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Global Regularity for a Class of Generalized Magnetohydrodynamic Equations

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Abstract. It remains unknown whether or not smooth solutions of the 3D incompressible MHD equations can develop finite-time singularities. One major difficulty is due to the fact that the dissipation given by the Laplacian operator is insufficient to control the nonlinearity and for this reason the 3D MHD equations are sometimes regarded as "supercritical". This paper presents a global regularity result for the generalized MHD equations with a class of hyperdissipation. This result is inspired by a recent work of Terence Tao on a generalized Navier–Stokes equations (T. Tao, Global regularity for a logarithmically supercritical hyperdissipative Navier–Stokes equations, arXiv: 0906.3070v3 [math.AP] 20 June 2009), but the result for the MHD equations is not completely parallel to that for the Navier–Stokes equations. Besov space techniques are employed to establish the result for the MHD equations.

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1. Introduction

This work aims at the global regularity problem concerning the generalized incompressible magnetohydrodynamic (GMHD) equations of the form

$$\begin{cases} \partial_t u + u\cdot\nabla u + \mathcal{L}_1^2 u = -\nabla p + b\cdot\nabla b, \quad x\in\mathbf{R}^d,\ t>0,\\ \partial_t b + u\cdot\nabla b + \mathcal{L}_2^2 b = b\cdot\nabla u, \quad x\in\mathbf{R}^d,\ t>0, \end{cases} \tag{1.1}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are multiplier operators with symbols given by m_1 and m_2, namely

$$\widehat{\mathcal{L}_1 u}(\xi) = m_1(\xi)\,\widehat{u}(\xi), \quad \widehat{\mathcal{L}_2\, b}(\xi) = m_2(\xi)\,\widehat{b}(\xi).$$

When

$$\mathcal{L}_1^2\, u = -\Delta u, \quad \mathcal{L}_2^2\, b = -\Delta b,$$

(1.1) becomes the standard incompressible MHD equations. The 3D MHD equations govern the dynamics of the velocity field u and the magnetic field b in electrically conducting fluids such as plasmas [2,16]. The fundamental issue of whether or not any classical solution of the 3D MHD equations can develop finite time singularities has attracted a lot of attention and important progress has been made (see e.g., [3,4,8–11,13–15,17,20–23,25]).

However, a complete solution of the global regularity issue for the 3D MHD equations appears to be beyond the reach of current techniques. One major difficulty is that the dissipation is insufficient to control the nonlinearity when applying the standard techniques to establish global a priori bounds. The d-D MHD equations for $d\ge 3$ may be regarded as supercritical in the sense that we need much "stronger" dissipation than the Laplacian $-\Delta$. In fact, if

$$\mathcal{L}_1^2\, u = (-\Delta)^{\gamma_1} u \quad \text{and} \quad \mathcal{L}_2^2\, b = (-\Delta)^{\gamma_2} b, \qquad \gamma_1 \ge \frac{1}{2}+\frac{d}{4}, \quad \gamma_2 \ge \frac{1}{2}+\frac{d}{4}, \tag{1.2}$$

then (1.1) has a global smooth solution for any sufficiently smooth initial data [22]. This paper improves the global regularity result of [22] by reducing the dissipation in (1.2) by a logarithmic factor. More precisely, we have the following theorem.

Theorem 1.1. *Consider the initial-value problem (IVP) of* (1.1) *with the initial data*

$$u(x,0)=u_0(x) \quad \textit{and} \quad b(x,0)=b_0(x). \tag{1.3}$$

Assume $(u_0,b_0)\in H^s(\mathbf{R}^d)$ *with* $s>1+\frac{d}{2}$. *Assume the symbols* m_1 *and* m_2 *satisfy*

$$m_1(\xi)\ge \frac{|\xi|^\alpha}{g_1(\xi)} \quad \textit{and} \quad m_2(\xi)\ge \frac{|\xi|^\beta}{g_2(\xi)}, \tag{1.4}$$

where α *and* β *satisfy*

$$\alpha\ge \frac{1}{2}+\frac{d}{4}, \quad \beta>0, \quad \alpha+\beta\ge 1+\frac{d}{2} \tag{1.5}$$

and $g_1\ge 1$ *and* $g_2\ge 1$ *are radially symmetric, nondecreasing and satisfy*

$$\int_1^\infty \frac{ds}{s\,(g_1^2(s)+g_2^2(s))^2}=+\infty. \tag{1.6}$$

Then the IVP for the GMHD equations (1.1) *and* (1.3) *has a unique global classical solution* (u,b).

This study on the GMHD equations is partially motivated by a recent work of Tao [18], who established the global regularity of a generalized Navier–Stokes equations, namely (1.1) with $b\equiv 0$. But the result presented here for the GMHD equations is not completely parallel to that for the generalized Navier–Stokes equations. In fact, the condition that $\beta\ge\frac{1}{2}+\frac{d}{4}$ is not required and (1.5) implies that it suffices to assume $\beta>0$ when α is sufficiently large.

Theorem 1.1 is proven by Besov space techniques. Identifying H^s with the Besov space $B^s_{2,2}$, the norm $\|(u,b)\|_{H^s}$ can be estimated more dedicatedly than Sobolev type inequalities. We divide the rest of this paper into two major parts. Section 2 presents the proof of Theorem 1.1. The proof relies on an inequality for commutators. The appendix provides definitions, properties and some useful facts of Besov spaces.

2. Proof of Theorem 1.1

This section proves Theorem 1.1. For notational convenience, we will write L^p for $L^p(\mathbf{R}^d)$. To prove this theorem, we need a bound for a special type of commutators.

Lemma 2.1. *For any* $j\ge -1$, $p\in[1,\infty]$,

$$\begin{aligned}\|[\Delta_j, f\cdot\nabla]g\|_{L^p} &\le C\,\|\nabla f\|_{L^q}\,\|\nabla g\|_{L^r}\,\|x\,\Phi_j\|_{L^\sigma} &(2.1)\\ &\le C\,2^{j(-1+d(1-\frac{1}{\sigma}))}\,\|\nabla f\|_{L^q}\,\|\nabla g\|_{L^r}\,\|x\Phi_0\|_{L^\sigma}, &(2.2)\end{aligned}$$

where $[\Delta_j, f\cdot\nabla]g$ *denotes* $\Delta_j(f\cdot\nabla g)-f\cdot\nabla\Delta_j\, g$, Φ_j*'s are defined as in* (A.1) *and* q,r *and* σ *satisfy*

$$q,r,\sigma\in[1,\infty], \quad 1+\frac{1}{p}=\frac{1}{q}+\frac{1}{r}+\frac{1}{\sigma}, \quad \frac{1}{r}+\frac{1}{\sigma}+\frac{1}{d}>1.$$

In particular,

(1) *for any* $p\in[1,\infty]$,

$$\|[\Delta_j, f\cdot\nabla]g\|_{L^p}\le C\,2^{-j}\,\|\nabla f\|_{L^\infty}\,\|\nabla g\|_{L^p}\,\|x\Phi_0\|_{L^1}, \tag{2.3}$$

(2) *for any* $p\in[1,\infty]$, $r'\le p$ *and* $\frac{1}{r}+\frac{1}{r'}=1$,

$$\|[\Delta_j, f\cdot\nabla]g\|_{L^p}\le C\,2^{j(-1+\frac{d}{r})}\,\|\nabla f\|_{L^p}\|\nabla g\|_{L^r}\,\|x\Phi_0\|_{L^{r'}}. \tag{2.4}$$

Remark. The special case in (2.4) has previously been obtained by Hmidi et al. [12].

Proof of Lemma 2.1. According to the definition of Δ_j in (A.4),

$$\begin{aligned}[\Delta_j, f\cdot\nabla]g &= \Delta_j(f\cdot\nabla g) - f\cdot\nabla\Delta_j g\\ &= \int \Phi_j(x-y)(f(y)-f(x))\cdot\nabla g(y)\,dy. \end{aligned}\tag{2.5}$$

The special case in (2.3) can be established rather easily. In fact, by Young's inequality for convolution,

$$\begin{aligned}\|[\Delta_j, f\cdot\nabla]g\|_{L^p} &\le \|\nabla f\|_{L^\infty}\left\|\int |x-y|\Phi_j(x-y)\,|\nabla g(y)|\,dy\right\|_{L^p}\\ &\le \|\nabla f\|_{L^\infty}\,\|\nabla g\|_{L^r}\,\|x\Phi_j(x)\|_{L^\sigma}\end{aligned}$$

where $1+1/p = 1/r+1/\sigma$. For more general cases, we insert the identity

$$f(y)-f(x) = \int_0^1 (y-x)\cdot\nabla f(x+\theta(y-x))\,d\theta$$

in (2.5) and apply Hölder's inequality to find

$$|[\Delta_j, f\cdot\nabla]g| \le \|\nabla g\|_{L^r}\int_0^1\left[\int [|y-x|\Phi_j(y-x)]^{r'}\,|\nabla f(x+\theta(y-x))|^{r'}dy\right]^{\frac{1}{r'}}\,d\theta$$

where $\frac{1}{r}+\frac{1}{r'}=1$ and $r'\le p$. Taking the L^p-norm, we have

$$\begin{aligned}&\|[\Delta_j, f\cdot\nabla]g\|_{L^p}\\ &\quad\le \|\nabla g\|_{L^r}\int_0^1\left\|\left[\int [|y-x|\Phi_j(y-x)]^{r'}\,|\nabla f(x+\theta(y-x))|^{r'}dy\right]^{\frac{1}{r'}}\right\|_{L^p}\,d\theta\\ &\quad= \|\nabla g\|_{L^r}\int_0^1\left\|\int [|y-x|\Phi_j(y-x)]^{r'}\,|\nabla f(x+\theta(y-x))|^{r'}dy\right\|_{L^{p/r'}}^{1/r'}\,d\theta.\end{aligned}\tag{2.6}$$

Making the substitution $z=-\theta(y-x)$ and then applying Young's inequality for convolution, we obtain

$$\begin{aligned}&\int_0^1\left\|\int [|y-x|\Phi_j(y-x)]^{r'}\,|\nabla f(x+\theta(y-x))|^{r'}dy\right\|_{L^{p/r'}}^{1/r'}\,d\theta\\ &\quad= \int_0^1\left\|\int\left|\frac{z}{\theta}\Phi_j\left(\frac{z}{\theta}\right)\right|^{r'}\,|\nabla f(x-z)|^{r'}\frac{dz}{\theta^d}\right\|_{L^{p/r'}}^{1/r'}\,d\theta\\ &\quad= \int_0^1\left\|\left|\frac{z}{\theta}\Phi_j\left(\frac{z}{\theta}\right)\right|^{r'}\right\|_{L^{\sigma/r'}}^{1/r'}\,\||\nabla f|^{r'}\|_{L^{q/r'}}^{1/r'}\,\theta^{-d/r'}\,d\theta\\ &\quad= \||\nabla f|^{r'}\|_{L^{q/r'}}^{1/r'}\int_0^1\left[\int\left|\frac{z}{\theta}\Phi_j\left(\frac{z}{\theta}\right)\right|^{\sigma}\,dz\right]^{1/\sigma}\,\theta^{-d/r'}\,d\theta\\ &\quad= \|\nabla f\|_{L^q}\int_0^1\|x\Phi_j(x)\|_{L^\sigma}\,\theta^{d/\sigma-d/r'}\,d\theta\\ &\quad\le C\,\|\nabla f\|_{L^q}\,\|x\Phi_j(x)\|_{L^\sigma},\end{aligned}\tag{2.7}$$

where

$$1 + \frac{d}{\sigma} - \frac{d}{r'} > 0, \quad 1 + \frac{r'}{p} = \frac{r'}{\sigma} + \frac{r'}{q} \quad \text{or} \quad \frac{1}{r} + \frac{1}{\sigma} + \frac{1}{d} > 1, \quad 1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r} + \frac{1}{\sigma}.$$

Inserting (2.7) in (2.6) yields

$$\|[\Delta_j, f \cdot \nabla]g\|_{L^p} \le \|\nabla g\|_{L^r} \|\nabla f\|_{L^q} \|x\Phi_j\|_{L^\sigma},$$

which is (2.1). To show (2.2), it suffices to notice (A.1), namely

$$\Phi_j(x) = 2^{jd}\Phi_0(2^j x).$$

(2.4) follows by letting $p = q$ in (2.2). This completes the proof of Lemma 2.1. □

Proof of Theorem 1.1. If (u, b) solves (1.1), then clearly

$$\|u(t)\|_2^2 + \|b(t)\|_2^2 + 2\nu \int_0^t \|\mathcal{L}_1 u\|_2^2 \, d\tau + 2\eta \int_0^t \|\mathcal{L}_2 b\|_2^2 \, d\tau = \|u_0\|_2^2 + \|b_0\|_2^2, \quad t > 0.$$

For notational convenience, we write

$$A_1(t) \equiv \|\mathcal{L}_1 u(t)\|_2^2 \quad \text{and} \quad A_2(t) \equiv \|\mathcal{L}_2 b(t)\|_2^2.$$

Identifying L^2 as $B^0_{2,2}$, we have

$$A_1 = \sum_j \|\Delta_j \mathcal{L}_1 u\|_2^2 \quad \text{and} \quad A_2 = \sum_j \|\Delta_j \mathcal{L}_2 b\|_2^2.$$

where the summations are over $j \ge -1$.

The rest of the proof is devoted to bound $\|(u, b)\|_{H^s}$. Here H^s is identified with the Besov space $B^s_{2,2}$. Let $j \ge -1$ be an integer. Applying Δ_j to (1.1), we have

$$\partial_t \Delta_j u + \nu \mathcal{L}_1 \Delta_j u = -\mathbf{P}\Delta_j(u \cdot \nabla u) + \mathbf{P}\Delta_j(b \cdot \nabla b), \tag{2.8}$$

where $\mathbf{P} = I - \nabla\Delta^{-1}\nabla\cdot$ is the projection operator onto divergence free vector fields. Similarly, applying Δ_j to the second equation in (1.1) yields

$$\partial_t \Delta_j b + +\eta \mathcal{L}_2 \Delta_j b = -\Delta_j(u \cdot \nabla b) + \Delta_j(b \cdot \nabla u). \tag{2.9}$$

Dotting (2.8) and (2.9) by $2\Delta_j u$ and $2\Delta_j b$, respectively, and integrating with respect to x, we obtain

$$\frac{d}{dt}\|\Delta_j u\|_{L^2}^2 + 2\nu\|\mathcal{L}_1 \Delta_j u\|_{L^2}^2 = -2\int \Delta_j u \cdot \Delta_j(u \cdot \nabla u)\, dx + 2\int \Delta_j u \cdot \Delta_j(b \cdot \nabla b)\, dx,$$
$$\frac{d}{dt}\|\Delta_j b\|_{L^2}^2 + 2\eta\|\mathcal{L}_2 \Delta_j b\|_{L^2}^2 = -2\int \Delta_j b \cdot \Delta_j(u \cdot \nabla b)\, dx + 2\int \Delta_j b \cdot \Delta_j(b \cdot \nabla u)\, dx.$$

Multiplying each of these equations by 2^{2sj} and summing over all $j \ge -1$, we get

$$\frac{d}{dt}\left(\|u\|_{H^s}^2 + \|b\|_{H^s}^2\right) + 2\nu \sum_j 2^{2sj}\|\mathcal{L}_1 \Delta_j u\|_{L^2}^2 + 2\eta \sum_j 2^{2sj}\|\mathcal{L}_2 \Delta_j b\|_{L^2}^2 \tag{2.10}$$
$$= I_1 + I_2 + I_3 + I_4, \tag{2.11}$$

where

$$I_1 = -2\sum_j 2^{2sj}\int \Delta_j u\cdot\Delta_j(u\cdot\nabla u)\,dx, \tag{2.12}$$

$$I_2 = 2\sum_j 2^{2sj}\int \Delta_j u\cdot\Delta_j(b\cdot\nabla b)\,dx, \tag{2.13}$$

$$I_3 = -2\sum_j 2^{2sj}\int \Delta_j b\cdot\Delta_j(u\cdot\nabla b)\,dx, \tag{2.14}$$

$$I_4 = 2\sum_j 2^{2sj}\int \Delta_j b\cdot\Delta_j(b\cdot\nabla u)\,dx. \tag{2.15}$$

Since the estimates of these four terms are similar, we only provide the details for I_3. Employing Bony's notion of paraproducts, we can write

$$\begin{aligned}\Delta_j(u\cdot\nabla b) = &\sum_{|j-k|\le 3}\Delta_j(S_{k-1}u\cdot\nabla\Delta_k b) + \sum_{|j-k|\le 3}\Delta_j(\Delta_k u\cdot\nabla S_{k-1}b)\\ &+\sum_{k\ge j-1}\Delta_j(\Delta_k u\cdot\nabla\tilde{\Delta}_k b).\end{aligned} \tag{2.16}$$

where $\tilde{\Delta}_k b = (\Delta_{k-1}+\Delta_k+\Delta_{k+1})b$. The first term in (2.16) can be further written as

$$\begin{aligned}\sum_{|j-k|\le 3}\Delta_j(S_{k-1}u\cdot\nabla\Delta_k b) = &\sum_{|j-k|\le 3}[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k b\\ &+\sum_{|j-k|\le 3}(S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k b + S_j u\cdot\nabla\Delta_j b,\end{aligned} \tag{2.17}$$

where we have used the fact that $\sum_{|j-k|\le 3}\Delta_j\Delta_k b = \Delta_j b$. After inserting (2.17) in (2.16) and (2.16) in (2.14), we can split I_3 into five terms,

$$\begin{aligned}
I_{31} &= -2\sum_j 2^{2sj}\int \Delta_j b\cdot\sum_{|j-k|\le 3}[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k b\,dx,\\
I_{32} &= -2\sum_j 2^{2sj}\int \Delta_j b\cdot\sum_{|j-k|\le 3}(S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k b\,dx,\\
I_{33} &= -2\sum_j 2^{2sj}\int \Delta_j b\cdot S_j u\cdot\nabla\Delta_j b\,dx,\\
I_{34} &= -2\sum_j 2^{2sj}\int \Delta_j b\cdot\sum_{|j-k|\le 3}\Delta_j(\Delta_k u\cdot\nabla S_{k-1}b)\,dx,\\
I_{35} &= -2\sum_j 2^{2sj}\int \Delta_j b\cdot\sum_{k\ge j-1}\Delta_j(\Delta_k u\cdot\nabla\tilde{\Delta}_k b)\,dx.
\end{aligned}$$

By the divergence-free condition $\nabla\cdot u = 0$, we have

$$I_{33} = 0.$$

To estimate I_{31}, we first apply Hölder's inequality and then Lemma 2.1 to find that

$$|I_{31}| \le C\sum_j 2^{2sj}\|\Delta_j b\|_2\sum_{|j-k|\le 3}\|\nabla S_{k-1}u\|_{L^\infty}\,\|\nabla\Delta_k b\|_{L^2}\,2^{-j}\|x\Phi_0\|_{L^1}$$

Since the summation is over k satisfying $|j-k|\le 3$, we can replace the summation by a constant multiple of the term with $k=j$. Applying Bernstein's inequality to $\|\nabla\Delta_k b\|_{L^2}$ yields

$$\begin{aligned}
|I_{31}| &\le C\sum_j 2^{2sj}\|\Delta_j b\|_2^2\,\|\nabla S_{j-1}u\|_{L^\infty}\\
&\le C\sum_j 2^{2sj}\|\Delta_j b\|_2^2\sum_{m\le j-2}\|\nabla\Delta_m u\|_{L^\infty}\\
&\le C\sum_j 2^{2sj}\|\Delta_j b\|_2^2\sum_{m\le j-2}2^{m(1+\frac{d}{2})}\|\Delta_m u\|_{L^2}\\
&= C\sum_j 2^{sj}\,\frac{2^{\beta j}}{g_2(2^j)}\,\|\Delta_j b\|_{L^2}\\
&\qquad\times g_2(2^j)\,2^{sj}\|\Delta_j b\|_{L^2}2^{-\beta j}\sum_{m\le j-2}2^{m(1+\frac{d}{2})}\|\Delta_m u\|_{L^2}.
\end{aligned}$$

By the Cauchy–Schwartz inequality,

$$\begin{aligned}
|I_{31}| \le{}& \frac{\eta}{4}\sum_j 2^{2sj}\,\frac{2^{2\beta j}}{g_2^2(2^j)}\,\|\Delta_j b\|_{L^2}^2\\
&+C(\eta)\sum_j g_2^2(2^j)\,2^{2sj}\|\Delta_j b\|_{L^2}^2 2^{-2\beta j}\left[\sum_{m\le j-2}2^{m(1+\frac{d}{2})}\|\Delta_m u\|_{L^2}\right]^2\\
\le{}& \frac{\eta}{4}\sum_j 2^{2sj}\,\|\mathcal{L}_2\Delta_j b\|_{L^2}^2\\
&+C(\eta)\sum_j g_2^2(2^j)\,2^{2sj}\|\Delta_j b\|_{L^2}^2 2^{-2\beta j}\left[\sum_{m\le j-2}2^{m(1+\frac{d}{2})}\|\Delta_m u\|_{L^2}\right]^2\\
={}& \frac{\eta}{4}\sum_j 2^{2sj}\,\|\mathcal{L}_2\Delta_j b\|_{L^2}^2 + C(\eta)\,(I_{311}+I_{312}),
\end{aligned}\tag{2.18}$$

where

$$\begin{aligned}
I_{311} &= \sum_{j\le N} g_2^2(2^j)\,2^{2sj}\|\Delta_j b\|_{L^2}^2 2^{-2\beta j}\left[\sum_{m\le j-2}2^{m(1+\frac{d}{2})}\|\Delta_m u\|_{L^2}\right]^2,\\
I_{312} &= \sum_{j> N} g_2^2(2^j)\,2^{2sj}\|\Delta_j b\|_{L^2}^2 2^{-2\beta j}\left[\sum_{m\le j-2}2^{m(1+\frac{d}{2})}\|\Delta_m u\|_{L^2}\right]^2
\end{aligned}$$

for an integer N to be determined later. They can be bounded as follows.

$$\begin{aligned}
I_{311} &= \sum_{j\le N} g_2^2(2^j)\,2^{2sj}\|\Delta_j b\|_{L^2}^2\left[\sum_{m\le j-2}2^{\beta(m-j)}\,2^{\alpha m}\|\Delta_m u\|_{L^2}\right]^2.\\
&\le \sum_{j\le N} g_2^2(2^j)\,2^{2sj}\|\Delta_j b\|_{L^2}^2\sup_{m\le j-2}2^{2\alpha m}\|\Delta_m u\|_{L^2}^2\,C_0
\end{aligned}$$

where C_0 denotes the constant

$$C_0=\left[\sum_{m\le j-2}2^{\beta(m-j)}\right]^2.$$

Therefore,

$$\begin{aligned} I_{311} &\le C_0\, g_1^2(2^N)\, g_2^2(2^N) \sup_{j\le N} \frac{2^{2\alpha j}}{g_1^2(2^j)} \|\Delta_j u\|_{L^2}^2 \sum_{j\le N} 2^{2sj} \|\Delta_j b\|_{L^2}^2 \\ &\le C_0\, g_1^2(2^N)\, g_2^2(2^N) \sup_{j\le N} \|\mathcal{L}_1 \Delta_j u\|_{L^2}^2 \sum_{j\le N} 2^{2sj} \|\Delta_j b\|_{L^2}^2 \\ &= C_0\, g_1^2(2^N)\, g_2^2(2^N) \sup_{j\le N} \|\mathcal{L}_1 \Delta_j u\|_{L^2}^2 \|b\|_{H^s}^2. \end{aligned}$$

We now estimate I_{312}. Let $0<\delta<\beta$ and write I_{312} as

$$I_{312} = \sum_{j>N} g_2^2(2^j)\, 2^{-2j(\beta-\delta)}\, 2^{2sj} \|\Delta_j b\|_{L^2}^2 \left[\sum_{m\le j-2} 2^{(m-j)\delta} 2^{m(1+\frac{d}{2}-\delta)} \|\Delta_m u\|_{L^2} \right]^2.$$

According to (1.6), g_2 grows logarithmically and we have, for $j\ge N$ (provided that N is sufficiently large),

$$g_2^2(2^j)\, 2^{-2j(\beta-\delta)} \le g_2^2(2^N)\, 2^{-2N(\beta-\delta)}.$$

Therefore

$$\begin{aligned} I_{312} &\le C\, g_2^2(2^N)\, 2^{-2N(\beta-\delta)} \sum_{j>N} 2^{2sj} \|\Delta_j b\|_{L^2}^2 \sum_m 2^{2m(1+\frac{d}{2}-\delta)} \|\Delta_m u\|_{L^2}^2 \\ &\le C\, g_2^2(2^N)\, 2^{-2N(\beta-\delta)}\, \|b\|_{H^s}^2\, \|u\|_{H^s}^2. \end{aligned}$$

Inserting the estimates for I_{311} and I_{312} in (2.18), we find that

$$\begin{aligned} |I_{31}| &\le \frac{\eta}{4} \sum_j 2^{2sj}\, \|\mathcal{L}_2 \Delta_j b\|_{L^2}^2 + C\, g_1^2(2^N)\, g_2^2(2^N) \sup_{j\le N} \|\mathcal{L}_1 \Delta_j u\|_{L^2}^2 \|b\|_{H^s}^2 \\ &\quad + C\, g_2^2(2^N)\, 2^{-2N(\beta-\delta)}\, \|b\|_{H^s}^2\, \|u\|_{H^s}^2. \end{aligned}$$

The estimate for I_{32} is actually easier than I_{31}. Since the summation for k is just over $|k-j|\le 3$, the summation over k can be replaced by a multiple of its typical term with $k=j$. By Hölder's and Bernstein's inequalities that

$$\begin{aligned} |I_{32}| &\le C \sum_j 2^{2sj} \|\Delta_j b\|_{L^2}\, \|\Delta_j u\|_2\, \|\nabla \Delta_j b\|_{L^\infty} \\ &\le C \sum_j 2^{2sj}\, 2^{(1+\frac{d}{2})j}\, \|\Delta_j b\|_{L^2}^2\, \|\Delta_j u\|_2. \end{aligned}$$

As in the estimate of I_{31}, we have

$$|I_{32}| \le \frac{\eta}{4} \sum_j 2^{2sj}\, \|\mathcal{L}_2 \Delta_j b\|_{L^2}^2 + C(\eta)\, (I_{321} + I_{322})$$

where I_{321} and I_{322} are given by

$$\begin{aligned} I_{321} &= \sum_{j\le N} g_2^2(2^j)\, 2^{2sj} \|\Delta_j b\|_{L^2}^2 2^{2\alpha j} \|\Delta_j u\|_2^2, \\ I_{322} &= \sum_{j>N} g_2^2(2^j)\, 2^{2sj} \|\Delta_j b\|_{L^2}^2 2^{2\alpha j} \|\Delta_j u\|_2^2. \end{aligned}$$

As in the estimate for I_{311}, we have

$$I_{321} \le C\, g_1^2(2^N)\, g_2^2(2^N) \sup_{j\le N} \|\mathcal{L}_1 \Delta_j u\|_{L^2}^2 \|b\|_{H^s}^2$$

or

$$I_{321} \le C\, g_2^4(2^N) \sup_{j\le N} \|\mathcal{L}_2 \Delta_j b\|_{L^2}^2 \|u\|_{H^s}^2.$$

The estimate for I_{322} is also similar to that for I_{312}.

$$\begin{aligned} I_{322} &\le \sum_{j>N} g_2^2(2^j)\, 2^{-j(2s-1-\frac{d}{2})}\, 2^{2sj} \|\Delta_j b\|_{L^2}^2\, 2^{2sj} \|\Delta_j u\|_{L^2}^2 \\ &\le g_2^2(2^N)\, 2^{-N(2s-1-\frac{d}{2})}\, \|b\|_{H^s}^2\, \|u\|_{H^s}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} |I_{32}| \le{}& \frac{\eta}{4} \sum_j 2^{2sj}\, \|\mathcal{L}_2 \Delta_j b\|_{L^2}^2 + C\, g_2^4(2^N) \sup_{j\le N} \|\mathcal{L}_2 \Delta_j b\|_{L^2}^2 \|u\|_{H^s}^2 \\ &+ C\, g_2^2(2^N)\, 2^{-N(2s-1-\frac{d}{2})}\, \|b\|_{H^s}^2\, \|u\|_{H^s}^2. \end{aligned}$$

The estimates for I_{34} and I_{35} are very similar and we omit the details. To bound the parallel terms I_1, I_2 and I_4, we can decompose and estimate them like what we did to I_3. The only difference is that the term I_{23} decomposed from I_2 and I_{43} from I_4 are no longer zero by themselves, but $I_{23} + I_{43}$ is zero. The rest of the terms in I_1, I_2 and I_4 can be similarly bounded as the corresponding terms in I_3. Now, we write

$$E_s(t) = \|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2$$

and set $2^N = E_s(t)$. Collecting all the estimates, we find that

$$\begin{aligned} &\frac{d}{dt} E_s(t) + \nu \sum_j 2^{2sj} \|\mathcal{L}_1 \Delta_j u\|_2^2 + \eta \sum_j 2^{2sj} \|\mathcal{L}_2 \Delta_j b\|_2^2 \\ &\le C\, (g_1^2(E_s) + g_2^2(E_s))^2\, E_s\, (A_1(t) + A_2(t)). \end{aligned}$$

The conclusion of Theorem 1.1 then follows from (1.6) and a simple ODE argument. □

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Appendix

This appendix provides the definitions of Besov spaces and related facts. Part of the materials presented here can be found in [1,6,19]. We denote by $\mathcal{S}(\mathbf{R}^d)$ the usual Schwarz class and $\mathcal{S}'(\mathbf{R}^d)$ the space of tempered distributions. Let $\widehat{f}$ denote the Fourier transform of f, defined by the formula

$$\widehat{f}(\xi) = \int_{\mathbf{R}^d} e^{-ix\cdot\xi}\, f(x)\, dx.$$

The fractional Laplacian $(-\Delta)^\alpha$ with $\alpha \in \mathbf{R}$ is defined through the Fourier transform

$$\widehat{(-\Delta)^\alpha f} = |\xi|^{2\alpha}\, \widehat{f}(\xi).$$

For notational convenience, we sometimes write Λ for $(-\Delta)^{\frac{1}{2}}$. We define $\mathcal{S}_0$ to be the following subspace of $\mathcal{S}$,

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S}, \int_{\mathbf{R}^d} \phi(x) x^\gamma dx = 0,\ |\gamma| = 0, 1, 2, \dots \right\}.$$

Its dual $\mathcal{S}_0'$ is given by

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P},$$

where $\mathcal{P}$ is the space of polynomials. In other words, two distributions in $\mathcal{S}'$ are identified as the same in $\mathcal{S}_0'$ if their difference is a polynomial.

For $j \in Z$, we define

$$A_j = \{\xi \in \mathbf{R}^d : 2^{j-1} < |\xi| < 2^{j+1}\}.$$

Then there exists a sequence $\{\Phi_j\} \in \mathcal{S}(\mathbf{R}^d)$ such that

$$\operatorname{supp} \widehat{\Phi}_j \subset A_j, \quad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x). \tag{A.1}$$

and

$$\sum_{k=-\infty}^{\infty} \widehat{\Phi}_k(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbf{R}^d \setminus \{0\}, \\ 0 & \text{if } \xi = 0. \end{cases}$$

As a consequence, for any $f \in \mathcal{S}_0'$,

$$\sum_{k=-\infty}^{\infty} \Phi_k * f = f. \tag{A.2}$$

To define the homogeneous Besov space, we set

$$\Delta_j f = \Phi_j * f, \quad j = 0, \pm 1, \pm 2, \ldots. \tag{A.3}$$

Definition A.1. For $s \in \mathbf{R}$ and $1 \leq p, q \leq \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ is defined by

$$\mathring{B}^s_{p,q} = \left\{ f \in \mathcal{S}_0' : \|f\|_{\mathring{B}^s_{p,q}} < \infty \right\},$$

where

$$\|f\|_{\mathring{B}^s_{p,q}} = \begin{cases} \left(\sum_j \left(2^{js} \|\Delta_j f\|_{L^p}\right)^q \right)^{1/q} & \text{for } q < \infty, \\ \sup_j 2^{js} \|\Delta_j f\|_{L^p} & \text{for } q = \infty. \end{cases}$$

To define the inhomogeneous Besov space, we let $\Psi \in C_0^\infty(\mathbf{R}^d)$ be even and satisfy

$$\widehat{\Psi}(\xi) = 1 - \sum_{k=0}^{\infty} \widehat{\Phi}_k(\xi).$$

It is clear that for any $f \in \mathcal{S}'$,

$$\Psi * f + \sum_{k=0}^{\infty} \Phi_k * f = f.$$

We further set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \leq -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \ldots \end{cases} \tag{A.4}$$

Definition A.2. For $s \in \mathbf{R}$ and $1 \leq p, q \leq \infty$, the inhomogeneous Besov space $B^s_{p,q}$ is defined by

$$B^s_{p,q} = \left\{ f \in \mathcal{S}' : \|f\|_{B^s_{p,q}} < \infty \right\},$$

where

$$\|f\|_{B^s_{p,q}} \equiv \begin{cases} \left(\sum_{j=-1}^{\infty} \left(2^{js} \|\Delta_j f\|_{L^p}\right)^q \right)^{1/q}, & \text{if } q < \infty, \\ \sup_{-1 \leq j < \infty} 2^{js} \|\Delta_j f\|_{L^p}, & \text{if } q = \infty. \end{cases} \tag{A.5}$$

We caution that Δ_j with $j \le -1$ associated with the homogeneous Besov space $\mathring{B}^s_{p,q}$ are defined differently from those associated with the inhomogeneous Besov space $B^s_{p,q}$. Therefore, it will be understood that Δ_j with $j \le -1$ in the context of the homogeneous Besov space are given by (A.3) and by (A.4) in the context of the inhomogeneous Besov space. For Δ_j defined by either (A.3) or (A.4) and $S_j \equiv \sum_{k<j} \Delta_k$,

$$\Delta_j \Delta_k = 0 \quad \text{if } |j-k| \ge 2 \quad \text{and} \quad \Delta_j (S_{k-1} f\, \Delta_k f) = 0 \quad \text{if} |j-k| \ge 4.$$

The Besov spaces and the standard Sobolev spaces defined by

$$\mathring{W}^{s,p} = \Lambda^{-s} L^p \quad \text{and} \quad W^{s,p} = (1-\Delta)^{-s/2} L^p$$

obey the simple facts stated in the following lemma (see [1]).

Lemma A.3. *Assume that $s \in \mathbf{R}$ and $p, q \in [1,\infty]$.*

(1) *If $s > 0$, then $B^s_{p,q} \subset \mathring{B}^s_{p,q}$.*
(2) *If $s_1 \le s_2$, then $B^{s_2}_{p,q} \subset B^{s_1}_{p,q}$. This inclusion relation is false for the homogeneous Besov spaces.*
(3) *If $1 \le q_1 \le q_2 \le \infty$, then $\mathring{B}^s_{p,q_1} \subset \mathring{B}^s_{p,q_2}$ and $B^s_{p,q_1} \subset B^s_{p,q_2}$.*
(4) *If $1 \le p_1 \le p_2 \le \infty$ and $s_1 = s_2 + d(\frac{1}{p_1} - \frac{1}{p_2})$, then $\mathring{B}^{s_1}_{p_1,q}(\mathbf{R}^d) \subset \mathring{B}^{s_2}_{p_2,q}(\mathbf{R}^d)$.*
(5) *If $1 \le p_1 \le p_2 \le \infty$, $1 \le q_1, q_2 \le \infty$, and $s_1 > s_2 + d(\frac{1}{p_1} - \frac{1}{p_2})$, then $B^{s_1}_{p_1,q_1}(\mathbf{R}^d) \subset B^{s_2}_{p_2,q_2}(\mathbf{R}^d)$.*
(6) *If $1 < p < \infty$, then*

$$B^s_{p,\min(p,2)} \subset W^{s,p} \subset B^s_{p,\max(p,2)}, \qquad \mathring{B}^s_{p,\min(p,2)} \subset \mathring{W}^{s,p} \subset \mathring{B}^s_{p,\max(p,2)}.$$

We will need a Bernstein type inequality for fractional derivatives.

Proposition A.4. *Let $\alpha \ge 0$. Let $1 \le p \le q \le \infty$.*

(1) *If f satisfies*

$$supp\, \widehat{f} \subset \{\xi \in \mathbf{R}^d : \ |\xi| \le K 2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbf{R}^d)} \le C_1\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbf{R}^d)}.$$

(2) *If f satisfies*

$$supp\, \widehat{f} \subset \{\xi \in \mathbf{R}^d : \ K_1 2^j \le |\xi| \le K_2 2^j\} \tag{A.6}$$

for some integer j and constants $0 < K_1 \le K_2$, then

$$C_1\, 2^{2\alpha j} \|f\|_{L^q(\mathbf{R}^d)} \le \|(-\Delta)^\alpha f\|_{L^q(\mathbf{R}^d)} \le C_2\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbf{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

The following proposition provides a lower bound for an integral originated from the dissipative term in the process of L^p estimates (see [7,24]).

Proposition A.5. *Assume either $\alpha \ge 0$ and $p = 2$ or $0 \le \alpha \le 1$ and $2 < p < \infty$. Let j be an integer and $f \in \mathcal{S}'$. Then*

$$\int_{\mathbf{R}^d} |\Delta_j f|^{p-2} \Delta_j f\, \Lambda^{2\alpha} \Delta_j f\, dx \ge C\, 2^{2\alpha j} \|\Delta_j f\|^p_{L^p}$$

for some constant C depending on d, α and p.

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