Global L^p estimates for degenerate Ornstein-Uhlenbeck operators*

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Abstract

We consider a class of degenerate Ornstein-Uhlenbeck operators in $\mathbb{R}^N$, of the kind

$$\mathcal{A} \equiv \sum_{i,j=1}^{p_0} a_{ij} \partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j}$$

where $(a_{ij}), (b_{ij})$ are constant matrices, (a_{ij}) is symmetric positive definite on $\mathbb{R}^{p_0}$ ($p_0 \leq N$), and (b_{ij}) is such that $\mathcal{A}$ is hypoelliptic. For this class of operators we prove global L^p estimates ($1 < p < \infty$) of the kind:

$$\left\| \partial^2_{x_i x_j} u \right\|_{L^p(\mathbb{R}^N)} \leq c \left\{ \|\mathcal{A}u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)} \right\} \text{ for } i, j = 1, 2, ..., p_0$$

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and corresponding weak (1,1) estimates. This result seems to be the first case of global estimates, in Lebesgue L^p spaces, for complete Hörmander's operators

$$\sum X_i^2 + X_0,$$

proved in absence of a structure of homogeneous group. We obtain the previous estimates as a byproduct of the following one, which is of interest in its own:

$$\left\| \partial^2_{x_i x_j} u \right\|_{L^p(S)} \leq c \left\| Lu \right\|_{L^p(S)}$$

for any $u \in C_0^\infty(S)$, where S is the strip $\mathbb{R}^N \times [-1,1]$ and L is the Kolmogorov-Fokker-Planck operator $\mathcal{A} - \partial_t$. To get this estimate we crucially use the left translation invariance of L on a Lie group $\mathcal{K}$ in $\mathbb{R}^{N+1}$ and some results on singular integrals on nonhomogeneous spaces recently proved in [1].

1 Introduction

Problem and main result

Let us consider the class of degenerate Ornstein-Uhlenbeck operators in $\mathbb{R}^N$:

$$\mathcal{A} = div\left(A\nabla\right) + \langle x, B\nabla\rangle = \sum_{i,j=1}^{N} a_{ij}\partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j},$$

where A and B are constant $N \times N$ matrices, A is symmetric and positive semidefinite. If we define the matrix:

$$C(t) = \int_0^t E(s)\, A E^T(s)\, ds, \text{ where } E(s) = \exp\left(-sB^T\right) \tag{1}$$

then it can be proved (see [15]) the equivalence between the three conditions:
- the operator $\mathcal{A}$ is hypoelliptic;
- $C(t) > 0$ for any $t > 0$;
- the following Hörmander's condition holds:

$$\text{rank}\mathcal{L}\left(X_1, X_2, ..., X_N, Y_0\right) = N, \quad \text{for any } x \in \mathbb{R}^N,$$

where

$$Y_0 = \langle x, B\nabla\rangle \quad \text{and}$$

$$X_i = \sum_{j=1}^{N} a_{ij}\partial_{x_j} \quad i = 1, 2, ..., N.$$

Under one of these conditions it is proved in [15] that, for some basis of $\mathbb{R}^N$, the matrices A, B take the following form:

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2}$$

with $A_0 = (a_{ij})_{i,j=1}^{p_0}$ $p_0 \times p_0$ constant matrix ($p_0 \leq N$), symmetric and positive definite:

$$\nu \left|\xi\right|^2 \leq \sum_{i,j=1}^{p_0} a_{ij}\xi_i\xi_j \leq \frac{1}{\nu}\left|\xi\right|^2 \tag{3}$$

for any $\xi \in \mathbb{R}^{p_0}$, some positive constant ν;

$$B = \begin{bmatrix} * & B_1 & 0 & \ldots & 0 \\ * & * & B_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \ldots & B_r \\ * & * & * & \ldots & * \end{bmatrix} \tag{4}$$

where B_j is a $p_{j-1} \times p_j$ block with rank p_j, $j = 1, 2, ..., r$, $p_0 \geq p_1 \geq ... \geq p_r \geq 1$ and $p_0 + p_1 + ... + p_r = N$.

In this paper we consider hypoelliptic degenerate Ornstein-Uhlenbeck operators, with the matrices A, B already written as (2) and (4). For this class of operators, we will prove the following global L^p estimates:

Theorem 1 *For any $p \in (1, \infty)$ there exists a constant $c > 0$, depending on p, N, p_0, the matrix B and the number ν in (3) such that for any $u \in C_0^\infty\left(\mathbb{R}^N\right)$ one has:*

$$\left\|\partial^2_{x_i x_j} u\right\|_{L^p(\mathbb{R}^N)} \leq c\left\{\left\|\mathcal{A}u\right\|_{L^p(\mathbb{R}^N)} + \left\|u\right\|_{L^p(\mathbb{R}^N)}\right\} \text{ for } i,j = 1,2,...,p_0 \tag{5}$$

$$\left\|Y_0 u\right\|_{L^p(\mathbb{R}^N)} \leq c\left\{\left\|\mathcal{A}u\right\|_{L^p(\mathbb{R}^N)} + \left\|u\right\|_{L^p(\mathbb{R}^N)}\right\}. \tag{6}$$

Moreover, the following weak $(1,1)$ estimates hold:

$$\left|\left\{x \in \mathbb{R}^N : \left|\partial^2_{x_i x_j} u\left(x\right)\right| > \alpha\right\}\right| \leq \frac{c_1}{\alpha}\left\{\left\|\mathcal{A}u\right\|_{L^1(\mathbb{R}^N)} + \left\|u\right\|_{L^1(\mathbb{R}^N)}\right\} \tag{7}$$

$$\left|\left\{x \in \mathbb{R}^N : \left|Y_0 u\left(x\right)\right| > \alpha\right\}\right| \leq \frac{c_1}{\alpha}\left\{\left\|\mathcal{A}u\right\|_{L^1(\mathbb{R}^N)} + \left\|u\right\|_{L^1(\mathbb{R}^N)}\right\} \tag{8}$$

for any $\alpha > 0$, some constant c_1 depending on N, p_0, B and ν.

Global estimates in Hölder spaces analogous to (5)-(6) have been proved by Da Prato and Lunardi [6] in the nondegenerate case $p_0 = N$ (corresponding to the classical Ornstein-Uhlenbeck operator) and by Lunardi [17] in the degenerate case; L^p estimates in the nondegenerate case $p_0 = N$ have been proved by Metafune, Prüss, Rhandi and Schnaubelt [19] by a semigroup approach. Note that, even in the nondegenerate case, global estimates in L^p or Hölder spaces are not straightforward, due to the unboundedness of the first order coefficients. Under this regard, our weak (1,1) estimate seems to be new even in the nondegenerate case. L^2 estimates with respect to an invariant Gaussian measure have been proved by Lunardi [18] in the nondegenerate case, and by Farkas and Lunardi [10] in the degenerate case.

The operator $\mathcal{A}$ can be seen as the infinitesimal generator of the Ornstein-Uhlenbeck semigroup. This is the Markov semigroup associated to the stochastic differential equation:

$$d\xi(t) = B^T \xi(t)dt + \sqrt{2}\, A_0^{1/2} dW(t)\,, \quad t > 0, \quad \xi(0) = x, \tag{9}$$

where $W(t)$ is a standard Brownian motion taking values in $\mathbb{R}^{p_0}$. This equation can describe the random motion of a particle in a fluid (see [25]). Several interpretations in physics and finance for the operator $\mathcal{A}$ or its evolutionary counterpart L (see below) are explained in the survey by Pascucci [21]. Nonlocal Ornstein-Uhlenbeck operators are studied by Priola and Zabczyk [22]. In infinite dimension, Ornstein-Uhlenbeck type operators arise naturally in the study of stochastic P.D.E.s (see [7] and [8], [3] and the references therein).

Remark 2 *To make easier a comparison of our setting with that considered in several papers we have quoted so far, we point out the fact that the condition* $C(t) > 0$ *is equivalent to the condition*

$$Q_t \equiv \int_0^t \exp\left(sB^T\right) A \exp(sB)\, ds = \exp\left(tB^T\right)\, C(t)\, \exp(tB) > 0.$$

The operator Q_t *has also control theoretic meaning and is considered in [6], [7], [8], [10], [19], [22]. Also, note that it is enough to require that* $C(t)$ *or* Q_t *is positive definite for some* $t_0 > 0$ *in order to get that it is positive definite for any* $t > 0$*.*

Relation with the evolution operator

The evolution operator corresponding to $\mathcal{A}$,

$$L = \mathcal{A} - \partial_t,$$

is a Kolmogorov-Fokker-Planck ultraparabolic operator, which has been extensively studied in the last fifteen years. The largest part of the related literature is devoted to the case where an underlying structure of homogeneous group is present. In absence of this structure (that is, in the general situation we are interested in), this operator has been studied for instance by Lanconelli and Polidoro [15], Di Francesco and Polidoro [9], Cinti, Pascucci and Polidoro [4] (see also the survey [16], and references therein). In particular, it is proved in [15] that the operator L is left invariant with respect to the Lie-group translation

$$\begin{aligned} (x,t) \circ (\xi,\tau) &= (\xi + E(\tau)\, x, t+\tau)\,; \\ (\xi,\tau)^{-1} &= (-E(-\tau)\,\xi, -\tau)\,, \text{ where} \\ E(\tau) &= \exp\left(-\tau B^T\right). \end{aligned}$$

We will deduce global estimates (5) from an analogous estimate for L on the strip

$$S \equiv \mathbb{R}^N \times [-1,1]\,,$$

which can be of independent interest:

Theorem 3 *For any $p \in (1, \infty)$ there exists a constant $c > 0$ such that*

$$\left\| \partial^2_{x_i x_j} u \right\|_{L^p(S)} \leq c \, \| Lu \|_{L^p(S)} \quad \text{for } i, j = 1, 2, ..., p_0, \tag{10}$$

for any $u \in C_0^\infty(S)$. The constant c depends on the same parameters than the c in Theorem 1.

To get the above L^p estimates, we have to set the problem in the suitable geometric framework, which for this specific class of operators has been studied in detail in [15], [9], while for general Hörmander's operators, with or without an underlying structure of homogeneous group, has been investigated by Folland [11], Rothschild and Stein [23], respectively.

In particular, L^p estimates for the second order derivatives have been proved in [11] on the whole space, but assuming the existence of a homogeneous group, and in [23] in the general case, but only locally. Therefore our results cannot be deduced by the existing theories.

Actually, Theorem 1 seems to be the first case of global estimates, in Lebesgue L^p spaces, for hypoelliptic degenerate Ornstein-Uhlenbeck operators, and more generally for complete Hörmander's operators

$$\sum X_i^2 + X_0,$$

in absence of an underlying structure of homogeneous group. We also want to stress that the group $\mathcal{K} = \left(\mathbb{R}^{N+1}, \circ\right)$ is not in general nilpotent. Hence, in view of the results in [24], one cannot expect a global L^p estimate like (10) to be true on the whole $\mathbb{R}^{N+1}$ (instead that on a strip).

Our result can also be seen as a first step to study existence and uniqueness for the Cauchy problem related to L in L^p spaces, as well as to characterize the domain of the generator of the Ornstein-Uhlenbeck semigroup in L^p spaces. We plan to address these problems in the next future.

Strategy of the proof

Let us start noting that Theorem 3 easily implies Theorem 1, apart from the weak estimates (7), (8), which will be proved separately. Namely, let

$$\psi \in C_0^\infty(\mathbb{R})$$

be a cutoff function fixed once and for all, $\text{sprt}\, \psi \subset [-1, 1]$, $\int_{-1}^1 \psi(t)\, dt > 0$. If $u : \mathbb{R}^N \to \mathbb{R}$ is a C_0^∞ solution to the equation

$$\mathcal{A}u = f \text{ in } \mathbb{R}^N,$$

for some $f \in L^p\left(\mathbb{R}^N\right)$, let

$$U(x, t) = u(x)\, \psi(t);$$

then

$$LU(x,t) = f(x)\psi(t) - u(x)\psi'(t) \equiv F(x,t).$$

Therefore Theorem 3 applied to U gives

$$\left\| \partial^2_{x_i x_j} U \right\|_{L^p(S)} \leq c \|F\|_{L^p(S)} \quad \text{for } i,j = 1,2,...,p_0 \tag{11}$$

hence

$$\left\| \partial^2_{x_i x_j} u \right\|_{L^p(\mathbb{R}^N)} \leq c \left\{ \|f\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)} \right\}$$

with c also depending on ψ. Note that (6) follows from (5).

We would like to describe now the general strategy of the proof of Theorem 3, as well as the main difficulties encountered. A basic idea is that of linking the properties of L to those of another operator of the same kind, which not only is left invariant with respect to a suitable Lie group of translations, but is also homogeneous of degree 2 with respect to a family of dilations (which are group automorphisms). Such an operator L_0 (see (13)) always exists under our assumptions, by [15], and has been called “the principal part” of L. Note that the operator L_0 fits the assumptions of Folland’s theory [11]. However, to get the desired conclusion on L, this is not enough. Instead, we exploit the fact that, by results proved by [9], the operator L possesses a fundamental solution Γ with some good properties. First of all, Γ is translation invariant and has a fast decay at infinity, in space; this allows to reduce the desired L^p estimates to estimates on a singular integral operator whose kernel vanishes far off the pole. Second, this singular kernel, which has the form $\eta \cdot \partial^2_{x_i x_j}\Gamma$ where η is a radial cutoff function, satisfies “standard estimates” (in the language of singular integrals theory) with respect to a suitable “local quasisymmetric quasidistance” d, which is a key geometrical object in our study. Namely,

$$d(z,\zeta) = \left\| \zeta^{-1} \circ z \right\|$$

where $\zeta^{-1} \circ z$ is the Lie group operation related to the operator L, while $\|\cdot\|$ is a homogeneous norm related to the principal part operator L_0 (recall that L does not have an associated family of dilations, and therefore does not have a natural homogeneous norm). This “hybrid” quasidistance is not (and seemingly is not equivalent to) the control distance of any family of vector fields; even worse, it does not fulfill enough good properties in order to apply the standard theory of “singular integrals in spaces of homogeneous type” (in the sense of Coifman-Weiss [5]). Instead, we have to set the problem in a weaker abstract context (“bounded nonhomogeneous spaces”), and apply an ad hoc theory of singular integrals to get the desired L^p bound. The alluded ad hoc result has been proved by one of us in [1], in the spirit of the theory of singular integrals in nonhomogeneous spaces, which has been developed, since the late 1990’s, by Nazarov-Treil-Volberg and other authors. With this machinery at hand, we can prove the desired L^p estimate for the singular integral with kernel $\eta \cdot \partial^2_{x_i x_j}\Gamma$ *on* *a ball*. To get the desired estimate on the whole strip $\mathbb{R}^N \times [-1,1]$, still another nontrivial argument is needed, based on a covering lemma and exploiting both the existence of a group of translations, and the relevant properties of the quasidistance d.

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2 Background and known results

The principal part operator

Let us consider our operator L, with the matrices A, B written in the form (2), (4). We denote by B_0 the matrix obtained by annihilating every $*$ block in (4):

$$B_0 = \begin{bmatrix} 0 & B_1 & 0 & \ldots & 0 \\ 0 & 0 & B_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & B_r \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \tag{12}$$

with B_j as in (4). By *principal part* of L we mean the operator

$$L_0 = div\,(A\nabla) + \langle x, B_0 \nabla \rangle - \partial_t. \tag{13}$$

For any $\lambda > 0$, let us define the matrix of *dilations on* $\mathbb{R}^N$,

$$D(\lambda) = diag\left(\lambda I_{p_0}, \lambda^3 I_{p_1}, ..., \lambda^{2r+1} I_{p_r}\right)$$

where I_{p_j} denotes the $p_j \times p_j$ identity matrix, and the matrix of *dilations on* $\mathbb{R}^{N+1}$,

$$\delta(\lambda) = diag\left(\lambda I_{p_0}, \lambda^3 I_{p_1}, ..., \lambda^{2r+1} I_{p_r}, \lambda^2\right).$$

Note that

$$\det(\delta(\lambda)) = \lambda^{Q+2}$$

where

$$Q + 2 = p_0 + 3p_1 + ... + (2r+1)\,p_r + 2$$

is called *homogeneous dimension of* $\mathbb{R}^{N+1}$. Analogously,

$$\det(D(\lambda)) = \lambda^Q$$

and Q is called *homogeneous dimension of* $\mathbb{R}^N$. A remarkable fact proved in [15] is that the operator L_0 is homogeneous of degree two with respect to the dilations $\delta(\lambda)$, which by definition means that

$$L_0\,(u\,(\delta(\lambda)\,z)) = \lambda^2\,(L_0 u)\,(\delta(\lambda)\,z)$$

for any $u \in C_0^\infty\left(\mathbb{R}^{N+1}\right), z \in \mathbb{R}^{N+1}, \lambda > 0$.

If we define

$$C_0(t) = \int_0^t E_0(s)\, A E_0^T(s)\, ds, \text{ where } E_0(s) = \exp\left(-sB_0^T\right) \tag{14}$$

then the operator L_0 turns out to be left invariant with respect to the associated translations:

$$(x,t) \odot (\xi,\tau) = (\xi + E_0(\tau)\, x, t+\tau);$$
$$(\xi,\tau)^{-1} = (-E_0(-\tau)\,\xi, -\tau).$$

Moreover, the dilations $z \mapsto \delta(\lambda)\, z$ are automorphisms for the group $\left(\mathbb{R}^{N+1}, \odot\right)$

There is a natural homogeneous norm in $\mathbb{R}^{N+1}$, induced by these dilations:

$$\|(x,t)\| = \sum_{j=1}^N |x_j|^{1/q_j} + |t|^{1/2}$$

where q_j are positive integers such that $D(\lambda) = diag\left(\lambda^{q_1}, ..., \lambda^{q_N}\right)$. Clearly, we have

$$\|\delta(\lambda)\, z\| = \lambda \|z\| \quad \text{for any } \lambda > 0, z \in \mathbb{R}^{N+1}.$$

Other properties of $\|\cdot\|$ will be stated later.

Fundamental solution

The following theorem collects some important known result about the fundamental solution of L:

Theorem 4 *Under the assumptions stated in the Introduction, the operator L possesses a fundamental solution*

$$\Gamma(z,\zeta) = \gamma\left(\zeta^{-1} \circ z\right) \text{ for } z,\zeta \in \mathbb{R}^{N+1},$$

with

$$\gamma(z) = \begin{cases} 0 \text{ for } t \le 0 \\ \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\left\langle C^{-1}(t)\, x, x\right\rangle - t\, TrB\right) & \text{for } t > 0 \end{cases}$$

where $z = (x,t)$ and $C(t)$ is as in (1). Recall that $C(t)$ is positive definite for any $t > 0$; hence $\gamma \in C^\infty\left(\mathbb{R}^{N+1} \setminus \{0\}\right)$. The following representation formulas hold:

$$u(z) = -(\gamma * Lu)(z) = -\int_{\mathbb{R}^{N+1}} \gamma\left(\zeta^{-1} \circ z\right) Lu(\zeta)\, d\zeta; \tag{15}$$

$$\partial^2_{x_i x_j} u(z) = -PV\left(\partial^2_{x_i x_j} \gamma * Lu\right)(z) + c_{ij} Lu(z) \tag{16}$$

for any $u \in C_0^\infty\left(\mathbb{R}^{N+1}\right)$, $i,j=1,2,...,p_0$, *for suitable constants* c_{ij} *which we do not need to specify. The "principal value" in (16) must be understood as*

$$PV\left(\partial^2_{x_ix_j}\gamma * Lu\right)(z) \equiv \lim_{\varepsilon\to 0}\int_{\|\zeta^{-1}\circ z\|>\varepsilon}\left(\partial^2_{x_ix_j}\gamma\right)\left(\zeta^{-1}\circ z\right)Lu(\zeta)\,d\zeta.$$

The above theorem is proved in [12] (see also [15]), apart from (16) which is proved in [9, Proposition 2.11].

The fundamental solution $\Gamma_0(z,\zeta)=\gamma_0\left(\zeta^{-1}\circ z\right)$ of the principal part operator L_0 enjoys special properties; namely, for $t>0$

$$\gamma_0(x,t)=\frac{(4\pi)^{-N/2}}{\sqrt{\det C_0(t)}}\exp\left(-\frac{1}{4}\left\langle C_0^{-1}(t)\,x,x\right\rangle\right) \tag{17}$$

with $C_0(t)$ as in (14); moreover (see [15, p.42]),

$$C_0\left(\lambda^2 t\right)=D(\lambda)\,C_0(t)\,D(\lambda) \qquad \forall\lambda,t>0 \tag{18}$$

from which we can see that γ_0 is homogeneous of degree $-Q$:

$$\gamma_0(\delta(\lambda)(x,t))=\lambda^{-Q}\gamma_0(x,t) \quad \forall\lambda>0,(x,t)\in\mathbb{R}^{N+1}\setminus\{(0,0)\}.$$

Furthermore, the following relation links L to L_0 (see [15, Lemma 3.3]):

$$\langle C(t)\,x,x\rangle=\langle C_0(t)\,x,x\rangle(1+O(t)) \text{ for } t\to 0; \tag{19}$$

$$\left\langle C^{-1}(t)\,x,x\right\rangle=\left\langle C_0^{-1}(t)\,x,x\right\rangle(1+O(t)) \text{ for } t\to 0; \tag{20}$$

and (see [15, eqt. (3.14)]):

$$\det C(t)=\det C_0(t)\,(1+O(t)) \text{ for } t\to 0. \tag{21}$$

3 Estimate on the nonsingular part of the integral

We now localize the singular kernel appearing in (16) introducing a cutoff function

$$\begin{gathered}\eta\in C_0^\infty\left(\mathbb{R}^{N+1}\right) \text{ such that}\\ \eta(z)=1 \text{ for } \|z\|\le\rho_0/2;\\ \eta(z)=0 \text{ for } \|z\|\ge\rho_0,\end{gathered}$$

where $\rho_0<1$ will be fixed later.

Let us rewrite (16) as:

$$\begin{aligned}\partial^2_{x_ix_j}u &= -PV\left(\left(\eta\partial^2_{x_ix_j}\gamma\right)*Lu\right)-\left((1-\eta)\,\partial^2_{x_ix_j}\gamma*Lu\right)+c_{ij}Lu \qquad (22)\\ &\equiv -PV\left(k_0*Lu\right)-\left(k_\infty*Lu\right)+c_{ij}Lu\end{aligned}$$

having set:

$$k_0 = \eta\partial^2_{x_i x_j}\gamma \tag{23}$$
$$k_\infty = (1-\eta)\,\partial^2_{x_i x_j}\gamma$$

for any $i,j = 1,2,...,p_0$ (we will left implicitly understood the dependence of the kernels k_0, k_∞ on these indices i,j, as well as on the number ρ_0 appearing in the definition of the cutoff function η).

Since in k_∞ the singularity of $\partial^2_{x_i x_j}\gamma$ has been removed and $\partial^2_{x_i x_j}\gamma$ has a fast decay as $x \to \infty$, we can prove the following:

Proposition 5 *For any $\rho_0 > 0$ there exists $c = c(\rho_0) > 0$ such that for any $z \in S$*

$$\int_S \left|k_\infty\left(\zeta^{-1}\circ z\right)\right| d\zeta \le c \tag{24}$$

$$\int_S \left|k_\infty\left(z^{-1}\circ \zeta\right)\right| d\zeta \le c. \tag{25}$$

Note that this proposition immediately implies the following

Corollary 6 *For any $p \in [1,\infty]$ there exists a constant $c > 0$ only depending on p, N, p_0, ν and the matrix B such that:*

$$\left\|-\left(k_\infty * Lu\right) + c_{ij}Lu\right\|_{L^p(S)} \le c\left\|Lu\right\|_{L^p(S)} \text{ for any } u \in C_0^\infty(S), \tag{26}$$

any $i,j = 1,...,p_0$.

Before proving the proposition we need an easy lemma to handle a typical change of variables in convolutions. It turns out that the Lebesgue measure is invariant with respect to left translations, but not with respect to the inversion $\zeta \longmapsto \zeta^{-1}$:

Lemma 7 *If we set $\zeta^{-1}\circ z = w = (\xi,\tau)$, then the following identity holds for the Jacobian of the map $w \longmapsto \zeta$ (for fixed z):*

$$d\zeta = e^{\tau\, TrB}dw. \tag{27}$$

Proof of the Lemma. Setting

$$\zeta^{-1}\circ z = w;$$
$$\zeta = z\circ w^{-1},$$

let us compute the Jacobian matrix of the map $w \longmapsto z\circ w^{-1}$. If $z = (x,t)$, $w = (\xi,\tau)$ we have

$$z\circ w^{-1} = \left(-E(-\tau)\,\xi + E(-\tau)\,x, t-\tau\right),$$

and the Jacobian is

$$J = \begin{bmatrix} -E(-\tau) & * \\ 0 & -1 \end{bmatrix}$$

with determinant

$$\mathrm{Det} J = \mathrm{Det} \exp\left(\tau B^T\right) = e^{\tau \mathrm{Tr} B}.$$

■

Proof of Proposition 5. Since we are not interested in the exact dependence of the constant c on ρ_0, for the sake of simplicity we will prove the Proposition for $\rho_0 = 1$. An analogous proof can be done for any ρ_0, finding a constant c which depends on ρ_0.

Note that (25) immediately follows by (24), with the change of variables $z^{-1} \circ \zeta = w^{-1}$ and applying the above Lemma. So, let us prove (24).

Recalling that, for $t > 0$, we have

$$\gamma(x,t) = \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\left\langle C^{-1}(t)\, x, x\right\rangle - t\mathrm{Tr} B\right),$$

let us compute:

$$(\partial_{x_i}\gamma)(x,t) = -\frac{1}{2}\gamma(x,t)\left\langle C^{-1}(t)\, x, e_i\right\rangle$$

$$\left(\partial^2_{x_i x_j}\gamma\right)(x,t) = \frac{1}{2}\gamma(x,t)\left\{\frac{1}{2}\left\langle C^{-1}(t)\, x, e_j\right\rangle\left\langle C^{-1}(t)\, x, e_i\right\rangle - \left\langle C^{-1}(t)\, e_j, e_i\right\rangle\right\}$$

(where we have denoted by e_i the i-th unit vector in $\mathbb{R}^N$). Since the matrix $C^{-1}(t)$ is symmetric and positive definite, we can bound

$$\left|\left\langle C^{-1}(t)\, x, e_j\right\rangle\right| \leq \left\langle C^{-1}(t)\, x, x\right\rangle^{1/2}\left\langle C^{-1}(t)\, e_j, e_j\right\rangle^{1/2}.$$

By (20) and (18) we have:

$$\left\langle C^{-1}(t)\, e_j, e_j\right\rangle = \left\langle C_0^{-1}(t)\, e_j, e_j\right\rangle(1 + O(t)) \quad \text{for } t \to 0 \tag{28}$$

and

$$\left\langle C_0^{-1}(t)\, e_j, e_j\right\rangle = \left\langle C_0^{-1}(1)\, D\left(\frac{1}{\sqrt{t}}\right)e_j, D\left(\frac{1}{\sqrt{t}}\right)e_j\right\rangle \leq c\left|D\left(\frac{1}{\sqrt{t}}\right)e_j\right|^2 =$$

$$(\text{since } j \in \{1, 2, ..., p_0\}\,) = c\left|\frac{1}{\sqrt{t}}e_j\right|^2 = \frac{c}{t}.$$

This shows that

$$\left\langle C^{-1}(t)\, e_j, e_j\right\rangle \leq \frac{c}{t}(1 + O(t)), \text{ and}$$

$$\left|\left\langle C^{-1}(t)\, e_j, e_i\right\rangle\right| \leq \left\langle C^{-1}(t)\, e_j, e_j\right\rangle^{1/2}\left\langle C^{-1}(t)\, e_i, e_i\right\rangle^{1/2} \leq \frac{c}{t}(1 + O(t)),$$

for $t \to 0$. Therefore

$$\left|\partial^2_{x_i x_j}\gamma(x,t)\right| \leq \frac{1}{2}\gamma(x,t)\left\{\frac{c}{t}\left\langle C^{-1}(t)\, x, x\right\rangle + \frac{c}{t}\right\}(1 + O(t)) =$$

$$= \frac{c}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\left\langle C^{-1}(t)\, x, x\right\rangle - t\mathrm{Tr} B\right)\left\{\frac{1}{t}\left\langle C^{-1}(t)\, x, x\right\rangle + \frac{1}{t}\right\}(1 + O(t))$$

$$\leq \frac{c}{t\sqrt{\det C(t)}} \exp\left(-\frac{(1-\delta)}{4}\left\langle C^{-1}(t)\, x, x\right\rangle - t\mathrm{Tr} B\right)$$

for some $\delta > 0$, any $t \in [-1, 1]$, since

$$\exp\left(-\frac{1}{4}\left\langle C^{-1}(t)\,x, x\right\rangle\right)\left\{\left\langle C^{-1}(t)\,x, x\right\rangle + 1\right\} \leq c\exp\left(-\frac{(1-\delta)}{4}\left\langle C^{-1}(t)\,x, x\right\rangle\right),$$

which follows from

$$\begin{gathered}(4\alpha+1)\exp(-\alpha) \leq c\exp(-(1-\delta)\,\alpha) \qquad \forall\alpha \geq 0 \\ \text{and } \alpha = \frac{1}{4}\left\langle C^{-1}(t)\,x, x\right\rangle \geq 0.\end{gathered}$$

Let us rewrite the last inequality as

$$\begin{aligned}&\left|\partial^2_{x_i x_j}\gamma(x,t)\right| \leq \frac{c}{t}\gamma_\delta(x,t), \qquad (29)\\ &\text{with } \gamma_\delta(x,t) = \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}}\exp\left(-\frac{(1-\delta)}{4}\left\langle C^{-1}(t)\,x, x\right\rangle - t\mathrm{Tr}B\right).\end{aligned}$$

With this bound in hand, we can now evaluate the following integral, for $z \in S$. Using Lemma 7 we have:

$$\begin{aligned}&\int_S \left|\left((1-\eta)\,\partial^2_{x_i x_j}\gamma\right)\left(\zeta^{-1}\circ z\right)\right| d\zeta \leq \\ &\leq c\int_{\mathbb{R}^N\times(-2,2),\|\zeta'\|\geq 1/2}\left|\left((1-\eta)\,\partial^2_{x_i x_j}\gamma\right)(\zeta')\right| d\zeta' \\ &= c\int_{\mathbb{R}^N\times(-2,2),\|\zeta'\|\geq 1/2,\|(x,0)\|\leq 1/4}\left|\left((1-\eta)\,\partial^2_{x_i x_j}\gamma\right)(x,t)\right| dxdt + \\ &+ c\int_{\mathbb{R}^N\times(-2,2),\|\zeta'\|\geq 1/2,\|(x,0)\|> 1/4}\left|\left((1-\eta)\,\partial^2_{x_i x_j}\gamma\right)(x,t)\right| dxdt \\ &\equiv I + II.\end{aligned}$$

Now,

$$\begin{aligned}I &\leq c\int_{1/16\leq|t|\leq 2,\|(x,0)\|\leq 1/4}\frac{(4\pi)^{-N/2}}{t\sqrt{\det C(t)}}\exp\left(-\frac{(1-\delta)}{4}\left\langle C^{-1}(t)\,x, x\right\rangle - t\mathrm{Tr}B\right) dxdt \\ &\leq c\int_{|x|\leq c_1}\exp\left(-c_2|x|^2\right) dx \leq c\end{aligned}$$

where we used (20) and the fact that

$$\left\langle C_0^{-1}(t)\,x, x\right\rangle \geq c\left|D\left(\frac{1}{\sqrt{t}}\right)x\right|^2 \geq c|x|^2 \text{ since } |t| \leq 2$$

while, by (21),

$$t\sqrt{\det C(t)} \geq c_1\sqrt{\det C_0(t)} = c_2 t^{(Q+2)/2} \geq c_3 \text{ since } |t| \geq 1/16.$$

To handle II, we start noting that, if $\|(x,0)\| > 1/4$, by (20) we can write

$$\begin{aligned} &\exp\left(-c_1\left\langle C^{-1}(t)\,x,x\right\rangle\right) \leq \exp\left(-c_2\left\langle C_0^{-1}(t)\,x,x\right\rangle\right) \leq \\ &\leq \exp\left(-c_3\left|D\left(\frac{1}{\sqrt{t}}\right)x\right|^2\right) \leq \exp\left(-c_4\frac{|x|^2}{t}\right) \leq \\ &\leq \exp\left(-\frac{c_5}{t}\right) \leq c_6 t \end{aligned}$$

hence

$$\begin{aligned} II &\leq \int_{\mathbb{R}^N\times(-2,2),\|\zeta'\|\geq 1/2,\|(x,0)\|>1/4} \frac{c}{\sqrt{\det C(t)}}\exp\left(-c_7\left\langle C^{-1}(t)\,x,x\right\rangle\right)dxdt = \\ &\quad(\text{letting } C^{-1/2}(t)\,x = y) \\ &= \int_{\mathbb{R}^N\times(-2,2)} c\exp\left(-c_7\,|y|^2\right)dydt = c. \end{aligned}$$

■

By Corollary 6 and (22), our final goal will be achieved as soon as we will prove that

$$\|PV\,(k_0 * Lu)\|_{L^p(S)} \leq c\,\|Lu\|_{L^p(S)} \tag{30}$$

for any $u \in C_0^\infty(S)\,,\, i,j = 1,...,p_0, 1 < p < \infty$. The proof of (30) will be carried out in the following sections, and concluded with Theorem 22.

4 Estimates on the singular kernel

To prove the singular integral estimate (30), we have to introduce some more structure in our setting. Let:

$$d\,(z,\zeta) = \left\|\zeta^{-1}\circ z\right\|.$$

Recall that $\circ$ is the translation induced by the the operator L (or more precisely by the matrix B), and $\|\cdot\|$ the homogeneous norm induced by the dilations associated to the principal part operator L_0 (see §2). This object has been introduced and used in [9], and turns out to be the right geometric tool to describe the properties of the singular kernel γ_0. Namely, the following key properties have been proved in [9]:

Proposition 8 *(See Lemma 2.1 in [9]). For any compact set $K \subset \mathbb{R}^N$ there exists a constant $c_K \geq 1$ such that*

$$\begin{aligned} &\left\|z^{-1}\right\| \leq c_K\,\|z\| \ \text{ for every } z \in K\times[-1,1] \\ &\|z\circ\zeta\| \leq c_K\left\{\|z\| + \|\zeta\|\right\} \ \text{ for every } \zeta \in S, z \in K\times[-1,1]\,. \end{aligned}$$

In terms of d, the above inequalities imply the following:

$$\begin{aligned} &d\,(z,\zeta) \leq c d\,(\zeta,z) \ \ \forall z,\zeta \in S \text{ with } d\,(\zeta,z) \leq 1 \\ &d\,(z,\zeta) \leq c\left\{d\,(z,w) + d\,(w,\zeta)\right\} \ \ \forall z,\zeta,w \in S \text{ with } d\,(z,w) \leq 1, d\,(w,\zeta) \leq 1. \end{aligned}$$

Let us define the d-balls:

$$B(z,\rho)=\left\{\zeta\in\mathbb{R}^{N+1}:d(z,\zeta)<\rho\right\}.$$

Lemma 9 *The d-balls are open with respect to the Euclidean topology. Moreover, the topology induced by this family of balls (saying that a set Ω is open whenever for any $x\in\Omega$ there exists $\rho>0$ such that $B(x,\rho)\subset\Omega$) coincides with the Euclidean topology.*

Proof. Since the function $\zeta\mapsto d(z_0,\zeta)$ is continuous, $B(z_0,\rho)$ is open with respect to the Euclidean topology; in particular, $B(z_0,\rho)$ contains an Euclidean ball centered at z_0.

Conversely, fix an Euclidean ball $B^E(z_0,\rho)$ of center z_0 and radius $\rho>0$, and assume that z is point such that

$$\left\|z^{-1}\circ z_0\right\|<\varepsilon,$$

for some $\varepsilon>0$ to be chosen later. Then, letting $w=z^{-1}\circ z_0$, we have

$$|z_0-z|=\left|z_0-z_0\circ w^{-1}\right|<\rho \text{ for } \varepsilon \text{ small enough,}$$

because

$$\left|w^{-1}\right|\leq c\left\|w^{-1}\right\|\leq c\|w\|<c\varepsilon,$$

and the translation $\circ$ is a smooth operation. Hence

$$B^E(z_0,\rho)\supseteq B(z_0,\varepsilon),$$

so that the two topologies coincide. ■

The relevant information about the measure of d-balls are contained in the following:

Proposition 10 *(i) The following dimensional bound holds:*

$$|B(z,\rho)|\leq c\rho^{Q+2} \text{ for any } z\in S, 0<\rho<1.$$

(ii) The following doubling condition holds in S:

$$|B(z,2\rho)\cap S|\leq c\,|B(z,\rho)\cap S| \quad \text{for any } z\in S, 0<\rho<1.$$

Proof. Let us compute the integral

$$|B(z,\rho)|=\int_{\|\zeta^{-1}\circ z\|<\rho}d\zeta.$$

Setting $\zeta^{-1}\circ z=w$ and applying Lemma 7 we have, if $z=(x,t)$, $w=(\xi,\tau)$:

$$|B(z,\rho)|=\int_{\|(\xi,\tau)\|<\rho}e^{\tau\operatorname{Tr}B}d\xi d\tau$$

Since $z \in S$, in particular, $|t| \leq 1$, $|t-\tau| \leq \rho^2$, hence $|\tau| \leq 2$ and the last integral is

$$\leq e^{2\operatorname{Tr}B} \int_{\|w\|<\rho} dw$$

by the dilation $w = \delta(\rho) w'$

$$= e^{2\operatorname{Tr}B} \rho^{Q+2} \int_{\|(\xi,\tau)\|<1} d\xi d\tau = c\rho^{Q+2}$$

which proves (i).

To prove (ii), let $\zeta = (x', t')$, $z = (x, t)$, $w = (\xi, \tau)$, and assume, to fix ideas, $t \geq 0$. Then

$$\begin{aligned}
|B(z,\rho) \cap S| &= \int_{\|(\xi,\tau)\|<\rho, |t-\tau|<1} e^{\tau \operatorname{Tr}B} d\xi d\tau \\
&\geq c \int_{\|(\xi,\tau)\|<\rho, 0\leq\tau\leq 1} d\xi d\tau \text{ (since } \rho < 1) \\
&= \frac{c}{2} \int_{\|w\|<\rho} dw = c\rho^{Q+2} \int_{\|w'\|<1} dw' = c\rho^{Q+2} \\
&\geq c\,|B(z, 2\rho)| \geq c\,|B(z,2\rho) \cap S|
\end{aligned}$$

by (i). ■

We also need the following bounds of the fundamental solution Γ in terms of d:

Proposition 11 *(See Proposition 2.7 in [9]) The following "standard estimates" hold for Γ in terms of d: there exist $c > 0$ and $M > 1$ such that*

$$\left|\partial^2_{x_i x_j} \Gamma(z,\zeta)\right| \leq \frac{c}{d(z,\zeta)^{Q+2}} \quad \forall z, \zeta \in S$$

$$\left|\partial^2_{x_i x_j} \Gamma(\zeta, w) - \partial^2_{x_i x_j} \Gamma(z,w)\right| \leq c \frac{d(w,z)}{d(w,\zeta)^{Q+3}} \quad \forall z, \zeta, w \in S$$

with $Md(w,z) \leq d(w,\zeta) \leq 1$.

An easy computation shows that the previous estimates extend to the kernel $k_0 = \eta \partial^2_{x_i x_j} \gamma$:

Proposition 12 *There exists $c > 0$ and $M > 1$ such that*

$$\left|k_0\left(\zeta^{-1} \circ z\right)\right| \leq \frac{c}{d(z,\zeta)^{Q+2}} \quad \forall z, \zeta \in S$$

$$\left|k_0\left(w^{-1} \circ \zeta\right) - k_0\left(w^{-1} \circ z\right)\right| \leq c \frac{d(w,z)}{d(w,\zeta)^{Q+3}} \quad \forall z, \zeta, w \in S$$

with $Md(w,z) \leq d(w,\zeta) \leq 1$.

Remark 13 *We can always assume that M is large enough, so that the conditions*

$$Md(w,z) \leq d(w,\zeta) \leq 1$$

imply

$$c_1 d(z,\zeta) \leq d(w,\zeta) \leq c_2 d(z,\zeta)$$

for some absolute constants $c_1, c_2 > 0$.

We will also need the following:

Lemma 14 *There exists $c > 0$ such that*

$$\left| \int_{r_1 < \|\zeta^{-1} \circ z\| < r_2} k_0\left(\zeta^{-1} \circ z\right) d\zeta \right| \leq c$$

for any $z \in S, 0 < r_1 < r_2$. Moreover, for every $z \in S$, the limit

$$\lim_{\varepsilon \to 0^+} \int_{\|\zeta^{-1} \circ z\| > \varepsilon} k_0\left(\zeta^{-1} \circ z\right) d\zeta$$

exists, is finite, and independent of z.

Proof. The change of variables $w = \zeta^{-1} \circ z$ (see Lemma 7) shows that

$$\int_{r_1 < \|\zeta^{-1} \circ z\| < r_2} k_0\left(\zeta^{-1} \circ z\right) d\zeta = \int_{r_1 < \|w\| < r_2} k_0(w)\, e^{\tau \mathrm{Tr} B} dw = \tag{31}$$

$$(\text{for } r_2 \leq \frac{\rho_0}{2}) \quad = \int_{r_1 < \|w\| < r_2} \partial^2_{x_i x_j} \gamma(w)\, e^{\tau \mathrm{Tr} B} dw$$

with $w = (\xi, \tau)$. However, by the divergence theorem the last integral equals

$$\int_{\|w\| = r_2} \partial_{x_i} \gamma(w)\, e^{\tau \mathrm{Tr} B} \nu_j d\sigma(w) - \int_{\|w\| = r_1} \partial_{x_i} \gamma(w)\, e^{\tau \mathrm{Tr} B} \nu_j d\sigma(w)$$
$$\equiv I(r_2) - I(r_1).$$

It is shown in [15, Lemma 2.10] that

$$I(\rho) \to \int_{\|w\| = 1} \partial_{x_i} \gamma_0(w)\, e^{\tau \mathrm{Tr} B} \nu_j d\sigma(w) \text{ as } \rho \to 0$$

with γ_0 as in (17). Since, on the other hand, $I(\rho)$ is continuous for $\rho \in (0, 1/2]$, we conclude that $I(\rho)$ is bounded for $\rho \in \left[0, \frac{\rho_0}{2}\right]$. This implies the first statement in the Lemma if $r_2 \leq \rho_0/2$. Note that we can always assume $r_2 \leq \rho_0$, because $k_0(w) = 0$ for $\|w\| > \rho_0$. Then, if $\rho_0/2 \leq r_2 \leq \rho_0$, we can write

$$\left| \int_{\rho_0/2 \leq \|w\| < r_2} k_0(w)\, e^{\tau \mathrm{Tr} B} dw \right| \leq \int_{\rho_0/2 \leq \|w\| \leq \rho_0} c\, \|w\|^{-(2+Q)} dw = c.$$

The second statement follows by a similar argument. ■

5 L^p estimates on singular integrals on nonhomogeneous spaces

We now want to apply to our singular kernel an abstract result, proved in [1], which we are going to recall now.

Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a *quasisymmetric quasidistance* on X if there exists a constant $c_d \geqslant 1$ such that for any $x, y, z \in X$:

$$d(x, y) \geqslant 0 \text{ and } d(x, y) = 0 \Leftrightarrow x = y;$$

$$d(x, y) \leq c_d d(y, x); \tag{32}$$

$$d(x, y) \leqslant c_d (d(x, z) + d(z, y)). \tag{33}$$

If d is a quasisymmetric quasidistance, then

$$d^* (x, y) = d(x, y) + d(y, x)$$

is a quasidistance, equivalent to d; d^* will be called *the symmetrized quasidistance* of d.

Definition 15 *We will say that* (X, d, μ, k) *is a* nonhomogeneous space with Calderón-Zygmund kernel k *if:*

1. *(X, d) is a set endowed with a quasisymmetric quasidistance d, such that the d-balls are open with respect to the topology induced by d;*
2. *μ is a positive regular Borel measure on X, and there exist two positive constants A, n such that:*

$$\mu(B(x, \rho)) \leq A\rho^n \text{ for any } x \in X, \rho > 0; \tag{34}$$

3. *$k(x, y)$ is a real valued measurable kernel defined in $X \times X$, and there exists a positive constant β such that:*

$$|k(x, y)| \leq \frac{A}{d(x, y)^n} \text{ for any } x, y \in X; \tag{35}$$

$$|k(x, y) - k(x_0, y)| \leq A \frac{d(x_0, x)^\beta}{d(x_0, y)^{n+\beta}} \tag{36}$$

for any $x_0, x, y \in X$ with $d(x_0, y) \geq A d(x_0, x)$, where n, A are as in (34).

Theorem 16 *(See Theorem 3 in [1]). Let (X, d, μ, k) be a bounded and separable nonhomogeneous space with Calderón-Zygmund kernel k. Also, assume that*

(i) $k^*(x, y) \equiv k(y, x)$ satisfies (36);

(ii) there exists a constant $B > 0$ such that

$$\left| \int_{d(x,y)>\rho} k(x,y)\, d\mu(y) \right| + \left| \int_{d(x,y)>\rho} k^*(x,y)\, d\mu(y) \right| \leqslant B \tag{37}$$

for any $\rho > 0, x \in X$;

(iii) for a.e. $x \in X$, the limits

$$\lim_{\rho \to 0} \int_{d(x,y)>\rho} k(x,y)\, d\mu(y); \quad \lim_{\rho \to 0} \int_{d(x,y)>\rho} k^*(x,y)\, d\mu(y)$$

exist finite. Then the operator

$$Tf(x) \equiv \lim_{\varepsilon \to 0} T_\varepsilon f(x) \equiv \lim_{\varepsilon \to 0} \int_{d(x,y)>\varepsilon} k(x,y) f(y)\, d\mu(y)$$

is well defined for any $f \in L^1(X)$, and

$$\|Tf\|_{L^p(X)} \leq c_p \|f\|_{L^p(X)} \text{ for any } p \in (1, \infty);$$

moreover, T is weakly $(1,1)$ continuous. The constant c_p only depends on all the constants implicitly involved in the assumptions: $p, c_d, A, B, n, \beta, diam(X)$.

We will also need the notion of Hölder space in this context:

Definition 17 (Hölder spaces) *We will say that $f \in C^\alpha(X)$, for some $\alpha > 0$, if*

$$\|f\|_\alpha \equiv \|f\|_\infty + |f|_\alpha \equiv \sup_{x \in X} |f(x)| + \sup_{x,y \in X, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} < \infty.$$

Our aim now is to apply the previous abstract result to the singular integral T with kernel k_0 on a bounded domain, say a ball $B(z_0, R)$. More precisely, as we shall see later, what we need is an estimate of the kind

$$\|Tf\|_{L^p(B(z_0,R))} \leq c \|f\|_{L^p(B(z_0,R))}$$

for $1 < p < \infty$, where R is a small radius fixed once and for all, z_0 is any point in the strip S, and the constant c is independent from z_0. Note that, by Proposition 8, our d is actually a quasisymmetric quasidistance in $X = B(z_0, R)$, as soon as R is small enough; moreover, by Proposition 10 the Lebesgue measure of a d-ball satisfies the required dimensional bound (34) with $n = Q + 2$. Also, Proposition 12 and Lemma 14 suggest that the kernel k_0 satisfies the properties required by Theorem 16. However, there is a subtle problem with this last assertion. Namely, saying, for instance, that k_0 satisfies the cancellation property in $B(z_0, R)$ means that

$$\left| \int_{\zeta \in B(z_0,R): r_1 < d(z,\zeta) < r_2} k_0\left(\zeta^{-1} \circ z\right) d\zeta \right| \leq c$$

whereas what we know (see Lemma 14) is that

$$\left|\int_{\zeta\in\mathbb{R}^{N+1}:r_1<d(z,\zeta)<r_2} k_0\left(\zeta^{-1}\circ z\right)d\zeta\right|\leq c.$$

The problem is that restricting the kernel k_0 to the domain $B(z_0,R)$ has the effect of a rough cut on the kernel, which can harm the validity of the cancellation property. To realize how things can actually go wrong, take the restriction of the Hilbert transform on the interval $(0,1)$: the singular integral operator

$$Tf(x)=\lim_{\varepsilon\to 0}\int_{y\in(0,1),|x-y|>\varepsilon}\frac{f(y)}{x-y}dy$$

is not so friendly, because

$$T1(x)=\lim_{\varepsilon\to 0}\int_{y\in(0,1),|x-y|>\varepsilon}\frac{1}{x-y}dy=\log\left(\frac{x}{1-x}\right)\text{ for any } x\in(0,1)$$

so (37) does not hold in this case. A more cautious choice, then, consists in cutting the kernel smoothly, by a couple of Hölder continuous cutoff functions. Namely, we have the following

Proposition 18 *Let k_0 be the above kernel (see (23)). There exists a constant $R_0>0$ such that, for any $z_0\in S$, $R\leq R_0$, if a,b are two cutoff functions belonging to $C^\alpha\left(\mathbb{R}^{N+1}\right)$ for some $\alpha>0$, with $sprt\,a$, $sprt\,b\subset B(z_0,R)$, and we set*

$$k(x,y)=a(x)\,k_0\left(y^{-1}\circ x\right)b(y),\tag{38}$$

then:

(a) k satisfies (35),(36) and (37) in $B(z_0,R)$ (with possibly other constants). Explicitly, "(37) in $B(z_0,R)$" means

$$\left|\int_{y\in B(z_0,R):r_1<d(x,y)<r_2} k(x,y)\,d\mu(y)\right|\leq c.\tag{39}$$

(b) for any $x\in B(z_0,R)$ there exists

$$h(x)\equiv\lim_{\varepsilon\to 0}\int_{y\in B(z_0,R):d(x,y)>\varepsilon}k(x,y)\,d\mu(y).$$

Finally, all the constants appearing in the above estimates about k depend on z_0,R and the cutoff functions a,b only through the C^α norms of a,b.

Remark 19 *Since in this Proposition and its proof the distinction between space and time variables is irrelevant, changing for a moment our notation we have denoted by $x,y,x_0...$ the variables in $\mathbb{R}^{N+1}$, and by $d\mu$ the Lebesgue measure $dxdt$ in $\mathbb{R}^{N+1}$.*

Proof. We will apply several times the properties of the kernel k_0 proved in Proposition 12 and Lemma 14. Also, we will use twice the following simple fact:

$$\int_{d(x,y)<\rho} \frac{d\mu(y)}{d(x,y)^{Q+2-\alpha}} \leq c\rho^{\alpha} \text{ for any } \rho > 0 \tag{40}$$

which can be checked by a dilation argument.

We chose R_0 small enough so that $x, y \in B(z_0, R_0)$ imply

$$d(x,y) + d(y,x) \leq 1.$$

Let $0 < R \leq R_0$.

(a). Condition (35) for k in $B(z_0, R)$ obviously follows from the analogous property of k_0. As to (36), we can write

$$k(x,y) - k(x_0,y) = [a(x) - a(x_0)]\, k_0\left(y^{-1} \circ x\right) b(y) +$$
$$+ a(x_0)\left[k_0\left(y^{-1} \circ x\right) - k_0\left(y^{-1} \circ x_0\right)\right] b(y) = I + II.$$

Now, for $d(x_0,y) > Md(x_0,x)$

$$|I| \leq |a|_{\alpha}\, d(x,x_0)^{\alpha} \cdot \frac{c}{d(x,y)^{Q+2}} \|b\|_{\infty} \leq c\frac{d(x,x_0)^{\alpha}}{d(x_0,y)^{Q+2+\alpha}}.$$

We have implicitly used the fact that the functions $d(x_0,y), d(x,y)$ are bounded by some absolute constant (since $x_0, x, y \in B(z_0,R)$), and the equivalence between $d(x_0,y)$ and $d(x,y)$, which holds under the assumption $d(x_0,y) > Md(x_0,x)$ (see Remark 13).

Moreover, since k_0 satisfies (36),

$$|II| \leq \|a\|_{\infty} \|b\|_{\infty}\, c\frac{d(x,x_0)}{d(x_0,y)^{Q+3}},$$

hence (36) holds for k in $B(z_0,R)$, with $n = Q+2, \beta = \alpha$.

To check (39) let us start noting that, since $\operatorname{sprt} b \subset B(z_0,R)$, we can write, for any $x \in B(z_0,R)$ and $0 < r_1 < r_2$

$$\int_{y \in B(z_0,R): r_1 < d(x,y) < r_2} k(x,y)\, d\mu(y) =$$
$$= a(x) \int_{y \in B(z_0,R): r_1 < d(x,y) < r_2} k_0\left(y^{-1} \circ x\right) b(y)\, d\mu(y) =$$
$$= a(x) \int_{y \in \mathbb{R}^{N+1}: r_1 < d(x,y) < r_2} k_0\left(y^{-1} \circ x\right) b(y)\, d\mu(y).$$

Note that there exists some absolute constant $c > 0$ such that $b(y)$ vanishes if $x \in B(z_0,R)$ and $d(x,y) \geq cR$; hence we can assume $r_2 \leq cR$. Under this condition, the last integral equals

$$a(x) \int_{y \in \mathbb{R}^{N+1}: r_1 < d(x,y) < r_2} k_0\left(y^{-1} \circ x\right) [b(y) - b(x)]\, d\mu(y) +$$
$$+ a(x)\, b(x) \int_{y \in \mathbb{R}^{N+1}: r_1 < d(x,y) < r_2} k_0\left(y^{-1} \circ x\right) d\mu(y) \equiv I + II.$$

Now, by (40)

$$|I| \leq c \left\| a \right\|_\infty |b|_\alpha \int_{d(x,y)<r_2} \frac{d\,(x,y)^\alpha}{d\,(x,y)^{Q+2}} d\mu\,(y) = c \left\| a \right\|_\infty |b|_\alpha\, r_2^\alpha \leq c \left\| a \right\|_\infty |b|_\alpha\, R_0^\alpha$$

while, by Lemma 14,

$$|II| \leq \left\| a \right\|_\infty \left\| b \right\|_\infty \left| \int_{y\in\mathbb{R}^{N+1}:r_1<d(x,y)<r_2} k_0\left(y^{-1}\circ x\right) d\mu\,(y) \right| \leq c \left\| a \right\|_\infty \left\| b \right\|_\infty.$$

(b) To show the existence of $h\,(x)$ let us consider, for $0 < \varepsilon_1 < \varepsilon_2$ and a fixed $x \in B\,(z_0, R)$,

$$\begin{aligned}
&\int_{y\in B(z_0,R):d(x,y)>\varepsilon_1} k\,(x,y)\, d\mu\,(y) - \int_{y\in B(z_0,R):d(x,y)>\varepsilon_2} k\,(x,y)\, d\mu\,(y) \\
&= a\,(x) \int_{y\in B(z_0,R):\varepsilon_1<d(x,y)<\varepsilon_2} k_0\left(y^{-1}\circ x\right) b\,(y)\, d\mu\,(y) \\
&= a\,(x) \int_{y\in\mathbb{R}^{N+1}:\varepsilon_1<d(x,y)<\varepsilon_2} k_0\left(y^{-1}\circ x\right) b\,(y)\, d\mu\,(y) \\
&= a\,(x) \int_{y\in\mathbb{R}^{N+1}:\varepsilon_1<d(x,y)<\varepsilon_2} k_0\left(y^{-1}\circ x\right) \left[b\,(y) - b\,(x)\right] d\mu\,(y) + \\
&+ a\,(x)\, b\,(x) \int_{y\in\mathbb{R}^{N+1}:\varepsilon_1<d(x,y)<\varepsilon_2} k_0\left(y^{-1}\circ x\right) d\mu\,(y) \\
&\equiv I + II.
\end{aligned}$$

Now,

$$\begin{aligned}
|I| &\leq \left\| a \right\|_\infty \int_{d(x,y)<\varepsilon_2} \left| k_0\left(y^{-1}\circ x\right) \left[b\,(y) - b\,(x)\right] \right| d\mu\,(y) \\
&\leq c \left\| a \right\|_\infty |b|_\alpha \int_{d(x,y)<\varepsilon_2} \frac{d\,(x,y)^\alpha}{d\,(x,y)^{Q+2}} d\mu\,(y) \\
&\leq c \left\| a \right\|_\infty |b|_\alpha\, \varepsilon_2^\alpha
\end{aligned}$$

by (40). On the other hand,

$$|II| \leq \left\| a \right\|_\infty \left\| b \right\|_\infty \left| \int_{y\in\mathbb{R}^{N+1}:\varepsilon_1<d(x,y)<\varepsilon_2} k_0\left(y^{-1}\circ x\right) d\mu\,(y) \right|$$

which tends to zero as $\varepsilon_2 \to 0$, by Lemma 14. This proves the existence of the limit $h\,(x)$. ■

¿From Theorem 16, Proposition 18 and the previous discussion, we immediately have the following:

Corollary 20 *For any fixed* $z_0 \in S$, *let*

$$Tf(z) = PV \int_{B(z_0,R)} k(z,\zeta) f(\zeta) \, d\zeta,$$

with k, R *as in the previous Proposition. Then for any* $p \in (1,\infty)$ *there exists* $c > 0$ *such that*

$$\|Tf\|_{L^p(B(z_0,R))} \leq c \|f\|_{L^p(B(z_0,R))}$$

for any $f \in L^p(B(z_0,R))$. *The constant* c *depends on the cutoff functions* a, b *only through their* C^α *norms, and does not depend on* z_0 *and* R.

We still need the following covering argument:

Lemma 21 *For every* $r_0 > 0$ *and* $K > 1$ *there exist* $\rho \in (0, r_0)$, *a positive integer* M *and a sequence of points* $\{z_i\}_{i=1}^{\infty} \subset S$ *such that:*

$$S \subset \bigcup_{i=1}^{\infty} B(z_i, \rho);$$

$$\sum_{i=1}^{\infty} \chi_{B(z_i, K\rho)}(z) \leq M \quad \forall z \in S.$$

Note that the above statement is nontrivial since the space S is unbounded and there is not a simple relation between d and the Euclidean distance. Since this property is better proved in an abstract context, we postpone its proof to the next section, and proceed to conclude the proof of our main result:

Theorem 22 *For a suitable choice of the number* ρ_0 *appearing in the definition of the kernel* k_0 *(see §3), for any* $p \in (1,\infty)$, *there exists a positive constant* c, *depending on* p, N, p_0, ν *and the matrix* B *such that*

$$\|PV(k_0 * f)\|_{L^p(S)} \leq c \|f\|_{L^p(S)}$$

for any $f \in L^p(S)$.

Proof. Pick a cutoff function

$$\begin{gathered} A \in C_0^\alpha(S) \text{ such that:} \\ A(z) = 1 \text{ for } \|z\| < \rho_0; \\ A(z) = 0 \text{ for } \|z\| > 2\rho_0 \end{gathered}$$

where the number ρ_0, to be fixed later, is the same appearing in the definition of the cutoff function η and the kernel k_0 (see (23) in §3). Let

$$a_i(z) = A\left(z^{-1} \circ z_i\right) \text{ for } i = 1, 2, ...;$$

Since $k_0(\zeta^{-1} \circ z)$ vanishes for $d(z,\zeta) > \rho_0$, we have that

$$z \in B(z_i, \rho_0) \text{ and } k_0(\zeta^{-1} \circ z) \neq 0 \implies \zeta \in B(z_i, C\rho_0) \tag{41}$$

for some absolute constant C. Define a second cutoff function

$$\begin{gathered} B \in C_0^{\alpha}(S) \text{ such that:} \\ B(z) = 1 \text{ for } \|z\| < C\rho_0; \\ B(z) = 0 \text{ for } \|z\| > 2C\rho_0 \end{gathered}$$

where C is the constant appearing in (41). Let

$$b_i(z) = B\left(z^{-1} \circ z_i\right) \text{ for } i = 1, 2, ...$$

Note that:

$$\begin{aligned} \|a_i\|_{C^{\alpha}} &= \|A\|_{C^{\alpha}} \text{ for } i = 1, 2, ... \\ \|b_i\|_{C^{\alpha}} &= \|B\|_{C^{\alpha}} \text{ for } i = 1, 2, ... \end{aligned} \tag{42}$$

Set

$$k_i(z, \zeta) = k_0(\zeta^{-1} \circ z) a_i(z) b_i(\zeta).$$

Let now R_0 be as in Proposition 18; set $r_0 = R_0/2C$ and let us apply Lemma 21 for this r_0: there exists $\rho_0 < r_0$ such that

$$S \subset \bigcup_{i=1}^{\infty} B(z_i, \rho_0); \tag{43}$$

$$\sum_{i=1}^{\infty} \chi_{B(z_i, 2C\rho_0)}(z) \leq M \quad \forall z \in S. \tag{44}$$

We eventually chose this value for the constant ρ_0.

Recall that $Tf = PV(k_0 * f)$. By (43) we can write

$$\|Tf\|_{L^p(S)} \leq \sum_{i=1}^{\infty} \|Tf\|_{L^p(B(z_i, \rho_0))}. \tag{45}$$

On the other side, by (41) for any $z \in B(z_i, \rho_0)$ we have

$$\begin{aligned} Tf(z) &= PV \int_{\mathbb{R}^{N+1}} k_0\left(\zeta^{-1} \circ z\right) f(\zeta) d\zeta = \\ &= a_i(z) P.V. \int_{\mathbb{R}^{N+1}} k_0\left(\zeta^{-1} \circ z\right) b_i(\zeta) f(\zeta) d\zeta = \int_{B(z_i, 2C\rho_0)} k_i(z, \zeta) f(\zeta) d\zeta \equiv T_i f(z) \end{aligned}$$

hence

$$\sum_{i=1}^{\infty} \|Tf\|_{L^p(B(z_i, \rho_0))} = \sum_{i=1}^{\infty} \|T_i f\|_{L^p(B(z_i, \rho_0))}. \tag{46}$$

Since $2C\rho_0 \leq R_0$, the kernel k_i also satisfies the assumptions of Proposition 18. Hence by Corollary 20 we have

$$\|T_i f\|_{L^p(B(z_i, 2C\rho_0))} \leq c \|f\|_{L^p(B(z_i, 2C\rho_0))} \tag{47}$$

with c independent of i, by (42). By (44) to (47) and we conclude

$$\|Tf\|_{L^p(S)} \leq c \sum_{i=1}^{\infty} \|f\|_{L^p(B(z_i,2C\rho_0))} \leq cM \|f\|_{L^p(S)}$$

which ends the proof. ■

Conclusion of the proof of Theorems 1 and 3. Theorem 22 and Corollary 6 imply Theorem 3, by (22). As we have shown in §1, Theorem 3 in turn implies (5)-(6) in Theorem 1. To finish the proof of Theorem 1 we are left to prove the weak $(1,1)$-estimates (7)-(8). This will be done here.

Let $u \in C_0^\infty(S)$. By (22) in §3 we can write, for any $\alpha > 0$:

$$\begin{aligned}
&\left|\left\{z \in S : \left|\partial^2_{x_i x_j} u(z)\right| \geq \alpha\right\}\right| \leq \\
&\leq \left|\left\{z \in S : \left|PV(k_0 * Lu)(z)\right| \geq \frac{\alpha}{3}\right\}\right| + \\
&+ \left|\left\{z \in S : \left|(k_\infty * Lu)(z)\right| \geq \frac{\alpha}{3}\right\}\right| + \\
&+ \left|\left\{z \in S : \left|c_{ij} Lu(z)\right| \geq \frac{\alpha}{3}\right\}\right| \\
&\equiv A + B + C.
\end{aligned}$$

Now, by Corollary 6

$$B + C \leq \frac{3}{\alpha}\left\{\|k_\infty * Lu\|_{L^1(S)} + \|c_{ij} Lu\|_{L^1(S)}\right\} \leq \frac{c}{\alpha}\|Lu\|_{L^1(S)}.$$

To bound A, we revise as follows the proof of Theorem 22, writing (with the same meaning of symbols and letting $f \equiv Lu$):

$$\begin{aligned}
A &= \left|\left\{z \in S : |Tf(z)| \geq \frac{\alpha}{3}\right\}\right| \leq \\
&\leq \sum_{i=1}^{\infty}\left|\left\{z \in B(z_i,\rho_0) : |Tf(z)| \geq \frac{\alpha}{3}\right\}\right| = \\
&= \sum_{i=1}^{\infty}\left|\left\{z \in B(z_i,\rho_0) : |T_i f(z)| \geq \frac{\alpha}{3}\right\}\right| \leq \\
&\leq \sum_{i=1}^{\infty}\left|\left\{z \in B(z_i,2C\rho_0) : |T_i f(z)| \geq \frac{\alpha}{3}\right\}\right| \leq \\
&\leq \sum_{i=1}^{\infty}\frac{c}{\alpha}\|f\|_{L^1(B(z_i,2C\rho_0))} \leq \frac{cM}{\alpha}\|f\|_{L^1(S)}
\end{aligned}$$

where we used the fact that T_i is also weak $(1,1)$ continuous on $L^1(B(z_i,2C\rho_0))$, by Theorem 16. This proves the weak estimate on the strip:

$$\left|\left\{z \in S : \left|\partial^2_{x_i x_j} u(z)\right| \geq \alpha\right\}\right| \leq \frac{c}{\alpha}\|Lu\|_{L^1(S)}. \tag{48}$$

Next, we take a cutoff function $\psi \in C_0^\infty(-1,1)$ such that $\psi(t) \geq 1$ in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and, for any $u \in C_0^\infty\left(\mathbb{R}^N\right)$, apply (48) to ψu, getting

$$\begin{aligned}
&\left|\left\{x \in \mathbb{R}^N : \left|\partial^2_{x_i x_j} u(x)\right| \geq \alpha\right\}\right| \leq \\
&\leq \left|\left\{(x,t) \in \mathbb{R}^N \times \left[-\frac{1}{2}, \frac{1}{2}\right] : \left|\psi(t)\, \partial^2_{x_i x_j} u(x)\right| \geq \alpha\right\}\right| \leq \\
&\leq \left|\left\{z \in S : \left|\partial^2_{x_i x_j} (u\psi)(z)\right| \geq \alpha\right\}\right| \leq \\
&\leq \frac{c}{\alpha} \left\|L(u\psi)\right\|_{L^1(S)} \leq \\
&\leq \frac{c}{\alpha} \left\{\left\|\mathcal{A}u\right\|_{L^1(\mathbb{R}^N)} + \left\|u\right\|_{L^1(\mathbb{R}^N)}\right\}.
\end{aligned}$$

So we have proved (7); then (8) follows from (7) using the equation, and this ends the proof. ■

6 A covering lemma

To make our proof of Theorem 22 complete, we are left to prove Lemma 21. This is what we are doing to do now, by a general abstract argument.

Definition 23 *We say that* (X, d, μ) *is a* space of locally homogeneous type *if the following conditions hold:*

(i) $d : X \times X \to \mathbb{R}_+$ *is a function such that for some constant* $C > 0$

(i_1) For every $x, y \in X$, *if* $d(y,x) \leq 1$ *then* $d(x,y) \leq C d(y,x)$

(i_2) For every $x, y, z \in X$, *if* $d(x,z) \leq 1$ *and* $d(y,z) \leq 1$ *then*

$$d(x,y) \leq C\left(d(x,z) + d(z,y)\right).$$

(ii) μ *is a positive measure defined on a* σ*-algebra of subsets of* X *which contains the* d*-balls*

$$B(x,\rho) = \{y \in X : d(y,x) < \rho\}, \quad x \in X, \rho > 0.$$

(iii) There exists $R > 0$ *such that if* $0 < R_1 < R_2 \leq R$ *then there exists* $C = C(R_1, R_2)$ *such that*

$$0 < \mu(B(x, R_2)) \leq C\mu(B(x, R_1)) < \infty \quad \textit{for any } x \in X. \tag{49}$$

Remark 24 *Note that* $(S, d, dxdt)$ *is a space of locally homogeneous type. Namely, condition (i) follows from Proposition 8, (ii) follows from Lemma 9 and (iii) follows from Proposition 10. Hence the following theorem will imply Lemma 21, and therefore will conclude the proof of Theorem 3.*

Theorem 25 *Let (X, d, μ) be a space of locally homogeneous type. Then for every $r_0 > 0$ and $K > 1$, there exist $\rho \in (0, r_0)$, a positive integer M and a countable set $\{x_i\}_{i \in A} \subset X$ such that:*

1. $$\bigcup_{i \in A} B(x_i, \rho) = X;$$

2. $$\sum_{i \in A} \chi_{B(x_i, K\rho)} \leq M^2.$$

Proof. First of all, we claim that for any $\rho > 0$, X admits a maximal countable family of disjoint balls of radius ρ. Namely: the existence of a maximal family (of arbitrary cardinality) of disjoint balls of radius ρ follows by Zorn's Lemma; let us show that this family $\{B(x_\alpha, \rho)\}_\alpha$ must be countable. Otherwise, since for any fixed $x_0 \in X$,

$$X = \bigcup_{n=1}^{\infty} B(x_0, n),$$

at least one ball $B(x_0, n)$ should contain an uncountable family of disjoint balls of radius ρ. By (49), every such ball has positive measure, and this would imply that $B(x_0, n)$ has infinite measure, which contradicts (49). This proves the claim.

Then, let $\left\{B\left(x_i, \frac{\rho}{C(C+1)}\right)\right\}_{i \in A}$ be a countable maximal family of disjoint balls.

Fix $x \in X$. There exists $i \in A$ such that

$$B\left(x, \frac{\rho}{C(C+1)}\right) \cap B\left(x_i, \frac{\rho}{C(C+1)}\right) \neq \emptyset.$$

To estimate $d(x_i, x)$, we consider $y \in B\left(x, \frac{\rho}{C(C+1)}\right) \cap B\left(x_i, \frac{\rho}{C(C+1)}\right)$, and we find

$$d(x_i, x) \leq C(d(x_i, y) + d(y, x)) < C\left(\frac{\rho}{C(C+1)} + C \cdot \frac{\rho}{C(C+1)}\right) = \rho$$

where, to apply (i_1)-(i_2) in the definition of space of locally homogeneous type, we have assumed $\rho \leq 1$. This proves (1).

To prove (2), fix an arbitrary $i \in A$; we want to estimate how many $j \in A$ satisfy the property

$$B(x_i, K\rho) \cap B(x_j, K\rho) \neq \emptyset. \tag{50}$$

Fix x_i and x_j and suppose there exists $y \in B(x_i, K\rho) \cap B(x_j, K\rho)$. We assume $K\rho \leq 1$; hence

$$d(x_i, x_j) \leq C(d(x_i, y) + d(y, x_j)) \leq C(K\rho + CK\rho) = C(1 + C)K\rho,$$

and we assume $C(1+C)K\rho \leq 1$. Now suppose that for $j = 1, 2, ..., N$ we have (50); we want to estimate N.

Take $z \in B(x_j, K\rho)$. Since $K\rho \leq 1$ and $d(x_i, x_j) \leq 1$ we have

$$\begin{aligned} d(x_i, z) &\leq C(d(x_i, x_j) + d(x_j, z)) \leq C(C(1+C)K\rho + K\rho) = \\ &= K\rho\left(C\left(C^2 + C + 1\right)\right) \equiv K'\rho \end{aligned}$$

with $K' > 1$. The previous computation shows that

$$\bigcup_{j=1}^{N} B(x_j, K\rho) \subset B(x_i, K'\rho).$$

Since by construction the balls $B\left(x_i, \frac{\rho}{C(C+1)}\right)$ are pairwise disjoint, we have

$$\begin{aligned} \sum_{j=1}^{N} \mu\left(B\left(x_j, \frac{\rho}{C(C+1)}\right)\right) &= \mu\left(\bigcup_{j=1}^{N} B\left(x_j, \frac{\rho}{C(C+1)}\right)\right) \leq \\ &\leq \mu\left(\bigcup_{j=1}^{N} B(x_j, K\rho)\right) \leq \mu(B(x_i, K'\rho)). \end{aligned}$$

Assuming $K'\rho \leq R$ (with R as in (iii) of the above definition), we also have, for some constant M only depending on C and ρ,

$$\sum_{j=1}^{N} \mu\left(B\left(x_j, \frac{\rho}{C(C+1)}\right)\right) \leq M\mu\left(B\left(x_i, \frac{\rho}{C(C+1)}\right)\right).$$

Now fix any $j = 1, 2, ..., N$. Note that i satisfies (50); repeating the previous argument exchanging i with j we get

$$\mu\left(B\left(x_i, \frac{\rho}{C(C+1)}\right)\right) \leq \mu(B(x_j, K'\rho)) \leq M\mu\left(B\left(x_j, \frac{\rho}{C(C+1)}\right)\right).$$

We have found that

$$\sum_{k=1}^{N} \mu\left(B\left(x_k, \frac{\rho}{C(C+1)}\right)\right) \leq M^2\mu\left(B\left(x_j, \frac{\rho}{C(C+1)}\right)\right) \text{ for any } j = 1, 2, ..., N.$$

Letting

$$a = \min_{j=1,2,...,N} \mu\left(B\left(x_j, \frac{\rho}{C(C+1)}\right)\right)$$

we get

$$Na \leq \sum_{k=1}^{N} \mu\left(B\left(x_k, \frac{\rho}{C(C+1)}\right)\right) \leq M^2 a$$

and since, by (iii), $0 < a < \infty$, we infer $N \leq M^2$, which is (2), provided ρ satisfies all the conditions we have imposed so far:

$$\rho \leq 1;\ K\rho \leq 1; K'\rho \equiv K\rho\,(C\,(1 + C\,(1 + C))) \leq R.$$

∎

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