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GLOBAL LARGE SOLUTIONS TO INITIAL BOUNDARY VALUE PROBLEMS IN ONE-DIMENSIONAL NONLINEAR THERMOVISCOELASTICITY

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Abstract. Initial boundary value problems in one-dimensional nonlinear thermoviscoelasticity are considered, and the existence of global classical solutions is established by means of the Leray-Schauder fixed point theorem.

Introduction. In this paper we study the existence of global smooth solutions to initial boundary value problems in one-dimensional nonlinear thermoviscoelasticity. The conservation laws of mass, momentum, and energy for one-dimensional materials with the reference density $\rho_0 = 1$ are

$$\begin{gathered} u_t - v_x = 0, \\ v_t - \sigma_x = 0, \\ \left(e + \frac{v^2}{2}\right)_t - (\sigma v)_x + q_x = 0, \end{gathered} \tag{1.1}$$

and the second law of thermodynamics is expressed by the Clausius-Duhem inequality

$$\eta_t + \left(\frac{q}{\theta}\right)_x \geq 0, \tag{1.2}$$

where subscripts indicate partial differentiations, u is the deformation gradient, v is the velocity, e denotes the internal energy, σ is the stress, η stands for the specific entropy, θ for the temperature, and q for the heat flux.

For one-dimensional, homogeneous, thermoviscoelastic materials, e, σ, η, and q are given by the constitutive relations (see [1])

$$e = \hat{e}(u, \theta), \qquad \sigma = \hat{\sigma}(u, \theta, v_x), \qquad \eta = \hat{\eta}(u, \theta), \qquad q = \hat{q}(u, \theta, \theta_x), \tag{1.3}$$

which in order to be consistent with (1.2), must satisfy

$$\begin{gathered} \hat{\sigma}(u, \theta, 0) = \hat{\psi}_u(u, \theta), \qquad \hat{\eta}(u, \theta) = -\hat{\psi}_\theta(u, \theta), \\ (\hat{\sigma}(u, \theta, w) - \hat{\sigma}(u, \theta, 0))w \geq 0, \qquad \hat{q}(u, \theta, g)g \leq 0, \end{gathered} \tag{1.4}$$

where $\psi = e - \theta\eta$ is the Helmholtz free energy function.

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Here we consider that the reference configuration is the unit interval $[0, 1]$ and that the initial values of the deformation gradient, the velocity, and the temperature are the given functions

$$u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x), \qquad \theta(x, 0) = \theta_0(x), \qquad x \in [0, 1]. \tag{1.5}$$

As boundary conditions we consider

$$\begin{cases} \sigma(1, t) = -\gamma v(1, t), & \sigma(0, t) = \gamma v(0, t), \quad t \geq 0, \\ \theta(1, t) = \theta(0, t) = T_0, & t \geq 0, \end{cases} \tag{1.6}$$

where $\gamma = 0$ or $\gamma = 1$, and $T_0 > 0$ is the reference temperature. The boundary condition $(1.6)_1$ with $\gamma = 1$, boundary damping, represents that the endpoints of the interval $[0, 1]$ are connected to some sort of dash pot.

Recently, Dafermos [2], Dafermos and Hsiao [3] considered the following boundary conditions (stress free and thermally insulated):

$$\sigma(1, t) = \sigma(0, t) = 0, \qquad q(1, t) = q(0, t) = 0, \qquad t \geq 0, \tag{1.7}$$

and established the existence of global smooth solutions to (1.1), (1.5), (1.7) for a class of solid-like materials by applying the Leray-Schauder fixed point theorem. The techniques in [2] work when only one end of the body is stress-free while the other is fixed, say $v(1, t) = 0$, $\sigma(0, t) = 0$. For the initial value problem to (1.1) Zheng and Shen [16] proved the global existence of smooth solutions provided that the initial data are sufficiently small, and Kim [9] established the existence of global weak solutions in $L^1 \cap BV$. When the material is an ideal gas or a real gas, there are independent investigations on the existence of global classical solutions of (1.1) (see [5–8, 11–14] and the references cited therein).

The purpose of this paper is to show the existence of globally defined classical solutions to (1.1), (1.5), (1.6) for the same class of solid-like materials as in [2]. Our paper is mainly based on techniques in [2, 3], subject to the necessary modifications in the proof.

As in [2] we consider the problem of existence of solutions to (1.1), (1.5)–(1.6) for a class of materials with constitutive relations

$$e = \hat{e}(u, \theta), \qquad \sigma = -\hat{p}(u, \theta) + \hat{\mu}(u) v_x, \qquad q = -\hat{\kappa}(u, \theta)\theta_x, \tag{1.8}$$

where the viscosity $\hat{\mu}(u)u$ satisfies

$$\hat{\mu}(u)u \geq \mu_0 > 0, \qquad 0 < u < \infty \tag{1.9}$$

for some constant μ_0. We assume that $\hat{e}(u, \theta)$, $\hat{p}(u, \theta)$, $\hat{\mu}(u)$, and $\hat{\kappa}(u, \theta)$ are twice continuously differentiable for $0 < u < \infty$ and $0 \leq \theta < \infty$, and are interrelated by

$$\hat{e}_u(u, \theta) = -\hat{p}(u, \theta) + \theta \hat{p}_\theta(u, \theta) \tag{1.10}$$

so as to be consistent with (1.4). Moreover, we will be concerned with solid-like materials, so we require that $\hat{p}(u, \theta)$ be compressive for small u and tensile for large u, at any temperature, i.e., there are $0 < \tilde{u} \leq \widetilde{U} < \infty$ such that

$$\begin{aligned} \hat{p}(u, \theta) &\geq 0, \qquad 0 < u < \tilde{u}, \qquad 0 \leq \theta < \infty, \\ \hat{p}(u, \theta) &\leq 0, \qquad \widetilde{U} < u < \infty, \qquad 0 \leq \theta < \infty. \end{aligned} \tag{1.11}$$

Hence, the assumption (1.11) implies that there is a constant η_0 with $\tilde{u} \le \eta_0 \le \widetilde{U}$ such that

$$\hat{p}(\eta_0, T_0) = 0. \tag{1.12}$$

We impose the following monotone condition on $\hat{p}$:

$$\begin{aligned} -\hat{p}_u(u, T_0) &\ge 0 && \text{for any } \overline{u} \le u \le \overline{U} \text{ if } \gamma = 0 \text{ in } (1.6)_1, \\ -\hat{p}_u(u, T_0) &\ge p_0 > 0 && \text{for any } 0 < u < \infty \text{ if } \gamma = 1 \text{ in } (1.6)_1. \end{aligned} \tag{1.13}$$

Here p_0 is a constant, and

$$\overline{u} := \widehat{M}^{-1}\left(\min_{\lambda\in[0,1]}\left[\widehat{M}\left(\min\left\{\tilde{u}, (1-\lambda)\eta_0 + \lambda\min_{[0,1]}u_0(\cdot)\right\}\right) - 2E_0^{1/2}(\lambda)\right] - 1\right), \tag{1.14$_1$}$$

$$\overline{U} := \widehat{M}^{-1}\left(\max_{\lambda\in[0,1]}\left[\widehat{M}\left(\max\left\{\widetilde{U}, (1-\lambda)\eta_0 + \lambda\max_{[0,1]}u_0(\cdot)\right\}\right) + 2E_0^{1/2}(\lambda)\right] + 1\right), \tag{1.14$_2$}$$

$$E_0(\lambda) := \left(1 + \frac{2\gamma^2}{p_0}\right)\int_0^1 \left\{E((1-\lambda)\eta_0 + \lambda u_0(x), (1-\lambda)T_0 + \lambda\theta_0(x)) + \frac{\lambda^2 v_0^2(x)}{2}\right\} dx + \gamma^2\eta_0^2, \tag{1.14$_3$}$$

$$\widehat{M}(u) := \int_1^u \hat{\mu}(w)\,dw, \qquad E(u, \theta) := \hat{\psi}(u, \theta) - \hat{\psi}(\eta_0, T_0) - (\theta - T_0)\hat{\psi}_\theta(u, \theta), \tag{1.14$_4$}$$

and $\hat{\psi}(u, \theta)$ is the Helmholtz free energy function. In view of (1.9), $\widehat{M}(u)$ is a strictly increasing function that maps $(0, \infty)$ onto $(-\infty, \infty)$. Using (1.11) we can show that u is *a priori* bounded, $\overline{u} < u(x, t) < \overline{U}$ (cf. Lemma 2.3), and hence no restrictions are necessary on the behavior of $\hat{e}(u, \theta)$, $\hat{p}(u, \theta)$, and $\hat{\kappa}(u, \theta)$ at $u = 0+$ and $u = \infty$. As concerns the temperature, we impose the following growth conditions upon $\hat{e}(u, \theta)$, $\hat{p}(u, \theta)$, and $\hat{\kappa}(u, \theta)$. There are positive constants ν and N possibly depending on $\overline{u}$ and/or $\overline{U}$ such that for any $\overline{u} \le u \le \overline{U}$, $0 \le \theta < \infty$,

$$\hat{e}(u, 0) \ge 0, \qquad \nu \le \hat{e}_\theta(u, \theta) \le N(1 + \theta^{1/3}), \tag{1.15}$$

$$|\hat{p}_u(u, \theta)| \le N(1 + \theta^{4/3}), \qquad |\hat{p}_\theta(u, \theta)| \le N(1 + \theta^{1/3}), \tag{1.16}$$

$$\nu \le \hat{\kappa}(u, \theta) \le N, \qquad |\hat{\kappa}_u(u, \theta)| \le N, \qquad |\hat{\kappa}_\theta(u, \theta)| \le N, \qquad |\hat{\kappa}_{uu}(u, \theta)| \le N. \tag{1.17}$$

We use the familiar notations $C^\alpha[0, 1]$ for the Banach space of functions on $[0, 1]$ that are uniformly Hölder continuous with exponent α and $C^{\alpha,\alpha/2}(Q_T)$ for those functions on $Q_T := [0, 1] \times [0, T]$ that are uniformly Hölder continuous with exponent α in x and $\alpha/2$ in t. The norms of $C^\alpha[0, 1]$ and $C^{\alpha,\alpha/2}(Q_T)$ will be denoted by $\|\cdot\|_\alpha$ and $|||\cdot|||_\alpha$, respectively. The main result of this paper is

THEOREM 1.1. Let $u_0(x)$, $u_0'(x)$, $v_0(x)$, $v_0'(x)$, $v_0''(x)$, $\theta_0(x)$, $\theta_0'(x)$, and $\theta_0''(x)$ be in $C^\alpha[0, 1]$ for some $\alpha \in (0, 1)$. Let $u_0(x) > 0$, $\theta_0(x) > 0$ for $x \in [0, 1]$, and assume that the initial data are compatible with the boundary conditions (1.6). Then there exists a unique solution $\{u(x, t), v(x, t), \theta(x, t)\}$ to (1.1), (1.5), (1.6) on $[0, 1] \times [0, \infty)$ such that for every $T > 0$ the functions $u, u_x, u_t, u_{xt}, v, v_x, v_t, v_{xx}, \theta, \theta_x, \theta_t, \theta_{xx}$ are in $C^{\alpha, \alpha/2}(Q_T)$ and $u_{tt}, v_{xt}, \theta_{xt}$ are in $L^2(Q_T)$. Furthermore, $\theta(x, t) > 0$, $\overline{u} < u(x, t) < \overline{U}$ for $0 \le x \le 1$, $0 \le t < \infty$, where $\overline{u}, \overline{U}$ are the same as in (1.14).

We will apply the Leray-Schauder fixed point theorem to prove the theorem in Sec. 3. Section 2 is devoted to the necessary *a priori* estimates.

REMARK 1.1. The techniques in this paper work for the boundary conditions:

$$\sigma(1, t) = -v(1, t), \qquad \sigma(0, t) = v(0, t), \qquad t \ge 0,$$
$$q(1, t) = q(0, t) = 0, \qquad t \ge 0$$

and for the boundary conditions that one end of the interval $[0, 1]$ is stress-free while the other has the boundary damping, say $\sigma(1, t) = 0$, $\sigma(0, t) = v(0, t)$, and an identical existence theorem can be obtained.

REMARK 1.2. The slower the growth rate of $\hat{e}(u, \theta)$, the faster the growth rate of $\hat{p}(u, \theta)$ that can be treated by the same procedure here. Furthermore, if the specific heat and/or the heat conductivity grows with the temperature, say $\hat{e}_\theta(u, \theta) \ge \nu(1 + \theta^s)$ and/or $\hat{\kappa}(u, \theta) \ge \nu(1 + \theta^s)$, then higher growth rates of $\hat{e}(u, \theta)$ and $\hat{p}(u, \theta)$ may be tolerated.

2. A priori estimates. Let $T > 0$ be arbitrary but fixed. Throughout this paper, Λ will denote a generic constant which may depend at most on T, T_0, ν, μ_0, N, $\overline{u}$, $\overline{U}$, and upper bounds of the $C^\alpha[0, 1]$ norm of u_0, u_0', v_0, v_0', v_0'', θ_0, θ_0', and θ_0''. Our aim in this section is to show the following.

THEOREM 2.1. *Let $\{u(x, t), v(x, t), \theta(x, t)\}$ be a solution of (1.1), (1.5), (1.6) on $[0, 1]\times[0, \infty)$ in the function class indicated in Theorem 1.1. Then $u, v, v_x, \theta, \theta_x$ can be a priori bounded in $C^{1/3, 1/6}(Q_T)$, i.e.,*

$$|||\, u \,|||_{1/3} + |||\, v \,|||_{1/3} + |||\, v_x \,|||_{1/3} + |||\, \theta \,|||_{1/3} + |||\, \theta_x \,|||_{1/3} \le \Lambda .$$

Furthermore, $\theta(t, x) > 0$, $\overline{u} < u(t, x) < \overline{U}$ for any $x \in [0, 1]$ and $t \ge 0$.

The proof of Theorem 2.1 is broken into a sequence of lemmas. The first observation is that using σ in (1.8) we can write $(1.1)_2$ in the form

$$v_t + \hat{p}(u, \theta)_x = (\hat{\mu}(u) v_x)_x, \tag{2.1}$$

and utilizing $(1.1)_3$, $(1.1)_2$, (1.10), (1.8), and $(1.1)_1$ we have

$$\hat{e}_\theta(u, \theta)\theta_t + \theta \hat{p}_\theta(u, \theta) v_x - \hat{\mu}(u) v_x^2 = (\hat{\kappa}(u, \theta)\theta_x)_x . \tag{2.2}$$

If we apply the maximum principle [15, III.3] to (2.2), recalling $\theta_0(x) > 0$, we infer

LEMMA 2.1.

$$\theta(x, t) > 0 \quad \text{for } 0 \le x \le 1, \ 0 \le t < \infty .$$

Now we derive an estimate on u, v, and θ by exploiting some relations associated with the second law of thermodynamics.

LEMMA 2.2. If $\overline{u} \le u(t, x) \le \overline{U}$ for all $x \in [0, 1]$ and $t \in [0, \tau]$, $\tau > 0$, then

$$\begin{aligned} &\frac{\nu}{2}\int_0^1 \frac{(\theta(x, t) - T_0)^2}{\theta(x, t) + T_0}\, dx + \frac{\omega_\gamma}{2}\int_0^1 (u(x, t) - \eta_0)^2\, dx + \frac{1}{2}\int_0^1 v^2(x, t)\, dx \\ &\quad \le \int_0^1 \left(E(u_0(x), \theta_0(x)) + \frac{v_0^2(x)}{2}\right) dx \equiv e_0, \qquad \forall 0 \le t \le \tau, \end{aligned} \tag{2.3}$$

where $\omega_\gamma = 0$ for $\gamma = 0$ and $\omega_\gamma = p_0$ for $\gamma = 1$, $E(u, \theta)$ and p_0 are the same as in $(1.14)_4$ and in $(1.13)_2$, respectively.

Proof. Recalling the definition $(1.14)_4$ of $E(u, \theta)$, making use of $(1.1)_1$, $(1.1)_2$, (2.2), and (1.4), noting that $-\hat{p}(u, \theta) = \hat{\psi}_u(u, \theta)$ and $\hat{e}_\theta(u, \theta) = -\theta\partial^2\hat{\psi}(u, \theta)/\partial\theta^2$, we obtain after a calculation that

$$\partial_t\left(E(u, \theta) + \frac{v^2}{2}\right) + \frac{T_0\hat{\mu}(u)}{\theta}v_x^2 + \frac{T_0\hat{\kappa}(u, \theta)}{\theta^2}\theta_x^2 = \partial_x\left(\sigma v + \frac{(\theta - T_0)}{\theta}\hat{\kappa}(u, \theta)\theta_x\right). \tag{2.4}$$

We integrate (2.4) over $[0, 1]\times[0, t]$ $(0 \le t \le T)$ and apply the boundary conditions (1.6) to arrive at

$$\begin{aligned} &\int_0^1 \left(E(u, \theta) + \frac{v^2}{2}\right)(x, t)\, dx + T_0\int_0^t\int_0^1 \left(\frac{\hat{\mu}(u)}{\theta}v_x^2 + \frac{\hat{\kappa}(u, \theta)}{\theta^2}\theta_x^2\right) dx\, ds \\ &\quad + \gamma\int_0^t (v^2(1, s) + v^2(0, s))\, ds = \int_0^1 \left(E(u_0(x), \theta_0(x)) + \frac{v_0^2(x)}{2}\right) dx. \end{aligned} \tag{2.5}$$

Recalling $\hat{\psi}_{uu} = -\hat{p}_u$, if we use the mean value theorem (or the Taylor theorem), (1.15), and (1.12)–(1.13), we see that

$$E(u, \theta) - \hat{\psi}(u, T_0) + \hat{\psi}(\eta_0, T_0) \ge \frac{\nu}{2}\frac{(\theta - T_0)^2}{(\theta + T_0)} \tag{2.6}$$

and

$$\hat{\psi}(u, T_0) - \hat{\psi}(\eta_0, T_0) \ge \frac{\omega_\gamma}{2}(u - \eta_0)^2$$

for $\overline{u} \le u \le \overline{U}$, which adding to (2.6) gives

$$E(u, \theta) \ge \frac{\nu}{2}\frac{(\theta - T_0)^2}{(\theta + T_0)} + \frac{\omega_\gamma}{2}(u - \eta_0)^2 \quad \text{for } \overline{u} \le u \le \overline{U}. \tag{2.7}$$

Inserting (2.7) into (2.5) yields the lemma. □

Next, we want to bound the deformation gradient $u(x, t)$. To this end we rewrite (2.1), using $(1.14)_4$, as follows:

$$v_t + \hat{p}(u, \theta)_x = \widehat{M}(u)_{tx}. \tag{2.8}$$

LEMMA 2.3. We have

$$\overline{u} < u(x, t) < \overline{U}, \qquad 0 \le x \le 1, \qquad 0 \le t < \infty, \tag{2.9}$$

where $\overline{u}$ and $\overline{U}$ are defined by $(1.14)_1$ and $(1.14)_2$, respectively.

Proof. If we integrate (2.8) over $[0, y] \times [s, \tau]$ and $[y, 1] \times [s, \tau]$, $0 \le y \le 1$, $0 \le s < \tau$, respectively, and apply the boundary condition $(1.6)_1$, we obtain

$$\widehat{M}(u(y, \tau)) - \widehat{M}(u(y, s)) = \int_s^\tau p(y, t)\, dt + \gamma \int_s^\tau v(0, t)\, dt + \int_0^y (v(x, \tau) - v(x, s))\, dx \tag{2.10}$$

and

$$\widehat{M}(u(y, \tau)) - \widehat{M}(u(y, s)) = \int_s^\tau p(y, t)\, dt - \gamma \int_s^\tau v(1, t)\, dt - \int_y^1 (v(x, \tau) - v(x, s))\, dx, \tag{2.11}$$

where $p(y, t) = \hat{p}(u(y, t), \theta(y, t))$. We add (2.11) to (2.10) and take $u_t = v_x$ into account to deduce

$$\begin{aligned}\widehat{M}(u(y, \tau)) - \widehat{M}(u(y, s)) &= \int_s^\tau p(y, t)\, dt - \frac{\gamma}{2} \int_s^\tau \int_0^1 v_x(x, t)\, dx\, dt \\ &\quad + \frac{1}{2}\left(\int_0^y - \int_y^1\right)(v(x, \tau) - v(x, s))\, dx \\ &= \int_s^\tau p(y, t)\, dt - \frac{\gamma}{2}\int_0^1 (u(x, \tau) - u(x, s))\, dx \\ &\quad + \frac{1}{2}\left(\int_0^y - \int_y^1\right)(v(x, \tau) - v(x, s))\, dx.\end{aligned} \tag{2.12}$$

By Lemma 2.2 and the Schwarz inequality, recalling the definition $(1.14)_3$, we see that if $\bar{u} \le u(x, t) \le \overline{U}$ for $0 \le x \le 1$, $0 \le t \le \tau$, then we have

$$\begin{aligned}&\frac{\gamma}{2}\left|\int_0^1 (u(x, \tau) - u(x, s))\, dx\right| + \frac{1}{2}\left|\left(\int_0^y - \int_y^1\right)(v(\tau, x) - v(s, x))\, dx\right| \\ &\le \gamma \max_{[0, \tau]}\left(\int_0^1 u^2(x, \cdot)\, dx\right)^{1/2} + \max_{[0, \tau]}\left(\int_0^1 v^2(x, \cdot)\, dx\right)^{1/2} \\ &\le \gamma(2\eta_0^2 + 4e_0/p_0)^{1/2} + (2e_0)^{1/2} \le 2((1 + 2\gamma^2/p_0)e_0 + \gamma^2\eta_0^2)^{1/2} \equiv 2E_0^{1/2}(1).\end{aligned} \tag{2.13}$$

In particular, $(1.14)_1$ and $(1.14)_2$ yield $\bar{u} < u_0(x) < \overline{U}$, $0 \le x \le 1$. Thus, if $\bar{u} < u(x, t) < \overline{U}$ is violated on $[0, 1] \times [0, \infty)$, then there are $\tau > 0$ and $y \in [0, 1]$ such that

$$\bar{u} < u(x, t) < \overline{U} \quad \text{for } x \in [0, 1], 0 \le t < \tau, \quad \text{but } u(y, \tau) = \bar{u} \quad \text{or} \quad u(y, \tau) = \overline{U}. \tag{2.14}$$

Note that $\bar{u} < \tilde{u}$. If $u(y, \tau) = \bar{u}$, then either $u(y, t) < \tilde{u}$ for $0 \le t \le \tau$, or $u(y, t) < \tilde{u}$ for $0 \le s < t \le \tau$, but $u(y, s) = \tilde{u}$. Recalling that, on account of (2.14), $\bar{u} \le u(x, t) \le \overline{U}$ for $0 \le x \le 1$ and $0 \le t \le \tau$, in the former case we apply (2.12) with $s = 0$ and utilize (1.11) and (2.13) to deduce

$$\widehat{M}(u(y, \tau)) > \widehat{M}(u_0(y)) - 2E_0^{1/2}(1) - 1, \tag{2.15}$$

while in the latter case (2.12) combined with (1.11) and (2.13) implies

$$\widehat{M}(u(y, \tau)) > \widehat{M}(\tilde{u}) - 2E_0^{1/2}(1) - 1. \tag{2.16}$$

In either case, by $(1.14)_1$, $\widehat{M}(u(y,\tau)) > \widehat{M}(\overline{u})$ which contradicts $u(y,\tau) = \overline{u}$. Hence, $\overline{u} < u(t,x)$, $0 \le x \le 1$, $0 \le t < \infty$.

Similarly, we can show that $u(y,\tau) = \overline{U}$ is a contradiction. This shows $u(t,x) < \overline{U}$ for $0 \le x \le 1$, $0 \le t < \infty$. □

As a result of Lemmas 2.1–2.3 one gets

$$\max_{[0,T]} \int_0^1 \theta(x,t)\,dx \le \Lambda. \tag{2.17}$$

By (1.15) and Lemma 2.1 we find that $\int_0^1 e(x,t)\,dx \ge 0$. Using this fact, we integrate $(1.1)_3$ over $[0,1]\times[0,T]$ and apply the boundary condition $(1.6)_1$ to infer

$$\int_0^T (q(1,t) - q(0,t))\,dt \le \int_0^1 \left(\hat{e}(u_0(x), \theta_0(x)) + \frac{v_0^2}{2}\right)(x)\,dx \le \Lambda, \tag{2.18}$$

where $q(x,t) = -\hat{\kappa}(u(x,t), \theta(x,t))\theta_x(x,t)$.

We now proceed to get estimates. We have

LEMMA 2.4.

$$\int_0^T \int_0^1 \theta^{-4/3}\theta_x^2\,dx\,dt \le \Lambda, \tag{2.19}$$

$$\int_0^T \int_0^1 \theta^{8/3}\,dx\,dt \le \Lambda, \tag{2.20}$$

$$\int_0^T \max_{[0,1]} \theta^{5/3}(\cdot\,,t)\,dt \le \Lambda. \tag{2.21}$$

Proof. Define $\widehat{H}(u,\theta) := \int_0^\theta \xi^{-1/3}\hat{e}_\theta(u,\xi)\,d\xi$. By (1.15) and (1.10),

$$\begin{gathered} |\widehat{H}(u,\theta)| \le 2N(1+\theta), \qquad \widehat{H}_\theta(u,\theta) = \theta^{-1/3}\hat{e}_\theta(u,\theta), \\ \widehat{H}_u(u,\theta) = \theta^{2/3}\hat{p}_\theta(u,\theta) - \widehat{G}(u,\theta), \end{gathered} \tag{2.22}$$

where

$$\widehat{G}(u,\theta) := \frac{2}{3}\int_0^\theta \xi^{-1/3}\hat{p}_\theta(u,\xi)\,d\xi. \tag{2.23}$$

Multiplication of (2.2) with $\theta^{-1/3}$ and use of (2.22)–(2.23) yield

$$H_t + \widehat{G}(u,\theta)v_x - \theta^{-1/3}\hat{\mu}(u)v_x^2 - \theta^{-1/3}(\hat{\kappa}(u,\theta)\theta_x)_x = 0. \tag{2.24}$$

Here $H(x,t) := \widehat{H}(u(x,t), \theta(x,t))$. In view of (2.23) and (1.16), $|\widehat{G}(u,\theta)| \le 2N(1+\theta)$. So if we integrate (2.24) over $[0,1]\times[0,T]$, integrate by parts with respect to x, and utilize $(1.6)_2$, (1.9), Lemma 2.3, (1.17), (2.18), (2.22), (2.17), and the Schwarz inequality, we arrive at

$$\begin{aligned} &\mu_0\overline{U}^{-1}\int_0^T\int_0^1 \theta^{-1/3}v_x^2\,dx\,dt + \frac{\nu}{3}\int_0^T\int_0^1 \theta^{-4/3}\theta_x^2\,dx\,dt \\ &\quad\le \Lambda + \int_0^T\int_0^1 \widehat{G}(u,\theta)v_x\,dx\,dt + T_0^{-1/3}\int_0^T (q(1,t) - q(0,t))\,dt \\ &\quad\le \Lambda + 4N^2\overline{U}\mu_0^{-1}\int_0^T\int_0^1 \theta^{7/3}\,dx\,dt + \frac{1}{2}\mu_0\overline{U}^{-1}\int_0^T\int_0^1 \theta^{-1/3}v_x^2\,dx\,dt. \end{aligned} \tag{2.25}$$

By (2.17), the Sobolev imbedding theorem $W^{1,1} \hookrightarrow L^\infty$, and the inequality $ab \le a^p/p + b^q/q \ (a, b \ge 0,\ p, q > 1,\ 1/p + 1/q = 1)$, we find

$$\begin{aligned}\int_0^T \int_0^1 \theta^{8/3}\,dx\,dt &\le \Lambda \int_0^T \max_{[0,1]} \theta^{5/3}(\cdot, t)\,dt \\ &\le \Lambda \int_0^T \int_0^1 (\theta^{5/3} + \theta^{2/3}|\theta_x|)\,dx\,dt \\ &\le \Lambda + \frac{1}{2}\int_0^T \int_0^1 \theta^{8/3}\,dx\,dt + \Lambda \int_0^T \int_0^1 \theta^{-4/3}\theta_x^2\,dx\,dt.\end{aligned} \tag{2.26}$$

This gives

$$\begin{aligned}4N^2\overline{U}\mu_0^{-1} \int_0^t \int_0^1 \theta^{7/3}\,dx\,dt &\le \Lambda + \frac{\nu}{12\Lambda}\int_0^T \int_0^1 \theta^{8/3}\,dx\,dt \\ &\le \Lambda + \frac{\nu}{6}\int_0^T \int_0^1 \theta^{-4/3}\theta_x^2\,dx\,dt,\end{aligned} \tag{2.27}$$

which together with (2.25) proves (2.19). (2.26) thus implies (2.20) and (2.21). □

The following lemma can be proved by multiplying (2.1) with v, integrating over $[0, 1] \times [0, T]$, and using Lemma 2.4; its proof is the same as in [2, Lemma 2.2] and we shall not repeat it here.

Lemma 2.5.

$$\int_0^T \int_0^1 v_x^2(x, t)\,dx\,dt \le \Lambda.$$

In a sequel we establish the higher-order estimates of solutions. Following [2], we define

$$Y := \max_{[0,T]} \int_0^1 \theta_x^2(x, t)\,dx, \qquad Z := \max_{[0,T]} \int_0^1 v_{xx}^2(x, t)\,dx. \tag{2.28}$$

It follows from the Sobolev imbedding theorem $W^{1,1} \hookrightarrow L^\infty$, (2.17), and the Schwarz inequality that

$$\begin{aligned}\theta^{3/2}(y, t) &\le \Lambda\left(\max_{Q_T}\theta\right)^{1/2} + \int_0^1 \theta^{1/2}(x, t)|\theta_x(x, t)|\,dx \\ &\le \Lambda + \frac{1}{2}\left(\max_{Q_T}\theta\right)^{3/2} + \Lambda Y^{1/2}, \qquad \forall 0 \le y \le 1,\ 0 \le t \le T,\end{aligned} \tag{2.29}$$

which yields

$$\max_{Q_T}\theta(x, t) \le \Lambda + \Lambda Y^{1/3}. \tag{2.30}$$

By virtue of the interpolation inequality and Lemma 2.2 one has

$$\begin{aligned}\int_0^1 v_x^2(x, t)\,dx &\le \Lambda \int_0^1 v^2(x, t)\,dx + \Lambda\left(\int_0^1 v^2(x, t)\,dx\right)^{1/2}\left(\int_0^1 v_{xx}^2(x, t)\,dx\right)^{1/2} \\ &\le \Lambda Z^{1/2} + \Lambda, \qquad \forall t \in [0, T],\end{aligned}$$

which in conjunction with the Sobolev imbedding theorem and the Schwarz inequality implies

$$v_x^2(x, t) \le \Lambda\left(\int_0^1 v_x^2(y, t)\, dy + \int_0^1 |v_x(y, t)|\, |v_{xx}(y, t)|\, dy\right)$$
$$\le \Lambda + \Lambda Z^{3/4}, \qquad \forall x \in [0, 1],\ t \in [0, T].$$

Therefore,

$$\max_{Q_T} |v_x(x, t)| \le \Lambda + \Lambda Z^{3/8}. \tag{2.31}$$

LEMMA 2.6.

$$\max_{[0, T]} \int_0^1 u_x^2(x, t)\, dx \le \Lambda + \Lambda Y^{1/9}. \tag{2.32}$$

The proof of Lemma 2.6 is completely the same as that of Lemma 2.3 in [2]. So, we omit its proof here.

LEMMA 2.7.

$$Y \le \Lambda + \Lambda Z^{3/4}, \tag{2.33}$$

$$\int_0^T \int_0^1 \theta_t^2(x, t)\, dx\, dt \le \Lambda + \Lambda Z^{3/4}. \tag{2.34}$$

Proof. Let

$$\widehat{Q}(u, \theta) := \int_0^\theta \hat{\kappa}(u, \xi)\, d\xi. \tag{2.35}$$

Setting $Q(x, t) = \widehat{Q}(u(x, t), \theta(x, t))$, we multiply (2.2) by Q_t, integrate over $[0, 1] \times [0, t]$, $0 \le t \le T$, and integrate by parts with respect to x to get

$$\begin{aligned} &\int_0^t \int_0^1 (\hat{e}_\theta \theta_t + \theta \hat{p}_\theta v_x - \hat{\mu} v_x^2) Q_t\, dx\, d\tau + \int_0^t \int_0^1 \hat{\kappa} \theta_x Q_{xt}\, dx\, d\tau \\ &\quad - \int_0^t (\hat{\kappa} \theta_x Q_t)(1, \tau)\, d\tau + \int_0^t (\hat{\kappa} \theta_x Q_t)(0, \tau)\, d\tau = 0. \end{aligned} \tag{2.36}$$

We now have to estimate every term in (2.36). Note that

$$Q_t = \widehat{Q}_u v_x + \hat{\kappa} \theta_t, \qquad Q_{xt} = (\hat{\kappa} \theta_x)_t + \widehat{Q}_u v_{xx} + \widehat{Q}_{uu} v_x u_x + \hat{\kappa}_u u_x \theta_t. \tag{2.37}$$

It follows from the fact that, in view of (1.17), $|\widehat{Q}_u|, |\widehat{Q}_{uu}| \le N\theta$, the assumptions (1.15) and (1.17), (2.31), and (2.20) that

$$\begin{aligned} \int_0^t \int_0^1 \hat{e}_\theta \theta_t Q_t\, dx\, d\tau &\ge \nu^2 \int_0^t \int_0^1 \theta_t^2\, dx\, d\tau - \left|\int_0^t \int_0^1 \hat{e}_\theta \theta_t \widehat{Q}_u v_x\, dx\, d\tau\right| \\ &\ge \frac{\nu^2}{2} \int_0^t \int_0^1 \theta_t^2\, dx\, d\tau - \Lambda \max_{Q_T} v_x^2 \int_0^t \int_0^1 (1 + \theta^{8/3})\, dx\, d\tau \\ &\ge \frac{\nu^2}{2} \int_0^t \int_0^1 \theta_t^2\, dx\, d\tau - (\Lambda + \Lambda Z^{3/4}). \end{aligned} \tag{2.38}$$

Making use of the inequality $ab \le a^p/p + b^q/q$ $(a, b \ge 0;\ p, q > 1,\ 1/p + 1/q = 1)$ and following the same arguments as those used for (2.51)–(2.53) in [2], we obtain

$$\int_0^t \int_0^1 \hat{\kappa}\theta_x(\hat{\kappa}\theta_x)_t\,dx\,d\tau \ge \frac{\nu^2}{2}\int_0^1 \theta_x^2(x, t)\,dx - \Lambda \tag{2.39}$$

and

$$\begin{aligned}&\left|\int_0^t\int_0^1(\theta\hat{p}_\theta v_x - \hat{\mu}v_x^2)\widehat{Q}_u v_x\,dx\,d\tau\right| + \left|\int_0^t\int_0^1(\theta\hat{p}_\theta v_x - \hat{\mu}v_x^2)\hat{\kappa}\theta_t\,dx\,d\tau\right|\\ &\quad + \left|\int_0^t\int_0^1\hat{\kappa}\theta_x\widehat{Q}_u v_{xx}\,dx\,d\tau\right| + \left|\int_0^t\int_0^1\hat{\kappa}\theta_x\widehat{Q}_{uu}v_x u_x\,dx\,d\tau\right|\\ &\quad \le \Lambda + \Lambda Z^{3/4} + \frac{\nu^2}{16}\int_0^t\int_0^1\theta_t^2\,dx\,d\tau + \frac{\nu^2}{12}Y.\end{aligned} \tag{2.40}$$

By the Schwarz inequality,

$$\begin{aligned}\left|\int_0^t\int_0^1\hat{\kappa}\theta_x\hat{\kappa}_u u_x\theta_t\,dx\,d\tau\right| &\le \frac{\nu^2}{16}\int_0^t\int_0^1\theta_t^2\,dx\,d\tau + \Lambda\int_0^t\int_0^1(\hat{\kappa}\theta_x)^2u_x^2\,dx\,d\tau\\ &\le \frac{\nu^2}{16}\int_0^t\int_0^1\theta_t^2\,dx\,d\tau + \Lambda + \frac{\nu^2}{24}Y + \Lambda\left(\int_0^t\int_0^1[(\hat{\kappa}\theta_x)_x]^2\,dx\,d\tau\right)^{3/4},\end{aligned} \tag{2.41}$$

where we have used the following estimate, which follows from the Sobolev imbedding theorem $W^{1,1} \hookrightarrow L^\infty$, Lemma 2.6, (2.19), and (2.30):

$$\begin{aligned}\Lambda\int_0^t\int_0^1(\hat{\kappa}\theta_x)^2u_x^2\,dx\,d\tau &\le \Lambda\int_0^t\max_{[0,1]}(\hat{\kappa}\theta_x)^2\int_0^1 u_x^2\,dx\,d\tau\\ &\le (\Lambda + \Lambda Y^{1/9})\left(\int_0^t\int_0^1(\hat{\kappa}\theta_x)^2 + \int_0^t\int_0^1|\hat{\kappa}\theta_x|\,|(\hat{\kappa}\theta_x)_x|\,dx\,d\tau\right)\\ &\le (\Lambda + \Lambda Y^{1/9})\left\{\Lambda\max_{Q_T}\theta^{4/3} + \max_{Q_T}\theta^{2/3}\left(\int_0^t\int_0^1\theta^{-4/3}\theta_x^2\right)^{1/2}\left(\int_0^t\int_0^1[(\hat{\kappa}\theta_x)_x]^2\right)^{1/2}\right\}\\ &\le \Lambda + \frac{\nu^2}{24}Y + \Lambda\left(\int_0^t\int_0^1[(\hat{\kappa}\theta_x)_x]^2\,dx\,d\tau\right)^{3/4}.\end{aligned} \tag{2.42}$$

We now estimate the boundary terms in (2.36). Let $\zeta = 1$ or $\zeta = 0$. In view of (2.9) and $(1.6)_2$, we have that $|\widehat{Q}_u(u(\zeta, t), \theta(\zeta, t))| \le \Lambda$. So by virtue of $(1.6)_2$, (2.37), (2.31), (2.19), and (2.30), we find that (cf. the proof of (2.42))

$$\begin{aligned}\left|\int_0^t(\hat{\kappa}\theta_x Q_t)(\zeta, \tau)\,d\tau\right| &= \left|\int_0^t(\hat{\kappa}\theta_x\widehat{Q}_u v_x)(\zeta, \tau)\,d\tau\right|\\ &\le \Lambda\max_{Q_T}|v_x|^2 + \Lambda\int_0^t\max_{[0,1]}(\hat{\kappa}\theta_x)^2\,d\tau\\ &\le \Lambda + \Lambda Z^{3/4} + \frac{\nu^2}{40}Y + \Lambda\left(\int_0^t\int_0^1[(\hat{\kappa}\theta_x)_x]^2\,dx\,d\tau\right)^{3/4}.\end{aligned} \tag{2.43}$$

Note that by (2.30)–(2.31), (2.20), and Lemma 2.5,

$$\int_0^t\int_0^1 \hat{e}_\theta^2\theta_t^2\,dx\,d\tau + \int_0^t\int_0^1 (\theta\hat{p}_\theta v_x - \hat{\mu}v_x^2)^2\,dx\,d\tau \le (\Lambda + \Lambda Y^{2/9})\int_0^t\int_0^1 \theta_t^2\,dx\,d\tau + \Lambda + \Lambda Z^{3/4},$$

which together with (2.2) gives

$$\Lambda\left(\int_0^t\int_0^1 [(\hat{\kappa}\theta_x)_x]^2\,dx\,d\tau\right)^{3/4} \le \Lambda + \frac{\nu^2}{40}Y + \frac{\nu^2}{24}\int_0^t\int_0^1 \theta_t^2\,dx\,d\tau + \Lambda Z^{3/4}. \quad (2.44)$$

Combining (2.36)–(2.41) and (2.43)–(2.44), we get

$$\frac{\nu^2}{4}\int_0^t\int_0^1 \theta_t^2\,dx\,d\tau + \frac{\nu^2}{2}\int_0^1 \theta_x^2(x, t)\,dx \le \Lambda + \Lambda Z^{3/4} + \frac{\nu^2}{4}Y, \qquad \forall t \in [0, T],$$

which shows (2.33) and (2.34). □

Differentiate (2.1) formally with respect to t, multiply by v_t, and integrate over $[0, 1]\times[0, t]$, $t \in (0, T]$. Integrating by parts with respect to x, we infer by the same procedure as in [2, Lemma 2.5] that

LEMMA 2.8.

$$\max_{[0,T]}\int_0^1 v_t^2(x, t)\,dx + \int_0^T\int_0^1 v_{xt}^2\,dx\,dt \le \Lambda + \Lambda Z^{11/12}. \quad (2.45)$$

The proof of the following lemma can be found in [2, Lemma 2.6] and will thus be omitted here.

LEMMA 2.9.

$$\max_{[0,T]}\int_0^1 v_t^2(x, t)\,dx \le \Lambda, \qquad \int_0^T\int_0^1 v_{xt}^2\,dx\,dt \le \Lambda, \qquad \max_{[0,T]}\int_0^1 v_{xx}^2(x, t)\,dx \le \Lambda. \quad (2.46)$$

We now want to bound derivatives of θ. We have

LEMMA 2.10.

$$\max_{[0,T]}\int_0^1 \theta_t^2(x, t)\,dx \le \Lambda, \qquad \int_0^T\int_0^1 \theta_{xt}^2\,dx\,dt \le \Lambda, \qquad \max_{[0,T]}\int_0^1 \theta_{xx}^2(x, t)\,dx \le \Lambda. \quad (2.47)$$

Proof. With the help of (2.28), (2.46), (2.31), (2.33), and (2.30) we have

$$\max_{Q_T}|v_x| \le \Lambda, \qquad \max_{Q_T}\theta \le \Lambda. \quad (2.48)$$

We differentiate formally (2.2) with respect to t, multiply by $\hat{e}_\theta\theta_t$, and integrate over $[0, 1]\times[0, t]$, $0 \le t \le T$. Keeping in mind that θ_t vanishes on the boundary, we integrate by parts with respect to x. Utilizing (2.48), after a lengthy calculation, which is recorded in [3, Lemma 3.6] and thus need not be reproduced here, we obtain (2.47). □

Proof of Theorem 2.1. We use (2.47) and the Schwarz inequality to see that $\theta(x, t)$, respectively $\theta_x(x, t)$, is uniformly Hölder continuous in t, respectively in x with exponent $1/2$. A standard interpolation property [10, II, Lemma 3.1; or 3, Lemma 3.3] implies that $\theta_x(x, t)$ is also uniformly Hölder continuous in t with exponent $1/6$; hence, $|||\, \theta_x \,|||_{1/3} \le \Lambda$. This immediately yields $|||\, \theta \,|||_{1/3} \le \Lambda$. Similarly, using (2.46), we conclude that $|||\, v_x \,|||_{1/3} \le \Lambda$ and thereby $|||\, v \,|||_{1/3} \le \Lambda$, $|||\, u \,|||_{1/3} \le \Lambda$. □

3. Proof of Theorem 1.1. In this section we prove Theorem 1.1 with the help of the Leray-Schauder fixed point theorem which we recall here for the reader's convenience.

THEOREM 3.1. Let $\mathscr{B}$ be a Banach space and $P: [0, 1] \times \mathscr{B} \to \mathscr{B}$ be a map with the following properties:

(i) for any fixed $\lambda \in [0, 1]$, $P(\lambda, \cdot): \mathscr{B} \to \mathscr{B}$ is completely continuous;

(ii) for every bounded subset $\mathscr{M}$ of $\mathscr{B}$, the family of maps $P(\cdot, \chi): [0, 1] \to \mathscr{B}$, $\chi \in \mathscr{M}$ is uniformly equicontinuous;

(iii) there is a bounded subset $\mathscr{M}$ of $\mathscr{B}$ such that any fixed point in $\mathscr{B}$ of $P(\lambda, \cdot)$, $\lambda \in [0, 1]$, is contained in $\mathscr{M}$; .

(iv) $P(0, \cdot)$ has precisely one fixed point in $\mathscr{B}$.

Then, $P(1, \cdot)$ has at least one fixed point in $\mathscr{B}$.

For our purposes, $\mathscr{B}$ will be the Banach space of functions $\{u(x, t), v(x, t), \theta(x, t)\}$ on Q_T with $u, v, v_x, \theta, \theta_x$ in $C^{1/3, 1/6}(Q_T)$ with norm

$$|||\, (u, v, \theta) \,|||_{\mathscr{B}} := |||\, u \,|||_{1/3} + |||\, v \,|||_{1/3} + |||\, \theta \,|||_{1/3} + |||\, v_x \,|||_{1/3} + |||\, \theta_x \,|||_{1/3} .$$

For $\lambda \in [0, 1]$ we define $P(\lambda, \cdot)$ as the map that carries $\{\tilde{u}, \tilde{v}, \tilde{\theta}\} \in \mathscr{B}$ into $\{u, v, \theta\} \in \mathscr{B}$ by solving the (linearized) system

$$\begin{gathered} u_t - v_x = 0, \\ v_t - \tilde{\mu}(\tilde{u}) v_{xx} + (\tilde{p}_u(\tilde{u}, \tilde{\theta}) + \tilde{\mu}_u(\tilde{u})\tilde{v}_x) u_x = -\tilde{p}_\theta(\tilde{u}, \tilde{\theta})\tilde{\theta}_x, \\ \tilde{e}_\theta(\tilde{u}, \tilde{\theta})\theta_t - \tilde{\kappa}(\tilde{u}, \tilde{\theta})\theta_{xx} - \tilde{\kappa}_u(\tilde{u}, \tilde{\theta})\tilde{\theta}_x u_x = -\tilde{\theta}\tilde{p}_\theta(\tilde{u}, \tilde{\theta})\tilde{v}_x + \tilde{\mu}(\tilde{u})\tilde{v}_x^2 + \tilde{\kappa}_\theta(\tilde{u}, \tilde{\theta})\tilde{\theta}_x^2 \end{gathered} \tag{3.1}$$

with boundary and initial conditions

$$\begin{gathered} (-\tilde{p}(\tilde{u}, \tilde{\theta}) + \tilde{\mu}(\tilde{u}) v_x)(1, t) = -\gamma\tilde{v}(1, t), \qquad (-\tilde{p}(\tilde{u}, \tilde{\theta}) + \tilde{\mu}(\tilde{u}) v_x)(0, t) = \gamma\tilde{v}(0, t), \\ \theta(1, t) = \theta(0, t) = T_0, \end{gathered} \tag{3.2}$$

$$\begin{gathered} u(x, 0) = (1 - \lambda)\eta_0 + \lambda u_0(x), \qquad v(x, 0) = \lambda v_0(x), \\ \theta(x, 0) = (1 - \lambda) T_0 + \lambda \theta_0(x), \end{gathered} \tag{3.3}$$

where η_0 is defined by (1.12). To solve (3.1)–(3.3) for arbitrary $\tilde{u}, \tilde{\theta} \in C^{1/3, 1/6}(Q_T)$, we have to define $\hat{e}, \hat{p}, \hat{\mu}, \hat{\kappa}$ on $\mathbb{R}^2$. Recalling that a smooth solution $u(x, t)$ of the original system (1.1), (1.5)–(1.6) will be bounded from below by $\overline{u}$ and from above by $\overline{U}$ (cf. Lemma 2.3), we construct C^2-smooth functions $\tilde{e}(u, \theta)$, $\tilde{p}(u, \theta)$, $\tilde{\mu}(u)$, $\tilde{\kappa}(u, \theta)$ on $\mathbb{R}^2$ which coincide with $\hat{e}(u, \theta)$, $\hat{p}(u, \theta)$, $\hat{\mu}(u)$, $\hat{\kappa}(u, \theta)$ for $\overline{u} \le u \le \overline{U}$, $0 \le \theta < \infty$, and which satisfy the following conditions:

$$\begin{gathered} \tilde{\mu}(u) \ge \tilde{\mu}_0 > 0, \qquad \tilde{e}_\theta(u, \theta), \\ \tilde{\kappa}(u, \theta) \ge \tilde{\nu} > 0 \quad \text{for } -\infty < u < \infty, \ -\infty < \theta < \infty. \end{gathered} \tag{3.4}$$

Under these assumptions (3.1)–(3.3) will have a unique solution not only in $\mathscr{B}$ but in a better function space. More precisely,

LEMMA 3.2. There is a unique solution $\{u(x, t), v(x, t), \theta(x, t)\}$ of (3.1)–(3.3) such that $u, u_t, u_x, v, v_t, v_x, v_{xx}, \theta, \theta_t, \theta_x, \theta_{xx}$ are all in $C^{\beta, \beta/2}(Q_T)$ with $\beta = \min\{\alpha, 1/3\}$, and the $C^{\beta, \beta/2}(Q_T)$ norms of these functions can be a priori bounded in terms of C (depending only on $\tilde{\mu}_0, \tilde{\nu}, T$, and on the $C^{\beta, \beta/2}(Q_T)$ norms of the coefficients) times the norms of the initial data and the right-hand sides of (3.1).

The existence of a solution to (3.1)–(3.3) can be established by the method of continuity (see [4]) which connects (3.1) to a system without u_x-terms in $(3.1)_2$ and $(3.1)_3$. The uniqueness is obvious. The a priori estimate in Lemma 3.2 follows from the classical Schauder-Friedman estimate (cf. [4, 10]).

Applying Lemma 3.2 we see that any solution to (3.1)–(3.3) will be in the space indicated in Lemma 3.2. It then follows from the interpolation property [10, II, Lemma 3.1; or 3, Lemma 3.3] that the solution $\{u, v, \theta\}$ to (3.1)–(3.3) is in a Hölder space which is compactly imbedded in $\mathscr{B}$ (in fact $u, v, v_x, \theta, \theta_x \in C^{1, 1/2}(Q_T)$). Thus, $P(\lambda, \cdot): \mathscr{B} \to \mathscr{B}$ is not only well defined, but also completely continuous as required in (i) of Theorem 3.1. Using the a priori estimate in Lemma 3.2, we can easily show that the family $P(\cdot, \{\tilde{u}, \tilde{v}, \tilde{\theta}\}): [0, 1] \to \mathscr{B}$, with $\{\tilde{u}, \tilde{v}, \tilde{\theta}\}$ in any fixed bounded subset of $\mathscr{B}$, is uniformly equicontinuous, so that (ii) of Theorem 3.1 holds.

To show (iii) we note that any fixed point $\{u, v, \theta\}$ of P will solve the system $(1.1)_1$, (2.1), (2.2), (3.3), (1.6) with $\hat{e}, \hat{p}, \hat{\mu}$, and $\hat{\kappa}$ replaced by $\tilde{e}, \tilde{p}, \tilde{\mu}$, and $\tilde{\kappa}$. Recalling that $\tilde{e} = \hat{e}$, $\tilde{p} = \hat{p}$, $\tilde{\mu} = \hat{\mu}$, $\tilde{\kappa} = \hat{\kappa}$ for $\overline{u} \le u \le \overline{U}$ and $\theta \ge 0$, Lemmas 2.1 and 2.3 imply that $\theta > 0$ and $\overline{u} \le u \le \overline{U}$; thus, $\tilde{e}, \tilde{p}, \tilde{\mu}, \tilde{\kappa}$ coincide with $\hat{e}, \hat{p}, \hat{\mu}, \hat{\kappa}$. Hence, any fixed point of P will be a solution of the original system (1.1), (3.3), (1.6), where the assumptions (1.15)–(1.17) are satisfied, and (iii) thus follows from Theorem 2.1.

To verify (iv) we easily see by virtue of (1.12) that $u(x, t) = \eta_0$, $v(x, t) = 0$, $\theta(x, t) = T_0$ is a fixed point of $P(0, \cdot)$. This solution is unique. The uniqueness of any fixed point of $P(\lambda, \cdot)$ for $\lambda \in [0, 1]$ can be shown in a standard fashion which is outlined in [3, Lemma 3.8]. Hence, Theorem 3.1 implies that $P(1, \cdot)$ has at least one fixed point; this point is unique, i.e., there is a unique solution of (1.1), (1.5), (1.6) on $[0, 1] \times [0, T]$ in the function class indicated in Lemma 3.2. To complete the proof of Theorem 1.1 it remains to show that the solution of (1.1), (1.5), (1.6) has derivatives not only in $C^{\beta, \beta/2}$ with $\beta = \min\{\alpha, 1/3\}$ but also in $C^{\alpha, \alpha/2}$. This can be done by noting that $u, v, v_x, \theta, \theta_x$ are in $C^{1, 1/2}(Q_T)$ and by another application of Lemma 3.2 with $\beta = \min\{\alpha, 1\}$. This completes the proof of Theorem 1.1.

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