

Global Regular Solutions for The Nonhomogeneous Carrier Equation

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We study in a $n + 1$ -dimensional cylinder Q global solvability of the mixed problem for the nonhomogeneous Carrier equation

$$u_{tt} - M(x, t, \|u(t)\|^2)\Delta u + g(x, t, u_t) = f(x, t)$$

without restrictions on a size of initial data and $f(x, t)$. For any natural n , we prove existence, uniqueness and the exponential decay of the energy for global generalized solutions. When $n = 2$, we prove $C^\infty(Q)$ -regularity of solutions.

Key words: Carrier equation; Global smooth solutions; Existence

1 INTRODUCTION

G. Carrier in [2] derived the equation which models vibrations of an elastic string when changes in tension are not small

$$\rho u_{tt} - \left(1 + \frac{EA}{LT_0} \int_0^L u^2 dx\right) u_{xx} = 0. \quad (1.1)$$

Here $u(x, t)$ is the x -derivative of the deformation, T_0 is the tension in the rest position, E is the Young's modulus, A is the cross-section of a string, L is the length of a string and ρ is the density of a material. Clearly, if properties of a material vary with x and t , then we have a hyperbolic quasilinear equation of the type

$$u_{tt} - M(x, t, \|u(t)\|^2)\Delta u = 0.$$

Moreover, a material of a string has internal friction that can be a nonlinear function of u_t and can also depend on x, t . In the present paper, we study the existence of global solutions for the nonhomogeneous Carrier equation,

$$u_{tt} - M(x, t, \|u(t)\|^2)\Delta u + g(x, t, u_t) = f(x, t). \quad (1.2)$$

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We must say that dependence of $M(x, t, \|u(t)\|^2)$ on x, t brings essential difficulties in proving global a priori estimates; and there are not published results on the existence of global solutions for the nonhomogeneous Kirchhoff-Carrier equation without restriction on the size of solutions. Of course, there are results on local in time solutions for general Kirchhoff-Carrier type equations, see [4]. When functions $M(\lambda)$ and $g(u_t)$ do not depend on x, t , we proved in [3] the existence of global solutions to the mixed problem for the nonhomogeneous damped Carrier equation without restrictions on a size of initial data.

Here we consider for (1.2) the mixed problem with the Dirichlet boundary conditions and prove existence and uniqueness of global solutions without restrictions on a size of initial data and $f(x, t)$. Moreover, we prove the exponential decay of the energy.

Existence of global regular solutions for quasilinear hyperbolic equations with a nonlinear damping, when initial data are large, was proved in [5]. Our paper consists of 3 main chapters. In chapter 2, we prove the existence of global strong solutions for any space dimension n and for large initial data. In chapter 3, using ideas of Nakao [8], under suitable restrictions for $n, g(u_t), M(x, t, \|u(t)\|^2)$, we prove the exponential decay of the energy. From the physical point of a view, this stability result corresponds to the case of thin strings when the ratio of the cross-section of a string to its length is sufficiently small.

In chapter 4, we prove that for $n = 2$ strong solutions are smooth even for large initial data and $f(x, t)$. In chapter 5, we explain assumptions that we impose to prove our results and give some simple examples of the Carrier type equations in order to illustrate the practicality of them.

2 STRONG SOLUTIONS

Let $x \in D \subset R^n$, where D is a bounded domain with a sufficiently smooth boundary Γ . In $Q = D \times (0, T)$ we consider the following problem,

$$u_{tt} - M(x, t, \|u(t)\|^2)\Delta u + g(x, t, u_t) = f(x, t), \quad (2.1)$$

$$u|_{\Gamma \times (0, T) = S} = 0, \quad (2.2)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x). \quad (2.3)$$

Here and in the sequel, we use standard functional spaces [6], otherwise necessary definitions will be given such as

$$(u, v)(t) = \int_D u(x, t)v(x, t) dx, \quad \|u(t)\|^2 = (u, u)(t),$$

M_t, M_{x_i} denote partial derivatives of the function $M(x, t, \|u(t)\|^2)$ with respect to corresponding parameters and $M_\lambda = (\partial/\partial\lambda)M(x, t, \lambda)$.

ASSUMPTIONS 2

(2.1) $M \in C^1(Q \times R^+) \cap C(\bar{Q} \times R^+)$, $R^+ = [0, \infty)$.

(2.2) There exists a continuous function $\phi(\lambda)$ such that for all $(x, t) \in \bar{Q}$, $0 < m_0 \leq \phi(\lambda) \leq M(x, t, \|u(t)\|^2) \leq C_1\phi(\lambda)$.

(2.3) $|M_\lambda \lambda^{1/2}| \leq k_0 M$.

(2.4) $|M_t(x, t, \lambda)| \leq C_2 M(x, t, \lambda)$.

$$(2.5) \quad g(x, t, u_t) \in C^1(Q \times R^1), \quad |g| + |g_t| + \sum_{i=1}^n |g_{x_i}| \leq C_3(1 + |u_t|^{\rho+1}).$$

$$(2.6) \quad g_\lambda(x, t, \lambda) \geq C_4|\lambda|^\rho, \quad g(x, t, 0) = 0.$$

Here $k_0, C_1, \dots, C_4, m_0$ are positive constants, $\rho > 1$.

THEOREM 2.1 *Let assumptions 2 hold; $u_0 \in H^2(D) \cap H_0^1(D)$, $u_1 \in H_0^1(D) \cap L^{2\rho+2}(D)$. Then for any $f \in H^1(0, T; L^2(D))$ there exists a unique strong solution to (2.1)–(2.3) from the class:*

$$u \in L^\infty(0, T; H^2(D) \cap H_0^1(D)),$$

$$u_t \in L^\infty(0, T; H_0^1(D)) \cap L^{\rho+2}(Q),$$

$$u_{tt} \in L^\infty(0, T; L^2(D)).$$

Proof Uniqueness of a strong solution can be proved by the standard way [3]. Here we prove only the existence part of theorem 2.1 constructing approximate solutions to (2.1)–(2.3) in the form,

$$u^N(x, t) = \sum_{j=1}^N g_j^N(t) w_j(x),$$

$$\Delta w_j + \lambda_j w_j = 0 \text{ in } D, \quad w_j|_\Gamma = 0, \quad (2.4)$$

where $g_j^N(t)$ are solutions to the following Cauchy problem,

$$\begin{aligned} (Lu^N, w_j)(t) &= \frac{u_{tt}^N}{M(x, t, \|u_{(t)}^N\|^2)}, w_j(t) + (\nabla u^N, \nabla w_j)(t) \\ &+ \left(\frac{g(x, t, u_t^N)}{M(x, t, \|u_{(t)}^N\|^2)}, w_j \right)(t) = \left(\frac{f}{M(x, t, \|u_{(t)}^N\|^2)}, w_j \right)(t), \end{aligned}$$

$$g_j^N(0) = (u_0, w_j)(0), \quad g_{jt}^N(0) = (u_1, w_j)(0), \quad j = 1, \dots, N. \quad (2.5)$$

Due to assumptions 2, problem (2.5) has solutions on some interval $(0, T_N)$; and we need a priori estimates in order to prove that $T_N = T$ and to pass to the limit in (2.5) as $N \rightarrow \infty$. In the sequel, we will omit the index N remembering that we work with approximations (2.4), (2.5). Also, we will not write independent variables (x, t) in functions

$$M(x, t, \|u(t)\|^2), \quad g(x, t, u_t).$$

First Estimate

Substituting in (2.5) $w_j = 2u_t$, and omitting the index N we obtain

$$\begin{aligned} \frac{d}{dt} [(u_t^2, M^{-1})(t) + \|\nabla u(t)\|^2] + 2(g, u_t M^{-1})(t) + 2(M_t M^{-2}(u, u_t)(t), u_t^2)(t) \\ + (M_t M^{-2}, u_t^2)(t) = 2(f, u_t M^{-1})(t). \end{aligned} \quad (2.6)$$

Using assumption 2.3, we get

$$I_1 = 2|(M_\lambda M^{-2}(u, u_t)(t), u_t^2)(t)| \leq 2k_0 \|u_t(t)\| (u_t^2, M^{-1})(t).$$

Now assumption 2.2 and the Young inequality imply

$$I_1 \leq \frac{2k_0}{\phi(\|u(t)\|^2)} \|u_t(t)\|^3 \leq \frac{c \|u_t(t)\|_{L^{\rho+2}(D)}^{\rho+2} + C(\epsilon)}{\phi(\|u(t)\|^2)},$$

where ϵ is an arbitrary positive number.

The same arguments and assumption 2.6 give

$$I_2 = 2(g, M^{-1}u_t) \geq \frac{2C_4}{c_1 \phi(\|u(t)\|^2)} \|u_t(t)\|_{L^{\rho+2}(D)}^{\rho+2}.$$

Substituting I_1, I_2 into (2.6), choosing $\epsilon > 0$ sufficiently small and taking into account assumption 2.4, we come to the inequality,

$$\begin{aligned} \frac{d}{dt} [(u_t^2, M^{-1})(t) + \|\nabla u(t)\|^2] + \|u_t(t)\|_{L^{\rho+2}(D)}^{\rho+2} \phi^{-1}(\|u(t)\|^2) \\ \leq C(1 + (u_t^2, M^{-1})(t) + \|f(t)\|^2), \end{aligned} \quad (2.7)$$

where the constant C does not depend on t, N , but can depend on T . By Gronwall's lemma,

$$((u_t^N)^2, M^{-1})(t) + \|\nabla u^N(t)\|^2 \leq C \quad \forall t \in (0, T).$$

From here, $\|u^N(t)\| \leq C$ and

$$m_0 \leq M(x, t, \|u^N(t)\|^2) \leq M_1 < \infty. \quad (2.8)$$

Returning to (2.7), we have

$$\|u_t^N(t)\|^2 + \|\nabla u^N(t)\|^2 + \int_0^t \|u_s^N(s)\|_{L^{\rho+2}(D)}^{\rho+2} ds \leq C \quad \forall t \in (0, T). \quad (2.9)$$

Second Estimate

Derivating (2.5) and substituting $w_j = u_{tt}^N$, we obtain

$$\begin{aligned} \frac{d}{dt} [(u_{tt}^2, M^{-1})(t) + \|\nabla u_t(t)\|^2] - 2(M_\lambda M^{-2}(u, u_t)(t), u_{tt}^2)(t) \\ - (M_t, M^{-2}u_{tt}^2)(t) + 2(g_{u_t}, M^{-1}u_{tt}^2)(t) + 2(g_t, M^{-1}u_{tt})(t) \\ = 2(g, u_{tt}M^{-2}(M_t + 2M_\lambda(u, u_t)(t)))(t) \\ + 2([f_t M^{-1} - M_t M^{-2}f - 2M_\lambda M^{-2}(u, u_t)(t)f], u_{tt})(t). \end{aligned}$$

Taking into account assumptions 2.5, 2.6 and estimate (2.9), we transform this equality to the inequality,

$$\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2 \leq C \left(1 + \int_0^t [\|u_{ss}(s)\|^2 + \|f(s)\|^2 + \|f_s(s)\|^2] ds \right),$$

whence,

$$\|u_{tt}^N(t)\| + \|\nabla u_t^N(t)\| \leq C, \quad (2.10)$$

where the constant C does not depend on N, t .

Third Estimate

Due to (2.4), we rewrite (2.5) in the form,

$$\begin{aligned} -(Lu, \Delta u_t)(t) &= (\nabla[u_{tt}M^{-1}], \nabla u_t)(t) + \frac{1}{2} \frac{d}{dt} \|\Delta u(t)\|^2 \\ &+ (\nabla[gM^{-1}], \nabla u_t)(t) = -(fM^{-1}, \Delta u_t)(t). \end{aligned} \quad (2.11)$$

The first term in (2.11) transform as follows,

$$\begin{aligned} I_1 &= (\nabla[u_{tt}M^{-1}], \nabla u_t) = \frac{1}{2} \frac{d}{dt} (|\nabla u_t|^2, M^{-1})(t) \\ &+ \frac{1}{2} (|\nabla u_t|^2, M^{-2}[M_t + M_\lambda 2(u, u_t)](t))(t) \\ &- \left(u_{tt}, \sum_{i=1}^n M_{x_i} M^{-2}, \nabla u_t \right)(t). \end{aligned}$$

Taking into account estimates (2.9), (2.10), we get

$$I_1 \geq \frac{1}{2} \frac{d}{dt} (|\nabla u_t|^2, M^{-1})(t) - C. \quad (2.12)$$

On the other hand, assumptions 2.5, 2.6 and estimates (2.9), (2.10) imply

$$\begin{aligned} I_2 &= (\nabla[gM^{-1}], \nabla u_t) = \left(\sum_{i=1}^n g_{x_i} M^{-1}, \nabla u_t \right) + (g_u, M^{-1}, |\nabla u_t|^2) - \left(\sum_{i=1}^n M_{x_i} M^{-2} g, \nabla u_t \right) \\ &\geq C_4(|u_t|^\rho |\nabla u_t|^2, M^{-1}) - C(1 + |u_t|^\rho |\nabla u_t|, M^{-1}). \end{aligned}$$

Using Young's inequality, and (2.9), we obtain

$$I_2 \geq \frac{1}{2} C_4(|u_t|^\rho |\nabla u_t|^2, M^{-1}) - C([1 + |\nabla u_t|^2 + |u_t|^{\rho+2}], M^{-1}). \quad (2.13)$$

Here all the constants do not depend on N, t .

The last term in (2.11) we treat in the manner,

$$I_3 = -(fM^{-1}, \Delta u_t)(t) = -\frac{d}{dt}(fM^{-1}, \Delta u)(t) + (f_t M^{-1}, \Delta u)(t) \\ - (fM^{-2}[M_t + 2(u, u_t)(t)M_t], \Delta u)(t).$$

With the help of (2.9) we reduce it to the inequality,

$$I_3 \leq -\frac{d}{dt}(fM^{-1}, \Delta u)(t) + C[\|\Delta u(t)\|^2 + \|f(t)\|^2 + \|f_t(t)\|^2]. \quad (2.14)$$

Substituting (2.12)–(2.14) into (2.11), we come to the inequality,

$$\frac{d}{dt}[(|\nabla u_t|^2, M^{-1})(t) + \|\Delta u(t)\|^2] \leq -2\frac{d}{dt}(fM^{-1}, \Delta u)(t) \\ + C[1 + \|f(t)\|^2 + \|f_t(t)\|^2 + \|\Delta u(t)\|^2 + \|u_t(t)\|_{L^{\rho+2}(D)}^{\rho+2}].$$

Integrating this inequality and exploiting (2.9), (2.10) and Gronwall's lemma, we obtain

$$\|\Delta u(t)\| \leq C \quad \forall t \in [0, \infty). \quad (2.15)$$

Remembering that (2.9), (2.10), (2.15) are estimates of Galerkin approximations (2.4), uniform in N , we can pass to the limit in (2.5) as $N \rightarrow \infty$, therewith to prove the existence result of Theorem 2.1. \blacksquare

3 STABILITY

To prove our stability result, we use the method of Nakao based on the following lemma, see [8].

LEMMA 3.1 *Let $\phi(t)$ be a nonnegative function on $[0, \infty)$ satisfying*

$$\sup_{t \leq \tau < t+1} \phi(\tau) \leq K_0(\phi(t) - \phi(t+1)), \quad K_0 > 0.$$

Then it follows that

$$\phi(t) \leq C\phi(0)e^{\lambda t}, \quad \lambda > 0.$$

ASSUMPTIONS 3

$$(3.1) \quad M(x, t, \lambda) \in C^1(D \times R^+ \times R^+),$$

$$(3.2) \quad g(x, t, u_t)u_t \geq \alpha u_t^2 + \beta |u_t|^{\rho+2},$$

$$(3.3) \quad |g(x, t, u_t)| \leq C_2(|u_t| + |u_t|^{\rho+1}),$$

$$(3.4) \quad \text{There exists a continuous function } \phi(\lambda) \text{ such that } 0 < m_0 \leq \phi(\lambda) \leq M(x, t, \lambda) \\ \leq C_1\phi(\lambda),$$

$$(3.5) \quad |M_\lambda| \lambda^{1/2} \leq k_0 M, \text{ where } k_0 \leq (\alpha\rho/\rho - 1)^{\rho-1} (\beta / (C_1 4^{(2\rho-1)/\rho} (\mu_D)^{(2\rho-1)/2})),$$

$$(3.6) \quad M_t(x, t, \lambda) \geq 0,$$

where $\rho > 1$; α, β, C_1, C_2 are positive constants; μ_D is the Lebesgue measure of D .

THEOREM 3.1 *Let $f = 0$, assumptions 3 hold; $1 < \rho < 4/(n-2)$ if $n > 2$, and $\rho > 1$ when $n = 1, 2$. Then there exist positive constants K and θ such that strong solutions to (2.1)–(2.3) satisfy the following inequality,*

$$\|u_t(t)\|^2 + \|u(t)\|_{H_0^1(D)}^2 \leq K e^{-\theta t}.$$

Proof Obviously, conditions of theorem 3.1 guarantee the existence of strong solutions to (2.1)–(2.3) for all finite $t > 0$. From the identity,

$$(u_{tt}M^{-1}, u_t)(t) + (\nabla u, \nabla u_t)(t) + (gM^{-1}, u_t)(t) = 0,$$

integrating by parts and taking into account assumption 3.3, we obtain

$$\begin{aligned} \frac{d}{dt} [(u_t^2, M^{-1})(t) + \|\nabla u(t)\|^2] + (u_t^2, M_t M^{-2})(t) + 2\alpha(u_t^2, M^{-1})(t) \\ + 2\beta(|u_t|^{\rho+2}, M^{-1})(t) + 4(u_t^2, M_\lambda M^{-2})(t)(u, u_t)(t) \leq 0. \end{aligned} \quad (3.1)$$

As in section 2, using assumptions 3.4, 3.5 and Young's inequality, we get

$$\begin{aligned} I_1 = 4|(u_t^2, M_\lambda M^{-2})(t)(u, u_t)(t)| &\leq \frac{4k_0}{\phi(\|u(t)\|^2)} \|u_t(t)\|^3 \\ &\leq \frac{4\epsilon\mu_D^{1/2\rho}}{\rho\phi(\|u(t)\|^2)} \|u_t(t)\|_{L^{\rho+2}(D)}^{\rho+2} \\ &\quad + \frac{4k_0^{\rho/\rho-1}(\rho-1)}{\epsilon^{\rho/\rho-1}\rho\phi(\|u(t)\|^2)} \|u_t(t)\|^2 \quad \forall \epsilon > 0. \end{aligned}$$

On the other hand, assumption 3.4 yields

$$\begin{aligned} I_2 = 2\alpha(u_t^2, M^{-1})(t) &\geq \frac{2\alpha}{C_1\phi(\|u(t)\|^2)} \|u_t(t)\|^2, \\ I_3 = 2\beta(|u_t|^{\rho+2}, M^{-1})(t) &\geq \frac{2\beta}{C_1\phi(\|u(t)\|^2)} \|u_t(t)\|_{L^{\rho+2}(D)}^{\rho+2}. \end{aligned}$$

Substituting $I_1 - I_3$ into (3.1) and taking into account assumption 3.6, we get

$$\begin{aligned} \frac{d}{dt} [(u_t^2, M^{-1})(t) + \|\nabla u(t)\|^2] + \left(\frac{2\beta}{C_1} - \frac{4\epsilon\mu_D^{1/2\rho}}{\rho} \right) \frac{\|u_t(t)\|_{L^{\rho+2}(D)}^{\rho+2}}{\phi(\|u(t)\|^2)} \\ + \left(\frac{2\alpha}{C_1} - \frac{4(\rho-1)k_0^{\rho/\rho-1}}{\rho\epsilon^{\rho/\rho-1}} \right) \frac{\|u_t(t)\|^2}{\phi(\|u(t)\|^2)} \leq 0. \end{aligned} \quad (3.2)$$

Putting $\epsilon = (\rho\beta/4C_1\mu_D^{\rho/2})$ and using assumption 3.5, we come to the inequality,

$$\frac{d}{dt}[(u_t^2, M^{-1})(t) + \|\nabla u(t)\|^2] + \frac{\beta\|u_t(t)\|_{L^{\rho+2}(D)}^{\rho+2}}{C_1\phi(\|u(t)\|^2)} + \frac{\alpha\|u_t(t)\|^2}{C_1\phi(\|u(t)\|^2)} \leq 0. \quad (3.3)$$

Denoting

$$\psi(t) = (u_t^2, M^{-1})(t) + \|\nabla u(t)\|^2,$$

we conclude from (3.3) that $\psi(t)$ is a nonincreasing positive function and

$$\psi(t) + \frac{\beta}{C_1} \int_0^t \frac{\|u_s(s)\|_{L^{\rho+2}(D)}^{\rho+2}}{\phi(\|u(s)\|^2)} ds + \frac{\alpha}{C_1} \int_0^t \frac{\|u_s(s)\|^2}{\phi(\|u(s)\|^2)} ds \leq \psi(0). \quad (3.4)$$

Hence, by Poincare's inequality,

$$\|\nabla u(t)\| \leq \psi^{1/2}(t) \leq \psi^{1/2}(0),$$

$$\|u(t)\| \leq C_D \psi^{1/2}(0) \quad \forall t \geq 0, \quad (3.5)$$

where the constant C_D depends only on D . Due to assumption 3.4.

$$\phi(\|u(t)\|^2) \leq C_\phi < \infty \quad (3.6)$$

and it follows from (3.4) that

$$\int_t^{t+1} \|u_s(s)\|_{L^{\rho+2}(D)}^{\rho+2} ds + \int_t^{t+1} \|u_s(s)\|^2 ds \leq C_3 F^2(t), \quad (3.7)$$

where C_3 is a positive constant depending on $\alpha, \beta, C_1, C_2, C_\phi$ and

$$F^2(t) = \psi(t) - \psi(t+1). \quad (3.8)$$

By the Mean Value Theorem for integrals, there exist $t_1 \in [t, t + \frac{1}{4}]$ and $t_2 \in [t + \frac{3}{4}, t + 1]$ such that

$$|u_t(t_i)| \leq C_4 F(t), \quad i = 1, 2. \quad (3.9)$$

Multiplying the identity

$$Lu = \frac{u_{tt}}{M} - \Delta u + \frac{g}{M} = 0$$

by u and integrating over $D \times (t_1, t_2)$, we obtain

$$\begin{aligned} \int_{t_1}^{t_2} \|\nabla u(s)\|^2 ds &= (u_t, uM^{-1})(t_1) - (u_t, uM^{-1})(t_2) \\ &\quad + \int_{t_1}^{t_2} (u_s^2, M^{-1})(s) ds - \int_{t_1}^{t_2} (u, u_s M_s M^{-2})(s) ds \\ &\quad - 2 \int_{t_1}^{t_2} (u_s, u)(s) (u_s, u M_\lambda M^{-2})(s) ds \\ &\quad - \int_{t_1}^{t_2} (g, uM^{-1})(s) ds. \end{aligned}$$

Taking into account assumption 3.4, estimates (3.5)–(3.7), (3.9), we transform this equality to the following inequality,

$$\int_{t_1}^{t_2} \|\nabla u(s)\|^2 ds \leq \epsilon \sup_{t \leq s \leq t+1} \psi(s) + C_5(\epsilon) F^2(t) + C_6 \int_{t_1}^{t_2} |(g, u)(s)| ds \quad \forall \epsilon > 0. \quad (3.10)$$

Using assumption 3.3; (3.9) and embedding $H_0^1(D) \hookrightarrow L^{\rho+2}(D)$, we obtain

$$\begin{aligned} C_6 \int_{t_1}^{t_2} |(g, u)(s)| ds &\leq C_7 \int_{t_1}^{t_2} |(|u_s(s)| + |u_s(s)|^{\rho+1}, u)(s)| ds \\ &\leq C_8 \sup_{t \leq s \leq t+1} \|u(s)\|_{H_0^1(D)} \int_{t_1}^{t_2} [\|u_s(s)\| + \|u_s(s)\|_{L^{\rho+2}(D)}^{\rho+1}] ds \\ &\leq C_9 \sup_{t \leq s \leq t+1} \|\nabla u(s)\| [F^{2(\rho+1)/(\rho+2)}(t) + F(t)] \\ &\leq \epsilon \sup_{t \leq s \leq t+1} \psi(s) + C_{10}(\epsilon) [F^2(t) + F^{4(\rho+1)/(\rho+2)}(t)], \end{aligned}$$

where ϵ is the same as in (3.10). Substituting this expression into (3.10), we get

$$\int_{t_1}^{t_2} \|\nabla u(s)\|^2 ds \leq 2\epsilon \sup_{t \leq s \leq t+1} \psi(s) + C_{11} [F^2(t) + F^{4(\rho+1)/(\rho+2)}(t)],$$

whence, by the definition of $\psi(t)$ and (3.7),

$$\int_{t_1}^{t_2} \psi(s) ds \leq \epsilon C_{12} \sup_{t \leq s \leq t+1} \psi(s) + C_{13}(\epsilon) [F^2(t) + F^{4(\rho+1)/(\rho+2)}(t)].$$

By the Mean Value Theorem,

$$\psi(\bar{t}) \leq 2\epsilon C_{12} \sup_{t \leq s \leq t+1} \psi(s) + 2C_{13}(\epsilon) [F^2(t) + F^{4(\rho+1)/(\rho+2)}(t)],$$

where $\bar{t} \in (t_1, t_2)$.

Integrating over $D \times (\bar{t}, t)$ the following identity,

$$(u_{tt} M^{-1} - \Delta u + g M^{-1}) u_t = 0,$$

and acting as above, we obtain

$$\psi(t) \leq cC_{14} \sup_{t \leq s \leq t+1} \psi(s) + C_{15}(c)[F^2(t) + F^{4(\rho+1)/(\rho+2)}(t)].$$

Putting ($c = 1/2C_{14}$) and recalling that $d/dt \psi(t) \leq 0$, we come to the inequality,

$$\sup_{t \leq s \leq t+1} \psi(s) \leq CF^2(t)[1 + F^{2\rho/(\rho+2)}(t)].$$

Exploiting definition of $F^2(t)$, its boundness and lemma 3.1, we get

$$\psi(t) \leq K e^{-\theta t},$$

where K and θ are positive constants.

This proves theorem 3.1. ■

Remark 3.1 Assumption 3.5 means that k_0 in (1.1) is small. Recalling that $k_0 = (EA/LT_0)$, where A is the cross-section area of a string and L its length, we conclude that for a small ratio A/L (the case of a thin string), k_0 is also small.

4 SMOOTH SOLUTIONS

In this section we prove that for $n = 2$, under some restrictions on the growth of $g(u_t)$, global strong solutions to (2.1)–(2.3) are smooth solutions provided functions $M(x, t, \|u(t)\|^2)$, $g(x, t, u_t)$, $f(x, t)$ and u_0, u_1 are smooth. Obviously, it is sufficient to prove that $u \in H^k(Q)$ where $k \geq 3$ is an arbitrary natural number.

ASSUMPTIONS 4

- (4.1) $M \in C^k(\bar{Q} \times R^+)$, $g \in C^k(\bar{Q} \times R^1)$, $k \geq 3$.
- (4.2) There exists a continuous function $\phi(\lambda)$ such that for all $x, t, \in \bar{Q}$, $0 < m_0 \leq \phi(\lambda) \leq M(x, t, \lambda) \leq C_1\phi(\lambda)$.
- (4.3) $|M_\lambda(x, t, \lambda)|\lambda^{1/2} \leq k_0M(x, t, \lambda)$, $|M_t(x, t, \lambda)| \leq C_2M(x, t, \lambda)$.
- (4.4) $|g(x, t, u_t)| + |g_t(x, t, u_t)| + |g_{tt}(x, t, u_t)| \leq C_3(1 + |u_t|^{\rho+1})$.
- (4.5) $g(x, t, 0) = 0$.
- (4.6) $g_\lambda(x, t, \lambda) \geq C_4|\lambda|^\rho$, $g_{\lambda\lambda}^2 \leq C_5(1 + g_\lambda)$; $|g_{\lambda t}| \leq C_6g_\lambda$. Here $m_0, k_0, C_1, \dots, C_6$ are positive constants, $\rho > 1$.
- (4.7) Compatibility conditions: initial data u_0, u_1 and Eq. (2.1) satisfy at $t = 0$, ∂D the following conditions,

$$\partial_t^j [f + M\Delta u - g] = 0, \quad j = 0, \dots, k - 3.$$

THEOREM 4.1 *Let $n = 2$, $k \geq 3$ be a natural number and assumptions 4 hold. Given $u_0 \in H^k(D) \cap H_0^1(D)$, $u_1 \in H^{k-1}(D) \cap H_0^1(D)$ when k is even and $u_0 \in H^k(D) \cap H_0^1(D)$, $u_1 \in H^k(D) \cap H_0^1(D)$ when k is odd. If $f \in H^{k-1}(Q)$, then there exists a unique solution to (4.1)–(4.3), $u \in H^k(Q)$.*

Proof To prove this theorem we use again approximations 2.4. Actually, under assumptions 4, theorem 2.1 provides a priori estimates (2.9), (2.10), (2.15). More necessary estimates will be proved in the form of lemmas 4.1, 4.2.

LEMMA 4.1 *Under assumptions 4, the following inequalities hold:*

$$\|u_m^N(t)\| + \|\Delta u_t^N(t)\| \leq C(\|f\|_{H^2(0,T;L^2(D))} + \|u_1\|_{H^2(D)} + \|u_0\|_{H^2(D)}), \sup_{\bar{Q}} |u_t^N(x, t)| \leq C.$$

Proof Considering the scalar product,

$$((Lu^N)_t, u_m^N)(t) = (\partial_t^2(fM^{-1}(x, t, \|u^N(t)\|^2)), u_m^N)(t),$$

we come, omitting N , to the equality,

$$\begin{aligned} \frac{d}{dt} [\|\nabla u_t(t)\|^2 + (u_m^2, M^{-1})(t)] &+ 3(u_m^2, \partial_t M^{-1})(t) \\ &+ 2(u_{tt}u_t, \partial_t^2 M^{-1})(t) + 2(\partial_t^2 g, u_{tt}M^{-1})(t) \\ &+ 4(\partial_t g, \partial_t M^{-1}, u_{tt})(t) + 2(g\partial_t^2 M^{-1}, u_{tt})(t) \\ &= 2([f_t M^{-1} + 2f_t \partial_t M^{-1} + f \partial_t^2 M^{-1}], u_{tt})(t). \end{aligned} \quad (4.1)$$

In the sequel, we will need estimates for the following expressions:

$$\partial_t M^{-1} = -[M_t(x, t, \|u(t)\|^2) + M_\lambda(x, t, \|u(t)\|^2)2(u, u_t)(t)]M^{-2},$$

where $M_\lambda = M_\lambda(x, t, \lambda(t))$;

$$\begin{aligned} \partial_t^2 M^{-1} &= \partial_t(\partial_t M^{-1}) = -[M_{tt} + 4M_{\lambda t}(u, u_t)(t) \\ &+ 4M_{\lambda\lambda}(u, u_t)^2(t)]M^{-2} - 2M_\lambda[\|u(t)\|^2 + (u, u_{tt})(t)]M^{-2} \\ &+ 2[M_t + 2M_\lambda(u, u_t)(t)]^2 M^{-3}. \end{aligned}$$

Estimates (2.9), (2.10) yield

$$|\partial_t M^{-1}| + |\partial_t^2 M^{-1}| \leq C. \quad (4.2)$$

We have

$$I_1 = \partial_t g(x, t, u_t) = g_t + g^{(1)}u_{tt}, \quad g^{(i)} = \partial_\eta^i g(x, t, \eta).$$

$$I_2 = \partial_t^2 g(x, t, u_t) = g_{tt} + 2g_t^{(1)}u_{tt} + g^{(2)}u_{tt}^2 + g^{(1)}u_{ttt}.$$

We treat the 4th term in (4.1) as follows,

$$I_3 = 2(\partial_t^2 g(x, t, u_t), u_{tt}M^{-1}) = 2(g_{tt}M^{-1}, u_{tt}) \\ + 4(g_t^{(1)}u_{tt}M^{-1}, u_{tt}) + (2(g^{(2)}u_{tt}^2M^{-1}, u_{tt}) + 2(g^{(1)}M^{-1}, u_{tt}^2)).$$

Taking into account (4.2) and assumption 4.6, we obtain

$$I_{31} = 2(g^{(2)}u_{tt}^2M^{-1}, u_{tt}) + 2(g^{(1)}M^{-1}, u_{tt}^2) \\ \geq (2 - c)(g^{(1)}M^{-1}, u_{tt}^2) - C(c)[(M^{-1}, u_{tt}^2) + \|u_{tt}\|_{L^4(D)}^4],$$

where c is an arbitrary positive number.

Analogously,

$$I_{32} = 4(g_t^{(1)}u_{tt}M^{-1}, u_{tt}) \leq c(g^{(1)}M^{-1}, u_{tt}^2) + C(c)(g^{(1)}, u_{tt}^2);$$

$$I_{33} = 2(g_{tt}M^{-1}, u_{tt}) \leq (M^{-1}, u_{tt}^2) + (M^{-1}, g_{tt}^2).$$

Due to assumption 4.4 and estimates (2.9), (2.10),

$$I_3 \geq (2 - 3c)(g^{(1)}M^{-1}, u_{tt}^2) - C(c)[1 + (M^{-1}, u_{tt}^2) + \|u_{tt}\|_{L^4(D)}^4].$$

When $n = 2$, the Gagliardo–Nirenberg inequality and (2.10) imply

$$\|u_{tt}\|^4 \leq C\|u_{tt}\|^2\|\nabla u_{tt}\|^2 \leq C\|\nabla u_{tt}\|^2,$$

hence, putting $c = 1/3$, we obtain

$$I_3 \geq (g^{(1)}M^{-1}, u_{tt}^2) - C[1 + (M^{-1}, u_{tt}^2) + \|\nabla u_{tt}\|^2]. \quad (4.3)$$

Substituting (4.2), (4.3) into (4.1), we reduce it to the inequality,

$$\frac{d}{dt}[(M^{-1}, u_{tt}^2)(t) + \|\nabla u_{tt}\|^2] \leq C[\|f(t)\|^2 + \|f_t(t)\|^2 \\ + \|f_{tt}(t)\|^2 + \|\nabla u_{tt}(t)\|^2 + (M^{-1}, u_{tt}^2)(t) + 1].$$

Integrating this inequality and making use of Gronwall's lemma, we get

$$\|u_{tt}(t)\|^2 + \|\nabla u_{tt}(t)\|^2 \leq C \quad \forall t \in (0, T). \quad (4.4)$$

Rewriting the system,

$$((Lu^N)_t, w_j)(t) = ((fM^{-1})_t, w_j)(t), \quad j = 1, \dots, n$$

in the form.

$$(\nabla u_t^N, \nabla w_j)(t) = ((u_{ttt}^N - u_{tt}^N \partial_t M^{-1} - M^{-1} \partial_t g - g \partial_t M^{-1} + \partial_t (fM^{-1}) w_j)(t),$$

we may conclude that

$$-\Delta(P_N u_t(t)) = F(t) \in L^2(D),$$

where P_N is a projectional operator: $L^2(D) \rightarrow [w_1, \dots, w_N]$. Hence

$$u_t^N \in H^2(D) \cap H_0^1(D). \quad (4.5)$$

In the case $n = 2$, it yields

$$u_t^N \in C(\bar{D}) \quad (4.6)$$

uniformly in N , $t \in (0, T)$. ■

LEMMA 4.2 *Let for any natural $l \geq 3$ the following inequality holds,*

$$\|\partial_t^l u^N(t)\| + \|\nabla \partial_t^{l-1} u^N(t)\| \leq C. \quad (4.7)$$

Then

$$\|\partial_t^{l+1} u^N(t)\| + \|\nabla \partial_t^l u^N(t)\| \leq C.$$

Proof We consider the equality,

$$(\partial_t^l (Lu^N), \partial_t^{l+1} u^N)(t) = (\partial_t^l (fM^{-1}), \partial_t^{l+1} u^N)(t).$$

Omitting the index N , we write it in the form,

$$\begin{aligned} & (\partial_t^l (u_{tt} M^{-1}), \partial_t^{l+1} u)(t) + (\nabla \partial_t^l u, \nabla \partial_t^{l+1} u)(t) \\ & + (\partial_t^l (gM^{-1}), \partial_t^{l+1} u)(t) = (\partial_t^l (fM^{-1}), \partial_t^{l+1} u)(t). \end{aligned} \quad (4.8)$$

Using the chain rule, we obtain

$$\partial_t^l (fg) = \sum_{j=0}^l C_j^l \partial_t^j f \partial_t^{l-j} g. \quad (4.9)$$

where C_j^l are binomial coefficients. On the other hand,

$$\partial_t^l g(t, u_t(t)) = \sum_{j=0}^l C_j^l \bar{\partial}_t^{l-j} g D_t^j g(u_t). \quad (4.10)$$

Here

$$\bar{\partial}_t^s g = \frac{\partial^s}{\partial t^s} g(t, u_t),$$

$$D_t^j g = \sum_{\rho=1}^j C_j^\rho g^{(\rho)}(t, u_t) (\partial_t u_t)^{s_1} (\partial_t^2 u_t)^{s_2} \cdots (\partial_t^j u_t)^{s_j}, \quad (4.11)$$

$$g^{(\rho)}(t, u_t) = \partial_{u_t}^\rho g(t, u_t); \quad s_1 + \cdots + s_j = \rho, \quad 1s_1 + 2s_2 + \cdots + js_j = j, \quad (4.12)$$

C_j^ρ are numerical coefficients. For more details on the chain rule formulas see [1].

Now we apply (4.9)–(4.12) to all of the terms in (4.8):

$$\begin{aligned} I_1 &= (\partial_t^l (u_t M^{-1}), \partial_t^{l+1} u) = \left(\sum_{j=0}^l C_j^l (\partial_t^j u_t) \partial_t^{l-j} M^{-1}, \partial_t^{l+1} u \right) \\ &= (M^{-1} \partial_t^{l+2} u, \partial_t^{l+1} u) + (|\partial_t^{l+1} u|^2, \partial_t M^{-1}) \\ &\quad + \sum_{j=0}^{l-2} C_j^l (\partial_t^j u_t (\partial_t^{l-j} M^{-1}), \partial_t^{l+1} u) = \frac{1}{2} \frac{d}{dt} (|\partial_t^{l+1} u|^2, M^{-1}) \\ &\quad + \left(l - \frac{1}{2} \right) (|\partial_t^{l+1} u|^2, \partial_t M^{-1}) + \sum_{j=0}^{l-2} C_j^l (\partial_t^{j+2} u, (\partial_t^{l-j} M^{-1}) \partial_t^{l+1} u). \end{aligned} \quad (4.13)$$

It can be seen that the highest derivative of $M^{-1}(x, t, \|u(t)\|^2)$ is $\partial_t^l(M^{-1})$ and its highest derivative of $\|u(t)\|$ is $\partial_t^l \|u(t)\|$ which is bounded by (4.7). It means that

$$|\partial_t^j M^{-1}(x, t, \|u(t)\|^2)| \leq C, \quad j = 0, \dots, l, \quad (4.14)$$

hence

$$I_1 \geq \frac{1}{2} \frac{d}{dt} (M^{-1}, |\partial_t^{l+1} u|^2)(t) - C[1 + (M^{-1}, |\partial_t^{l+1} u|^2)(t)]. \quad (4.15)$$

We treat in the same manner the term

$$(\partial_t^l (g M^{-1}), \partial_t^{l+1} u) = \left(\sum_{j=0}^l C_j^l (\partial_t^j g(x, t, u_t)) \partial_t^{l-j} M^{-1}, \partial_t^{l+1} u \right), \quad (4.16)$$

where $\partial_t^j g(x, t, u_t(t))$ is defined by (4.10)–(4.12).

Straight calculations in (4.16) show that for $l \geq 3$ the highest derivatives $\partial_t^{l+1}u$ and $\partial_t^l u$ enter in the expression for $\partial_t^l g(x, t, u_t)$ linearly. It means that we can rewrite (4.16) as follows.

$$\begin{aligned} (\partial_t^l(gM^{-1}), \partial_t^{l+1}u) &= (M^{-1}g^{(1)}, |\partial_t^{l+1}u|^2) + (M^{-1}g^{(2)}\partial_t^l u, \partial_t^{l+1}u) \\ &\quad + (\Phi(x, t, u_t, \|u(t)\|^2), \partial_t^{l+1}u), \end{aligned} \quad (4.17)$$

where a continuous function $\Phi(x, t, u_t, \|u(t)\|^2)$ contains derivatives up to the order $\partial_t^{l-1}u$.

Lemma 4.1 yields that $u_t \in C(\bar{D})$, hence, the functions $g^{(\rho)}(x, t, u_t)$, $\rho = 1, \dots, l$ are bounded. Taking into account (4.7) and assumption 4.6, we transform (4.17) to the form.

$$\begin{aligned} (\partial_t^l(M^{-1}), \partial_t^{l+1}u) &\geq C_0(\|u_t\|^\rho, |\partial_t^{l+1}u|^2) - C[\|\partial_t^{l+1}u(t)\|^2 + \|\partial_t^l u(t)\|^2 + 1] \\ &\geq C_0(\|u_t\|^\rho, |\partial_t^{l+1}u|^2) - C[1 + \|\partial_t^{l+1}u(t)\|^2], \end{aligned}$$

where C_0 is a positive constant.

Using this and (4.15), we reduce (4.8) to the inequality,

$$\frac{d}{dt}[(M^{-1}, |\partial_t^{l+1}u|^2)(t) + \|\nabla \partial_t^l u(t)\|^2] \leq C[1 + \|\partial_t^l f(t)\|^2 + (M^{-1}|\partial_t^{l+1}u(t)|^2)(t)].$$

Integration and the Gronwall lemma give

$$\|\partial_t^{l+1}u^N(t)\| + \|\nabla \partial_t^l u^N(t)\| \leq C,$$

where the constant C does not depend on N , $t \in (0, T)$. ■

Passing to the limit as $N \rightarrow \infty$, we prove

LEMMA 4.3 *Let all the conditions of theorem 4.1 hold. Then for any $f \in H^{k-1}(0, T; L^2(D))$ there exists a unique solution to (2.1)–(2.3) from the class,*

$$u, u_t \in L^\infty(0, T; H^2(D) \cap H_0^1(D)),$$

$$\partial_t^j u \in L^\infty(0, T; H_0^1(D)), \quad j = 2, \dots, k-1,$$

$$\partial_t^k u \in L^\infty(0, T; L^2(D)).$$

Proof The proof is obvious, we omit it. ■

Now, returning to (2.1), we write it in the form.

$$(\nabla \partial_t^j u, \nabla \phi)(t) = (\partial_t^j [M^{-1}(f - g - u_t)], \phi)(t), \quad j = 2, \dots, k-2.$$

where ϕ is an arbitrary function from $H_0^1(D)$. Because from assumption 4.7, $\partial_t^j u|_{\partial D} = 0$, from the theory of elliptic problems it easily follows that

$$\partial_t^{k-j} u \in L^\infty(0, T; H^j(D) \cap H_0^1(D)), \quad j = 1, \dots, k,$$

$$\partial_t^k u \in L^\infty(0, T; L^2(D)).$$

This completes the proof of theorem 4.1. ■

5 COMMENTS

Here we want to explain assumptions that we made and to give simple examples of equations that satisfy these assumptions. We have 3 types of results:

- (1) Existence and uniqueness of global strong solutions for all finite $t > 0$.
- (2) The exponential energy decay of strong solutions.
- (3) Regularity result when solutions belong to $H^k(Q)$ for any $k \in \mathbb{N}$.

It is clear that these results are different, hence, the conditions that guarantee them are also different.

ASSUMPTIONS 2 They guarantee the existence of global strong solutions for any finite $T > 0$. The following equation

$$u_{tt} - (1 + a(x, t)) \int_0^1 u^2(x, t) dx u_{xx} + b(x, t) |u_t|^\rho u_t = f(x, t),$$

where $a, b \in C^1(D \times [0, \infty))$; $0 < a_0 \leq a(x, t)$; $0 < b_0 < b(x, t)$; $\rho > 1$, satisfy assumptions 2. We have

$$\begin{aligned} \lambda &= \int_0^1 u^2 dx; \quad \phi(\|u(t)\|^2) = 1 + \int_0^1 u^2 dx \geq 1, \quad M(x, t, \|u(t)\|^2) \\ &\leq \max \left(1, \max_{\bar{Q}}(a(x, t)) \right) \phi(\|u(t)\|^2), \quad \lambda^{1/2} M_\lambda = a(x, t) \lambda^{1/2} \\ &\leq \max_{\bar{Q}} a^{1/2}(x, t) M(x, t, \lambda), \quad g(x, t, u_t) u_t = b(x, t) |u_t|^{\rho+2} \\ &\geq b_0 |u_t|^{\rho+2}; \quad g_{u_t}(x, t, u_t) \geq b_0 |u_t|^\rho. \end{aligned}$$

The restriction $\rho > 1$ implies the existence of global solutions when the initial data are not small. The case $\rho = 1$ also can be considered if k_0 from assumption 2.3 is sufficiently small, see for details [3]. It can be easily seen that assumption 2.3 is fulfilled for any polynomial $M(\lambda)$. On the other hand, assumptions 2 are not uniform in $t \in (0, \infty)$, hence, do not guarantee any decay of the energy when $t \rightarrow \infty$. For this purpose we have

ASSUMPTIONS 3 which are satisfied by the following equation.

$$u_{tt} - \left(1 + \frac{\delta}{1 + e^{-t}} \int_0^1 u^2(x, t) dx\right) u_{xx} + au_t + \beta |u_t|^\rho u_t = f(x, t),$$

where α, β, δ are positive numbers such that

$$\delta < \left(\frac{\rho}{\rho - 1}\right)^{2(\rho-1)} \frac{\alpha^{2(\rho-1)} \beta^2}{2^{(\delta\rho-4)/\rho} C_1}, \quad \rho > 1.$$

The nonlinear damping, $\beta > 0$, implies the existence of global solutions and due to the linear damping, $\alpha > 0$, we prove the exponential decay of the energy.

ASSUMPTIONS 4 Here we prove the existence of smooth solutions for any finite $T > 0$ when the initial data are not small. Actually, it is an answer to the question of J.-L. Lions [7]: what conditions on ρ , and n guarantee the existence of global smooth solutions to the problem

$$u_{tt} - \Delta u + |u_t|^\rho u_t = f(x, t), \quad \Delta = \sum_1^n \frac{\partial^2}{\partial x_i^2}.$$

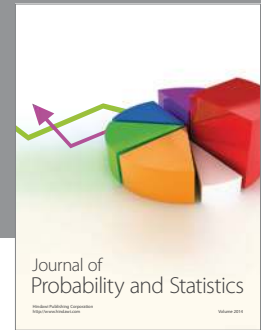
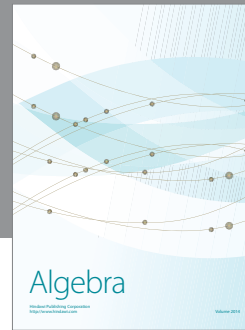
The answer is given by theorem 4.1 when $\rho = 2$, $n = 2$. The following equation illustrates assumptions 4:

$$u_{tt} - (1 + a(x, t)) \int_0^1 u^2(x, t) dx (u_{x_1 x_1} + u_{x_2 x_2}) + b(x, t) u_t^3 = f(x, t),$$

where $a, b \in C^\infty(Q)$; $a_0 \leq a(x, t)$, $b_0 \leq b(x, t)$, $f \in H^k(Q)$.

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