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# Global optimization of concave functions subject to separable quadratic constraints and of all-quadratic separable problems.



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*ABSTRACT*

In this paper we propose different methods for finding the global minimum of a concave function subject to quadratic separable constraints. The first method is of the branch and bound type, and is based on rectangular partitions to obtain upper and lower bounds. Convergence of the proposed algorithm is also proved. For computational purposes, different procedures that accelerate the convergence of the proposed algorithm are analysed.

The second method is based on piecewise linear approximations of the constraint functions. When the constraints are convex the problem is reduced to global concave minimization subject to linear constraints. In the case of non-convex constraints we use zero-one integer variables to linearize the constraints. The number of integer variables depends only on the concave parts of the constraint functions.

* Parts of the present paper were prepared while the second author was visiting Georgia Tech and Penn State University.

**Introduction**

In this paper, we are concerned with determining the global minimum of problems of the form:

$$\text{global min } f(x)$$

$$\text{s.t. } g_i(x) := \sum_{k=1}^{n} \left(\frac{1}{2} p_{ik} x_k^2 + q_{ik} x_k + r_{ik}\right) \le 0, \; i=1, \ldots, m \qquad (1)$$

$$\underline{m}_k \le x_k \le \overline{m}_k, \; k=1, \ldots, n$$

where $p_{ik}$, $q_{ik}$, $r_{ik}$, $\underline{m}_k$, $\overline{m}_k$, $(i=1, \ldots, m$ and $k=1, \ldots, n)$ are given real numbers, $f(x)$ is a real valued concave function defined on an open convex set containing the rectangle $M_0=\{x: \underline{m} \le x \le \overline{m}\}$, $\underline{m}=(\underline{m}_1, \ldots, \underline{m}_n)^T$, $\overline{m}=(\overline{m}_1, \ldots, \overline{m}_n)^T$. Denote by D the feasible set of problem (1). Clearly, $D$ is compact and $f$ is continuous on $D$ so that the global minimum of $f$ over $D$ exists.

Note that the second problem addressed in the title of the paper can be included under problem (1), since

$$\text{global min} f_0(x) = \sum_{k=1}^{n} \left(\frac{1}{2} p_{0k} x_k^2 + q_{0k} x_k + r_{0k}\right)$$

$$\text{s.t. } x \in D$$

is equivalent to

$$\text{global min } t$$

$$\text{s.t. } x \in D, \; f_0(x) \le t$$

with the additional variable $t \in R$.

Problems of the form (1) are very difficult to solve. Sahni [21], proved that the problem (with a linear objective function and quadratic constraints) is NP-hard. Interest in this problem is apparent from its practical applications as well as from the theoretical point of view. Many practical problems that can be formulated as in (1), include location problems, production planning, and minimization of chance-constrained risks [18]. Recently, it has been shown that some VLSI chip design problems lead to the formulation of quadratic programs with quadratic constraints [11].

In this paper, we propose different methods for the solution of (1). From the computational point of view we are interested in algorithms that determine an $\varepsilon$-approximate solution. First we develop a branch and bound algorithm which is based on rectangular partitions to obtain upper and lower bounds on the global minimum. Next, we use piecewise linear approximations of the constraint functions. When all $g_i(x)$ are convex we reduce the approximate piecewise linear program to concave minimization subject to convex constraints. For nonconvex constraints we use zero-one integer variables to linearize the piecewise linear functions that approximate the concave parts of the constraint functions.

There is little work done on nonconvex problems with quadratic constraints. In [4], a dual method is developed for minimizing a convex quadratic function subject to convex quadratic constraints. This method may be used to obtain approximate solutions of (1), when a convex approximation of the objective function can be computed. Problems with a linear objective function and quadratic constraints have been considered by Kabe [10]. Reeves [20] proposed a branch and bound type algorithm for (1), where the objective function is separable. This algorithm is based on locating local minima using linear programming techniques (see also [12]). Other methods for related problems are given in [2], [3], [6], [13], [22] and [23].

**Branch and bound algorithm**

The algorithm proposed for solving problem (1) will be a Branch-and Bound (BB) procedure consisting of

- more and more refined rectangular partitions of the initial rectangle $M_0$,
- lower bounds $\beta(M) \le \min f(M)$ associated with each partition element (rectangle) $M$ generated by the procedure,
- upper bounds $\alpha_r \ge \min f(D)$ determined in each step $r$ of the algorithm,
- certain deletion rules by which some partition sets $M$ are deleted since we know that $M \cap D = \emptyset$ or that $\min f(D)$ cannot be attained in M.

We briefly discuss the four elements of the procedure before presenting a complete statement of the algorithm.

**Subdivision of rectangles**

**Definion 1:** Let $M \subset R^n$ be an $n$-dimensional rectangle ($n$-rectangle) and let $I$ be a finite set of indices. A set $\{M_i : i \in I\}$ of $n$-rectangles $M_i \subset M$ is said to be a rectangular partition of $M$ if we have

$$M = \bigcup_{i \in I} M_i, \quad M_i \cap M_j = \partial M_i \cap \partial M_j \quad \text{for all} \quad i,j \in I,\ i \ne j, \tag{2}$$

where $\partial M_i$ denotes the boundary of $M_i$.

**Definition 2:** Let $\{M_q\}$ be an infinite decreasing (nested) sequence of rectangular partition sets generated by the algorithm. The underlying subdivision procedure of rectangles is called exhaustive if the sequence of diameters $d(M_q)$ of $M_q$ satisfies

$$\lim_{q \to \infty} d(M_q) = 0. \tag{3}$$

Note that an $n$-rectangle $M=\{x : a \leq x \leq b\}$, $a,b \in R^n$, $a<b$, is uniquely determined by its "lower left" vertex $a$, and its "upper right" vertex $b$. Each of the $2^n$ vertices of $M$ is of the form

$$a + l$$

where $l$ is a vector with components 0 or $(b_i - a_i)$ $(i=1, \ldots, n)$, and for the diameter $d(M)$ of $M$ we have

$$d(M) = \| b-a \|,$$

where $\| . \|$ denotes the Euclidean norm in $R^n$.

A very simple frequently used subdivision, called bisection (e.g. Horst [8]), consists of subdividing an $n$-rectangle $M=\{x : a \leq x \leq b\}$ into two $n$-rectangles by a cutting hyperplane through $(a+b)/2$, perpendicular to the longest edge of $M$. It is well known that bisection is exhaustive in the sense of Definition 2.

**Lower and upper bounds**

Let $M$ be an $n$-rectangle and denote by $V(M)$ the vertex set of $M$. Then it is well known that, by concavity of $f$, we have $\min f(M) = \min f(V(M))$ and for the lower bound $\beta(M)$ we set

$$\beta(M) = \min f(V(M)). \tag{4}$$

Obviously, we have

$$\beta(M) \leq \min f(D \cap M)$$

whenever $D \cap M \neq \emptyset$.

For the upper bounds $\alpha_r \geq \min f(D)$ in step $r$ of the algorithm, we always choose

$$\alpha_r = \min f(S_r), \tag{5}$$

where $S_r$ is the set of feasible points calculated until step $r$, that is, we have

$$\alpha_r = f(x^r), \tag{6}$$

where $x^r$ is the best feasible point determined so far. If feasible points are not available, i.e. $S_r = \emptyset$, we set $\alpha_r = \infty$.

**Deletion**

A partition set $M$ is deleted of course whenever we have $\beta(M)>\alpha_s$ for some iteration step $s$, because then, clearly, the global minimum of $D$ cannot be attained in $M$.

A more difficult question is that of properly deleting infeasible partition sets $M$, i.e., sets satisfying $M\cap D=\emptyset$. Usually $M$ is known by its vertex set $V(M)$ and $V(M)\cap D=\emptyset$ does not imply $M\cap D=\emptyset$. Therefore, from the information at hand, we will not be able to delete all partition sets $M$ satisfying $M\cap D=\emptyset$, and we have to apply a "deletion by infeasibilty" rule that is "certain in the limit" in the sense of fathoming "enough" infeasible sets to guarantee convergence of the algorithm to a global minimum of $f$ on $D$.

In order to derive such a deleting rule, note that each constraint function $g_i$ is obviously Lipschitzian on $M$, that is, there is a constant $L_i=L_i(M)>0$ such that we have

$$|g_i(z)-g_i(x)| \le L_i \|z-x\|, \text{ for all } x, z\in M. \tag{7}$$

An upper bound for $L_i$ is given by any number $A_i$ satisfying

$$A_i \ge \max\{\|\nabla g_i(y)\| : y\in M\}. \tag{8}$$

Let $M=\{x: a_k \le x_k \le b_k,\ k=1,\ldots,n\}$. Using monotonicity and separability, we see that

$$A_i = A_i^* = \max\{\|\nabla g_i(y)\| : y\in M\} \tag{9}$$

is given by

$$A_i^* = [\sum_{k=1}^{n}(\max_{a_k\le y_k\le b_k} |p_{ik}y_k+q_{ik}|)^2]^{1/2} = [\sum_{k\in N_i^1}(p_{ik}a_k+q_{ik})^2 + \sum_{k\in N_i^2}(p_{ik}b_k+q_{ik})^2]^{1/2} \tag{10}$$

where

$$N_i^1 = \{k: -\frac{q_{ik}}{p_{ik}} \ge \frac{a_k+b_k}{2}\},\quad N_i^2=\{k: -\frac{q_{ik}}{p_{ik}} < \frac{a_k+b_k}{2}\}.$$

Let $V'(M)$ be any nonempty subset of the vertex set $V(M)$; for example $V'(M)=\{a,b\}$ if $M=\{x: a\le x\le b\}$. Denote again by $d(M)=\|a-b\|$ the diameter of $M$. Then we propose the following:

**Deletion Rule (DR):** Delete a partition set $M$ whenever there is an $i\in\{1,\ldots,m\}$ satisfying

$$\max\{g_i(x): x\in V'(M)\} - A_i d(M) > 0 \tag{11}$$

where $A_i\ge L_i$ (see (8), (9) and (10)).

**Lemma 1:** Let the subdivision procedure be exhaustive and apply the Deletion Rule (DR). Then every infinite decreasing sequence $\{M_q\}$ of partition sets generated by the algorithm satisfies

$$M_q \underset{q\to\infty}{\to} \{\bar{x}\}, \ \bar{x}\in D.$$

**Proof:** The definition of an exhaustive subdivision implies that there is a point $\bar{x}$ satisfying $M_q \underset{q\to\infty}{\to} \{\bar{x}\}$. We have to show that $\bar{x}\in D$. Let $A_i(M_q)$ be an overestimator of the Lipschitzian $L_i(M_q)$ of $g_i$ on $M_q$. Since $\{M_q\}$ is a decreasing sequence, we may assume that $A_i(M_{q+1}) \le A_i(M_q)$ holds and that

$$A \ge A_i(M_q), \ \text{for all } \ q, i=1, \ldots, m \tag{12}$$

is at hand.

Apply deletion rule (DR) and suppose that we have $\bar{x}\notin D$. Since $M_q \to \{\bar{x}\}$, by continuity of $g_i$ $(i=1, \ldots, m)$ and by (12), it follows that, for each sequence of nonempty sets $V'(M_q)\subset V(M_q)$, we have

$$\max \{g_i(x):x\in V'(M_q)\} - A_i(M_q)d(M_q) \underset{q\to\infty}{\to} g_i(\bar{x}), \ i=1, \ldots, m \tag{13}$$

As $\bar{x}\notin D$, there is at least one $i\in\{1, \ldots, m\}$ satisfying $g_i(\bar{x})>0$. Taking into account boundedness of $\{A_i(M_q)\}$, $d(M_q)\to 0$ and continuity of $g_i$, we see from (13) that there is a positive integer $q_0$ such that

$$\max \{g_i(x):x\in V'(M_q)\} - A_i(M_q)d(M_q) > 0, \ \text{for all } \ q\ge q_0$$

holds. This is a contradiction to the deletion rule (DR).

**Remark:** Note that, by $A_i \ge L_i$, we have $g_i(z) \ge g_i(x) - L_i \| z-x \| \ge g_i(x) - A_i d(M)$ for all $x,z\in M$ and (11) implies $g_i(z) > 0$ for all $z\in M$, hence deletion rule (DR) does, in fact, only fathom infeasible sets $M$.

**The algorithm**

**Step 0.**

Let $M_0=M$, choose $S_{M_0}\subset D$ (possibly empty) and determine $\beta(M_0)=\min f(V(M))$, $\alpha_0=\min f(S_{M_0})$ ($\alpha_0=\infty$ if $S_{M_0}=\emptyset$). Let $I_0=\{M_0\}$, and $\beta_0=\beta(M_0)$. If $\alpha_0<\infty$ then choose $x^0\in \mathit{argmin} f(S_{M_0})$ (i.e. $f(x^0)=\alpha_0$). If $\alpha_0-\beta_0=0$ ($\le\varepsilon>0$ for practical purposes) then stop; $\alpha_0=\beta_0=\min f(D)$ ($\alpha_0-\beta_0\le\varepsilon$), $x^0$ is an $\varepsilon$-approximate solution. Otherwise set $r=1$ and go to step r.

**Step r.**

At the the begining of step $r$ we have the current rectangular partition $I_{r-1}$ of a subset of $M_0$ still under consideration. Furthermore, for every $M\in I_{r-1}$ we have $S_M\subset M\cap D$ and bounds

$\beta(M)$, $\alpha(M)$ satisfying

$$\beta(M) = \min f(M) \le \alpha(M).$$

Moreover, we have the current lower and upper bounds $\beta_{r-1}$, $\alpha_{r-1}$ satisfying

$$\beta_{r-1} \le \min f(D) \le \alpha_{r-1}.$$

Finally, if $\alpha_{r-1}<\infty$, then we have a point $x^{r-1}\in D$ satisfying $f(x^{r-1})=\alpha_{r-1}$ (the best feasible point obtained so far).

r.1 Delete all $M\in I_{r-1}$ satisfying $\beta(M)\ge\alpha_{r-1}$. Let $R_r$ be the collection of the remaining rectangles in the partition $I_{r-1}$.

r.2 Select a nonempty collection of sets $P_r\subset R_r$ satisfying

$$\mathit{argmin}\ \{\beta(M): M\in I_{r-1}\}\subset P_r \tag{14}$$

and subdivide every member of $P_r$ by bisection (or any other exhaustive subdivision yielding rectangular partitions). Let $P_r'$ be the collection of all new partition elements.

r.3 Fathom any $M\in P_r'$ satisfying deletion rule (DR). Let $I_r'$ be the collection of all remaining members of $P_r'$.

r.4 Assign to each $M\in I_r'$ the set $S_M\subset M\cap D$ of feasible points in $M$ known so far and

$$\beta(M) = \min f(V(M)),\quad \alpha(M) = \min f(S_M)\quad (\alpha(M)=\infty \text{ if } S_M=\varnothing).$$

r.5 Set $I_r = (R_r - P_r)\cup I_r'$. Compute
$\alpha_r = \inf\{\alpha(M): M\in I_r\}$,
$\beta_r = \min\{\beta(M): M\in I_r\}$,
If $\alpha_r<\infty$, then let $x^r\in D$ such that $f(x^r)=\alpha_r$.

r.6 If $\alpha_r-\beta_r=0$ $(\le\varepsilon)$, then stop; $x^r$ is an $\varepsilon$-approximate solution. Otherwise go to step $r+1$.

The convergence of the algorithm is based on the following theorem.

**Theorem 1:** (i) The sequence of lower bounds $\beta_r$ satisfies

$$\beta := \lim_{r\to\infty} \beta_r = \min f(D).$$

(ii) Assume that we have $S_M\neq\varnothing$ for all partition sets $M$. Then

$$\beta = \lim_{r\to\infty} \beta_r = \min f(D) = \lim_{r\to\infty} \alpha_r =: \alpha$$

holds and every accumulation point of the sequence $\{x^r\}$ solves problem (1).

**Proof:** (i) Using a standard argument on finiteness of each partition $I_r$ (e.g. Horst [8], Horst and Tuy [9]), we see by the selection rule (14) that there must be a decreasing sequence $\{M_q\}$ of successively refined partition sets satisfying

$$\beta(M_q) = \beta_q. \tag{15}$$

Clearly, by construction, $\{\beta_r\}$ is a nondecreasing sequence bounded from above by $\min f(D)$ such that

$$\beta := \lim_{r \to \infty} \beta_r \le \min f(D) \tag{16}$$

exists.

From Lemma 1 we see that $\lim_{q \to \infty} M_q = \{\bar{x}\}$ and $\bar{x} \in D$ hold. But $\beta(M_q) = \min f(V(M_q)) = f(v^q)$ where $v^q \in V(M_q) \subset M_q$, hence continuity of $f$ implies that we have

$$\lim_{q \to \infty} \beta(M_q) = f(\bar{x}). \tag{17}$$

Combining (15), (16), (17) yields

$$\beta = f(\bar{x}) \le \min f(D)$$

which is only possible if $f(\bar{x}) = \min f(D)$ since $\bar{x} \in D$.

(ii) Whenever $S_M \neq \emptyset$, i.e., $\alpha(M) < \infty$ for all partition elements $M$, then the algorithm can be regarded as a realization of the general (BB)- scheme presented in Horst [8] or in Horst and Tuy [9] and (ii) follows from the theory developed in these papers if we show that the bounding operation is "consistent", i.e., if $\alpha(M_q) - \beta(M_q) \to 0$ as $q \to \infty$, for every decreasing sequence $\{M_q\}$.

We have already seen that $M_q \underset{q \to \infty}{\to} \{\bar{x}\}$, $\bar{x} \in D$, and $\beta(M_q) \underset{q \to \infty}{\to} f(\bar{x})$ hold. But $\alpha(M_q) = \min f(S_{M_q})$, $\emptyset \neq S_{M_q} \subset M_q$, and continuity of $f$ imply that likewise we have $\alpha(M_q) \underset{q \to \infty}{\to} f(\bar{x})$.

**Remark:** For problem (1) it may be difficult to obtain enough feasible points such that $S_M \neq \emptyset$ for all partition elements $M$. In that case, we propose to consider the iteration sequence $\{\tilde{x}^r\}$ defined by $f(\tilde{x}^r) = \beta_r$. Note that $\tilde{x}^r$ is not necessarily feasible for problem (1). However, it can be proved in a similar way as above that every accumulation point of $\{\tilde{x}^r\}$ solves problem (1).

**Computational considerations and acceleration of convergence**

The general algorithm was shown to converge using only simple calculations to make decisions on partitioning, deleting and bounding. A closer examination presented here leads to procedures that allow improved bounding, at the expense of solving additional subproblems to extract the information needed for these decisions.

Let us assume that $M$ has not been deleted by the deletion rule (11). We may still have $M \cap D = \emptyset$. Currently, the algorithm assigns a lower bound on the minimum of $f$ over $M \cap D$ as

$$\beta(M) = \min f(M) = \min f(V(M)) \leq f(M \cap D).$$

Since, from **Step r.1**, $M$ is deleted whenever $\beta(M) \geq \alpha_{r-1}$, a tighter lower bound would invoke this test sooner. Assuming $M \cap D \neq \emptyset$, a mechanism needs to be devised for identifying points in $S_M \subset M \cap D$. Otherwise, as mentioned earlier, it is conceivable for $S_M = \emptyset$ for all $M \in I_{r-1}$ (in this case, $\alpha_{r-1} = \infty$). These two situations shall be addressed next. In particular, we shall attempt to obtain a better bound than $\beta(M)$ by using only linear programming calculations. In the process, we will be able to identify, at times, when $M \cap D = \emptyset$ or possibly even uncover feasible points of $M \cap D \neq \emptyset$ for inclusion in $S_M$.

We begin by proposing a simple linearization of the constraints in

$$G = \{x : g_i(x) = \sum_{k=1}^{n} (\frac{1}{2} p_{ik} x_k^2 + q_{ik} x_k + r_{ik}) \leq 0,\ i=1, \ldots, m\}.$$

Note that $M \cap D = M \cap G$ since $M \subset M_0$ and $D = M_0 \cap G$. Let $g_{ik}(x_k) = \frac{1}{2} p_{ik} x_k^2 + q_{ik} x_k + r_{ik}$ and let $Vex_A f$ denote the convex envelope of $f$ over $A$. Then we have [5]

$$Vex_M g_i(x) = \sum_{k=1}^{n} Vex_{\mu_k} g_{ik}(x_k) \leq g_i(x) \tag{18}$$

where

$$\mu_k = \{x_k : a_k \leq x_k \leq b_k\},\ k=1, \ldots, n.$$

Each $g_{ik}(x_k)$ is linearized by one of the following three cases:

**Case 1:** $p_{ik} = 0$. Then $g_{ik}(x_k)$ is linear

**Case 2:** $p_{ik} < 0$. Then $g_{ik}(x_k)$ is concave and $Vex_{\mu_k} g_{ik}(x_k) = \alpha_{ik} x_k + \beta_{ik}$ where

$$\alpha_{ik} = [g_{ik}(b_k) - g_{ik}(a_k)]/(b_k - a_k),\ \ \beta_{ik} = g_{ik}(b_k) - \alpha_{ik} b_k = g_{ik}(a_k) - \alpha_{ik} a_k.$$

Replace $g_{ik}(x_k)$ by $l_{ik}(x_k) = \alpha_{ik} x_k + \beta_{ik}$.

**Case 3:** $p_{ik} > 0$. Then $g_{ik}$ is convex. Compute $\hat{x}_k = -q_{ik}/p_{ik}$ which minimizes $g_{ik}$. If $g_{ik}(\hat{x}_k) > 0$, replace $g_{ik}(x_k)$ by the constant $g_{ik}(\hat{x}_k)$ in the constraint $g_i(x) \le 0$. (This effectively enlarges the region feasible to that constraint). Otherwise, continue.

**Case 3a:** If $\hat{x}_k < a_k$, compute

$$\rho_{ik}^1 = \frac{1}{p_{ik}}[-q_{ik} + (q_{ik}{}^2 - 2p_{ik}r_{ik})^{1/2}].$$

Replace $g_{ik}(x_k)$ by the linear support of its graph at $\rho_{ik}^1$. This is given by

$$l_{ik}(x_k) = \alpha_{ik}x_k + \beta_{ik} \le g_{ik}(x_k),$$

where $\alpha_{ik} = p_{ik}\rho_{ik}^1 + q_{ik} > 0$ and $\beta_{ik} = -\alpha_{ik}\rho_{ik}^1$.

**Case 3b:** If $\hat{x}_k > b_k$, replace $g_{ik}(x_k)$ by the linear support of its graph at

$$\rho_{ik}^2 = \frac{1}{p_{ik}}[-q_{ik} - (q_{ik}^2 - 2p_{ik}r_{ik})^{1/2}];$$

namely,

$$l_{ik}(x_k) = \alpha_{ik}x_k + \beta_{ik} \le g_{ik}(x_k),$$

where $\alpha_{ik} = p_{ik}\rho_{ik}^2 + q_{ik} < 0$ and $\beta_{ik} = -\alpha_{ik}\rho_{ik}^2$.

**Case 3c:** If $a_k \le \hat{x}_k \le b_k$, compute $\rho_{ik}^1$ and $\rho_{ik}^2$, as above, and replace $g_{ik}(x_k)$ by the maximum of the supports at $\rho_{ik}^1$ and $\rho_{ik}^2$,

$$\max \{l_{ik}^1(x_k), l_{ik}^2(x_k)\}$$

where

$$l_{ik}^j(x_k) = \alpha_{ik}^j x_k + \beta_{ik}^j, \ \ j = 1, 2$$

$$\alpha_{ik}^j = p_{ik}\rho_{ik}^j + q_{ik}, \ \ j = 1, 2$$

$$\beta_{ik}^j = -\alpha_{ik}^j\rho_{ik}^j, \ \ j = 1, 2.$$

Let $t_i$ be the number of terms in $g_i(x)$ that fall into Case 3c. That is, $t_i$ is the cardinality of the index set

$$K_i = \{k : p_{ik} > 0, \ g_{ik}(\hat{x}_k) \le 0, \ a_k \le \hat{x}_k \le b_k, \ \ k=1, \ldots, n\}.$$

Using the above linearization, for each constraint $i$ we have

$$l_i(x) := \sum_{k \notin K_i} l_{ik}(x_k) + \sum_{k \in K_i} \max \{l_{ik}^1(x_k), l_{ik}^2(x_k)\} \le g_i(x),$$

where we take $l_{ik}(x_k) = g_{ik}(\hat{x}_k)$ in the case $p_{ik} > 0$ and $g_{ik}(\hat{x}_k) > 0$. Now, we have $l_i(x)$ is a linear underestimate of $g_i(x)$ when $K_i = \emptyset$, and it is piecewise linear and convex, otherwise.

In particular, the region defined by $l_i(x) \le 0$ is equivalent to the polyhedral set

$$\sum_{k \notin K_i} l_{ik}(x_k) + \sum_{k \in K_i} z_{ik} \le 0$$

$$l_{ik}^1(x_k) \le z_{ik}, \quad k \in K_i$$

$$l_{ik}^2(x_k) \le z_{ik}, \quad k \in K_i,$$

which involves $t_i$ new variables $(z_{i1}, \ldots, z_{it_i})$ and $2t_i$ additional constraints.

Performing the above linearization for every constraint $i$, let $L(G)$ denote the resultant polyhedral set and let $L_x(G)$ denote its projection onto the $n$-dimensional space of $x$. It is clear that

$$L_x(G) \supset conv(G) \supset G$$

where $conv(G)$ denotes the convex hull of $G$. Also, it is clear that if $(\bar{x}, \bar{z})$ minimizes $f(x)$ over $L(G) \cap M$, then $\bar{x}$ minimizes $f(x)$ over $L_x(G) \cap M$. If $L_x(G) \cap M = \emptyset$, then $G \cap M = M \cap D = \emptyset$.

Since $f$ is concave (with no assumed exploitable structure), too much effort is required to find the global minimizer $\bar{x}$. Instead, we seek an $\hat{x} \in L_x(G) \cap M$ that satisfies $f(\hat{x}) \le f(\bar{x})$. Consider the problem

$$\min f(x) \tag{19}$$

$$\text{s.t. } (x, z) \in L(G) \cap M.$$

Since the feasible set is compact, the algorithm of Falk and Hoffman [7] applies. But we only run the algorithm until the best lower bound occurs at a point $(\hat{x}, \hat{z})$ that satisfies $\hat{x} \in L_x(G) \cap M$. Their method guarantees that $f(\hat{x}) \le f(\bar{x})$, as desired. Thus, we obtain

$$\beta(M) = \min f(M) \le f(\hat{x}) \le \min f(M \cap D),$$

thereby achieving a higher lower bound than $\beta(M)$. If $\hat{x} \in G$, then $S_M \neq \emptyset$. Otherwise, schemes can be developed to find at least one point in $L_x(G) \cap G$, starting from $\hat{x}$, and these points are added to the set $S_M$.

**Bilinear constraint approach**

An alternative linearization of $G$ can be derived from the bilinear constraints that arise from the quadratic constraints in the following way. Each constraint function can be written as

$$g_i(x) = \sum_{k=1}^{n} \left(\frac{1}{2} p_{ik} x_k^2 + q_{ik} x_k + r_{ik}\right) = \sum_{k=1}^{n} (x_k y_{ik} + r_{ik})$$

where $y_{ik} = \frac{1}{2}p_{ik}x_k + q_{ik}$ for every $i$ and all $k$. We also have from Theorems 2 and 3 of Al-Khayyal and Falk [1]

$$Vex_{\Omega_i} g_i(x) = \sum_{k=1}^{n} [Vex_{\Omega_{ik}}(x_k y_{ik}) + r_{ik}] \tag{20}$$

where

$$\Omega_i = \Omega_{i1} \times \Omega_{i2} \times \cdots \times \Omega_{in}$$

$$\Omega_{ik} = \{(x_k, y_{ik}) \colon a_k \le x_k \le b_k,\ c_{ik} \le y_{ik} \le d_{ik}\}$$

$$Vex_{\Omega_{ik}}(x_k y_{ik}) = \max\{l_{ik}^1(x_k, y_{ik}),\ l_{ik}^2(x_k, y_{ik})\}$$

$$l_{ik}^1(x_k, y_{ik}) = c_{ik}x_k + a_k y_{ik} - a_k c_{ik}$$

$$l_{ik}^2(x_k, y_{ik}) = d_{ik}x_k + b_k y_{ik} - b_k d_{ik}$$

$$c_{ik} = \frac{1}{2}\min\{p_{ik}a_k,\ p_{ik}b_k\} + q_{ik}$$

$$d_{ik} = \frac{1}{2}\max\{p_{ik}a_k,\ p_{ik}b_k\} + q_{ik}.$$

Let $y_i = (y_{i1}, \ldots, y_{in})^T$. Since $Vex_{\Omega_i} g_i(x)$ is piecewise linear and convex, the feasible region defined by

$$Vex_{\Omega_i} g_i(x) \le 0$$

$$y_{ik} = \frac{1}{2}p_{ik}x_k + q_{ik},\ \ k=1, \ldots, n$$

$$(x, y_i) \in \Omega_i,$$

is equivalent to the polyhedral region involving $n$ additional variables $(z_{i1}, \ldots, z_{in})$ and $2n$ additional linear inequality constraints:

$$\sum_{k=1}^{n} z_{ik} \le 0$$

$$y_{ik} = \frac{1}{2}p_{ik}x_k + q_{ik},\ k=1, \ldots, n$$

$$l_{ik}^1(x_k, y_{ik}) + r_{ik} \le z_{ik},\ k=1, \ldots, n$$

$$l_{ik}^2(x_k, y_{ik}) + r_{ik} \le z_{ik},\ k=1, \ldots, n$$

$$(x, y_i) \in \Omega_i.$$

This is done for each constraint $i=1, \ldots, m$ to yield a polyhedral set $P(G)$ whose projection onto $R^n$ (the space of $x$-variables), denoted by $P_x(G)$ contains the convex hull of $G$; that is, we have

$$G \subseteq conv(G) \subseteq P_x(G).$$

Hence,

$$\min\{f(x): (x, y, z) \in P(G) \cap M\} \le \min\{f(x): x \in G \cap M\}.$$

The discussion at the end of the preceding section may now be followed with the set $L(G)$ replaced by $P(G)$ and the vector $(x, z)$ replaced by $(x, y, z)$.

**Piecewise linear approximation**

In [17], [19], a computational algorithm is proposed for the solution of separable concave problems subject to linear constraints. In that paper an approximate piecewise linear program is considered that provides an $\varepsilon$-approximate solution for any given tolerance $\varepsilon > 0$. The concave piecewise linear program is solved by using an equivalent mixed zero-one integer linear programming formulation (see also [14]) having a simple structure. The same techniques have been extended for problems with an indefinite quadratic objective function [15].

For problem (1) we assume that the objective function $f(x)$ is continuous. For each interval $[\underline{m}_k, \overline{m}_k]$ choose a fixed grid of points by partitioning it into $n_k$ subintervals of length

$$h_k = (\overline{m}_k - \underline{m}_k)/n_k. \tag{21}$$

Let $\gamma_i(x) = \sum_{k=1}^{n} \gamma_{ik}(x_k)$ be the piecewise linear function that approximates $g_i(x) = \sum_{k=1}^{n} g_{ik}(x_k)$. Each piecewise linear function $\gamma_{ik}(x_k)$ interpolates $g_{ik}(x_k)$ at the grid points $x_k{}^j = \underline{m}_k + jh_k$ for $j=0, \ldots, n_k$. We may assume that $n_k > 1$, since for $n_k = 1$ we obtain a linear function interpolating each $g_{ik}(x_k)$ at the endpoints $\underline{m}_k$ and $\overline{m}_k$. Then the separable piecewise linear program

$$\text{global min } f(x)$$

$$\text{s.t. } \gamma_i(x) = \sum_{k=1}^{n} \gamma_{ik}(x_k) \le 0, \quad i=1, \ldots, m \tag{22}$$

$$\underline{m}_k \le x_k \le \overline{m}_k, \quad k=1, \ldots, n,$$

will provide an approximate solution to the original problem (1). The approximation error depends on the mesh sizes $h_k$, $k=1, \ldots, n$, and the curvature of the objective and constraint functions.

Piecewise linear interpolation preserves monotonicity ,convexity and concavity. Next we prove that when the constraint functions $g_i(x)$ are convex then the piecewise linear approximate problem (22) is reduced to an equivalent one with linear constraints.

**Convex constraints**

If $g(x)$ is a piecewise linear convex function, then $g(x)$ can be defined as the pointwise maximum of $k$ affine functions $l_1(x), \ldots, l_k(x)$, that is

$$g(x) = \max\ \{l_1(x), \ldots, l_k(x)\}. \tag{23}$$

Moreover, given the set

$$S = \{x: \sum_{j=1}^{n} \max\{l_1{}^j(x), \ldots, l_k{}^j(x)\} \le 0\}$$

we associate the following set (with $n$ additional variables):

$$\bar{S} = \{(x,y): y_j \ge l_i{}^j(x),\ \ i=1, \ldots, k,\ j=1, \ldots, n \text{ and } \sum_{j=1}^{n} y_j \le 0\}.$$

Then it is easy to see that if $(x,y)$ solves the problem $\min\{f(x): (x,y)\in\bar{S}\}$, then $x$ solves $\min\{f(x): x\in S\}$.

Let $l_{ik}{}^j(x_k)$ be the affine function that coincides with $\gamma_{ik}(x_k)$ in the interval $[x_k{}^{j-1}\ x_k{}^j]$, where $x_k{}^j = \underline{m}_k + jh_k$, $j=1, \ldots, n_k$. Then by (23)

$$\gamma_i(x) = \sum_{k=1}^{n} \gamma_{ik}(x_{ik}) = \sum_{k=1}^{n} \max\ \{l_{ik}{}^1(x_k), \ldots, l_{ik}{}^{n_k}(x_k)\},\ i=1, \ldots, m,$$

where

$$l_{ik}{}^j = (\gamma_{ik}(x_k{}^j) - \gamma_{ik}(x_k{}^{j-1}))(x_k - \underline{m}_k - (j-1)h_k)/h_k + \gamma_{ik}(x_k{}^j),\ \ j=1, \ldots, n_k.$$

Hence, problem (22) can be solved by solving the following problem under linear constraints:

$$\text{global min } f(x)$$

$$\text{s.t. } \sum_{k=1}^{n} y_{ik} + r_{ik} \le 0,\ i=1, \ldots, m \tag{24}$$

$$y_{ik} \ge l_{ik}{}^j(x_k),\ \ k=1, \ldots, n,\ j=1, \ldots, n_k,\ i=1, \ldots, m$$

$$\underline{m}_k \le x_k \le \overline{m}_k,\ \ k=1, \ldots, n$$

If $(x^*, y^*)$ is a solution of problem (24), then $x^*$ solves problem (22). The number of additional variables $y_{ik}$ introduced is independent of the number of partitions used and is equal to

$$Y = mn.$$

The number of additional constraints depends on the partition set and is given by

$$C = mn \sum_{k=1}^{n} n_k.$$

When $f(x)$ is a concave quadratic function problem (21) is a global concave minimization under linear constraints. A method such as the one described in [17] can be used to obtain a guranteed $\varepsilon$-approximate solution.

If the objective function is indefinite quadratic, we have the same approach since indefinite quadratic programming is reduced to concave programming with convex constraints. This is true because if $f(x)$ is indefinite, then $f(x) = f_1(x) + f_2(x)$ where $f_1(x)$ is concave and $f_2(x)$ is convex. Then any problem of the form

$$\text{global} \min_{x \in S} f(x) = f_1(x) + f_2(x)$$

is equivalent to the concave minimization problem

$$\text{global min } f_1(x)$$

$$\text{s.t. } x \in S,\ f_2(x) \le y$$

with $y$ an additional real variable.

**Nonconvex constraints**

When some of the constraint functions $g_i$'s are nonconvex the separable piecewise linear program is more difficult to solve. It is of interest to note that problems with integer 0–1 restrictions can be formulated as problems with concave (reverse convex) constraints. For instance, the constraint $x_k \in \{0, 1\}$ can be written as

$$-x_k^2 + x_k \le 0, \ \ 0 \le x_k \le 1.$$

For problems with nonconvex constraints we obtain an equivalent formulation with linear constraints and zero-one integer variables of special structure. To illustrate the idea behind this formulation consider one of the functions $\gamma_{ik}(x_k)$ which is assumed to be concave. Let

$$x_k = \underline{m}_k + \sum_{j=1}^{n_k} h_k \omega_j^{\ k} \tag{25}$$

where

$$0 \le \omega_j^{\ k} \le 1,\ j=1, \ldots, n_k$$

$$\omega_{j+1}^{\ k} \le z_j^{\ k} \le \omega_{j+1}^{\ k},\ \ z_j^{\ k} \in \{0,1\},\ \ j=1, \ldots, n_k-1. \tag{26}$$

Under this transformation we obtain a linearization of the piecewise linear concave function $\gamma_{ik}(x_k)$ by setting

$$\gamma_{ik}(x_k) = \gamma_{ik}(\underline{m}_k) + \sum_{j=1}^{n_k} \omega_j^{\ k}(\gamma_{ik}(\underline{m}_k + jh_k) - \gamma_{ik}(\underline{m}_k + (j-1)h_k)). \tag{27}$$

For each function $\gamma_{ik}$ we introduce $n_k - 1$ zero-one integer variables. However, because of (26) we must have

$$z^k = (z_1{}^k, \ldots, z_{n_k-1}{}^k) = (1, 1, \ldots, 1, 0, 0, \ldots, 0). \tag{28}$$

This property of zero-one variables makes the integer problem easier to solve, since $z^k$ takes only $n_k$ possible values, instead of $2^{n_k-1}$ values for the general case when (28) does not hold.

Applying the above arguments for all constraint functions we have that, in case of concave functions, problem (22) is eqivalent to the following problem:

$$\text{global min } f(\omega)$$

$$\sum_{k=1}^{n} [\ \gamma_{ik}(\underline{m}_k) + \sum_{j=1}^{n_k} \omega_j{}^k(\gamma_{ik}(\underline{m}_k + jh_k) - \gamma_{ik}(\underline{m}_k + (j-1)h_k))\ ] + r_{ik} \le 0,\ \ i=1, \ldots, m$$

$$0 \le \omega_j{}^k \le 1,\ \ k=1, \ldots, n,\ j=1, \ldots, n_k \tag{29}$$

$$\omega_{j+1}{}^k \le z_j{}^k \le \omega_{j+1}{}^k,\ \ z_j{}^k \in \{0,1\},\ \ k=1, \ldots, n\ \ j=1, \ldots, n_k-1$$

$$0 \le \sum_{j=1}^{n_k} h_k \omega_j{}^k \le \overline{m}_k - \underline{m}_k,\ \ k=1, \ldots, n.$$

If $g_{ik}(x_{ik})$ is convex we may use the same transformation as in (25), (26) and (27). However, in the presence of convexity it is easy to see (because of the monotone increasing slopes) that we do not need to use integer variables. Hence it is clear that an all-quadratic separable problem can be solved by solving an equivalent concave zero-one program with linear constraints.

## Concluding remarks

This paper presents different methods for the global minimization of concave functions subject to quadratic separable constraints. The first method is a branch and bound algorithm based on rectangular partitions to obtain upper and lower bounds of the global minimum. Various techniques that improve branching and bounding have been considered.

We also discussed bilinear programming techniques to approximate the feasible domain using convex envelopes. From the practical point of view, an approximate solution is desired. An alternative method for computing an ε-approximate solution is given based on the use of piecewise linear approximations of the constraint functions. For convex constraints the approximate problem can be formulated as a concave minimization problem subject to linear constraints. For nonconvex constraints the resulting problem has linear and integer 0–1 constraints.

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