qquad\square$

4. Proof of Theorem 1.1: the second part

This section is devoted to the proof of (1.4), and then the proof of Theorem 1.1 will be completed. As a convention, we use the capital letter C to denote different

suitable constants. Let V satisfy the assumption (1.2). As a result we have, with $f(x) = \left(1 + |\partial_x V(x)|^2\right)^{\frac{1}{2}}$,

$$\forall x \in \mathbb{R}^n, \quad |\partial_x f(x)| \leq C f(x)^s \quad \text{with } s < \frac{4}{3}. \tag{4.1}$$

In view of (2.3), to prove (1.4) one only has to show

Proposition 4.1. *If $V(x)$ satisfies the assumption* (1.2), *then*

$$\forall u \in C_0^\infty(\mathbb{R}^{2n}), \quad \left\| (1 - \triangle_x)^{\frac{\delta}{2}} u \right\|_{L^2} \leq C \left\{ \| Pu \|_{L^2} + \| u \|_{L^2} \right\}, \tag{4.2}$$

where δ equals to $\frac{2}{3}$ if $s \leq \frac{2}{3}$, $\frac{4}{3} - s$ if $\frac{2}{3} < s \leq \frac{10}{9}$, and $\frac{2}{3} - \frac{s}{2}$ if $\frac{10}{9} < s < \frac{4}{3}$.

We will use localization arguments to prove the above proposition. Firstly let's recall some standard results concerning the partition of unity. For more detail we refer to [6] for instance. Let g be a metric of the following form

$$g_x = f(x)^s |dx|^2, \quad x \in \mathbb{R}^n, \tag{4.3}$$

where s is the real number given in (4.1).

Lemma 4.2. *Suppose f satisfies the condition* (4.1). *Then the metric g defined by* (4.3) *is slowly varying, i.e., we can find two constants $C_*, r > 0$ such that if $g_x(x - y) \leq r^2$ then*

$$C_*^{-2} \leq \frac{g_x}{g_y} \leq C_*^2.$$

Proof. We only need to show that

$$\exists r, C_* > 0, \ \forall x, y \in \mathbb{R}^n, \quad |x - y| \leq r f(x)^{-\frac{s}{2}} \implies C_*^{-1} \leq \frac{f(x)^{\frac{s}{2}}}{f(y)^{\frac{s}{2}}} \leq C_*. \tag{4.4}$$

Making use of (3.1) and the fact that $s < \frac{4}{3}$, we have

$$\forall x \in \mathbb{R}^n, \quad \left| \partial_x \left(f(x)^{-\frac{s}{2}} \right) \right| \leq f(x)^{-\frac{s}{2}-1} |\partial_x f(x)| \leq C f(x)^{\frac{s}{2}-1} \leq C$$

with C the constant in (3.1). As a consequence, one can find a constant $\tilde{C}$ depending only on C and the dimension n, such that

$$\forall x, y \in \mathbb{R}^n, \quad \left| f(x)^{-\frac{s}{2}} - f(y)^{-\frac{s}{2}} \right| \leq \tilde{C} |x - y|,$$

from which we conclude that if $|x - y| \leq r f(x)^{-\frac{s}{2}}$ then

$$\left| f(x)^{-\frac{s}{2}} - f(y)^{-\frac{s}{2}} \right| \leq r \tilde{C} f(x)^{-\frac{s}{2}}.$$

Thus

$$\left| \frac{f(x)^{\frac{s}{2}}}{f(y)^{\frac{s}{2}}} - 1 \right| \le r\tilde{C}.$$

This gives (4.4) if we choose $r = \frac{\tilde{C}}{2}$ and $C_* = 2$. □

Let g be the metric given by (4.3). We denote by $S(1, g)$ the class of smooth real-valued functions $a(x)$ satisfying the following condition:

$$\forall \gamma \in \mathbb{Z}_+^n, \quad \forall x \in \mathbb{R}^n, \quad \left| \partial^\gamma a(x) \right| \le C_\gamma f(x)^{\frac{s|\gamma|}{2}}.$$

The space $S(1, g)$ endowed with the seminorms

$$|a|_{k,S(1,g)} = \sup_{x \in \mathbb{R}^n, |\gamma| = k} f(x)^{-\frac{sk}{2}} \left| \partial^\gamma a(x) \right|, \quad k \ge 0,$$

becomes a Fréchet space.

The main feature of a slowly varying metric is that it allows us to introduce some partitions of unity related to the metric. We state it as the following lemma.

Lemma 4.3 ([6, Lemma 18.4.4.]). *Let g be a slowly varying metric. We can find a constant $r_0 > 0$ and a sequence $x_\mu \in \mathbb{R}^n$, $\mu \ge 1$, such that the union of the balls*

$$\Omega_{\mu, r_0} = \left\{ x \in \mathbb{R}^n; \quad g_{x_\mu} \left(x - x_\mu \right) < r_0^2 \right\}$$

coves the whole space $\mathbb{R}^n$. Moreover there exists a positive integer N, depending only on r_0, such that the intersection of more than N balls is always empty. One can choose a family of nonnegative functions $\left\{ \varphi_\mu \right\}_{\mu \ge 1}$ uniformly bounded in $S(1, g)$ such that

$$\text{supp } \varphi_\mu \subset \Omega_{\mu, r_0}, \quad \sum_{\mu \ge 1} \varphi_\mu^2 = 1 \quad \text{and} \quad \sup_{\mu \ge 1} \left| \partial_x \varphi_\mu(x) \right| \le C f(x)^{\frac{s}{2}}. \quad (4.5)$$

Here by uniformly bounded in $S(1, g)$, we mean

$$\sup_\mu \left| \varphi_\mu \right|_{k, S(1,g)} \le C_k, \quad k \ge 0.$$

Remark 4.4. If we choose r_0 small enough such that $r_0 \le r$ with r the constant given in Lemma 4.2, then there exists a constant C, such that for any $\mu \ge 1$ one has

$$\forall x, y \in \text{supp } \varphi_\mu, \quad C^{-1} f(y) \le f(x) \le C f(y). \quad (4.6)$$

Lemma 4.5. *Let $V(x)$ satisfy the assumption (1.2), and let $\{\varphi_\mu\}_{\mu\geq 1}$ be the partition of unity given above. Then for any $u \in C_0^\infty(\mathbb{R}^{2n})$ we have*

$$\sum_{\mu\geq 1}\left\|(y\cdot\partial_x\varphi_\mu)\,u\right\|_{L^2}^2 \leq C\left\|\Lambda_y f(x)^{\frac{s}{2}}u\right\|_{L^2}^2 \tag{4.7}$$

and

$$\sum_{\mu\geq 1}\left\|\varphi_\mu(x)\left(\partial_x V(x)-\partial_x V(x_\mu)\right)\cdot\partial_y u\right\|_{L^2}^2 \leq C\left\|\Lambda_y f(x)^{\frac{s}{2}}u\right\|_{L^2}^2. \tag{4.8}$$

Proof. Firstly we show (4.7). Observe

$$\left\|(y\cdot\partial_x\varphi_\mu)\,u\right\|_{L^2}^2 = \left\langle(y\cdot\partial_x\varphi_\mu)^2\,u,\ u\right\rangle_{L^2},$$

and by Lemma 4.3, we see that $\sum_{\mu\geq 1}|\partial_x\varphi_\mu|^2$ is a sum of at most N terms and hence bounded from above by f^s. As a result,

$$\sum_{\mu\geq 1}\left(y\cdot\partial_x\varphi_\mu(x)\right)^2 \leq C\,|y|^2\sum_{\mu\geq 1}|\partial_x\varphi_\mu|^2 \leq C\,|y|^2\,f^s.$$

Then (4.7) follows. Next we estimate (4.8). Note that $|x-x_\mu| \leq C f(x_\mu)^{-\frac{s}{2}}$ for any $x \in \operatorname{supp}\varphi_\mu$, and hence we can deduce from (1.2) and (4.6) that

$$\sum_{\mu\geq 1}\varphi_\mu(x)^2\left|\partial_x V(x)-\partial_x V(x_\mu)\right|^2 \leq C\sum_{\mu\geq 1}\varphi_\mu(x)^2 f(x)^{2s}\left|x-x_\mu\right|^2 \leq Cf(x)^s.$$

This along with the inequality

$$\sum_{\mu\geq 1}\left\|\varphi_\mu\left(\partial_x V(x)-\partial_x V(x_\mu)\right)\cdot\partial_y u\right\|_{L^2}^2$$

$$\leq \left\langle\sum_{\mu\geq 1}\varphi_\mu(x)^2\left|\partial_x V(x)-\partial_x V(x_\mu)\right|^2\left|\partial_y u\right|,\ \left|\partial_y u\right|\right\rangle_{L^2}$$

implies (4.8). Then the proof is completed. $\qquad\square$

Lemma 4.6. *Let $\{\varphi_\mu\}_{\mu\geq 1}$ be the partition given in Lemma 4.3, and let $a\in{]0,1/2[}$ be a real number. Then there exists a constant C, depending on the integer N given in Lemma 4.3, such that for any $u \in C_0^\infty(\mathbb{R}^{2n})$ we have*

$$\left\|(1-\triangle_x)^a\,u\right\|_{L^2}^2 \leq C\sum_{\mu\geq 1}\left\|(1-\triangle_x)^a\,\varphi_\mu u\right\|_{L^2}^2 + C\left\|Pu\right\|_{L^2}^2 + C\left\|u\right\|_{L^2}^2. \tag{4.9}$$

In order to prove Lemma 4.6 we need the following technical lemma.

Lemma 4.7. *Let $b \in]0, 1[$ be a real number and $|D_x|^b$ be the Fourier multiplier defined by, with $u \in C_0^\infty(\mathbb{R}^n)$,*

$$|D_x|^b\, u(x) = \mathcal{F}^{-1}\left(|\xi|^b\, \hat{u}(\xi)\right).$$

Let $\left\{\varphi_\mu\right\}_{\mu \geq 1}$ be the partition given in Lemma 4.3. Then there exists a constant C such that for any $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$\left\|\sum_{\mu \geq 1}\left[|D_x|^b,\ f^{-s/2}\varphi_\mu\right]\varphi_\mu u\right\|_{L^2(\mathbb{R}^n)} \leq C\,\|u\|_{L^2(\mathbb{R}^n)} \tag{4.10}$$

and

$$\left\|\left[|D_x|^b,\ f^{-s/2}\right]u\right\|_{L^2(\mathbb{R}^n)} \leq C\,\|u\|_{L^2(\mathbb{R}^n)}. \tag{4.11}$$

Recall here $f(x) = \left(1 + |\partial_x V(x)|^2\right)^{1/2}$ and s is the real number given in (4.1).

Proof. In the proof we use C to denote different suitable positive constants, and for simplicity we use the notation

$$\omega_\mu = f^{-s/2}\varphi_\mu.$$

In view of Lemma 4.3 and the estimate (4.1), we have

$$\sup_{x \in \mathbb{R}^n}\left(\sum_{\mu \geq 1}\left|\varphi_\mu(x)\right|^2\right)^{\frac{1}{2}} + \sup_{x,x' \in \mathbb{R}^n}\left(\sum_{\mu \geq 1}\left|\omega_\mu(x) - \omega_\mu(x')\right|^2\right)^{\frac{1}{2}}$$
$$+ \sup_{x \in \mathbb{R}^n}\left(\sum_{\mu \geq 1}\left|\partial_x\omega_\mu(x)\right|^2\right)^{\frac{1}{2}} \leq C. \tag{4.12}$$

Next we will show the following relation:

$$\forall\, u \in C_0^\infty(\mathbb{R}^n), \quad |D_x|^b\, u(x) = C_b \int_{\mathbb{R}^n} \frac{u(x) - u(x - \tilde{x})}{|\tilde{x}|^{n+b}}\, d\tilde{x} \tag{4.13}$$

with $C_b \neq 0$ being a complex constant depending only on the real number b and the dimension n. In fact, the inverse Fourier transform implies

$$\int_{\mathbb{R}^n} \frac{u(x) - u(x - \tilde{x})}{|\tilde{x}|^{n+b}}\, d\tilde{x} = \int_{\mathbb{R}^n} \hat{u}(\xi)\, e^{i\,x \cdot \xi}\left(\int_{\mathbb{R}^n} \frac{1 - e^{-i\,\tilde{x} \cdot \xi}}{|\tilde{x}|^{n+b}}\, d\tilde{x}\right) d\xi.$$

On the other hand, we can verify that

$$\int_{\mathbb{R}^n} \frac{1 - e^{-i\,\tilde{x} \cdot \xi}}{|\tilde{x}|^{n+b}}\, d\tilde{x} = |\xi|^b \int_{\mathbb{R}^n} \frac{1 - e^{-i\,z \cdot \frac{\xi}{|\xi|}}}{|z|^{n+b}}\, dz.$$

Observe that $\int_{\mathbb{R}^n} \frac{1-e^{-i\,z\cdot\frac{\xi}{|\xi|}}}{|z|^{n+b}}\,dz \neq 0$ is a complex constant depending only on b and the dimension n, but independent of ξ. Then the above two equalities give (4.13). Now we use (4.13) to get

$$|D_x|^b\,(\omega_\mu\,\varphi_\mu\,u)(x) = C_b \int_{\mathbb{R}^n} \frac{\omega_\mu(x)\varphi_\mu(x)u(x) - \omega_\mu(x-\tilde{x})\varphi_\mu(x-\tilde{x})u(x-\tilde{x})}{|\tilde{x}|^{n+b}}\,d\tilde{x}$$

$$= \omega_\mu(x)|D_x|^b\,(\varphi_\mu\,u)(x) + C_b \int_{\mathbb{R}^n} \frac{\varphi_\mu(x-\tilde{x})u(x-\tilde{x})\big(\omega_\mu(x)-\omega_\mu(x-\tilde{x})\big)}{|\tilde{x}|^{n+b}}\,d\tilde{x},$$

which gives

$$\begin{aligned}
&\Big[|D_x|^b,\ \omega_\mu\Big](\varphi_\mu\,u)(x) \\
&= C_b \int_{\mathbb{R}^n} \frac{\varphi_\mu(x-\tilde{x})u(x-\tilde{x})\big(\omega_\mu(x)-\omega_\mu(x-\tilde{x})\big)}{|\tilde{x}|^{n+b}}\,d\tilde{x}.
\end{aligned} \tag{4.14}$$

Let ρ be the characteristic function of the unit ball $\{x \in \mathbb{R}^n;\ |x| \leq 1\}$. We compute

$$\left\| \sum_{\mu \geq 1} \Big[|D_x|^b,\ \omega_\mu\Big]\varphi_\mu u \right\|_{L^2}^2$$

$$= |C_b|^2 \int_{\mathbb{R}^n} \left(\sum_{\mu \geq 1} \int_{\mathbb{R}^n} \frac{u(x-\tilde{x})\varphi_\mu(x-\tilde{x})\big(\omega_\mu(x)-\omega_\mu(x-\tilde{x})\big)}{|\tilde{x}|^{n+b}}\,d\tilde{x} \right)^2 dx$$

$$\leq 2|C_b|^2 \int_{\mathbb{R}^n} \left(\sum_{\mu \geq 1} \int_{\mathbb{R}^n} \frac{\rho(\tilde{x})u(x-\tilde{x})\varphi_\mu(x-\tilde{x})\big(\omega_\mu(x)-\omega_\mu(x-\tilde{x})\big)}{|\tilde{x}|^{n+b}}\,d\tilde{x} \right)^2 dx$$

$$+ 2|C_b|^2 \int_{\mathbb{R}^n} \left(\sum_{\mu \geq 1} \int_{\mathbb{R}^n} \frac{(1-\rho(\tilde{x}))\,u(x-\tilde{x})\varphi_\mu(x-\tilde{x})\big(\omega_\mu(x)-\omega_\mu(x-\tilde{x})\big)}{|\tilde{x}|^{n+b}}\,d\tilde{x} \right)^2 dx$$

$$=: \mathcal{A}_1 + \mathcal{A}_2.$$

Now we treat the terms $\mathcal{A}_1$ and $\mathcal{A}_2$. Cauchy's inequality yields

$$\left| \sum_{\mu \geq 1} \varphi_\mu(x-\tilde{x})\big(\omega_\mu(x)-\omega_\mu(x-\tilde{x})\big) \right|$$

$$\leq \left(\sum_{\mu \geq 1} \big|\varphi_\mu(x-\tilde{x})\big|^2 \right)^{\frac{1}{2}} \left(\sum_{\mu \geq 1} \big|\omega_\mu(x)-\omega_\mu(x-\tilde{x})\big|^2 \right)^{\frac{1}{2}}.$$

This along with (4.12) gives that for any $x, \tilde{x} \in \mathbb{R}^n$, we have

$$\left| \sum_{\mu \geq 1} \varphi_\mu(x-\tilde{x})\big(\omega_\mu(x)-\omega_\mu(x-\tilde{x})\big) \right| \leq C$$

and hence
$$\mathcal{A}_2 \le C \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{(1 - \rho(\tilde{x})) \, |u(x - \tilde{x})|}{|\tilde{x}|^{n+b}} \, d\tilde{x} \right)^2 dx.$$

Moreover, using the relation
$$\omega_\mu(x) - \omega_\mu(x - \tilde{x}) = \int_0^1 \partial_x \omega_\mu \big(tx + (1 - t)(x - \tilde{x}) \big) \cdot \tilde{x} \, dt$$

and the inequality (4.12) yields that for any $x, \tilde{x} \in \mathbb{R}^n$ we have
$$\left(\sum_{\mu \ge 1} \big| \omega_\mu(x) - \omega_\mu(x - \tilde{x}) \big|^2 \right)^{\frac{1}{2}} \le C \, |\tilde{x}|$$

and hence
$$\left| \sum_{\mu \ge 1} \varphi_\mu(x - \tilde{x}) \big(\omega_\mu(x) - \omega_\mu(x - \tilde{x}) \big) \right| \le C \, |\tilde{x}|,$$

which implies
$$\mathcal{A}_1 \le C \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{\rho(\tilde{x}) \, |u(x - \tilde{x})|}{|\tilde{x}|^{n+b-1}} \, d\tilde{x} \right)^2 dx.$$

Combining these inequalities gives
$$\left\| \sum_{\mu \ge 1} \Big[|D_x|^b, \, \omega_\mu \Big] \varphi_\mu u \right\|_{L^2}^2 \le C \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{\rho(\tilde{x}) \, |u(x - \tilde{x})|}{|\tilde{x}|^{n+b-1}} \, d\tilde{x} \right)^2 dx$$
$$+ C \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{(1 - \rho(\tilde{x})) \, |u(x - \tilde{x})|}{|\tilde{x}|^{n+b}} \, d\tilde{x} \right)^2 dx.$$

Moreover, for the terms on the right side of the above inequality, we can use Young's inequality for convolutions and the fact that ρ is the characteristic function of the unit ball, to get
$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{\rho(\tilde{x}) \, |u(x - \tilde{x})|}{|\tilde{x}|^{n+b-1}} \, d\tilde{x} \right)^2 dx \le C \, \|u\|_{L^2(\mathbb{R}^n)}^2 \left\| \frac{\rho}{|x|^{n+b-1}} \right\|_{L^1(\mathbb{R}^n)}^2 \le C \, \|u\|_{L^2(\mathbb{R}^n)}^2$$

and
$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{(1 - \rho(\tilde{x})) \, |u(x - \tilde{x})|}{|\tilde{x}|^{n+b}} \, d\tilde{x} \right)^2 dx \le C \|u\|_{L^2(\mathbb{R}^n)}^2 \left\| \frac{1 - \rho}{|x|^{n+b}} \right\|_{L^1(\mathbb{R}^n)}^2 \le C \|u\|_{L^2(\mathbb{R}^n)}^2.$$

We combine these inequalities to get the desired estimate (4.10). The estimate (4.11), which is easier to treat, can be obtained via the similar arguments as above. This completes the proof. $\qquad\square$

Proof of Lemma 4.6. We only need show that, with $b \in]0, 1[$,

$$
\begin{aligned}
&\forall\, u \in C_0^\infty(\mathbb{R}^{2n}), \\
&\left\| |D_x|^b u \right\|_{L^2}^2 \le C \sum_{\mu \ge 1} \left\| |D_x|^b \varphi_\mu u \right\|_{L^2}^2 + C \left\| Pu \right\|_{L^2}^2 + C \left\| u \right\|_{L^2}^2 .
\end{aligned}
\tag{4.15}
$$

By (4.5), we see $\left\| |D_x|^b u \right\|_{L^2}^2 = \left\| \sum_{\mu \ge 1} |D_x|^b \varphi_\mu^2 u \right\|_{L^2}^2$. Thus

$$
\begin{aligned}
\left\| |D_x|^b u \right\|_{L^2}^2 \le\ &2 \left\| \sum_{\mu \ge 1} \left[|D_x|^b, f^{-\frac{s}{2}} \varphi_\mu \right] \varphi_\mu f^{\frac{s}{2}} u \right\|_{L^2}^2 \\
&+ 2 \left\| \sum_{\mu \ge 1} f^{-\frac{s}{2}} \varphi_\mu |D_x|^b \varphi_\mu f^{\frac{s}{2}} u \right\|_{L^2}^2 .
\end{aligned}
\tag{4.16}
$$

In view of (4.10) we have

$$
\left\| \sum_{\mu \ge 1} \left[|D_x|^b, f^{-\frac{s}{2}} \varphi_\mu \right] \varphi_\mu f^{\frac{s}{2}} u \right\|_{L^2}^2 \le C \left\| f^{\frac{s}{2}} u \right\|_{L^2}^2 \le C \left\| P u \right\|_{L^2}^2 + C \left\| u \right\|_{L^2}^2, \tag{4.17}
$$

the last inequality following from (3.2). It remains to handle the second term on the right side of (4.16). For each $\mu \ge 1$, set

$$
I_\mu = \left\{ \nu \ge 1; \quad \operatorname{supp} \varphi_\nu \cap \operatorname{supp} \varphi_\mu \ne \emptyset \right\}.
$$

Then I_μ is a finite set and has at most N elements. Recall here N is the integer given in Lemma 4.3 such that the intersection of more than N balls is always empty. Direct calculus give that for any $u \in C_0^\infty(\mathbb{R}^{2n})$ we have

$$
\begin{aligned}
&\left\| \sum_{\mu \ge 1} f^{-\frac{s}{2}} \varphi_\mu |D_x|^b \varphi_\mu f^{\frac{s}{2}} u \right\|_{L^2}^2 \\
&= \sum_{\mu \ge 1} \sum_{\nu \in I_\mu} \left\langle \varphi_\mu f^{-\frac{s}{2}} |D_x|^b \varphi_\mu f^{\frac{s}{2}} u, \ \varphi_\nu f^{-\frac{s}{2}} |D_x|^b \varphi_\nu f^{\frac{s}{2}} u \right\rangle_{L^2} \\
&\le \sum_{\mu \ge 1} \sum_{\nu \in I_\mu} \left\| \varphi_\mu f^{-\frac{s}{2}} |D_x|^b \varphi_\mu f^{\frac{s}{2}} u \right\|_{L^2}^2 + \sum_{\mu \ge 1} \sum_{\nu \in I_\mu} \left\| \varphi_\nu f^{-\frac{s}{2}} |D_x|^b \varphi_\nu f^{\frac{s}{2}} u \right\|_{L^2}^2 \\
&= 2 \sum_{\mu \ge 1} \sum_{\nu \in I_\mu} \left\| \varphi_\mu f^{-\frac{s}{2}} |D_x|^b \varphi_\mu f^{\frac{s}{2}} u \right\|_{L^2}^2 \\
&\le 2 \sum_{\mu \ge 1} \sum_{\nu \in I_\mu} \left\| f^{-\frac{s}{2}} |D_x|^b \varphi_\mu f^{\frac{s}{2}} u \right\|_{L^2}^2 .
\end{aligned}
$$

Since I_μ has at most N elements then it follows from the above inequalities that

$$\left\| \sum_{\mu \geq 1} f^{-\frac{s}{2}} \varphi_\mu \, |D_x|^b \, \varphi_\mu \, f^{\frac{s}{2}} u \right\|_{L^2}^2 \leq 2N \sum_{\mu \geq 1} \left\| f^{-\frac{s}{2}} \, |D_x|^b \, \varphi_\mu f^{\frac{s}{2}} u \right\|_{L^2}^2. \qquad (4.18)$$

One the other hand, one can verify that

$$\begin{aligned}
\sum_{\mu \geq 1} \left\| f^{-\frac{s}{2}} \, |D_x|^b \, \varphi_\mu f^{\frac{s}{2}} u \right\|_{L^2}^2 &\leq 2 \sum_{\mu \geq 1} \left\| \left[|D_x|^b , \; f^{-\frac{s}{2}} \right] \varphi_\mu f^{\frac{s}{2}} u \right\|_{L^2}^2 \\
&\quad + 2 \sum_{\mu \geq 1} \left\| |D_x|^b \, f^{-\frac{s}{2}} \varphi_\mu f^{\frac{s}{2}} u \right\|_{L^2}^2 \\
&\leq C \sum_{\mu \geq 1} \left\| \varphi_\mu f^{\frac{s}{2}} u \right\|_{L^2}^2 + C \sum_{\mu \geq 1} \left\| |D_x|^b \, \varphi_\mu u \right\|_{L^2}^2 \\
&\leq C \, \|P u\|_{L^2}^2 + C \, \|u\|_{L^2}^2 + C \sum_{\mu \geq 1} \left\| |D_x|^b \, \varphi_\mu u \right\|_{L^2}^2 ,
\end{aligned}$$

the second inequality using (4.11) and the last inequality using (3.2). These inequalities along with (4.18) gives, with $u \in C_0^\infty(\mathbb{R}^{2n})$,

$$\left\| \sum_{\mu \geq 1} f^{-\frac{s}{2}} \varphi_\mu \, |D_x|^b \, \varphi_\mu \, f^{\frac{s}{2}} u \right\|_{L^2}^2 \leq C \sum_{\mu \geq 1} \left\| |D_x|^b \, \varphi_\mu u \right\|_{L^2}^2 + C \, \|P u\|_{L^2}^2 + C \, \|u\|_{L^2}^2 .$$

This along with (4.16) and (4.17) yields the desired estimate (4.15) , completing the proof of Lemma 4.6. $\qquad \square$

4.1. End of the proof of Theorem 1.1

In this subsection we prove Proposition 4.1. Let $\left\{ \varphi_\mu \right\}_{\mu \geq 1}$ be the partition of unity given in Lemma 4.3. For each $\mu \geq 1$, define the operator R_μ by

$$R_\mu = -y \cdot \partial_x \varphi_\mu(x) - \varphi_\mu \left(\partial_x V(x) - \partial_x V(x_\mu) \right) \cdot \partial_y. \qquad (4.19)$$

We associate with each $x_\mu \in \mathbb{R}^n$ the operator

$$P_{x_\mu} = y \cdot \partial_x - \partial_x V(x_\mu) \cdot \partial_y - \triangle_y + \frac{|y|^2}{4} - \frac{n}{2}.$$

Then we have

$$\varphi_\mu P u = P_{x_\mu} \, \varphi_\mu \, u + R_\mu u$$

with R_μ the operator given in (4.19). This gives

$$\sum_{\mu \geq 1} \| P_{x_\mu} \, \varphi_\mu \, u \|_{L^2}^2 \leq 2 \sum_{\mu \geq 1} \left(\| \varphi_\mu \, Pu \|_{L^2}^2 + \| R_\mu u \|_{L^2}^2 \right)$$
$$\leq 2 \| Pu \|_{L^2}^2 + 2 \sum_{\mu \geq 1} \| R_\mu u \|_{L^2}^2 . \tag{4.20}$$

Proposition 4.8. *There is a constant C independent of x_μ, such that for any $u \in C_0^\infty(\mathbb{R}^{2n})$, one has*

$$\left| \partial_x V(x_\mu) \right|^{\frac{4}{3}} \| u \|_{L^2}^2 + \left\| (1 - \triangle_x)^{\frac{1}{3}} u \right\|_{L^2}^2 \leq C \left\{ \| P_{x_\mu} u \|_{L^2}^2 + \| u \|_{L^2}^2 \right\}, \tag{4.21}$$

or equivalently,

$$\left\| \tilde{\Lambda}_{x_\mu}^{\frac{2}{3}} u \right\|_{L^2}^2 \leq C \left\{ \| P_{x_\mu} u \|_{L^2}^2 + \| u \|_{L^2}^2 \right\}, \tag{4.22}$$

where $\tilde{\Lambda}_{x_\mu} = \left(1 + \frac{1}{2} \left| \partial_x V(x_\mu) \right|^2 - \triangle_x \right)^{\frac{1}{2}}$.

The above proposition can be proven in the same way as [3, Proposition 5.22], by taking Fourier analysis in the variable x and then reducing the problem to a semi-classical problem. We refer to [3] and references therein for more details.

Lemma 4.9. *Suppose $V(x)$ satisfies the assumption (1.2). Let R_μ be the operator given in (4.19). Then*

$$\forall \, u \in C_0^\infty(\mathbb{R}^{2n}), \ \sum_{\mu \geq 1} \| R_\mu u \|_{L^2}^2 \leq C \left\{ \left\| Pf(x)^{\tilde{s}} u \right\|_{L^2}^2 + \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right\}, \tag{4.23}$$

where $\tilde{s} = \frac{2}{3} - \delta$ with δ given in (1.4), i.e., $\tilde{s}$ equals to 0 if $s \leq \frac{2}{3}$, $s - \frac{2}{3}$ if $\frac{2}{3} < s \leq \frac{10}{9}$, and $\frac{s}{2}$ if $\frac{10}{9} < s < \frac{4}{3}$.

Proof. As a convention, we use the capital letter C to denote different suitable constants. Since $V(x)$ satisfies (1.2), then (4.1) holds. Observe $\sum_{\mu \geq 1} \| R_\mu u \|_{L^2}^2$ is bounded from above by

$$2 \sum_{\mu \geq 1} \left\| \left(y \cdot \partial_x \varphi_\mu \right) u \right\|_{L^2}^2 + 2 \sum_{\mu \geq 1} \left\| \varphi_\mu(x) \left(\partial_x V(x) - \partial_x V(x_\mu) \right) \cdot \partial_y u \right\|_{L^2}^2 .$$

Then in view of (4.7) and (4.8), we have

$$\sum_{\mu \geq 1} \| R_\mu u \|_{L^2}^2 \leq C \left\| \Lambda_y f(x)^{\frac{s}{2}} u \right\|_{L^2}^2 .$$

So we only have to treat the term $\left\|\Lambda_y f(x)^{\frac{s}{2}} u\right\|_{L^2}^2$. It follows from (2.3) that

$$\left\|\Lambda_y f(x)^{\frac{s}{2}} u\right\|_{L^2}^2 \leq C \left\{ \left|\left\langle Pf(x)^{\frac{s}{2}} u,\ f(x)^{\frac{s}{2}} u\right\rangle_{L^2}\right| + \left\|f(x)^{\frac{s}{2}} u\right\|_{L^2}^2 \right\}.$$

Since $\frac{s}{2} < \frac{2}{3}$ then by (3.2) we have

$$\forall\, u \in C_0^\infty\left(\mathbb{R}^{2n}\right),\ \left\|f(x)^{\frac{s}{2}} u\right\|_{L^2} \leq \left\|f(x)^{\frac{2}{3}} u\right\|_{L^2} \leq C \left\{ \|Pu\|_{L^2} + \|u\|_{L^2} \right\}. \quad (4.24)$$

The above two inequalities yield that for any $u \in C_0^\infty\left(\mathbb{R}^{2n}\right)$,

$$\left\|\Lambda_y f(x)^{\frac{s}{2}} u\right\|_{L^2}^2 \leq C \left\{ \left|\left\langle Pf(x)^{\frac{s}{2}} u,\ f(x)^{\frac{s}{2}} u\right\rangle_{L^2}\right| + \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}. \quad (4.25)$$

a) Firstly let us consider the case when $s \leq \frac{2}{3}$. In such a case, we have

$$\left|\left\langle Pf(x)^{\frac{s}{2}} u,\ f(x)^{\frac{s}{2}} u\right\rangle_{L^2}\right| \leq \left|\langle Pu,\ f(x)^s u\rangle_{L^2}\right| + \left|\left\langle \left[P,\ f(x)^{\frac{s}{2}}\right] u,\ f(x)^{\frac{s}{2}} u\right\rangle_{L^2}\right|$$

$$\leq \|Pu\|_{L^2}^2 + \left\|f(x)^{\frac{2}{3}} u\right\|_{L^2}^2 + \left|\left\langle \left[P, f(x)^{\frac{s}{2}}\right] u, f(x)^{\frac{s}{2}} u\right\rangle_{L^2}\right|$$

$$\leq C \|Pu\|_{L^2}^2 + C \|u\|_{L^2}^2 + \left|\left\langle \left[P, f(x)^{\frac{s}{2}}\right] u, f(x)^{\frac{s}{2}} u\right\rangle_{L^2}\right|,$$

the last inequality using (3.2). This along with (4.25) gives

$$\left\|\Lambda_y f(x)^{\frac{s}{2}} u\right\|_{L^2}^2 \leq C \|Pu\|_{L^2}^2 + C \|u\|_{L^2}^2 + \left|\left\langle \left[P,\ f(x)^{\frac{s}{2}}\right] u,\ f(x)^{\frac{s}{2}} u\right\rangle_{L^2}\right|.$$

On the other hand using (4.1) with $s \leq \frac{2}{3}$ implies, for any $\varepsilon > 0$,

$$\left|\left\langle \left[P,\ f(x)^{\frac{s}{2}}\right] u,\ f(x)^{\frac{s}{2}} u\right\rangle_{L^2}\right| \leq C \left\|\Lambda_y f(x)^{\frac{s}{2}} u\right\|_{L^2} \|u\|_{L^2}$$

$$\leq \varepsilon \left\|\Lambda_y f(x)^{\frac{s}{2}} u\right\|_{L^2}^2 + C_\varepsilon \|u\|_{L^2}^2.$$

Combining the above two inequalities and taking $\varepsilon \leq \frac{1}{2}$, we get

$$\left\|\Lambda_y f(x)^{\frac{s}{2}} u\right\|_{L^2}^2 \leq C \|Pu\|_{L^2}^2 + C \|u\|_{L^2}^2.$$

Since $\sum_{\mu \geq 1} \|R_\mu u\|_{L^2}^2 \leq C \left\|\Lambda_y f(x)^{\frac{s}{2}} u\right\|_{L^2}^2$, then the above estimate gives the validity of (4.23) for $s \leq \frac{2}{3}$.

b) Next we shall prove (4.23) for $\frac{2}{3} < s < \frac{4}{3}$. If $\frac{10}{9} < s < \frac{4}{3}$, then it follows from (4.25) and (4.24) that

$$\forall C_0^\infty \left(\mathbb{R}^{2n} \right), \quad \left\| \Lambda_y f(x)^{\frac{s}{2}} u \right\|_{L^2}^2 \leq C \left\| P f(x)^{\frac{s}{2}} u \right\|_{L^2}^2 + \|Pu\|_{L^2} + C \|u\|_{L^2}^2. \quad (4.26)$$

This gives the validity of (4.23) for $s \in]\frac{10}{9}, \frac{4}{3}[$.

Now we focus on the case when $\frac{2}{3} < s \leq \frac{10}{9}$. Observe that

$$\left| \left\langle P f(x)^{\frac{s}{2}} u, \ f(x)^{\frac{s}{2}} u \right\rangle_{L^2} \right| = \left| \left\langle P f(x)^{\frac{s}{2}} u, \ f(x)^{\frac{2}{3} + \left(\frac{s}{2} - \frac{2}{3} \right)} u \right\rangle_{L^2} \right|$$

$$\leq \left| \left\langle P f(x)^{s - \frac{2}{3}} u, \ f(x)^{\frac{2}{3}} u \right\rangle_{L^2} \right| + \left| \left\langle \left[P, \ f(x)^{\frac{s}{2} - \frac{2}{3}} \right] f(x)^{\frac{s}{2}} u, \ f(x)^{\frac{2}{3}} u \right\rangle_{L^2} \right|.$$

Moreover since $f(x)$ satisfies (4.1), then

$$\left| \left[P, \ f(x)^{\frac{s}{2} - \frac{2}{3}} \right] f(x)^{\frac{s}{2}} u \right| \leq C \, |y| \, f(x)^{2s - \frac{5}{3}} \, |u| \, ,$$

and thus

$$\left| \left\langle \left[P, \ f(x)^{\frac{s}{2} - \frac{2}{3}} \right] f(x)^{\frac{s}{2}} u, \ f(x)^{\frac{2}{3}} u \right\rangle_{L^2} \right| \leq \varepsilon \left\| \Lambda_y f(x)^{2s - \frac{5}{3}} u \right\|_{L^2}^2 + C_\varepsilon \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2.$$

Combination of the above three inequalities gives

$$\left| \left\langle P f(x)^{\frac{s}{2}} u, \ f(x)^{\frac{s}{2}} u \right\rangle_{L^2} \right| \leq \varepsilon \left\| \Lambda_y f(x)^{2s - \frac{5}{3}} u \right\|_{L^2}^2$$

$$+ C_\varepsilon \left\{ \left\| P f(x)^{s - \frac{2}{3}} u \right\|_{L^2}^2 + \left\| f(x)^{\frac{2}{3}} u \right\|_{L^2}^2 \right\}.$$

Moreover since $2s - \frac{5}{3} \leq \frac{s}{2}$ for $s \leq \frac{10}{9}$, then

$$\left\| \Lambda_y f(x)^{2s - \frac{5}{3}} u \right\|_{L^2}^2 \leq \left\| \Lambda_y f(x)^{\frac{s}{2}} u \right\|_{L^2}^2,$$

and hence by (3.2) we obtain

$$\left| \left\langle P f(x)^{\frac{s}{2}} u, \ f(x)^{\frac{s}{2}} u \right\rangle_{L^2} \right| \leq \varepsilon \left\| \Lambda_y f(x)^{\frac{s}{2}} u \right\|_{L^2}^2$$

$$+ C_\varepsilon \left\{ \left\| P f(x)^{s - \frac{2}{3}} u \right\|_{L^2}^2 + \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}.$$

Inserting the above inequality into (4.25) and then taking ε small enough, we get the desired estimate (4.23) for $\frac{2}{3} < s < \frac{10}{9}$. Thus the proof of Lemma 4.9 is completed. $\square$

Now we are ready to prove the main result of this section.

Proof of Proposition 4.1. Now we want to show that

$$\left\| (1 - \triangle_x)^{\frac{\delta}{2}} u \right\|_{L^2}^2 \le C \left\{ \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right\}. \tag{4.27}$$

Recall δ equals to $\frac{2}{3}$ if $s \le \frac{2}{3}$, $\frac{4}{3} - s$ if $\frac{2}{3} < s \le \frac{10}{9}$, and $\frac{2}{3} - \frac{s}{2}$ if $\frac{10}{9} < s < \frac{4}{3}$. Using the estimates (4.20) and (4.23) gives that

$$\forall u \in C_0^\infty(\mathbb{R}^{2n}), \ \sum_{\mu \ge 1} \| P_{x_\mu} \varphi_\mu u \|_{L^2}^2 \le C \left\{ \left\| P f(x)^{\tilde{s}} u \right\|_{L^2}^2 + \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right\}, \tag{4.28}$$

where $\tilde{s} = \frac{2}{3} - \delta$. We can verify that

$$-\tilde{s} + s - 1 \le 0. \tag{4.29}$$

Firstly let us consider the case that $s \le \frac{2}{3}$. Then $\tilde{s} = 0$ and (4.28) becomes

$$\forall \ u \in C_0^\infty(\mathbb{R}^{2n}), \quad \sum_{\mu \ge 1} \| P_{x_\mu} \varphi_\mu u \|_{L^2}^2 \le C \left\{ \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right\}.$$

On the other hand, using (4.9) with $a = \frac{1}{3}$ and then using (4.21), we have

$$\left\| (1 - \triangle_x)^{\frac{1}{3}} u \right\|_{L^2}^2 \le C \sum_{\mu \ge 1} \left\| (1 - \triangle_x)^{\frac{1}{3}} \varphi_\mu u \right\|_{L^2}^2 + C \| Pu \|_{L^2}^2 + C \| u \|_{L^2}^2$$

$$\le C \sum_{\mu \ge 1} \| P_{x_\mu} \varphi_\mu u \|_{L^2}^2 + C \sum_{\mu \ge 1} \| \varphi_\mu u \|_{L^2}^2 + C \| Pu \|_{L^2}^2 + C \| u \|_{L^2}^2.$$

As a result, it follows from these inequalities that

$$\forall \ u \in C_0^\infty(\mathbb{R}^{2n}) \quad \left\| (1 - \triangle_x)^{\frac{1}{3}} u \right\|_{L^2}^2 \le C \left\{ \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right\}.$$

This gives the validity of (4.27) for $s \le \frac{2}{3}$.

Now we consider the case when $\frac{2}{3} < s < \frac{4}{3}$. Note that $\delta = \frac{2}{3} - \tilde{s}$. Then we use (4.9) with $a = \frac{\delta}{2}$ to get

$$\left\| (1 - \triangle_x)^{\frac{\delta}{2}} u \right\|_{L^2}^2 \le C \sum_{\mu \ge 1} \left\| (1 - \triangle_x)^{\frac{\delta}{2}} \varphi_\mu u \right\|_{L^2}^2 + C \| Pu \|_{L^2}^2 + C \| u \|_{L^2}^2$$

$$\le C \sum_{\mu \ge 1} \left\| \left(1 + \frac{1}{2} | \partial_x V(x_\mu) |^2 - \triangle_x \right)^{\frac{\delta}{2}} \varphi_\mu u \right\|_{L^2}^2 + C \| Pu \|_{L^2}^2 + C \| u \|_{L^2}^2$$

$$= C \sum_{\mu \ge 1} \left\| \left(1 + \frac{1}{2} | \partial_x V(x_\mu) |^2 - \triangle_x \right)^{\frac{1}{3}} \left(1 + \frac{1}{2} | \partial_x V(x_\mu) |^2 - \triangle_x \right)^{-\frac{\tilde{s}}{2}} \varphi_\mu u \right\|_{L^2}^2$$

$$+ C \| Pu \|_{L^2}^2 + C \| u \|_{L^2}^2$$

$$\le C \sum_{\mu \ge 1} \left\| \left(1 + \frac{1}{2} | \partial_x V(x_\mu) |^2 - \triangle_x \right)^{\frac{1}{3}} f(x_\mu)^{-\tilde{s}} \varphi_\mu u \right\|_{L^2}^2 + C \| Pu \|_{L^2}^2 + C \| u \|_{L^2}^2.$$

Consequently, using (4.22) yields

$$\left\|(1 - \triangle_x)^{\frac{\delta}{2}} u\right\|_{L^2}^2 \le C \sum_{\mu \ge 1} \left\| P_{x_\mu} f(x_\mu)^{-\tilde{s}} \varphi_\mu u \right\|_{L^2}^2 + C \|Pu\|_{L^2}^2 + C \|u\|_{L^2}^2.$$

Thus (4.27) will follow if we can show that

$$\sum_{\mu \ge 1} \left\| P_{x_\mu} f(x_\mu)^{-\tilde{s}} \varphi_\mu u \right\|_{L^2}^2 \le C \left\{ \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}. \qquad (4.30)$$

To prove (4.30), we write

$$f(x_\mu)^{-\tilde{s}} \varphi_\mu = \left(f(x)^{\tilde{s}} f(x_\mu)^{-\tilde{s}} \right) \varphi_\mu f(x)^{-\tilde{s}}.$$

Then

$$\sum_{\mu \ge 1} \left\| P_{x_\mu} f(x_\mu)^{-\tilde{s}} \varphi_\mu u \right\|_{L^2}^2 \le (I) + (II)$$

with (I), (II) given by

$$(I) = 2 \sum_{\mu \ge 1} \left\| \left(f(x)^{\tilde{s}} f(x_\mu)^{-\tilde{s}} \right) P_{x_\mu} \varphi_\mu f(x)^{-\tilde{s}} u \right\|_{L^2}^2$$

and

$$(II) = 2 \sum_{\mu \ge 1} \left\| \left[P_{x_\mu}, \ f(x)^{\tilde{s}} f(x_\mu)^{-\tilde{s}} \right] \varphi_\mu f(x)^{-\tilde{s}} u \right\|_{L^2}^2.$$

By (4.6), we see

$$(I) \le C \sum_{\mu \ge 1} \left\| P_{x_\mu} \varphi_\mu f(x)^{-\tilde{s}} u \right\|_{L^2}^2.$$

This along with (4.28) gives

$$(I) \le C \left\{ \|Pu\|_{L^2}^2 + \left\| Pf(x)^{-\tilde{s}} u \right\|_{L^2}^2 + \left\| f(x)^{-\tilde{s}} u \right\|_{L^2}^2 \right\}. \qquad (4.31)$$

By use of (4.1) and (4.29), we have

$$\left\| [P, \ f(x)^{-\tilde{s}}] u \right\|_{L^2}^2 \le C \left\| f(x)^{-\tilde{s}+s-1} |y| u \right\|_{L^2}^2 \le C \| |y| u \|_{L^2}^2,$$

and hence

$$\left\| Pf(x)^{-\tilde{s}} u \right\|_{L^2}^2 \le 2 \|Pu\|_{L^2}^2 + 2 \left\| [P, \ f(x)^{-\tilde{s}}] u \right\|_{L^2}^2 \le C \left\{ \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}.$$

This along with (4.31) gives

$$I \leq C \left\{ \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}.$$

Now it remains to treat the term (II). The equality

$$\left[P_{x_\mu}, \ f(x)^{\tilde{s}} f(x_\mu)^{-\tilde{s}} \right] = \left(y \cdot \partial_x \left(f(x)^{\tilde{s}} \right) \right) f(x_\mu)^{-\tilde{s}}$$

gives

$$(II) = 2 \sum_{\mu \geq 1} \left\| \left(y \cdot \partial_x \left(f(x)^{\tilde{s}} \right) \right) f(x_\mu)^{-\tilde{s}} f(x)^{-\tilde{s}} \varphi_\mu u \right\|_{L^2}^2. \qquad (4.32)$$

By (4.6), (4.1) and (4.29), we have

$$\left| \partial_x \left(f(x)^{\tilde{s}} \right) \right| f(x_\mu)^{-\tilde{s}} f(x)^{-\tilde{s}} \varphi_\mu \leq C f(x)^{s-1-\tilde{s}} \leq C.$$

So

$$\sum_{\mu \geq 1} \left\| \left(y \cdot \partial_x \left(f(x)^{\tilde{s}} \right) \right) f(x_\mu)^{-\tilde{s}} f(x)^{-\tilde{s}} \varphi_\mu \, u \right\|_{L^2}^2 \leq C \sum_{\mu \geq 1} \left\| \varphi_\mu \, |y| \, u \right\|_{L^2}^2 \leq C \left\| \Lambda_y u \right\|_{L^2}^2.$$

This along with (4.32) gives

$$(II) \leq C \left\| \Lambda_y u \right\|_{L^2}^2 \leq C \left\{ \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}.$$

Combining the estimate on the term (I), we get the required inequality (4.30). The proof of Proposition 4.1 is thus completed. $\square$

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