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Peter Constantin, Ciprian Foias

Institutions: Indiana University

Published on: 01 Jan 1985 - Communications on Pure and Applied Mathematics (Wiley)

Topics: Lyapunov exponent, Lyapunov equation and Navier–Stokes equations

GLOBAL LYAPUNOV EXPONENTS, KAPLAN-YORKE FORMULAS AND THE DIMENSION OF THE ATTRACTORS FOR 2D NAVIER-STOKES EQUATIONS

by

P. CONSTANTIN and C. FOIAS
Indiana University

This work was supported by the U.S. Dept. of Energy under Contract DE-AC02-82ER12049.

1. Introduction.

We study the fractal and Hausdorff dimensions of the universal attractor for the Navier-Stokes equations in two space dimensions. The finite dimensionality of the attractors for the Navier-Stokes equation was first implicitely proven in [16] and explicitely in [10]. The subject has been investigated recently by several authors ([15], [1], [21], [2]).

For a large class of dissipative systems, including the Navier-Stokes system, the finite dimensionality of the attractors, the existence of a finite number of determing modes [9], the exponential decay of finite dimensional volume elements in the phase space [3], and the squeezing property [10], seem to indicate that the chaotic behavior of the flow has a finite number of degrees of freedom. It is important, for theoretical as well as for practical reasons to estimate this number in terms of physically significant quantities.

In [8] the generalized Grashof number (corresponding to the Grashof number in the Bénard convection) was defined by

$$G = \frac{|f|}{\nu^2 \lambda_1} \tag{1.1}$$

where $|f|$ is the L^2 norm of the driving forces, supposed to be time independent (see Eq. (2.1) and (2.5)), ν is the viscosity of the flow and λ_1 is the smalles eigenvalue of the Stokes operator. G is a non-dimensional number. The long time behavior of the solutions of the Navier-Stokes equations is determined by the behavior of a finite number, N , of explicit parameters, the determinant modes. An estimate of N of the form

(1.2) $$N \le c_0 G((\log G)^{\frac{1}{2}} + 1)$$

in the case of periodic boundary conditions (periodic case) and

(1.3) $$N \le c_1 G^2$$

in the case of homogeneous boundary conditions (aperiodic case) was obtained in [8] . (The constants c_0, c_1, $c_2 \ldots$ are dimensionless. They might depend on the shape of the domain filled with fluid but not on its size.)

The previously known estimates for the fractal and Hausdorff dimensions D of the universal attractor (largest bounded invariant set) for the Navier-Stokes equations, in terms of the Grashof number were of the form

(1.4) $D \le c_3 G^2(\log G+1)$ in the periodic case [11] and

(1.5) $D \le c_4 G^2 e^{c_5 G^4}$ in the aperiodic case.

In this work we prove estimates of the form

(1.6) $D \le c_6 G((\log G)^{\frac{1}{2}} + 1)$ in the periodic case

(1.7) $D \le c_7 G^2$ in the aperiodic case.

The paper is organized as follows. In section 2 we recall the functional setting and give the precise definition of the universal attractor. In section 3 we establish the equations governing the transport of finite dimensional volume elements (in the phase space) under the action of the Navier-Stokes system. Section 4 is devoted to the definition and study of global Lyapunov exponents. In section 5 we prove the existence of a critical dimension $N_0(f)$ which enjoys the property that every N dimensional volume element in the phase space, for $N \geq N_0 + 1$ decays exponentially in time. We give estimates for $N_0(f)$ in terms of the Grashof number, separately for the periodic and aperiodic cases. The techniques are inspired from those of [8], [7] and adapted for the traces of operators which appear in the equations obtained in section 3. Recently, Kaplan and Yorke ([13], [5]) were lead by numerical evidence to conjecture that the information dimension of the attractor of a dynamical system is equal to an expression given in terms of (local) Lyapunov exponents. In section 6 we prove that the Kaplan-Yorke expressions, with the global Lyapunov exponents we introduced in section 4 replacing the local ones, yield upper bounds for the fractal and Hausdorff dimensions of the universal attractor. In particular it follows that these dimensions do not exceed $N_0(f) + 1$. The proof relies on an idea first used in [4] for dynamical systems. In the last section we give an example (an infinite dimensional version of a system appearing in [19]) for which the theorems in section 6 give the upper bound $G + 1$ for the dimensions of the universal attractor. We show that this attractor has at least dimension $G - 3$. Thus, our formulation of the Kaplan-Yorke conjecture leads to sharp (up to absolute constants) upper bounds for the dimensions of the attractors of systems in the class of Navier-Stokes abstract equations we treat.

One of the authors enjoyed the hospitality of the Institute for Mathematics and its Applications (Univ. of Minnesota) and wishes to express his thanks to Professor G. Sell and H. Weinberger for very useful discusssions which lead us to the introduction and use of the global Lyapunov exponents.

2. Preliminaries.

Let Ω be an open bounded set in $\mathbb{R}^2$ with sufficiently smooth boundary Γ. The Navier-Stokes equations for the velocities $u(x,t) = (u_1(x,t), u_2(x,t))$ and the pressure $p(x,t)$ are

$$(2.1)\qquad \begin{cases} \dfrac{\partial u^i}{\partial t} - \nu\Delta u^i + (u\cdot\nabla)u^i + \dfrac{\partial p}{\partial x_i} = F^i \quad \text{in } \Omega \quad i = 1, 2 \\ \operatorname{div} u = 0 . \end{cases}$$

They are supplemented by the conditions

$$(2.2)\qquad \begin{cases} u(x,t) = 0 \quad \text{for } x \text{ in } \Gamma \\ u(x,0) = u_0(x) \quad \text{in } \Omega . \end{cases}$$

The functions $F = (F^1, F^2)$ and u_0 are given. Let us denote by $\mathbb{H}^j(\Omega) = (H^j(\Omega))^2$ $j = 1,2$, where $H^j(\Omega)$, $j = 1,2$ are the usual Sobolev spaces and by $\mathbb{L}^2(\Omega) = (L^2(\Omega))^2$. We consider the linear space

$$\mathcal{V} = \{\phi \in (C_0^\infty(\Omega))^2 \mid \operatorname{div} \phi = 0\}$$

and denote by H and V respectively the closures of $\mathcal{V}$ in $\mathbb{L}^2(\Omega)$ and $\mathbb{H}^1(\Omega)$. The scalar product and norm in H are denoted by $(\cdot,\cdot)$ and $|\cdot|$ respectively. The scalar product in V is

$$((u,v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx \tag{2.3}$$

and the corresponding norm will be denoted by $\|\cdot\|$.

We denote by P the orthogonal projection of $\mathbb{L}^2(\Omega)$ onto H . We define $A = -P\Delta$, the operator with domain $\mathcal{D}(A) = \mathbb{H}^2(\Omega) \cap V$ acting in H . We use the same notation for the (bounded) extension of A to an operator from V to its dual V' . It is well known that, as an operator in H , A is selfadjoint and that its spectrum consists of a sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \to \infty$$

of eigenvalues counted according to their multiplicites.

There exists an orthonormal basis of H formed with eigenvectors (w_m) $m = 1,2,\ldots$ for A :

$$Aw_m = \lambda_m w_m \quad , \; m \geq 1 \; .$$

We denote, for every $m \geq 1$ by P_m the orthogonal projection of H onto the span of $w_1, w_2, \ldots, w_m$.

For u, v in $H^1(\Omega)$ we define $B(u,v) \in V'$ by

(2.4) $$(B(u,v),w) = \sum_{j,k=1}^{2} \int_\Omega u_j \frac{\partial v_k}{\partial x_j} w_k \, dx \,, \quad \text{for every} \quad w \in V \,.$$

We refer to [23] for the various inequalities concerning $B(u,v)$ that will be used.

The equations (2.1), (2.2) are equivalent to

(2.5) $$\begin{cases} \frac{du}{dt} + \nu A u + B(u,u) = f \quad \text{in} \quad V' \\ \\ u(0) = u_0 \end{cases}$$

where $f = PF$ and $u(0)$ can be defined in a suitable natural way (see [23] for details).

Throughtout the work f will be a fixed element of H. For $u_0 \in H$ the problem (2.5) has a unique solution, $u(t)$, defined for all $t \geq 0$. We denote $u(t)$ by $S(t)u_0$. It is known that $S(t)u_0$ is a real analytic $\mathcal{D}(A)$ valued function on $(0,\infty)$. The range of $S(t)u_0$ $t \geq 0$ is bounded in V. Moreover, if one considers the set

(2.6) $$X = \bigcap_{t>0} S(t) B_\rho^V(0)$$

where $B_\rho^V(0) = \{u \in V \mid \|u\| < \rho\}$ and ρ is given by

(2.7) $$\rho > c_0 \nu \lambda_1^{\frac{1}{2}} G e^{c_1 G^4}$$

then, it can be easily checked that X has the following properties

(i) $S(t)X = X$ for every $t \geq 0$

(ii) X is bounded in H

(iii) For every $u_0 \in H$, $\lim \mathrm{dist}_V(S(t)u_0, X) = 0$.

X is the largest set in H enjoying (i) and (ii). X will be called the universal attractor for $(S(t))$.

When the homogeneous boundary conditions (2.2) are replaced by periodic ones and the data u_0, F are periodic (with the same period L) then the problem (2.1), (2.2) admits a similar abstract treatment to the one described for the homogeneous boundary conditions case. The description of the spaces H, V, of the form $B(\cdot,\cdot)$ and of the operator A are carried out in detail in [23]. Let us only notice that in this case

$$(B(u,u),Au) = 0 \quad \forall u \in \mathcal{D}(A)$$

and consequently that in the definition of X in (2.6) we can take instead of (2.7)

$$\rho > c_0 \nu \lambda_1^{1/2} G \tag{2.8}$$

3. The transport of finite dimensional manifolds by the solutions of the Navier-Stokes equation.

In this section we establish the equations governing the transport of finite dimensional volume elements under the action of $S(t)$.

Let ϕ be a smooth function defined on an open set D of $\mathbb{R}^N$ and taking values in V. Let us denote by Σ_0 the image of ϕ and by $\Sigma_t = S(t)\Sigma_0$.

Let us denote by $(\cdot;\cdot)$ and $|\cdot|$ the scalar product and norm in $\Lambda^N H$. The volume element in Σ_t is

$$\left|\frac{\partial}{\partial\alpha_1}(S(t)\phi(\alpha)) \wedge \frac{\partial}{\partial\alpha_2}(S(t)\phi(\alpha)) \wedge \dots \wedge \frac{\partial}{\partial\alpha_N}(S(t)\phi(\alpha))\right| d\alpha_1 \dots d\alpha_N$$

If we put $u_0 = \phi(\alpha)$, the functions $v_i(t) = \frac{\partial}{\partial\alpha_i}(S(t)\phi(\alpha))$, $i=1,,,N$

satisfy the equation

$$\frac{dv}{dt} + A(t)v(t) = 0 \tag{3.1}$$

for $A(t)$ given by

$$A(t) = \nu A + B(S(t)u_0,\cdot) + B(\cdot,S(t)u_0), \tag{3.2}$$

Using the same technique as in [6] one can prove the following

Theorem 3.1. Let u_0 be an element of V, ξ an element of H. The problem

$$\frac{dv}{dt} + A(t)v(t) = 0 \tag{3.1}$$

$$v(0) = \xi \tag{3.3}$$

with $A(t)$ given by (3.2) has a unique solution, $v(t)$, satisfying

(i) v is a real analytic $\mathcal{D}(A)$ - valued function on $t > 0$

(ii) there exists a constant k depending only on $|f|$ such that

$$|v(t)| \leq e^{kt}|\xi| \; ; \; t \geq 0$$

(iii) $$\|v(t)\| \leq \frac{e^{kt}}{\sqrt{t}} |\xi| \; ; \; t > 0 \, .$$

We need now some algebraic tools. First, let us recall the useful formula

(3.4) $$(v_1\wedge\ldots\wedge v_N \; ; \; w_1\wedge\ldots\wedge w_N) = \det[(v_i, w_j)_{i,j=1,\ldots,N}]$$

where v_i , w_j are elements in H and det stands for determinant.

Suppose $v_1,\ldots,v_N$ are elements in H . We denote by $P(v_1,\ldots,v_N)$ the orthogonal projection of H onto the span of v_i, $i = 1,\ldots,N$.

Lemma 3.2. For any $w \in H$

(3.5) $$|v_1\wedge\ldots\wedge v_N|^2 \, P(v_1,\ldots,v_N)w$$

$$= \sum_{j=1}^{N} (-1)^{j-1} (w \wedge v_1\wedge\ldots\wedge \hat{v}_j\wedge\ldots\wedge v_N \; ; \; v_1\wedge\ldots\wedge v_N) v_j$$

where $\widehat{(\;)}$ means $(\;)$ is omitted.

Proof. Obviously

$$P(v_1,v_2,\dots,v_N)w = \sum_{j=1}^{N} \gamma_j(w)v_j$$ and the γ_j satisfy

$$(w,v_j) = \sum_{k=1}^{N} \gamma_k(v_j,v_k) \ .$$

(3.5) follows from the Cramer rule and (3.4).

Let now T be a linear operator in H, $T : D_T \to H$. Let $v_1,\dots,v_N$ be elements of H. One can define T_N in $\Lambda^N H$ by

$$T_N = T\wedge I\wedge\dots\wedge I + I\wedge T\wedge I\wedge\dots\wedge I + \dots + I\wedge I\wedge\dots\wedge T \ .$$

Lemma 3.3. The following formula holds

$$(3.6)\quad (T_N(v_1\wedge\dots\wedge v_N) \ ; \ v_1\wedge\dots\wedge v_N) = |v_1\wedge\dots\wedge v_N|^2 \cdot \mathrm{Trace}(TP(v_1,\dots,v_N))$$

provided $v_1,\dots,v_N \in \mathcal{D}_T$.

Proof. Indeed let w_j be an orthonormal basis for H such that the span of v_i, $i = 1,..N$ is spanned by w_ℓ $\ell = 1,\dots,L$.

From (3.5) we obtain, for every ℓ, $1 \le \ell \le L$

$$|v_1\wedge\dots\wedge v_N|^2 w_\ell = |v_1\wedge\dots\wedge v_N|^2 \ P(v_1,\dots,v_N)w_\ell = \sum_{j=1}^{N}(-1)^{j-1} \cdot$$

$$\cdot (w_\ell\wedge v_1\wedge\dots\wedge\hat{v}_j\wedge\dots\wedge v_N \ ; \ v_1\wedge\dots\wedge v_N)v_j$$

$$= \sum_{j=1}^{N}\sum_{k=1}^{N}(-1)^{j-1}(-1)^{k-1}(w_\ell,v_k) \cdot$$

$$\cdot (v_1\wedge v_2\dots\wedge\hat{v}_j\wedge\dots\wedge v_N \ ; \ v_1\wedge\dots\wedge\hat{v}_k\wedge\dots v_N)v_j \ .$$

Therefore

$$|v_1\wedge\ldots\wedge v_N|^2 \ \mathrm{Trace}(TP(v_1,\ldots,v_N)) = |v_1\wedge\ldots\wedge v_N|^2 \sum_{\ell=1}^{L} (Tw_\ell, w_\ell) =$$

$$\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{\ell=1}^{L} (-1)^{j+k-2} \cdot (Tv_j, w_\ell)(w_\ell, v_k) \ .$$

$$\cdot\ (u_1\wedge\ldots\wedge\hat{v}_j\wedge\ldots v_N \ ;\ v_1\wedge\ldots\hat{v}_k\wedge\ldots v_N) = \sum_{j=1}^{N}\sum_{k=1}^{N} (-1)^{j+k-2}(Tv_j, v_k) \ .$$

$$\cdot\ (v_1\wedge\ldots\wedge\hat{v}_j\wedge\ldots\wedge v_N \ ;\ v_1\wedge\ldots\wedge\hat{v}_k\wedge\ldots\wedge v_N) = \sum_{j=1}^{N} (-1)^{j-1} \ .$$

$$\cdot\ (Tv_j\wedge v_1\wedge\ldots\wedge v_N \ ;\ v_1\wedge\ldots v_N) = (T_N(v_1\wedge\ldots\wedge v_N) \ ;\ v_1\wedge\ldots\wedge v_N) \ .$$

<u>Remark 3.4</u>. Suppose that T is antisymmetric on the (v_i) $i = 1,\ldots,N$ i.e. $(Tv_i, v_j) + (v_i, Tv_j) = 0$, $1 \le i$, $j \le N$. Then, it follows from (3.6) that

$$(T_N(v_1\wedge\ldots\wedge v_N) \ ;\ v_1\wedge\ldots\wedge v_N) = 0 \ .$$

Let us consider N solutions v_i of (3.1) with $A(t)$ given by (3.2).

<u>Lemma 3.5</u>. The Wronskian $|v_1\wedge\ldots\wedge v_N|$ is either identically zero or never vanishes and satisfies

$$(3.7)\qquad \frac{d}{dt}\log|v_1\wedge\ldots\wedge v_N| + \mathrm{Trace}\ (A(t)P(v_1(t),\ldots,v_N(t)) = 0$$

$$(3.8)\qquad \frac{d}{dt}\log|v_1\wedge\ldots\wedge v_N| + \nu(\lambda_1+\ldots+\lambda_N)$$

$$+ \operatorname{Trace}(B(\cdot, S(t)u_0)P(v_1(t),\ldots,v_N(t))) \le 0 .$$

Proof. Let us remark first that from (3.1) follows that $\frac{d}{dt}(v_1(t)\wedge\ldots\wedge v_N(t)) + A_N(t)(v_1(t)\wedge\ldots\wedge v_N(t)) = 0$. Therefore $\frac{1}{2}\frac{d}{dt}|v_1\wedge\ldots\wedge v_N|^2 + (A_N(t)v_1\wedge\ldots\wedge v_N \,;\, v_1\wedge\ldots\wedge v_N) = 0$. From (3.6) we have

$$(3.9)\quad \frac{1}{2}\frac{d}{dt}|v_1\wedge\ldots\wedge v_N|^2 + |v_1\wedge\ldots\wedge v_N|^2 \operatorname{Trace}(A(t)P(v_1(t),\ldots,v_N(t)) = 0 .$$

If $|v_1(t_0)\wedge\ldots\wedge v_N(t_0)| = 0$ for some $t_0 \ge 0$ then from (3.9) it follows that $|v_1(t)\wedge\ldots\wedge v_N(t)|^2 = 0$ for all $t \ge t_0$ and from the analicity part of Thm. 3.1 it follows that $|v_1\wedge\ldots\wedge v_N|$ is identically zero. If not, then we can divide in (3.9) by $|v_1\wedge\ldots\wedge v_N|^2$ and get (3.7). To prove (3.8) we remark that for every $\eta_1,\ldots,\eta_N$ in $\mathcal{D}(A)$ we have

$$(3.10)\quad ((\nu A)_N(\eta_1\wedge\ldots\wedge\eta_N) \,;\, \eta_1\wedge\ldots\wedge\eta_N) \ge \nu(\lambda_1+\ldots+\lambda_N)|\eta_1\wedge\ldots\wedge\eta_N|^2 .$$

Also by Remark 3.4 the term corresponding to $B(S(t)u_0,\cdot)$ in the trace part of formulae (3.7), (3.9) vanishes, so that (3.8) follows from (3.10) and (3.7).

4. Global Lyapunov exponents.

Definition 4.1. Let $t > 0$, $u_0 \in V$. We define the linear operator $S'(t,u_0) : H \to H$ by

$$S'(t,u_0)\xi = v(t,u_0,\xi) \tag{4.1}$$

where v is the solution to problem (3.1), (3.3) with $A(t)$ given by (3.2).

From Theorem 3.1 we deduce that $S'(t,u_0)$ is a bounded operator from H to V and therefore compact as an operator in H .

Let us denote by $M(t,u_0) := [S'(t,u_0)^*S'(t,u_0)]^{\frac{1}{2}}$. $M(t,u_0)$ is a compact, non-negative operator in H . Let us denote by $m_j(t,u_0)$ the eigenvalues of $M(t,u_0)$ counted according to their multiplicites:

$$0 < \ldots \le m_N(t,u_0) \le m_{N-1}(t,u_0) \le \ldots \le m_1(t,u_0)$$

Let us consider an orthonormal family $\phi_j(t,u_0)$ of eigenvectors for $M(t,u_0)$ corresponding to the eigenvalues $m_j(t,u_0)$. From the uniqueness part in Thm. 3.1 it follows that

$$S'(t+s,u_0)\xi = S'(t,S(s)u_0)S'(s,u_0)\xi \tag{4.2}$$

for any $\xi \in H$, $t,s \ge 0$.

Therefore, if $S'(s,u_0)\xi = 0$ for some $s > 0$ it remains zero for any $\tau \geq s$. From the analicity part in Thm. 3.1 it follows that $S'(\tau,u_0)\xi = 0$ for every $\tau > 0$ and from the strong continuity in H it follows that $\xi = 0$. Therefore $M(t,u_0)$ is injective for every $t > 0$ and $\phi_j(t,u_0)$ form a basis. We denote by $\psi_j(t,u_0)$ the vectors

$$\psi_j(t,u_0) = S'(t,u_0)\,\phi_j(t,u_0)\,;$$

then

$$(\psi_j(t,u_0),\psi_k(t,u_0)) = \delta_{jk}\, m_j(t,u_0) m_k(t,u_0)$$

where δ_{jk} is the Kronecker symbol. We have then a representation of $S'(t,u_0)$ of the form:

$$S'(t,u_0) = \sum_{j=1}^{\infty} (\phi_j(t,u_0),\cdot)\psi_j(t,u_0) \quad . \tag{4.3}$$

Lemma 4.2. There exists a positive continuous function $c(t)$ defined for every $t > 0$, depending on $|f|$, such that, for every u, $u_0 \in V$ the following estimate holds

$$|S(t,u)-S(t,u_0)-S'(t,u_0)(u-u_0)| \leq c(t)|u-u_0|^{5/4} . \tag{4.4}$$

The proof follows from standard energy estimates and will not be given here. (See for instance [6] for similar results).
The inequality is true for 3-D Navier Stokes equations, too.

We remark that (4.4) implies in particular that $S'(t,u_0)$ is the Fréchet derivative $\frac{DS(t)}{Du_0}$ of $S(t)$.

Combining (4.3) with Lemma 4.2 we obtain

<u>Lemma 4.3</u>. For any u , $u_0 \in V$ and any $N \geq 1$

$$(4.5) \qquad |S(t,u)-(S(t,u_0) + \sum_{j=1}^{N} (\phi_j(t,u_0), u-u_0)\psi_j(t,u_0))|$$

$$\leq (m_{N+1}(t,u_0) + c(t)|u-u_0|^{1/4})|u-u_0| .$$

The geometrical interpretation of (4.3) and (4.4) is that, up to an error of $r^{5/4}$, $S(t)$ transforms a ball in H centered at u_0 and of radius r into an infinite dimensional ellipsoid, centered at $S(t)u_0$ and with axes of lengths $rm_j(t,u_0)$ $(j = 1,2,...)$. The N dimensional ellipsoid

$$(4.6) \quad \sum_N^r(t,u_0) := \{S(t,u_0) + \sum_{j=1}^{N} (\phi_j(t,u_0), u-u_0)\psi_j(t,u_0) \mid |u-u_0| < r\}$$

has volume less than the corresponding box:

$$(4.7) \qquad N\text{-vol} \sum_N^r (t,u_0) \leq 2^N m_1(t,u_0)\dots m_N(t,u_0)r^N .$$

The classical Lyapunov exponents are numbers $\mu_j(u_0)$ such that, asymptotically , $e^{t\mu_j(u_0)} \sim m_j(t,u_0)$ (More precisely $\mu_j(u_0)$ would be $\lim_{t\to\infty} \frac{1}{t} \log m_j(t,u_0)$ if the limit existed).

We want to define global Lyapunov exponents. In order to do so we first denote by $P_N(t,u_0)$ the quantity

$$P_N(t,u_0) = m_1(t,u_0)\dots m_N(t,u_0) .$$

This quantity, which is modulo a $(2r)^N$ factor the volume of the box appearing in (4.7) can be expressed as

$$(4.8)\qquad P_N(t,u_0) = \sup_{\substack{\xi_i \in H, i=1,\dots,N \\ |\xi_i|\le 1}} |S'(t,u_0)\xi_1 \wedge \dots \wedge S'(t,u_0)\xi_N|$$

(See [20] for a similar approach for dynamical systems).

Indeed, we can compute $|S'(t,u_0)\xi_1 \wedge \dots \wedge S'(t,u_0)\xi_N|^2$ using (3.4):

$$(4.9)\quad |S'(t,u_0)\xi_1 \wedge \dots \wedge S'(t,u_0)\xi_N|^2 = \det[(S'(t,u_0)\xi_i, S'(t,u_0)\xi_j)]_{1\le i,j\le N} =$$

$$= \det[(M(t,u_0)\xi_i, M(t,u_0)\xi_j)]_{1\le i,j\le N} = |M(t,u_0)\xi_1 \wedge \dots \wedge M(t,u_0)\xi_N|^2 =$$

$$= \Big| \sum_{1\le j_1<j_2<\dots<j_N} m_{j_1}(t,u_0)\dots m_{j_N}(t,u_0)$$

$$\times \det[(\xi_i, \phi_{j_k})]_{1\le i,k\le N}\ \phi_{j_1} \wedge \dots \wedge \phi_{j_N}\Big|^2 =$$

$$= \sum_{1\le j_1<\dots<j_N} m_{j_1}^2(t,u_0)\dots m_{j_N}^2(t,u_0)\ \det{}^2[(\xi_i,\phi_{j_k})]_{1\le i,k\le N}$$

$$\le P_N^2(t,u_0)|\xi_1\wedge\dots\wedge\xi_N|^2 .$$

So $|S'(t,u_0)\xi_1\wedge\dots\wedge S'(t,u_0)\xi_N| \le P_N(t,u_0)$ if $|\xi_i| \le 1$, $1 \le i \le N$. Since, for $\xi_i = \phi_i(t,u_0)$ there is equality, (4.8) is proven. It follows, from (4.8) and (4.2) that

$$(4.10) \qquad P_N(t+s,u_0) \le P_N(t,S(s)u_0)P_N(s,u_0)$$

for any $t, s \ge 0$ and $u_0 \in V$

We start defining the Lyapunov exponents. Let ρ satisfy (2.7) or (2.8). Then $S(s)B_\rho^V(0) \subset B_\rho^V(0)$ for any $s \ge 0$ ($B_\rho^V(0)$ is the ball in V centered in 0 of radius ρ). We define

$$P_N(t) = \sup_{u_0\in B_\rho^V(0)} P_N(t,u_0), \qquad N \ge 1$$

$$\bar{m}_j(t) = \sup_{u_0\in B_\rho^V(0)} m_j(t,u_0), \qquad j \ge 1 .$$

The finiteness of $P_N(t)$, $m_j(t)$ follow from that of $\bar{m}_1(t)$ which is a consequence of (4.8) for $N = 1$ and of point (ii) of Thm. 3.1. We note the inequality which is thus obtained:

$$(4.11) \qquad \bar{m}_1(t) \le e^{kt} \quad (k \text{ given in Thm. 3.1}).$$

From (4.10) it follows that the function $\log P_N(t)$ is subadditive. The functions $P_N(t,u_0)$, $m_j(t,u_0)$ are never zero from the injectivity of $M(t,u_0)$. Therefore we can define

Definition 4.4. For every $j \geq 1$, $N \geq 1$, we put

(4.12) $$\bar{\mu}_j = \limsup_{t\to\infty} \frac{1}{t} \log \bar{m}_j(t)$$

(4.13) $$\pi_N = \lim_{t\to\infty} \frac{1}{t} \log P_N(t) .$$

We define inductively μ_j by

(4.14) $$\mu_1 = \pi_1 , \mu_N = \pi_N - \pi_{N-1} , N \geq 2$$

We note that if the classical $\mu_j(u_0)$ exists then $\mu_j(u_0) \leq \bar{\mu}_j$. Also it is clear that the sequence $\bar{\mu}_j$ is nonincreasing and that

(4.15) $$\mu_j \leq \bar{\mu}_j \leq \frac{\mu_1 + \ldots + \mu_j}{j} , 1 \leq j < \infty$$

Proposition 4.5. For every $N \geq 1$ there exists a positive continuous function $C_N(t)$ defined for every $t > 0$ such that, for every u_1, $u_2 \in V$,

(4.16) $$|P_N^2(t,u_1) - P_N^2(t,u_2)| \leq C_N(t) | u_1 - u_2 |^{1/2} .$$

Proof. Let $\xi_1,\dots\xi_N$ be vectors in H with $|\xi_i| \le 1$. Since (*) $\|M(t,u_j)\|_{L(H,H)} \le e^{kt}$, $j = 1,2$ (Thm. 3.1,ii) we infer that

$$|\det[(M(t,u_1)\xi_i,M(t,u_1)\xi_j)]_{1\le i,j\le N}$$

$$- \det[(M(t,u_2)\xi_i,M(t,u_2)\xi_j)]_{1\le i,j\le N}|$$

$$\le 2Ne^{Nkt}e^{(N-1)kt}\|M(t,u_1)-M(t,u_2)\|_{L(H,H)}$$

In virtue of (4.8) we obtain easily

$$|P_N^2(t,u_1)-P_N^2(t,u_2)| \le 2Ne^{(2N-1)kt}\|M(t,u_1)-M(t,u_2)\|_{L(H,H)}$$

For any two bounded nonnegative selfadjoint operators T, S

$$\|T^{1/2}-S^{1/2}\|_{L(H,H)} \le \frac{4\sqrt{2}}{\pi}\|T-S\|^{1/2}_{L(H,H)}$$

(See [14], lemma V. 3.43. p. 284). Therefore

$$\|M(t,u_1)-M(t,u_2)\|_{L(H,H)} \le \frac{8}{\pi}e^{kt/2}\|S'(t,u_1)-S'(t,u_2)\|^{1/2}_{L(H,H)}.$$

One can estimate $\|S'(t,u_1)-S'(t,u_2)\|_{L(H,H)} \le \tilde{C}(t)|u_1-u_2|$ with a suitable $\tilde{C}(t)$ by standard energy inequalities and (4.16) follows. As a consequence of Prop. 4.5 we note

Lemma 4.6. For every $t > 0$, $N \ge 1$

(*) Here after $\|\cdot\|_{L(H,H)}$ denotes the norm of the bounded linear operators on H.

$$\inf_{u_0 \in B_\rho^V(0)} m_N(t,u_0) > 0 .$$

Proof. Indeed, suppose $\inf_{u_0 \in B_\rho^V(0)} m_N(t,u_0) = 0$ for some $t > 0$, $N \geq 1$. Then, by $P_N^2(t,u_0) \leq e^{2(N-1)kt} m_N^2(t,u_0)$ (see (4.11)), the compacity in H of $B_\rho^V(0)$ and the continuity of $P_N^2(t,\cdot)$ (Prop. 4.5) there exists $u_0 \in B_\rho^V(0)$ such that $P_N^2(t,u_0) = 0$. This implies $m_j(t,u_0) = 0$ for some $1 \leq j \leq N$ and thus contradicts the injectivity of $M(t,u_0)$.

5. Estimates for the critical dimension.

Let us rewrite the equations of transport of volume elements (eq. (3.7) or (3.9)) in the form

$$(5.1) \qquad \frac{d}{dt} |v_1 \wedge \ldots \wedge v_N|^2 + |v_1 \wedge \ldots \wedge v_N|^2 [2\nu \, \mathrm{Trace}(AP(v_1,\ldots,v_N)) + 2 \, \mathrm{Trace}(B(\cdot,S(t)u_0)P(v_1,\ldots,v_N))] = 0 .$$

where v_i, $1 \leq i \leq N$, satisfy (3.1), (3.3) with $A(t)$ given by (3.2), $u_0 \in B_\rho^V(0)$ and $N > 0$.

In this section we shall prove the existence of (and estimate) a critical dimension $N_0 = N_0(f)$ enjoying the property that every N-volume $|v_1 \wedge \ldots \wedge v_N|$ decays exponentially if $N \geq N_0 + 1$. We shall treat separately the periodic boundary conditions case and the homogeneous boundary conditions case.

A. The periodic case.

This case is special because of the absence of boundary layers which is reflected, in particular, in the identity $B(w,w,Aw) \equiv 0$ for all $w \in \mathcal{D}(A)$. [See [23] for details]. An easy but important consequence of this identity is the inequality

$$\frac{1}{t}\int_0^t |Au|^2 d\tau \leq \frac{|f|^2}{\nu^2} + \frac{\rho^2}{\nu}\frac{1}{t} \tag{5.2}$$

valid for any $t > 0$ and any solution $u(\tau) = S(\tau)u_0$ of the Navier-Stokes equations with initial data $u_0 \in B_\rho^V(0)$.

We intend to obtain an estimate of the form

$$\frac{1}{t}\int_0^t 2[\nu\ \mathrm{Trace}(AP(v_1(\tau),\ldots,v_N(\tau)) + \mathrm{Trace}(B(\cdot,S(\tau)u_0)P(v_1(\tau),\ldots,v_N(\tau)))]d\tau \geq C_N(f) \quad \text{for } t \geq t_0,\ N \geq N_0, \tag{5.3}$$

with $C_N(f)$, $t_0(f,\rho)$ and $N_0(f)$ (large) positive constants. The dependence of these constants on N, f, ν will be specified later. From (5.3) it would follow that

$$|v_1 \wedge\ldots\wedge v_N|^2 \leq |\xi_1 \wedge\ldots\wedge \xi_N|^2 e^{-C_N t},\ t \geq t_0(\rho),\ N \geq N_0(f)$$

or, more precisely

$$(5.4) \quad \sup_{u_0 \in B_\rho^V(0)} \ \sup_{\substack{|\xi_i| \le 1 \\ 1 \le i \le N}} |S'(t,u_0)\xi_1 \wedge \ldots S'(t,u_0)\xi_N|^2$$

$$\le e^{-C_N t} \ , \ t \ge t_0 \ .$$

The way to obtain (5.3) is the following: We notice that the term involving A is very large, namely

$$2\nu \, \mathrm{Trace}(AP(v_1,\ldots v_N)) = 2\nu \frac{(A_N(v_1 \wedge \ldots \wedge v_N) ; \ v_1 \wedge \ldots \wedge u_N)}{| \ v_1 \wedge \ldots \wedge v_N|^2}$$

$$\ge 2\nu(\lambda_1 + \ldots + \lambda_N) \ .$$

We keep part of this term and use part of it to kill contributions from $\mathrm{Trace}(B(\cdot,S(t)u_0)P(v_1,\ldots v_N))$. Also it is useful to split this last term into two parts, corresponding to the finite spectrum of $u(t) = S(t)u_0$ and respectively the tail of the spectrum. Let us take λ a positive large number, to be fixed later. Let us denote by P_λ the projection in H on the eigenspace of A corresponding to eigenvalues $\lambda_j \le \lambda$ and $Q_\lambda = I - P_\lambda$. We shall use the estimate

$$(5.5) \qquad \|B(\cdot, P_\lambda u)\|_{L(H,H)} \le c[(\log \frac{\lambda}{\lambda_1})^{1/2} + 1]|AP_\lambda u|$$

for $u \in \mathcal{D}(A)$, which is an immediate consequence of

$$(5.6) \qquad |\nabla P_\lambda u|_{L^\infty} \le c_1[(\log \frac{\lambda}{\lambda_1})^{1/2} + 1]|AP_\lambda u| \ .$$

The proof of (5.6) is elementary in the periodic case since the eigenfunctions and eigenvalues of A are explicitly computable.

From (5.5) it follows that

$$(5.7)\qquad |\mathrm{Trace}(B(\cdot,P_\lambda u(\tau))P(v_1(\tau),\ldots u_N(\tau)))|$$

$$\leq cN\,[(\log \frac{\lambda}{\lambda_1})^{1/2}+1]\,|AP_\lambda u(\tau)|.$$

In order to estimate the tail part of $\mathrm{Trace}(B(\cdot,u(\tau))P(v_1,\ldots,v_N))$ we shall use the inequality

$$(5.8)\quad |\mathrm{Trace}(T_1T_2P)| \leq |\mathrm{Trace}(T_2^*T_2P)|^{1/2}\cdot|\mathrm{Trace}(T_1T_1^*P)|^{1/2}$$

valid for instance if P is a finite dimensional orthogonal projection on a subspace of the domain of T_2 and T_1 is bounded. We take first $T_1 = C_\lambda(u) = B(\cdot,Q_\lambda u)A^{-1/4}$ and $T_2 = A^{1/4}$. We remark that $C_\lambda(u)$ is bounded and

$$(5.9)\qquad \|C_\lambda(u)\|_{L(H,H)} \leq c\,\|Q_\lambda u\|^{1/2}\,|Q_\lambda Au|^{1/2}.$$

Indeed, (5.9) is a consequence of

$$(5.10)\quad |B(w,u)| \leq c|A^{1/4}w|\;\|u\|^{1/2}|Au|^{1/2} \quad \text{(see, for instance [23]).}$$

Using (5.8), (5.9) we proceed to estimate the term involving $Q_\lambda u$ as follows:

$$|\mathrm{Trace}(B(\cdot,Q_\lambda u(\tau))P(v_1(\tau),\ldots v_N(\tau)))|$$

$$= |\mathrm{Trace}(C_\lambda(u(\tau))A^{1/4}P(v_1(\tau),\ldots,v_N(\tau)))|$$

$$\leq |\mathrm{Trace}(C_\lambda(u(\tau))C_\lambda^*(u(\tau))P(v_1(\tau),\ldots,v_N(\tau)))|^{1/2}$$

$$\cdot\ |\mathrm{Trace}(A^{1/2}P(v_1(\tau)\ldots v_N(\tau))|^{1/2}$$

$$\leq\ N^{1/2}\ \|C_\lambda(u(\tau))\|_{L(H,H)}\ \ |\mathrm{Trace}(A^{1/2}P(v_1(\tau),\ldots,v_N(\tau)))|^{1/2}$$

$$\leq\ N^{3/4}c\|Q_\lambda u(\tau)\|^{1/2}\ |AQ_\lambda u(\tau)|^{1/2}\ |\mathrm{Trace}(AP(v_1(\lambda),\ldots,v_N(\tau))|^{1/4}$$

(We used (5.8) with $T_2 = A^{1/2}, T_1 = I$).

We use Young's inequality and obtain, using $\|Q_\lambda u\| \leq \lambda^{-1/2}|Au|$,

$$(5.11)\quad |\mathrm{Trace}(B(\cdot,Q_\lambda u(\tau))P(v_1(\tau),\ldots v_N(\tau)))|$$

$$\leq \frac{\nu}{2}\ |\mathrm{Trace}(AP(v_1(\tau),\ldots,v_N(\tau)))| + \frac{3c}{2^{7/3}}\ N\frac{|Au(\tau)|^{4/3}}{(\nu\lambda)^{1/3}}$$

We start proving (5.3). We use (5.7), (5.11) and get

$$\frac{1}{t}\int_0^t [\,2\nu\ \mathrm{Trace}(AP(v_1(\tau),\ldots,v_N(\tau))\ +\ 2\ \mathrm{Trace}(B(\cdot,P_\lambda S(\tau)u_0)$$

$$P(v_1(\tau),\ldots,v_N(\tau)))\ +\ 2\ \mathrm{Trace}(B(\cdot,Q_\lambda S(\tau)u_0)P(v_1(\tau),\ldots,v_N(\tau))\,]d\tau$$

$$\geq \nu(\lambda_1+\ldots+\lambda_N)\ -\ \frac{N}{t}\int_0^t \{2c[(\log\frac{\lambda}{\lambda_1})^{1/2}+1]|Au(\tau)|$$

$$+\ \frac{3c}{2^{4/3}}\ \frac{|Au(\tau)|^{4/3}}{(\nu\lambda)^{1/3}}\}\,d\tau\ \ \geq$$

$$\nu(\lambda_1+\ldots+\lambda_N) - 2cN\,[(\log \frac{\lambda}{\lambda_1})^{1/2} + 1]\,(\frac{1}{t}\int_0^t |Au(\tau)|^2)^{1/2}$$

$$- \frac{3c}{2^{4/3}} \frac{N}{(\nu\lambda)^{1/3}} (\frac{1}{t}\int_0^t |Au|^2\, d\tau)^{2/3} .$$

From (5.2) we obtain

$$E(t) := \frac{1}{t}\int_0^t [2\nu \text{ Trace}(AP(v_1(\tau),\ldots,v_N(\tau)) + 2 \text{ Trace } B(\cdot,S(\tau)u_0)$$

$$P(v_1(\tau),\ldots v_N(\tau))]d\tau \geq \nu(\lambda_1+\ldots+\lambda_N) - 2cN[(\log \frac{\lambda}{\lambda_1})^{1/2} + 1]$$

$$[\frac{|f|^2}{\nu^2} + \frac{\rho^2}{\nu}\frac{1}{t}]^{1/2} - \frac{3c}{2^{4/3}} \frac{N}{(\nu\lambda)^{1/3}} [\frac{|f|^2}{\nu^2} + \frac{\rho^2}{\nu}\frac{1}{t}]^{2/3} .$$

It is now time to recall the dimensionless constant that it is natural to use instead of $|f|$, namely the Grashof number

$$G = \frac{|f|}{\nu^2\lambda_1} .$$

Let us take $t \geq t_0$ where t_0 depends on $|f|$ and ρ only and is defined by

(5.12) $$t_0 = t_0(\rho,f) = G^{-2} \frac{\rho^2}{\nu^3\lambda_1^2}$$

It is well known that the asymptotic behavior of $\frac{\lambda_j}{\lambda_1}$ is j, therefore, there exists a constant c_1 such that

$$\nu(\lambda_1+\ldots+\lambda_N) \geq c_1\nu\lambda_1 N(N+1) .$$

We obtain, for $t \geq t_0(\rho, f)$,

$$E(t) \geq \nu\lambda_1 N\{c_1(N+1) - 2\sqrt{2}c[(\log\frac{\lambda}{\lambda_1})^{1/2}+1]G - \frac{3c}{2^{2/3}}(\frac{\lambda}{\lambda_1})^{-1/3} G^{4/3}\} .$$

Now we choose λ such that $\frac{\lambda}{\lambda_1} = G$. Taking $c_2 = 2\sqrt{2}c + \frac{3c}{2^{2/3}}$ one can write

$$E(t) \geq c_1\nu\lambda_1 N\{N + 1 - \frac{c_2}{c_1} G[(\log G)^{1/2} + 1]\} .$$

It is clear now that the choice for $N_0(f)$ given by

$$N_0(f) = \text{integer part of } \frac{c_2}{c_1} G[(\log G)^{1/2} + 1] \tag{5.13}$$

gives the inequality

$$E(t) \geq c_1\nu\lambda_1 N(N-N_0) \quad \text{for } t \geq t_0(\rho,f),\ N \geq N_0(f) .$$

We proved

Theorem 5.2. There exist $c_1 > 0$, $c_2 > 0$ such that the following inequality holds

$$\sup_{u_0 \in B_\rho^V(0)} \ \sup_{\substack{\xi_i, 1\leq i\leq N \\ |\xi_i|\leq 1}} |S'(t,u_0)\xi_1 \wedge \dots \wedge S'(t,u_0)\xi_N|^2 \tag{5.4'}$$

$$\leq e^{-c_1\nu\lambda_1 tN(N-N_0)}$$

for any $t \geq t_0(\rho,f)$ (t_0 given in (5.12)) and $N \geq N_0(f)$, ($N_0(f)$ given by (5.13)).

Corollary 5.3. a) For any $N \geq N_0(f)$, $t \geq t_0(f,\rho)$ (with N_0 given in (5.13) and t_0 in (5.12)),

(5.4")
$$P_N(t) \leq e^{-\frac{c_1}{2}\nu\lambda_1(N-N_0)Nt} .$$

b) For any $N \geq N_0(f)$

(5.4"')
$$\mu_1+\dots+\mu_N \leq -\frac{c_1}{2}\nu\lambda_1 N(N-N_0).$$

Proof. a) is a consequence of (4.8), the definition of $P_N(t)$, and the relation (5.4). b) is a consequence of a) and of the definition (4.4) of μ_j.

B. The general case.

In this subsection we give estimates for $N_0(f)$ which are valid for both the periodic and the aperiodic cases.

Let us take, as before, N solutions, v_i, $i = 1,\dots,N$, of the equation (3.1) with $A(t)$ given in (3.2) and initial data $\xi_i = v_i(0)$. Since we want to estimate $v_1\wedge\dots\wedge v_N$ we may assume that the initial data are linearly independent. Consequently, for any $t \geq 0$ we can find an orthonormal basis, $x_1(t),\dots,x_N(t)$, for the linear space spanned by $v_1(t),\dots,v_N(t)$ in H. Then

(5.14) $$\mathrm{Trace}(B(\cdot, S(t)u_0)P(v_1(t),\dots,v_N(t))) =$$

$$= \sum_{j=1}^{N} B(x_j(t), u(t), x_j(t))$$

where $u(t) = S(t)u_0$ and $u_0 \in B^V_\rho(0)$.

Using the estimate

(5.15) $$|B(w,u,w)| \le c|w| \; \|w\| \; \|u\| \qquad \text{(see [23])}$$

we obtain from (5.14)

(5.16) $$|\mathrm{Trace}(B(\cdot, S(t)u_0)P(v_1(t),\dots,v_N(t)))|$$

$$\le c\Big(\sum_{j=1}^{N} \|x_j\|\Big)\|u\| \le cN^{1/2}\Big(\sum_{j=1}^{N} \|x_j\|^2\Big)^{1/2} \|u\|$$

$$\le \frac{\nu}{2} \sum_{j=1}^{N} \|x_j\|^2 + \frac{c^2}{2} N \frac{\|u\|^2}{\nu} .$$

We notice that

(5.17) $$\mathrm{Trace}(AP(v_1(t),\dots,v_N(t))) = \sum_{j=1}^{N} \|x_j\|^2$$

and thus, for any fixed τ , the integrand of (5.3) satisfies

(5.18) $$2[\nu \; \mathrm{Trace}(AP(v_1(\tau),\dots,v_N(\tau))$$

$$+ \mathrm{Trace}(B(\cdot, S(\tau)u_0)P(v_1(\tau),\dots,v_N(\tau)))]$$

$$\ge \nu(\lambda_1+\dots+\lambda_N) - c^2 N \frac{\|u\|^2}{\nu} \ge c_1 \nu\lambda_1\Big(N+1 - \frac{c^2}{c_1} \frac{\|u\|^2}{\nu^2\lambda_1}\Big)N$$

since there exists a constant c_1 such that

$$\nu(\lambda_1+\dots+\lambda_N) \ge c_1\nu\lambda_1 N(N+1) \quad \text{(see [18])} .$$

Now, from the inequality

$$\text{(5.19)} \qquad \frac{1}{\nu^2\lambda_1}\frac{1}{t}\int_0^t \|u\|^2 d\tau \le \frac{|A^{-\frac{1}{2}}f|^2}{\nu^4\lambda_1} + \frac{\rho^2}{\nu^3\lambda_1^2 t}$$

we obtain

$$\text{(5.20)} \qquad \frac{1}{t}\int_0^t 2[\mathrm{Trace}(AP(v_1(\tau),\dots,v_N(\tau)))$$

$$+ \mathrm{Trace}(B(\cdot,S(\tau)u_0)P(v_1(\tau),\dots,v_N(\tau)))]d\tau$$

$$\ge c_1\nu\lambda_1 N[N+1 - \frac{c^2}{c_1}(\frac{|A^{-\frac{1}{2}}f|^2}{\nu^4\lambda_1} + \frac{\rho^2}{\nu^3\lambda_1^2 t})] .$$

We define $G_* = G_*(f)$ by

$$\text{(5.21)} \qquad G_* = \frac{|A^{-\frac{1}{2}}f|}{\nu^2\lambda_1^{\frac{1}{2}}} .$$

We remark that G_* is nondimensional and clearly $G_* \le G$. (Moreover G^* may be much smaller than G if f has only high wave number modes.) Let us define $N_0(f)$ by

$$\text{(5.22)} \qquad N_0(f) = \text{integer part of } \frac{2c^2}{c_1} G_*^2$$

and put

$$t_0(\rho,f) = G_*^{-2} \frac{\rho^2}{\nu^3\lambda_1^2} . \tag{5.23}$$

From (5.20) we obtain, as in 5.A,

Theorem 5.4. There exist nondimensional constants c , c_1 such that the following inequality holds

$$\sup_{u_0 \in B_\rho^V(0)} \sup_{\substack{|\xi_i| \le 1 \\ i \le i \le N}} |S'(t,u_0)\xi_1 \wedge \ldots \wedge S'(t,u_0)\xi_N|^2 \le e^{-c_1 \nu\lambda_1 N(N-N_0)t} \tag{5.24}$$

for any $t \ge t_0(\rho,f)$, $N \ge N_0(f)$. (t_0 and N_0 are defined in (5.23), (5.22)).

Corollary 5.5. a) For any $N \ge N_0(f)$, $t \ge t_0(\rho,f)$ (given in (5.22), (5.23))

$$P_N(t) \le e^{-\frac{c_1}{2}\nu\lambda_1 N(N-N_0)t} . \tag{5.25}$$

b) For any $N \ge N_0(f)$

$$\mu_1+\ldots+\mu_N \le -\frac{c_1}{2}\nu\lambda_1 N(N-N_0) . \tag{5.26}$$

6. Dimension of the attractor. The Kaplan-Yorke conjecture.

We start by recalling the definitions of the Hausdorff and fractal (or Kolmogorov-Mandelbrot) dimensions.

Definition 6.1. Let $X \subset H$ be a compact set. We define the Hausdorff dimension of X, $d_H(X)$ by

$$d_H(X) := \inf\{d > 0 \mid \mu_H^d(X) = 0\}$$

where

$$\mu_H^d(X) = \lim_{\substack{r\to 0\\ r>0}} \mu_{H,r}^d(X)$$

and

$$\mu_{H,r}^d(X) = \inf\{\sum_{i=1}^k r_i^d \mid X \subset \bigcup_{i=1}^k B_i \;,\; B_i \text{ open balls in } H \text{ of radii } r_i \le r\} .$$

Definition 6.2. We define the fractal dimension of the compact set $X \subset H$, $d_M(X)$ by

$$d_M(X) = \inf\{d > 0 \mid \mu_M^d(X) = 0\}$$

where

$$\mu_M^d(X) = \limsup_{r\to 0} r^d n_X(r) ,$$

$n_X(r)$ being the minimal number of balls of H of radii equal to r required to cover X. (See [17].)

Remark 6.3. The definition 6.2 agrees with the conventional one

$$d_M(X) = \limsup_{r\to 0} \frac{\log(n_X(r))}{\log(1/r)} .$$

Remark 6.4. $d_H(X) \le d_M(X)$. However, it might happen that $d_H(X)$ is finite while $d_M(X)$ is infinite. For example, consider $(e_k)_{k=1,2,...}$ an orthonormal family in H and take for X the continuous, piecewise linear curve joining the points $x(0) = 0$, $x(\frac{1}{k}) = \frac{1}{\log k} e_k$, $k \ge 2$, $x(1) = e_1$. The Hausdorff dimension of X is 1 but the fractal dimension is $+\infty$ since, if one takes $r_m = \frac{1}{\log m}\cdot\frac{1}{\sqrt{2}}$, $m \ge 2$ then

$$n_X(r_m) \ge m - 1 = e^{\frac{1}{\sqrt{2}\, r_m}} - 1 .$$

Indeed $x(\frac{1}{j})$, $x(\frac{1}{k})$ cannot belong to the same ball of radius r_m unless both j and k are larger or equal than m .

In [13] , Kaplan and Yorke conjectured that, for a finite dimensional dynamical system, the information dimension (which is smaller than the fractal dimension) of the attractor should be equal to the expression

$$d(X) = j_0 + \frac{\mu_1+\dots+\mu_{j_0}}{|\mu_{j_0+1}|} \tag{6.1}$$

where μ_i are (classical) Lyapunov exponents and j_0 is defined by

$$j_0 = \max\{j \mid \mu_1+\dots+\mu_j \ge 0\} . \tag{6.2}$$

We are going to prove in this section that expressions like (6.1) constructed with the global Lyapunov exponents μ_i , $\bar{\mu}_i$ are upper bounds for the Hausdorff and fractal dimension of any bounded, invariant under $S(t)$, set in H .

Let X be a set in H which satisfies

(i) X is bounded in H

(ii) $S(t)X = X$ for any $t \geq 0$.

Such a set is, for instance, the universal attractor

$$X = \bigcap_{t>0} S(t)B_\rho^V(0) \quad \text{where } \rho \text{ is given in (2.7) or (2.8).}$$

A set satisfying (i) and (ii) is compact in H .

The general idea of our approach to estimate the dimension of X , which appeared for the first time in [4] , is the following: Suppose X is covered by a finite number of balls. After a sufficiently long time these balls become slightly distorted infinite dimensional ellipsoids with axes of lengths given in terms of the Lyapunov exponents. Comparing the expressions defining the dimensions obtained by means of the initial covering with those corresponding to the covering by ellipsoids, the Kaplan-Yorke expressions appear naturally.

Lemma 6.5. Let $N \geq j_0$ be an integer and D be a number satisfying

(i) $N \leq D \leq N + 1$

(ii) $(D-N)\mu_{N+1} + \mu_1+\ldots+\mu_N < 0$.

Then $\mu_H^D(X) = 0$.

Proof. We note that $N \geq j_0$ implies $\mu_1+\ldots+\mu_{N+1} < 0$ so that $\bar{\mu}_{N+1} < 0$ by (4.15). Let us take $\varepsilon > 0$ such that

$$(D-N)\mu_{N+1} + \mu_1+\ldots+\mu_N + 3\varepsilon < 0 . \tag{6.3}$$

Let us choose $t = t(\varepsilon)$ large enough to satisfy

(6.4) $t \geq t_0(G,\rho_0)$ where $G = \frac{|f|}{\nu^2\lambda_1}$, $\rho_0 = \frac{\rho}{\nu\lambda_1^{1/2}}$ and $X \subset B_\rho^V(0)$

(6.5) $\frac{N \log 10}{t} + \frac{D \log 2}{t} < \varepsilon$

(6.6) $e^{-\varepsilon t} \leq 2^{-(D-N)-1}$

(6.7) $\bar{m}_{N+1}(t) \leq \frac{1}{8}$

(6.8) $\frac{\log P_N(t)}{t} \leq \mu_1+\ldots+\mu_N + \varepsilon$, $\frac{\log P_{N+1}(t)}{t} \leq \mu_1+\ldots+\mu_{N+1} + \varepsilon$.

Condition (6.7) can be satisfied since $\bar{\mu}_{N+1} < 0$ and (6.8) can be satisfied because of the definition of μ_j . Let us fix t enjoying (6.4)-(6.8). We choose $r_0 = r_0(\varepsilon)$ such that

$$1 + \frac{c(t)r_0^{1/4}}{\tilde{m}_{N+1}(t)} \leq 2 \tag{6.9}$$

where $c(t)$ appears in (4.4), (4.5) and $\tilde{m}_{N+1}(t) = \inf_{u_0 \in B_\rho^V(0)} (m_{N+1}(t,u_0))$.

By Lemma 4.6, $\tilde{m}_{N+1}(t) > 0$. Since $D - N \geq 0$, $1 + N - D \geq 0$ we have, using (6.8), (6.5) and (6.3) that

$$(1+N-D)\log P_N(t) + (D-N)\log P_{N+1}(t) + N\log 10 + D\log 2 \le -\varepsilon t .$$

From (6.6) it follows that

$$10^N 2^D P_N(t)^{1+N-D} \cdot P_{N+1}^{D-N}(t) \le 2^{-(D-N)-1}$$

and therefore

$$(6.10) \qquad 10^N 2^D [P_{N+1}(t,u_0)]^{D-N} [P_N(t,u_0)]^{1+N-D} \le 2^{-(D-N)-1}$$

for any $u_0 \in B_\rho^V(0)$.

Let us consider a ball of radius $r \le r_0$, centered at u_0 . By Lemma 4.3 we have that

$$(6.11) \qquad \operatorname{dist}(S(t)u, \textstyle\sum_N^r(t,u_0)) \le \theta(t,u_0,r)\cdot r$$

for any u in the ball $|u-u_0| \le r$. In (6.11) $\theta(t,u_0,r)$ is given by

$$(6.12) \qquad \theta(t,u_0,r) = m_{N+1}(t,u_0) + c(t)r^{1/4}$$

and $\sum_N^r(t,u_0)$ is the N dimensional ellipsoid defined in (4.6).

We remark that (6.9) implies that

$$(6.13) \qquad \theta(t,u_0,r) \le m_{N+1}(t,u_0)\left(1 + \frac{c(t)r^{1/4}}{\bar{m}_{N+1}(t)}\right) \le 2m_{N+1}(t,u_0)$$

and (6.7) together with (6.13) gives

(6.14) $$\theta(t,u_0,r) \le 2\bar{m}_{N+1}(t) \le \frac{1}{4} .$$

We know (see (4.6)) that the N-volume of $\Sigma_N^r(t,u_0)$ is smaller than $2^N P_N(t,u_0) r^N$. Thus, using a Vitali type argument [22], we infer that:

(6.15) the number of balls of radii $\theta(t,u_0,r)\cdot r$ necessary to cover $\Sigma_N^r(t,u_0)$ does not exceed

$$\frac{2^N P_N(t,u_0) r^N}{\omega_N \left[\frac{(\theta(t,u_0,r) r}{5}\right]^N} \le 10^N \ \theta(t,u_0,r)^{-N} P_N(t,u_0)$$

(ω_N is the volume of S^{N-1}).
From (6.11) and (6.15) we infer that

(6.16) the number of balls of radii $2\theta(t,u_0,r)\cdot r$ necessary to cover $S(t)B_H(u_0,r)$ does not exceed

$$10^N \theta(t,u_0,r)^{-N} P_N(t,u_0) .$$

We denoted by $B_H(u_0,r) = \{u \mid |u-u_0| \le r\}$. We remark that the radii appearing in (6.16) are smaller than $r/2$ (see (6.14)).

Let us take now a finite covering of X by balls $B_H(u_i,r_i)$, $1 \le i \le \ell$, $r_i \le r(\le r_0)$. From the invariance property of X

$$X \subset \bigcup_{i=1}^{\ell} S(t) B_H(u_i,r_i)$$

We cover each $S(t)B_H(u_i,r_i)$ by balls of radii $2\theta(t,u_i,r_i)r_i$ and obtain thus a new covering of X with balls of radii not larger than $r/2$. Using (6.16) we obtain

$$\mu^D_{H,\frac{r}{2}}(X) \le \sum_{i=1}^{\ell} 10^N \theta^{-N}(t,u_i,r_i)P_N(t,u_i)2^D\theta^D(t,u_i,r_i)r_i^D$$

$$\le \sum_{i=1}^{\ell} 2^D 10^N \left(1+\frac{c(t)r_i}{\tilde{m}_{N+1}(t)}\right)^{D-N} P_N(t,u_i)^{1+N-D}\, P_{N+1}(t,u_i)^{D-N}\, r_i^D .$$

Now (6.10) and (6.9) imply

$$\mu^D_{H,\frac{r}{2}}(X) \le \frac{1}{2}\sum_{i=1}^{\ell} r_i^D .$$

Since the covering $B_H(u_i,r_i)$ was arbitrary we obtained

$$(6.17) \qquad \mu^D_{H,\frac{r}{2}}(X) \le \frac{1}{2}\,\mu^D_{H,r}(X) \quad \text{for every} \quad r \le r_0$$

The function $r \to \mu^D_{H,r}(X)$ is nonincreasing so the proof of the lemma is complete.

<u>Theorem 6.6</u>. Let X be a invariant set which is bounded in H. Then

$$d_H(X) \le j_0 + \frac{\mu_1+\ldots+\mu_{j_0}}{|\mu_{j_0+1}|},$$

where j_0 is defined by (6.2).

<u>Proof</u>. If $D \ge j_0+1$, taking N = integer part of D we can apply Lemma 6.5 since (ii) is true in virtue of $\mu_{N+1} < 0$,

$\mu_1+\ldots+\mu_N < 0$. Therefore $\mu_H^D(X) = 0$ and so $d_H(X) \le D$. If D satisfies

$$j_0 + \frac{\mu_1+\ldots+\mu_{j_0}}{|\mu_{j_0+1}|} < D \le j_0 + 1$$

we can apply Lemma 6.5 for $N = j_0$ (since $|\mu_{j_0+1}| = -\mu_{j_0+1}$) and so $\mu_H^D(X) = 0$; thus $d_H(X) \le D$.

<u>Theorem 6.7</u> a) In the periodic case, if X is as above

$$d_H(X) \le \frac{c_2}{c_1} G\,[(\log G)^{1/2} + 1] + 2$$

where c_2 , c_1 are absolute constants (depending on the shape of Ω only).

b) In the general case,

$$d_H(X) \le \frac{2c^2}{c_1} G_*^2 + 2$$

where c , c_1 are absolute constants.

<u>Proof</u>. We remark that $0 \le \dfrac{\mu_1+\ldots+\mu_{j_0}}{|\mu_{j_0+1}|} < 1$ since

$$0 > \frac{\mu_1+\ldots+\mu_{j_0}+\mu_{j_0+1}}{|\mu_{j_0+1}|} = \frac{\mu_1+\ldots+\mu_{j_0}}{|\mu_{j_0+1}|} - 1 .$$

Since (5.4‴) in the periodic case and Corollary 5.5 (b) in the aperiodic case imply that $j_0 \le N_0(f)$, the estimates follow from Thm. 6.6 and (5.13) in the periodic case, respectively (5.22) in the aperiodic case.

We shall consider now the fractal dimension.

Lemma 6.8. Let N be a natural number, D a real number satisfying

(i) $D \geq N$

(ii) $(D-N)\bar{\mu}_{N+1} + \mu_1+\ldots+\mu_N < 0$

(iii) $\bar{\mu}_{N+1} < 0$.

Then $\mu_M^D(X) = 0$.

Proof. Let us take $\varepsilon > 0$ small enough such that

(6.18) $$(D-N)\bar{\mu}_{N+1} + \mu_1+\ldots+\mu_N + (D-N+2)\varepsilon \leq 0 \ .$$

Let us choose $t = t(\varepsilon)$ large enough to insure the validity of (6.4), (6.7),

(6.19) $$\frac{\log \bar{m}_{N+1}(t)}{t} \leq \bar{\mu}_{N+1} + \varepsilon \ ; \ \frac{\log P_N(t)}{t} \leq \mu_1+\ldots+\mu_N + \varepsilon$$

and

(6.20) $$\frac{(2D+1)\log 2 + N \log 5}{t} \leq \varepsilon \ .$$

Let us take $r \leq r_0$, where r_0 is defined by condition (6.9). Let us consider a ball $B_H(u_0,r)$ centered at $u_0 \in X$. We define $\tilde{\theta}(t)$ by

(6.21) $$\tilde{\theta}(t) = 2\bar{m}_{N+1}(t).$$

We remark that (6.7) gives $\tilde{\theta}(t) \le \frac{1}{4}$. Reasoning as in the proof of Lemma 6.5 we infer that

(6.22) the number of balls of radii $2\tilde{\theta}(t)r$ required to cover $S(t)B(u_0,r)$ does not exceed $5^N[\bar{m}_{N+1}(t)]^{-N} P_N(t)$.

We infer that

(6.23) $$n_X(2\tilde{\theta}(t)r) \le [5^N(\bar{m}_{N+1}(t))^{-N} P_N(t)]\, n_X(r)$$

(see Def. 6.2 for $n_X(\cdot)$).

Thus we obtain

(6.24) $$(2\tilde{\theta}(t)r)^D n_X(2\tilde{\theta}(t)r) \le [\, 4^D 5^N(\bar{m}_{N+1}(t))^{D-N} P_N(t)]\, r^D n_X(r)\,.$$

We remark that (6.18), (6.19) and (6.20) imply that

(6.25) $$4^D 5^N(\bar{m}_{N+1}(t))^{D-N} P_N(t) \le 1/2$$

so that (6.24) becomes

(6.26) $$(2\tilde{\theta}(t)r)^D n_X(2\tilde{\theta}(t)r) \le (1/2)\, r^D n_X(r) \quad \text{for every} \quad r \le r_0\,.$$

We remark that

$$\sup\{r^D n_X(r) \mid 2\tilde{\theta}(t)r_0 \le r \le r_0\} \le M = r_0^D n_X(2\tilde{\theta}(t)r_0) < \infty .$$

In virtue of the simple Lemma 6.9 given below, applied for $\alpha = 2\tilde{\theta}(t)$ and $\phi(r) = r^D n_X(r)$, the proof is complete.

Lemma 6.9 Let $\phi(r) \ge 0$ defined for $0 < r \le r_0$ be a function satisfying, for some α , $0 < \alpha < 1$

(i) $\sup\{\phi(r) \mid \alpha r_0 \le r \le r_0\} = M < \infty$

(ii) $\phi(\alpha r) \le \frac{1}{2}\phi(r)$ for every $0 < r \le r_0$.

Then $\lim_{r\to 0} \phi(r) = 0$.

Proof. From (ii) we have

(ii)' $\phi(\alpha^\ell r) \le (\frac{1}{2})^\ell \phi(r)$ for every $r \le r_0$, $\ell \ge 1$.

If $\alpha^{\ell+1} r_0 \le r \le \alpha^\ell r_0$ then $\alpha r_0 \le \alpha^{-\ell} r \le r_0$ so by (i) , $\phi(\alpha^{-\ell} r) \le M$ and by (ii)' $\phi(r) = \phi(\alpha^\ell \alpha^{-\ell} r) \le (\frac{1}{2})^\ell M$. It follows that, if $r \le \alpha^k r_0$, $\phi(r) \le (\frac{1}{2})^k M$ (since there exists $\ell \ge k$ with $\alpha^{\ell+1} r_0 \le r \le \alpha^\ell r_0$) .

Theorem 6.10. Let X be a bounded set in H which is invariant under $S(t)$. Then

$$d_M(X) \le j_0 + 1 .$$

Furthermore,

a) In the periodic case

$$d_M(X) \le cG[(\log G)^{1/2} + 1] + 2$$

with $c(=\frac{c_2}{c_1})$ a nondimensional constant depending on Ω only,

b) in the general case

$$d_M(X) \le cG_*^2 + 2$$

with $c(= \frac{2c^2}{c_1})$ a nondimensional constant.

Proof. It is enough to prove that $d_M(X) \le j_0 + 1$ since, as we noted in the proof of Thm. 6.8, $j_0 \le N_0(f)$ and a) and b) are the estimates that we obtained for $N_0(f) + 1$.

If $D \ge j_0 + 1$ then, taking $N = j_0 + 1$ we can apply Lemma 6.9 ((ii) is true since both $\overline{\mu}_{j_0+2}$ and $\mu_1+\ldots+\mu_{j_0+1}$ are negative) and conclude that

$$\mu_M^D(X) = 0 \ .$$

From the definition of the fractal dimension it follows that $d_M(X) \le D$.

Theorem 6.11. Let X be a bounded set in H which is invariant under $S(t)$. Then

$$d_M(x) \le j_0 + \frac{\mu_1+\dots+\mu_{j_0}}{|\bar{\mu}_{j_0+1}|} .$$

Proof. If $\dfrac{\mu_1+\dots+\mu_{j_0}}{|\bar{\mu}_{j_0+1}|} \ge 1$ the result is covered by Thm. 6.10. If $\dfrac{\mu_1+\dots+\mu_{j_0}}{|\bar{\mu}_{j_0+1}|} < 1$ then, since $\bar{\mu}_{j_0+1} < 0$, from $D > j_0 + \dfrac{\mu_1+\dots+\mu_{j_0}}{|\bar{\mu}_{j_0+1}|}$ it follows that the conditions (i), (ii), (iii) of Lemma 6.9 are fulfilled for $N = j_0$. Therefore $\mu_M^D(X) = 0$ and

$$d_M(X) \le j_0 + \frac{\mu_1+\dots+\mu_{j_0}}{|\bar{\mu}_{j_0+1}|} .$$

7. An example.

In this section we consider the following infinite dimensional generalization of a differential system considered in[19] :

$$(7.1)\quad \begin{cases} \dfrac{du_1}{dt} + \nu u_1 + u_2^2+\dots+u_n^2+\dots = f_1 \\[2ex] \dfrac{du_j}{dt} + \nu u_j - u_1 u_j = 0 \qquad (j = 2,3,\dots) \end{cases}$$

in $H = \ell^2$. This system is amenable to the same abstract treatment as the one used for the Navier-Stokes equations. Indeed setting

$$(7.2)\qquad A = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 2 & 0 & \dots \\ 0 & 0 & 3 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} , \quad f = f_1 e_1 \quad \text{where} \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$B(v,u) = -u_1 v + (u,v)e_1$ for $u, v \in H$, the system (7.1) becomes the abstract differential equation in (2.5).

Because of the special from (7.2) of $B(\cdot,\cdot)$ the relation (5.3) becomes in this case

$$(7.3) \quad \frac{1}{t}\, 2\int_0^t [\nu\, \mathrm{Trace}(AP(v_1(\tau),\dots,v_N(\tau)) + $$

$$+ \mathrm{Trace}((B(\cdot,S(\tau)u_0)P(v_1,\dots,v_N)))]d\tau$$

$$\geq \nu N(N+1) - \frac{2N}{t}\int_0^t |u(\tau)|d\tau$$

$$\geq \nu N[N+1 - G - \frac{2\rho}{t\nu^2}].$$

where $G = \frac{|f|}{\nu^2 \lambda_1} = \frac{|f_1|}{\nu^2}$ and ρ is subjected to a similar condition to (2.7).[1] Hence $N_0(f)$ which appears in (5.3) and in the proof of Thm. 6.10 can be taken

$$N_0(f) = \text{integer part of } G .$$

Thus Thm. 6.10 yields, in this case that the fractal dimension of the universal attractor X of the system (7.1) satisfies

$$(7.4) \qquad d_M(X) \leq G + 1 .$$

On the other hand one can obtain a lower bound for $d_H(X)$ as follows.

Let $w = (w_j), j = 1,2,\dots$ with $w_1 = \nu N$, $w_N = \sqrt{|f_1| - \nu^2 N}$, and $w_j = 0$ for $j \neq 1, N$, where N is chosen such that

$$\nu^2 N < |f_1| \leq \nu^2 (N+1) \tag{7.5}$$

Then w is a stationary solution of (7.1) such that the nonpositive eigenvalues of the operator

$$A(w) = \nu A + B(\cdot, w) + B(w, \cdot)$$

are precisely $-\nu, -2\nu, \ldots -(N-2)\nu$. Therefore the unstable manifold at w, Σ_w, has the dimension $N-2$ (see [12] p. 242). Clearly Σ_w is included in X and hence $d_H(X) \geq N-2$. From (7.5) it follows

$$d_H(X) \geq G - 3 . \tag{7.6}$$

Comparing (7.5) and (7.6) we see that our formulation of the Kaplan-Yorke conjecture, proven in this paper gives, for systems amenable to the abstract form (2.5), a sharp upper bound of the dimensions of their universal attractors.

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