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Global rigidity theorems of hypersurfaces

Haizhong Li

0. Introduction

This paper is a continuation of our previous paper [14]. In Section 1, we first study the Cheng–Yau's self-adjoint operator $\Box$ for a given Codazzi tensor field $\phi=\sum_{i,j}\phi_{ij}\omega_i\omega_j$ on an n-dimensional compact Riemannian manifold. We obtain a general rigidity theorem (see Theorem 1.1) which generalizes Cheng–Yau's works ([5]). One of our conditions is

$$|\nabla\phi|^2 \geq |\nabla(\operatorname{tr}\phi)|^2, \tag{0.1}$$

which is the natural generalization of one of the following two conditions,

(1) $\operatorname{tr}\phi=\text{constant}$,

(2) $(\operatorname{tr}\phi)^2-|\phi|^2=\text{constant}\geq 0$.

We also note that the condition (0.1) comes out naturally when we study the operator $\Box$. Let M be an n-dimensional hypersurface in an $(n+1)$-dimensional real space form $R^{n+1}(c)$. Observing that the second fundamental form tensor h_{ij} is a natural Codazzi tensor on M, in Section 2, we apply the study of Section 1 to these hypersurfaces and obtain general rigidity results (see Theorem 2.1 and Theorem 2.2) which unify some existing results. Condition (0.1) becomes in this case

$$|\nabla B|^2 \geq n^2|\nabla H|^2, \tag{0.2}$$

where $|\nabla B|^2=\sum_{i,j,k}h_{ijk}^2$, $H=(1/n)\sum_k h_{kk}$. Thus condition (0.2) is the natural generalization of one of the following two conditions,

(1) $H=\text{constant}$,

(2) $R-c=\text{constant}\geq 0$, where R is the normalized scalar curvature.

The case (1) has been studied by many authors (see [24], [26], [17] and [2]); case (2) has been studied by [5] and [14]. Our rigidity theorems unify some existing results. In Section 3, we check the geometric meaning of our condition (0.2) for the simplest case $n=2$. If M is a W-surface, then we find that condition (0.2) is equivalent to the concept "*special W-surface*" which was first introduced by S. S. Chern [6] for surfaces in $\mathbf{R}^3$. Thus condition (0.2) can be considered a natural generalization of the concept of *special W-surfaces* to higher dimensional hypersurfaces. Our results in this section generalize Chern's results. Let M be an n-dimensional spacelike hypersurface in an $(n+1)$-dimensional Lorentzian space form $R_1^{n+1}(c)$. Observing that the second fundamental form tensor h_{ij} is a natural Codazzi tensor on M, in Section 4, we apply the study of Section 1 to these hypersurfaces and obtain some rigidity theorems which naturally generalize the existing results of Akutagawa [1], Ramanathan [21] and Montiel [15] about Goddard's conjecture [10]. In this section, we also propose two problems related to Goddard's conjecture.

1. Cheng–Yau's self-adjoint operator $\square$

Let M be an n-dimensional Riemannian manifold, $e_1, \dots, e_n$ a local orthonormal frame field on M, and $\omega_1, \dots, \omega_n$ its dual coframe field. Then the structure equations of M are given by

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} = -\omega_{ji}, \tag{1.1}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \tag{1.2}$$

where

$$\Omega_{ij} = -\tfrac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{1.3}$$

and

$$R_{ijkl} + R_{ijlk} = 0,$$

where ω_{ij} is the Levi–Civita connection form and R_{ijkl} is the Riemannian curvature tensor of M.

For any C^2-function f defined on M, we define its gradient and Hessian by the following formulas

$$df = \sum_i f_i \omega_i, \tag{1.4}$$

$$\sum_j f_{ij} \omega_j = df_i + \sum_j f_j \omega_{ji}. \tag{1.5}$$

We know that $f_{ij} = f_{ji}$ by exterior differentiation of (1.4).

Let $\phi=\sum_{i,j}\phi_{ij}\omega_i\otimes\omega_j$ be a symmetric tensor defined on M. The covariant derivative of ϕ_{ij} is defined by (see [5])

$$\sum_k \phi_{ijk}\omega_k = d\phi_{ij}+\sum_k \phi_{kj}\omega_{ki}+\sum_k \phi_{ik}\omega_{kj}. \tag{1.6}$$

We call the symmetric tensor $\phi=\sum_{i,j}\phi_{ij}\omega_i\otimes\omega_j$ a *Codazzi tensor*, if (see, for example, [9] or [23])

$$\phi_{ijk}=\phi_{ikj}. \tag{1.7}$$

The second covariant derivative of ϕ_{ij} is defined by

$$\sum_l \phi_{ijkl}\omega_l = d\phi_{ijk}+\sum_m \phi_{mjk}\omega_{mi}+\sum_m \phi_{imk}\omega_{mj}+\sum_m \phi_{ijm}\omega_{mk}. \tag{1.8}$$

By exterior differentiation of (1.6), we obtain

$$\sum_{l,k} \phi_{ijkl}\omega_l\wedge\omega_k = \sum_m \phi_{mj}\Omega_{mi}+\sum_m \phi_{im}\Omega_{mj}. \tag{1.9}$$

Therefore we have the following Ricci identities

$$\phi_{ijkl}-\phi_{ijlk} = \sum_m \phi_{mj}R_{mikl}+\sum_m \phi_{im}R_{mjkl}. \tag{1.10}$$

Remark 1.1. The concept of Codazzi tensor on a Riemannian manifold is a natural generalization of the second fundamental form of a hypersurface in a real space form. The class of manifolds admitting Codazzi tensor fields is large (see [9], [19], [23]).

We first recall the definition of the following self-adjoint operator $\square$ introduced by Cheng–Yau in [5].

Definition 1.1. Let $\phi=\sum_{i,j}\phi_{ij}\omega_i\otimes\omega_j$ be a Codazzi tensor field on a Riemannian manifold M. We define the operator $\square$ associated to ϕ by

$$\square f = \sum_{i,j}\left(\left(\sum_k \phi_{kk}\right)\delta_{ij}-\phi_{ij}\right)f_{ij},$$

for any C^2-function f defined on M.

Proposition 1.1. *Let M be a compact orientable Riemannian manifold. Then the operator $\Box$ is self-adjoint.*

Proof. Let $\varphi_{ij}=(\sum_k \phi_{kk})\delta_{ij}-\phi_{ij}$. Then

$$\sum_j \varphi_{ijj}=\left(\sum_k \phi_{kk}\right)_i-\sum_j \phi_{ijj}=0,$$

where we make use of the fact that ϕ_{ij} is a Codazzi tensor field on M. We complete the proof of Proposition 1.1 by applying Proposition 1 of [5].

The Laplacian of the tensor ϕ_{ij} is defined to be $\sum_k \phi_{ijkk}$, and therefore

$$\begin{aligned}\Delta\phi_{ij}&=\sum_k \phi_{ijkk}\\&=\sum_k(\phi_{ijkk}-\phi_{ikjk})+\sum_k(\phi_{ikjk}-\phi_{ikkj})+\sum_k(\phi_{ikkj}-\phi_{kkij})+\sum_k \phi_{kkij}\\&=\sum_{m,k}\phi_{mk}R_{mijk}+\sum_{m,k}\phi_{im}R_{mkjk}+\sum_k(\phi_{ijkk}-\phi_{ikjk})\\&\quad+\sum_k(\phi_{ikkj}-\phi_{kkij})+\left(\sum_k \phi_{kk}\right)_{ij}.\end{aligned}\tag{1.11}$$

By use of (1.7), we have

$$\Delta\phi_{ij}=\left(\sum_k \phi_{kk}\right)_{ij}+\sum_{m,k}\phi_{mk}R_{mijk}+\sum_{m,k}\phi_{im}R_{mkjk}.\tag{1.12}$$

Let $|\phi|^2=\sum_{i,j}\phi_{ij}^2$, $|\nabla\phi|^2=\sum_{i,j,k}\phi_{ijk}^2$ and $\operatorname{tr}\phi=\sum_k\phi_{kk}$. Then equation (1.12) shows that

$$\tfrac{1}{2}\Delta|\phi|^2=|\nabla\phi|^2+\sum_{i,j}\phi_{ij}(\operatorname{tr}\phi)_{ij}+\sum_{i,j,m,k}\phi_{ij}\phi_{mk}R_{mijk}+\sum_{i,j,m,k}\phi_{ij}\phi_{im}R_{mkjk}.\tag{1.13}$$

Near a given point $p\in M$, we choose a local orthonormal frame field $\{e_1,\dots,e_n\}$ and its dual frame field $\{\omega_1,\dots,\omega_n\}$ such that $\phi=\sum_{i,j}\phi_{ij}\omega_i\otimes\omega_j$, $\phi_{ij}=\lambda_i\delta_{ij}$ at p. Then (1.13) is simplified to

$$\tfrac{1}{2}\Delta|\phi|^2=|\nabla\phi|^2+\sum_i\lambda_i(\operatorname{tr}\phi)_{ii}+\tfrac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2.\tag{1.14}$$

Denoting the second symmetric function of ϕ_{ij} by m, we have

$$m=\sum_{i\neq j}\lambda_i\lambda_j=(\operatorname{tr}\phi)^2-|\phi|^2.\tag{1.15}$$

Combining (1.14) with (1.15), we obtain

$$\tfrac{1}{2}\Delta(\operatorname{tr}\phi)^2=\tfrac{1}{2}\Delta m+|\nabla\phi|^2+\sum_i \lambda_i(\operatorname{tr}\phi)_{ii}+\tfrac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2. \tag{1.16}$$

From Definition 1.1 of $\Box$, we have by (1.16)

$$\Box(\operatorname{tr}\phi)=\tfrac{1}{2}\Delta m+|\nabla\phi|^2-|\nabla(\operatorname{tr}\phi)|^2+\tfrac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2. \tag{1.17}$$

Since $\Box$ is self-adjoint and M is compact, we get by integration of (1.17)

$$\int_M[|\nabla\phi|^2-|\nabla(\operatorname{tr}\phi)|^2]+\int_M\tfrac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2=0. \tag{1.18}$$

Our first result is the following theorem.

Theorem 1.1. *Let $\phi=\sum_{i,j}\phi_{ij}\omega_i\otimes\omega_j$ be a Codazzi tensor field on a Riemannian manifold M. We assume the following condition*

$$|\nabla\phi|^2\geq|\nabla(\operatorname{tr}\phi)|^2. \tag{1.19}$$

(1) *If M has positive sectional curvature, then all the eigenvalues of ϕ_{ij} are the same on M.*

(2) *If M has nonnegative sectional curvature, then we have $|\nabla\phi|^2=|\nabla(\operatorname{tr}\phi)|^2$ and $R_{ijij}=0$, when $\lambda_i\neq\lambda_j$ on M.*

The following two lemmas show that condition (1.19) is natural.

Lemma 1.1. *If*

$$\operatorname{tr}\phi=\text{constant}, \tag{1.20}$$

then (1.19) *holds.*

Lemma 1.2. *If the second symmetric function of ϕ_{ij} is a nonnegative constant, i.e.*

$$m=\sum_{i\neq j}\lambda_i\lambda_j=(\operatorname{tr}\phi)^2-|\phi|^2=\text{constant}\geq 0, \tag{1.21}$$

then (1.19) *holds.*

Proof. Taking the covariant derivative of (1.15) and noting m=constant, we have for each k

$$(\operatorname{tr}\phi)(\operatorname{tr}\phi)_k=\sum_{i,j}\phi_{ij}\phi_{ijk}.$$

It follows that

$$(1.22)\quad (\operatorname{tr}\phi)^2|\nabla(\operatorname{tr}\phi)|^2=\sum\nolimits_k\Big(\sum_{i,j}\phi_{ij}\phi_{ijk}\Big)^2\leq\Big(\sum_{i,j}\phi_{ij}^2\Big)\Big(\sum_{i,j,k}\phi_{ijk}^2\Big)=|\phi|^2|\nabla\phi|^2.$$

On the other hand, from $m=(\operatorname{tr}\phi)^2-|\phi|^2\geq 0$, we get (1.19). This completes the proof of Lemma 1.2.

Corollary 1.1. *Let $\phi=\sum_{i,j}\phi_{ij}\omega_i\otimes\omega_j$ be a Codazzi tensor field on a Riemannian manifold M.*

(1) *If M has positive sectional curvature and* (1.20) *or* (1.21) *holds, then all the eigenvalues of ϕ_{ij} are the same on M.*

(2) *If M has nonnegative sectional curvature and* (1.20) *or* (1.21) *holds, then M is the closure of $\bigcup o_i$, where each point of the open set o_i has a product neighborhood $N_1\times\ldots\times N_l$ such that the tangent space of each N_i is spanned by eigenvectors of ϕ_{ij} with the same eigenvalue. In particular, when M is locally irreducible, all the eigenvalues of ϕ_{ij} are the same.*

Proof. From Theorem 1.1, Lemma 1.1 and Lemma 1.2, we only need to prove (2) of Corollary 1.1. Under the assumptions equality holds in (1.19). We have

$$\phi_{ijk}=c_k\phi_{ij},$$

where c_k are some numbers. If $\phi_{ij}=\lambda_i\delta_{ij}$, we have

$$(\lambda_i-\lambda_j)\omega_{ij}=0,\quad i\neq j.$$

Using the fact that $\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2=0$, we can prove that M is the closure of $\bigcup o_i$, where each point of the open set o_i has a product neighborhood $N_1\times\ldots\times N_l$ such that the tangent space of each N_i is spanned by eigenvectors of ϕ_{ij} with the same eigenvalue. This completes the proof of Corollary 1.1.

In this paper, we also need the following algebraic lemma which was first used by Okumura [18] (also see [26], [2] and [14]).

Lemma 1.3. *Let μ_i, $i=1,\dots,n$, be real numbers such that* $\sum_i \mu_i=0$ *and* $\sum_i \mu_i^2=\beta^2$, *where* $\beta=\text{constant}\geq 0$. *Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{1.23}$$

and equality holds in (1.23) *if and only if* $(n-1)$ *of the* μ_i *are equal.*

Proof. We can obtain Lemma 1.3 by using the method of Lagrange's multipliers to find the critical points of $\sum_i \mu_i^3$ subject to the conditions $\sum_i \mu_i=0$ and $\sum_i \mu_i^2=\beta^2$. We omit it here.

2. Hypersurfaces in a real space form

Let $R^{n+1}(c)$ be an $(n+1)$-dimensional Riemannian manifold with constant sectional curvature c. We also call it a *real space form.* When $c>0$, $R^{n+1}(c)= S^{n+1}(c)$ (i.e. $(n+1)$-dimensional sphere space); when $c=0$, $R^{n+1}(c)=\mathbf{R}^{n+1}$ (i.e. $(n+1)$-dimensional Euclidean space); when $c<0$, $R^{n+1}(c)=H^{n+1}(c)$ (i.e. $(n+1)$-dimensional hyperbolic space). Let M be an n-dimensional compact hypersurface in $R^{n+1}(c)$. For any $p\in M$ we choose a local orthonormal frame $e_1,\dots,e_n,e_{n+1}$ in $R^{n+1}(c)$ around p such that $e_1,\dots,e_n$ are tangential to M. Take the corresponding dual coframe $\{\omega_1,\dots,\omega_n,\omega_{n+1}\}$. In this paper we make the following convention on the range of indices,

$$1\leq A,B,C\leq n+1; \quad 1\leq i,j,k\leq n.$$

The structure equations of $R^{n+1}(c)$ are

$$d\omega_A=\sum_B \omega_{AB}\wedge\omega_B, \quad \omega_{AB}=-\omega_{BA},$$

$$d\omega_{AB}=\sum_C \omega_{AC}\wedge\omega_{CB}-c\omega_A\wedge\omega_B.$$

If we denote by the same letters the restrictions of ω_A, ω_{AB} to M, we have

$$d\omega_i=\sum_j \omega_{ij}\wedge\omega_j, \quad \omega_{ij}=-\omega_{ji}, \tag{2.1}$$

$$d\omega_{ij}=\sum_k \omega_{ik}\wedge\omega_{kj}-\tfrac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l, \tag{2.2}$$

where R_{ijkl} is the curvature tensor of the induced metric on M.

Restricted to M, $\omega_{n+1}=0$, thus

$$0=d\omega_{n+1}=\sum_i \omega_{n+1i}\wedge\omega_i, \tag{2.3}$$

and by Cartan's lemma we can write

$$\omega_{in+1}=\sum_j h_{ij}\omega_j, \quad h_{ij}=h_{ji}. \tag{2.4}$$

The quadratic form $B=\sum_{i,j} h_{ij}\omega_i\otimes\omega_j$ is the second fundamental form of M. The Gauss equation is

$$R_{ijkl}=c(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+h_{ik}h_{jl}-h_{il}h_{jk}, \tag{2.5}$$

$$n(n-1)(R-c)=n^2H^2-|B|^2, \tag{2.6}$$

where R is the normalized scalar curvature, $H=(1/n)\sum_i h_{ii}$ the mean curvature and $|B|^2=\sum_{i,j} h_{ij}^2$ the norm square of the second fundamental form of M, respectively.

The Codazzi equation is

$$h_{ijk}=h_{ikj}, \tag{2.7}$$

where the covariant derivative of the second fundamental form is defined by

$$\sum_k h_{ijk}\omega_k=dh_{ij}+\sum_k h_{kj}\omega_{ki}+\sum_k h_{ik}\omega_{kj}. \tag{2.8}$$

Let $\phi_{ij}=h_{ij}$ in Section 1 and $h_{ij}=\lambda_i\delta_{ij}$. We have from (1.18)

$$\int_M [|\nabla B|^2-n^2|\nabla H|^2]+\int_M \tfrac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i-\lambda_j)^2=0. \tag{2.9}$$

By use of (2.5), we have

$$\tfrac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i-\lambda_j)^2=nc|B|^2-n^2H^2c-|B|^4+nH\sum_i \lambda_i^3. \tag{2.10}$$

Let $\mu_i=\lambda_i-H$ and $|Z|^2=\sum_i \mu_i^2$. We have

$$\sum_i \mu_i=0, \quad |Z|^2=|B|^2-nH^2, \tag{2.11}$$

$$\sum_i \lambda_i^3=\sum_i \mu_i^3+3H|Z|^2+nH^3. \tag{2.12}$$

Putting (2.11), (2.12) into (2.10), we get

$$\frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i-\lambda_j)^2=|Z|^2(nc+nH^2-|Z|^2)+nH\sum_i \mu_i^3. \tag{2.13}$$

By use of Lemma 1.3, we have

$$\begin{aligned}\frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i-\lambda_j)^2 \\ \geq (|B|^2-nH^2)\left(nc+2nH^2-|B|^2-\frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{|B|^2-nH^2}\right).\end{aligned} \tag{2.14}$$

Putting (2.14) into (2.9), we obtain the following key integral inequality

$$\begin{aligned}&\int_M \Bigg[|\nabla B|^2-n^2|\nabla H|^2+(|B|^2-nH^2)\\ &\qquad\times\left(nc+2nH^2-|B|^2-\frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{|B|^2-nH^2}\right)\Bigg]\\ &=\int_M [|\nabla B|^2-n^2|\nabla H|^2]+\int_M \Bigg[(|B|^2-nH^2)\\ &\qquad\times\left(\sqrt{|B|^2-nH^2}+\frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}\,|H|+\sqrt{nc+\frac{n^3H^2}{4(n-1)}}\right)\\ &\qquad\times\left(-\sqrt{|B|^2-nH^2}-\frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}\,|H|+\sqrt{nc+\frac{n^3H^2}{4(n-1)}}\right)\Bigg]\\ &\leq 0.\end{aligned} \tag{2.15}$$

Note that we assume $n^2H^2+4(n-1)c\geq 0$, if $c<0$.
From (2.15), we get the following result.

Theorem 2.1. *Let M be an n-dimensional compact hypersurface in an $(n+1)$-dimensional real space form $R^{n+1}(c)$. If*

$$|\nabla B|^2\geq n^2|\nabla H|^2 \tag{2.16}$$

and

$$nH^2\leq |B|^2\leq nc+\frac{n^3}{2(n-1)}H^2-\frac{n-2}{2(n-1)}\sqrt{n^4H^4+4(n-1)n^2H^2c}, \tag{2.17}$$

then either

$$|B|^2\equiv nH^2$$

and M *is a totally umbilical hypersurface;*

or

$$|B|^2\equiv nc+\frac{n^3}{2(n-1)}H^2-\frac{n-2}{2(n-1)}\sqrt{n^4H^4+4(n-1)n^2H^2c} \tag{2.18}$$

and M *has two different principal curvatures* λ_1 *and* λ_n, *i.e.*

$$\lambda_1=...=\lambda_k=\frac{nH+\sqrt{n^2H^2+4k(n-k)c}}{2k},$$

$$\lambda_{k+1}=...=\lambda_n=\frac{nH-\sqrt{n^2H^2+4k(n-k)c}}{2(n-k)}$$

for some k *with* $1\leq k\leq n$.

Corollary 2.1. ([2] and [26]) *Let* M *be an* n*-dimensional compact hypersurface in an* $(n+1)$*-dimensional real space form* $R^{n+1}(c)$ *with constant mean curvature* H. *If* (2.17) *holds, then either*

(1) $|B|^2\equiv nH^2$ *and* M *is totally umbilical; or*

(2) $|B|^2\equiv nc+n^3H^2/2(n-1)-(n-2)\sqrt{n^4H^4+4(n-1)n^2H^2c}\,/2(n-1)$,

and case (2) *happens if and only if*

(a) *when* $H=0$, *then* $c>0$ *and* M *is a Clifford torus in* $S^{n+1}(c)$,

(b) *when* $H\neq0$, *then* $c>0$ *and* $M=S^{n-1}\times S^1$.

Remark 2.1. Except the statement of classification the results (a) and (b) were first proved by Sun Ziqi in 1984 and published in 1987 (in Chinese) (see Theorem 1 of [26]) under the guidance of Professor C. K. Peng. A complete statement of Corollary 2.1 was rediscovered by H. Alencar and M. do Carmo independently in 1992 and published in 1994 (see Theorem 1.5 of [2]).

Proof of Corollary 2.1. From Lemma 1.1 and Theorem 2.1 it follows that either $|B|^2\equiv nH^2$ and M is totally umbilical, or

$$|B|^2\equiv nc+n^3H^2/2(n-1)-(n-2)\sqrt{n^4H^4+4(n-1)n^2H^2c}\,/2(n-1).$$

In the latter case, when $H=0$, we have $c>0$ and the conclusion comes from [7] or [12]; when $H\neq0$, we have $|\nabla B|=0$ and $n-1$ of the λ_i are equal by Lemma 1.3. Let $H>0$, without loss of generality, and $\lambda_1=...=\lambda_{n-1}\neq\lambda_n$. Then from $(n-1)\lambda_1+\lambda_n=nH$ and $R_{1n1n}=\lambda_1\lambda_n+c=0$,

$$\lambda_1=\frac{nH+\sqrt{n^2H^2+4(n-1)c}}{2(n-1)},\quad \lambda_n=\frac{nH-\sqrt{n^2H^2+4(n-1)c}}{2}.$$

When $c>0$, $M=S^{n-1}(1/\lambda_1)\times S^1(1/\lambda_n)$; the case $c\leq0$ does not happen since M is compact. This completes the proof of Corollary 2.1.

Corollary 2.2. ([14]) *Let M be an n-dimensional $(n\geq 3)$ compact hypersurface with constant normalized scalar curvture R in an $(n+1)$-dimensional real space form $R^{n+1}(c)$. Assume*

(1) $\bar{R}\equiv R-c\geq 0$,

(2) *the norm square $|B|^2$ of the second fundamental form of M satisfies*

$$n\bar{R}\leq |B|^2\leq \frac{n[n(n-1)\bar{R}^2+4(n-1)\bar{R}c+nc^2]}{(n-2)(n\bar{R}+2c)}. \tag{2.19}$$

Then either

$$|B|^2\equiv n\bar{R}, \tag{2.20}$$

and M is totally umbilical; *or*

$$|B|^2\equiv \frac{n[n(n-1)\bar{R}^2+4(n-1)\bar{R}c+nc^2]}{(n-2)(n\bar{R}+2c)}, \tag{2.21}$$

and (2.21) *holds if and only if $c>0$ and $M=S^{n-1}(1/\lambda_1)\times S^1(1/\lambda_n)$.*

Proof. Choosing $\phi_{ij}=h_{ij}$ in Lemma 1.2, we have from the Gauss equation $n^2H^2-|B|^2=n(n-1)\bar{R}\geq 0$,

$$|\nabla B|^2\geq n^2|\nabla H|^2. \tag{2.22}$$

Again from the Gauss equation (2.6), we find that condition (2.17) is equivalent to (2.19), noting that the cases $c\leq 0$ do not happen since M is compact. Thus we obtain Corollary 2.2 from Theorem 2.1.

Remark 2.2. When M is an n-dimensional embedded hypersurface in $R^{n+1}(c)$, Corollary 2.1 and Corollary 2.2 hold without the conditions (2.18) and (2.19), respectively (see [16], [22]) (in this case M is totally umbilical).

Remark 2.3. From the *main theorem* on p. 1052 of [13], we can prove that condition (2.17) or (2.19) implies $\mathrm{Ric}(M)\geq 0$. We also can prove that if

$$nH^2\leq |B|^2\leq nc+\frac{n^3H^2}{2(n-1)}-\frac{n-2}{2(n-1)}\sqrt{n^4H^4+4(n-1)n^2H^2c}-\varepsilon, \tag{2.17'}$$

or

$$n\bar{R}\leq |B|^2\leq \frac{n[n(n-1)\bar{R}^2+4(n-1)\bar{R}c+nc^2]}{(n-2)(n\bar{R}+2c)}-\varepsilon \tag{2.19'}$$

holds for some small positive number ε, then $\mathrm{Ric}(M)\geq a(\varepsilon)>0$. Thus from Bonnet–Myers' theorem, Theorem 2.1 and Corollary 2.1 hold if we substitute the condition "compact and (2.17)" by "complete and (2.17′)", Corollary 2.2 holds if we substitute the condition "compact and (2.19)" by "complete and (2.19′)". In this case, it is not necessary to refer to Omori and Yau's generalized maximum principle as many people do. (See [20], [27].)

Remark 2.4. Let M be an n-dimensional complete hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbf{R}^{n+1}$. In this case, (2.17) becomes

$$nH^2\leq|B|^2\leq\frac{n^2H^2}{n-1}, \tag{2.17″}$$

and (2.19) becomes

$$nR\leq|B|^2\leq\frac{n(n-1)}{n-2}R. \tag{2.19″}$$

From an inequality of Chen–Okumura [3], we know that (2.17″) or (2.19″) implies that the sectional curvature K of M is nonnegative, i.e., $K\geq 0$. Thus, from Hartman's theorem [11], we obtain the following result.

Proposition 2.1. *Let M be an n-dimensional complete hypersurface in an $(n+1)$-dimensional Euclidean space $\mathbf{R}^{n+1}$. If the mean curvature H is constant and (2.17″) holds, or if the normalized scalar curvature R is constant and (2.19″) holds, then either M is totally umbilical, or $M=S^{n-1}\times\mathbf{R}^1$.*

Choosing $\phi_{ij}=h_{ij}=\lambda_i\delta_{ij}$ in Theorem 1.1 and noting that $R_{ijij}=c+\lambda_i\lambda_j$, we obtain the following theorem.

Theorem 2.2. *Let M be an n-dimensional compact hypersurface in an $(n+1)$-dimensional real space form $R^{n+1}(c)$.*

(1) *If M has positive sectional curvature and (2.22) holds, then M is totally umbilical.*

(2) *If M has nonnegative sectional curvature and (2.22) holds, then either M is totally umbilical, or M has the following two different principal curvatures*

$$\lambda_1=\ldots=\lambda_k=\frac{nH+\sqrt{n^2H^2+4k(n-k)c}}{2k},$$
$$\lambda_{k+1}=\ldots=\lambda_n=\frac{nH-\sqrt{n^2H^2+4k(n-k)c}}{2(n-k)},$$

where $1\leq k\leq n$.

When H=constant, we have the following corollary.

Corollary 2.3. ([17]) *Let M be an n-dimensional compact hypersurface in an $(n+1)$-dimensional real space form $R^{n+1}(c)$ with constant mean curvature H. If M has nonnegative sectional curvature, then either M is totally umbilical, or $c>0$ and $M=S^{n-k}\times S^k$, $1\le k\le n$.*

Corollary 2.4. ([5]) *Let M be an n-dimensional compact hypersurface with nonnegative sectional curvature in an $(n+1)$-dimensional real space form $R^{n+1}(c)$. Suppose the normalized scalar curvature of M is constant and not less than c. Then M is either totally umbilical, or $c>0$ and $M=S^{n-k}\times S^k$, $1\le k\le n$.*

Proof. Since we assume $\bar{R}\equiv R-c=\text{constant}\ge 0$, we have by (2.6)

$$n^2H^2-|B|^2=\text{constant}\ge 0. \tag{2.23}$$

Thus (2.22) holds by Lemma 1.2. We conclude that there are at most two constant and distinct λ_i's (thus we complete the proof of Corollary 2.4) by Theorem 2.2 and the assumption $\sum_{i\neq j}\lambda_i\lambda_j=n(n-1)R=\text{constant}$.

Corollary 2.5. *Let M be an n-dimensional compact hypersurface with non-negative sectional curvature in an $(n+1)$-dimensional real space form $R^{n+1}(c)$ $(c\ge 0)$. Suppose the normalized scalar curvature R is proportional to the mean curvature H of M, that is*

$$R=aH,\quad a^2>\frac{4nc}{n-1}, \tag{2.24}$$

where a is a constant. Then M is either totally umbilical, or $c>0$ and $M=S^{n-k}\times S^k$, $1\le k\le n$.

Proof. By use of the Gauss equation (2.6) and the assumption (2.24), we have

$$|B|^2=n^2H^2+n(n-1)(c-aH). \tag{2.25}$$

Taking the covariant derivative of (2.25), we have for each k

$$2\sum_{i,j}h_{ij}h_{ijk}=(2n^2H-n(n-1)a)H_k.$$

It follows that

$$4|B|^2|\nabla hB|^2\ge 4\sum_k\left(\sum_{i,j}h_{ij}h_{ijk}\right)^2=(2n^2H-n(n-1)a)^2|\nabla H|^2. \tag{2.26}$$

By (2.24) and (2.25), we have

$$(2.27)\qquad \begin{aligned}(2n^2H-n(n-1)a)^2-4n^2|B|^2 &= (4n^4H^2+n^2(n-1)^2a^2-4n^3(n-1)Ha)\\ &\quad -n^2(4n^2H^2+4n(n-1)(c-aH))\\ &= n^2(n-1)((n-1)a^2-4nc)>0.\end{aligned}$$

Combining (2.26) with (2.27), we see that (2.22) holds. Thus we conclude that there are at most two constant and distinct λ_i's by Theorem 2.2 and the assumption (2.24). This completes the proof of Corollary 2.5.

3. Surfaces in a 3-dimensional real space form $R^3(c)$

In this section we will check the geometric meaning of the condition

$$(3.1)\qquad |\nabla B|^2 \geq n^2|\nabla H|^2$$

in the simplest case $n=2$.

Let M be a surface in a 3-dimensional real space form $R^3(c)$ with induced metric $ds^2=\omega_1^2+\omega_2^2$. In this case the Gauss equation (2.6) is

$$(3.2)\qquad K=c+\lambda_1\lambda_2,$$

that is,

$$(3.2')\qquad 2(K-c)=4H^2-|B|^2.$$

We have

$$|\nabla B|^2=h_{111}^2+3h_{112}^2+3h_{221}^2+h_{222}^2$$

and

$$\begin{aligned}4|\nabla H|^2 &= (h_{111}+h_{221})^2+(h_{112}+h_{222})^2\\ &= h_{111}^2+h_{221}^2+h_{112}^2+h_{222}^2+2h_{111}h_{221}+2h_{112}h_{222}.\end{aligned}$$

Thus we know that

$$(3.3)\qquad |\nabla B|^2\geq 4|\nabla H|^2$$

is equivalent to

$$(3.3')\qquad h_{112}^2+h_{122}^2\geq h_{111}h_{122}+h_{112}h_{222}.$$

We first recall a notion introduced by S. S. Chern for surfaces in 3-dimensional Euclidean space (see [6]).

Definition 3.1. Let M be a surface in a 3-dimensional real space form $R^3(c)$. At a point of M, let λ_1 and λ_2 denote the principal curvatures. We call M a *W-surface* if $d\lambda_1$ and $d\lambda_2$ are linear dependent, that is, if there exist functions f and g, not both zero, such that

$$f d\lambda_1 + g\lambda_2 = 0. \tag{3.4}$$

We call M a *special W-surface*, if the functions f and g in (3.4) can be chosen to be positive, $f>0$, $g>0$.

Now let M be a special W-surface, i.e. there exist functions $f>0$ and $g>0$ such that (3.4) holds.

By (2.8), it is a direct check that

$$h_{ii1} = (\lambda_i)_1, \quad h_{ii2} = (\lambda_i)_2, \quad i=1,2, \tag{3.5}$$

where $d\lambda_i = (\lambda_i)_1\omega_1 + (\lambda_i)_2\omega_2$, $i=1,2$.

The equation (3.4) can be written as

$$f(\lambda_1)_i + g(\lambda_2)_i = 0, \quad i=1,2, \tag{3.6}$$

where $f>0$, $g>0$ on M.

Combining (3.5) with (3.6), we have

$$f h_{111} + g h_{221} = 0, \quad f h_{112} + g h_{222} = 0. \tag{3.7}$$

Thus (3.3′) holds, i.e. (3.3) holds. From Theorem 2.2, we obtain the following theorem.

Theorem 3.1. *Let M be a compact special W-surface in a 3-dimensional real space form $R^3(c)$ with nonnegative sectional curvature. Then either M is totally umbilical, or M is flat.*

Proof. The last statement of Theorem 3.1 comes from $K\equiv 0$ when $\lambda_1 \neq \lambda_2$.

Corollary 3.1. *Let M be a compact surface in a 3-dimensional real space form $R^3(c)$ with nonnegative sectional curvature, i.e. $K\geq 0$. If*

$$a(K-c) + bH + d = 0, \tag{3.9}$$

a, b, d being constants such that $b^2-4ad>0$, then either M is totally umbilical, or M is flat.

Proof. Let $F(\lambda_1,\lambda_2) = a(K-c)+bH+d=0$. We have

$$\frac{\partial F}{\partial \lambda_1}\frac{\partial F}{\partial \lambda_2} = a^2(K-c)+abH+\frac{b^2}{4} > a^2(K-c)+abH+ad = 0.$$

This completes the proof of Corollary 3.1.

Corollary 3.2. *A convex special W-surface in the 3-dimensional Euclidean space $\mathbf{R}^3$ is a sphere.*

Proof. We only need to note that M is called convex, if $K>0$ on M.

Corollary 3.3. *Let M be a complete surface in $R^3(c)$ with constant Gauss curvature K. If $K>\max(c,0)$, then M is totally umbilical.*

Remark 3.1. For $n=2$, our condition (3.1) is almost equivalent to the concept *"special W-surface"* first introduced by S. S. Chern [6]. Thus condition (3.1) can be considered a natural generalization of the concept of *"special W-surface"* to the higher hypersurfaces in $R^{n+1}(c)$.

4. Spacelike hypersurfaces in a Lorentzian space form

Let $R_1^{n+1}(c)$ be an $(n+1)$-dimensional Lorentzian manifold of constant curvature c; we also call it a *Lorentzian space form.* When $c>0$, $R_1^{n+1}(c)=S_1^{n+1}(c)$ (i.e. $(n+1)$-dimensional de Sitter space); when $c=0$, $R_1^{n+1}(c)=\mathbf{L}^{n+1}$ (i.e. $(n+1)$-dimensional Lorentz–Minkowski space); when $c<0$, $R_1^{n+1}(c)=H_1^{n+1}(c)$ (i.e. $(n+1)$-dimensional anti de Sitter space) (see, for example, [21]).

Let M be an n-dimensional compact spacelike hypersurface in $R_1^{n+1}(c)$. For any $p\in M$ we choose a local orthonormal frame $e_1, \dots, e_n, e_{n+1}$ in $R_1^{n+1}(c)$ around p such that $e_1, \dots, e_n$ are tangential to M. Take the corresponding dual coframe $\{\omega_1, \dots, \omega_n, \omega_{n+1}\}$ with the matrix of connection one forms being ω_{ij}. The metric of $R_1^{n+1}(c)$ is given by $\overline{ds^2}=\sum_i \omega_i^2-\omega_{n+1}^2$. We make the convention on the range of indices that $1\leq i,j,k\leq n$.

A well-known argument [4] shows that the forms ω_{in+1} may be expressed as $\omega_{in+1}=\sum_j h_{ij}\omega_j$, $h_{ij}=h_{ji}$. The second fundamental form $B=\sum_{i,j} h_{ij}\omega_i\otimes\omega_j$. The mean curvature of M is given by $H=(1/n)\sum_i h_{ii}$.

The Gauss equations are

$$R_{ijkl}=c(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})-(h_{ik}h_{jl}-h_{il}h_{jk}), \tag{4.1}$$

$$R_{ij}=(n-1)c\delta_{ij}-nHh_{ij}+\sum_k h_{ik}h_{kj}, \tag{4.2}$$

$$n(n-1)(R-c)=-n^2H^2+|B|^2, \tag{4.3}$$

where R is the normalized scalar curvature, and $|B|^2=\sum_{i,j} h_{ij}^2$ the norm square of the second fundamental form of M, respectively.

The Codazzi equation is

$$h_{ijk}=h_{ikj}, \tag{4.4}$$

where the covariant derivative of h_{ij} is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}.$$

Let $\phi_{ij} = h_{ij} = \lambda_i \delta_{ij}$ in Section 1. We have from (1.18)

$$\int_M [|\nabla B|^2 - n^2|\nabla H|^2] + \int_M \tfrac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 = 0. \tag{4.5}$$

By use of (4.1), we have

$$\tfrac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 = nc|B|^2 - n^2H^2c + |B|^4 - nH\sum_i \lambda_i^3. \tag{4.6}$$

Let $\mu_i = \lambda_i - H$ and $|Z|^2 = \sum_i \mu_i^2$. We have

$$\sum_i \mu_i = 0, \quad |Z|^2 = |B|^2 - nH^2, \tag{4.7}$$

$$\sum_i \lambda_i^3 = \sum_i \mu_i^3 + 3H|Z|^2 + nH^3. \tag{4.8}$$

Putting (4.7), (4.8) into (4.6), we get

$$\tfrac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 = |Z|^2(nc - nH^2 + |Z|^2) - nH\sum_i \mu_i^3. \tag{4.9}$$

By use of Lemma 1.3 and (4.7) we have

$$\begin{aligned}\tfrac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 \\ \geq (|B|^2 - nH^2)\left(nc - 2nH^2 + |B|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{|B|^2 - nH^2}\right).\end{aligned} \tag{4.10}$$

Putting (4.10) into (4.5), we obtain

$$\begin{aligned}\int_M \Bigg[& |\nabla B|^2 - n^2|\nabla H|^2 \\ & + (|B|^2 - nH^2)\left(nc + |B|^2 - 2nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{|B|^2 - nH^2}\right)\Bigg] \leq 0.\end{aligned} \tag{4.11}$$

Note that

$$\begin{aligned} & nc - 2nH^2 + |B|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{|B|^2 - nH^2} \\ & = \left(\sqrt{|B|^2 - nH^2} - \frac{1}{2}(n-2)|H|\sqrt{\frac{n}{n-1}}\right)^2 + n\left(c - \frac{n^2}{4(n-1)}H^2\right).\end{aligned} \tag{4.12}$$

Theorem 4.1. *Let M be an n-dimensional compact spacelike hypersurface in de Sitter space $S_1^{n+1}(c)$. If $|\nabla B|^2 \geq n^2|\nabla H|^2$ and $H^2 \leq 4(n-1)c/n^2$, then M is totally umbilical.*

Proof. Under the assumptions of Theorem 4.1, we have from (4.11) and (4.12),

$$H^2 \equiv \frac{4(n-1)}{n^2}c, \quad R_{ijij} = c - \lambda_i\lambda_j = 0, \quad \text{when } \lambda_i \neq \lambda_j.$$

Thus M has at most two distinct constant principal curvatures. We conclude that M is totally umbilical from the compactness of M. This completes the proof of Theorem 4.1.

Theorem 4.2. *Let M be an n-dimensional complete spacelike hypersurface in de Sitter space $S_1^{n+1}(c)$. If $|\nabla B|^2 \geq n^2|\nabla H|^2$ and $H^2 \leq 4(n-1)c/n^2 - \varepsilon$, for some given small positive real number ε, then M is totally umbilical.*

Proof. From (4.2) and the assumption we obtain

$$(4.13) \qquad \begin{aligned} R_{ii} &= (n-1)c - nH\lambda_i + \lambda_i^2 = \left(\lambda_i - \tfrac{1}{2}nH\right)^2 + (n-1)c - \tfrac{1}{4}n^2H^2 \\ &\geq (n-1)c - \tfrac{1}{4}n^2H^2 \geq \tfrac{1}{4}n^2\varepsilon. \end{aligned}$$

This completes the proof of Theorem 4.2 if we apply Bonnet–Myers' theorem and Theorem 4.1.

Corollary 4.1. ([1], [21] or [15]) *Let M be an n-dimensional complete spacelike hypersurface in de Sitter space $S_1^{n+1}(c)$ with constant mean curvature H satisfying $H^2 < 4(n-1)/n^2$. Then M is totally umbilical.*

Proof. Since the constant mean curvature H satisfies $H^2 < 4(n-1)/n^2$, we can choose ε with $4(n-1)/n^2 - H^2 > \varepsilon > 0$. We obtain Corollary 4.1 from Theorem 4.2.

Remark 4.1. Theorem 4.2 is the best possible ($n>2$) since Corollary 4.1 is the best possible (see [15]).

Goddard [10] conjectured that complete spacelike hypersurfaces with constant mean curvature H must be totally umbilical. Later, Akutagawa [1] has proved that Goddard's conjecture is true when $H^2 < 4(n-1)c/n^2$, if $n>2$, and when $H^2 \leq c$, if $n=2$. (Ramanathan [21] has independently studied the case $n=2$.) It was pointed out that the conjecture is false by Akutagawa [1] and Ramanathan [21] when $H^2 > c$, in case $n=2$ and by Montiel [15] when $H^2 \geq 4(n-1)c/n^2$ in case $n>2$. Moreover, Montiel [15] solved Goddard's conjecture in the compact case without restrictions on the range of H. In this paper, we also prove the following theorem.

Theorem 4.3. *Let M be an n-dimensional compact spacelike hypersurface in de Sitter space $S_1^{n+1}(c)$ with constant normalized scalar curvature R. If*

$$\frac{n-2}{n}c \le R \le c, \tag{4.14}$$

then M is totally umbilical.

Proof. By (4.3) and (4.14)

$$n^2H^2 - |B|^2 \ge 0. \tag{4.15}$$

Choosing $\phi_{ij}=h_{ij}$ in Lemma 1.2, we have

$$|\nabla B|^2 \ge n^2|\nabla H|^2. \tag{4.16}$$

On the other hand, from (4.3), we know that

$$nc - 2nH^2 + |B|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{|B|^2 - nH^2} \ge 0 \tag{4.17}$$

is equivalent to
(4.18)

$$nc + \frac{n-2}{n}|B|^2 + 2(n-1)(R-c) - \frac{n-2}{n}\sqrt{(|B|^2+n(R-c))(|B|^2-n(n-1)(R-c))} \ge 0.$$

It is a direct check that the assumption $R \ge (n-2)c/n$ implies that (4.18) holds. Thus from (4.11), (4.16) and (4.17), we can prove that M is totally umbilical just as the proof of Theorem 4.1.

Comparing Theorem 4.3 with Montiel's result about constant mean curvature, we find that the following problem is very interesting.

Problem 1. Let M be an n-dimensional compact spacelike hypersurface in the $(n+1)$-dimensional de Sitter space $S_1^{n+1}(1)$ with constant scalar curvature. Is M totally umbilical?

Now we consider two examples.

Example 4.1 (see Example 2 of [15]). Consider the spacelike hypersurface embedded into $S_1^{n+1}(1)$ given by

$$M_r = \{x \in S_1^{n+1}(1) \mid -x_0^2 + x_1^2 = -\sinh^2 r\},$$

with r a positive real number. The hyperspace M_r is isometric to the Riemannian product $H^1(1-\coth^2 r) \times S^{n-1}(1-\tanh^2 r)$ of a 1-dimensional hyperbolic space and an $(n-1)$-dimensional sphere of constant sectional curvatures $1-\coth^2 r$ and $1-\tanh^2 r$, respectively. Then M has two distinct principal curvatures

$$\lambda_1 = \ldots = \lambda_{n-1} = \tanh r, \quad \lambda_n = \coth r,$$

and

$$R = 1 - \frac{1}{n}(2 + (n-2)\tanh^2 r).$$

Thus for any R satisfying

$$0 < R < \frac{n-2}{n}, \tag{4.19}$$

we can choose some r such that the hypersurface M_r above is complete, not totally umbilical and has constant scalar curvature R satisfying (4.19).

Example 4.2. (See [15].) Consider the spacelike hypersurface in $S_1^{n+1}(1)$ given by

$$M_r = H^{n-1}(1-\coth^2 r) \times S^1(1-\tanh^2 r) \quad (n > 2)$$

with r a positive real number. Then $\lambda_1 = \tanh r$, $\lambda_2 = \ldots = \lambda_n = \coth r$, $R = 1 - (2 + (n-2)\coth^2 r)/n$.

Thus for any R satisfying

$$R < 0, \tag{4.20}$$

we can choose some r such that the hypersurface M_r above is complete, not totally umbilical and has constant scalar curvature R satisfying (4.20).

Combining Theorem 4.3 with Examples 4.1 and 4.2, we find the following problem interesting.

Problem 2. Let M be an n-dimensional complete spacelike hypersurface in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(1)$ $(n \geq 3)$ with constant normalized scalar curvature R satisfying

$$\frac{n-2}{n} \leq R \leq 1.$$

Is M totally umbilical?

In this part of this paper, we consider the classification of the complete spacelike surfaces in the 3-dimensional de Sitter space $S_1^3(1)$ with constant Gauss curvature.

Proposition 4.1. *Let M be a complete spacelike surface in $S_1^3(1)$ with constant Gauss curvature K satisfying*

$$0 < K \leq 1. \tag{4.21}$$

Then M is totally umbilical.

Proof. Noting that $K>0$ implies that M is compact by Bonnet–Myers' theorem, we obtain Proposition 4.1 by applying Theorem 4.3 to the case $n=2$.

The following two examples show that there exist complete spacelike surfaces with constant Gauss curvature K, where K takes all possible values in the range $(-\infty, 0]$.

Example 4.3. (Cf. pp. 17–18 of [1].) Let M be a spacelike rotation surface in $S_1^3(1)$,

$$f(s,t) = (x^0(s), x(s)\cos t, x(s)\sin t, z(s)),$$

where

$$x^0(s) = (x(s)^2-1)^{1/2}\cosh\phi(s), \quad z(s) = (x(s)^2-1)^{1/2}\sinh\phi(s), \quad x(s) > 1,$$
$$\phi(s) = \int_0^s (-1+x(u)^2+x'(u)^2)^{1/2}(x'(u)^2-1)^{-1}\,du.$$

Then the principal curvature along the coordinate t (resp. s) (see the proposition on p. 18 of [1]) is given by

$$\lambda_1 = -(-1+x^2+(x')^2)^{1/2}/x,$$
$$\lambda_2 = -(x''+x)/(-1+x^2+(x')^2)^{1/2}.$$

By the Gauss equation $\lambda_1\lambda_2 = 1-K$, for any constant $K<0$, and the equation

$$x'' + Kx = 0,$$

has a solution $x(s) = A\cosh(\sqrt{-K}s)$, with a constant $A>1$. It is easily verified that the spacelike surface above is complete, not totally umbilical and with $K=$ constant <0.

Example 4.4. (See Example 11 of [21].) For $t>0$ define $f_t\colon R^2 \to S_1^3(1)$ by

$$(x_1,x_2) \mapsto \left(t\cosh\left(\frac{x_1}{t}\right), t\sinh\left(\frac{x_2}{t}\right),\right.$$
$$\left.(1+t^2)^{1/2}\cos\left(\frac{x_2}{(1+t^2)^{1/2}}\right), (1+t^2)^{1/2}\sin\left(\frac{x_2}{(1+t^2)^{1/2}}\right)\right).$$

These surfaces have been studied by Dajczer and Nomizu [8], who have proved that f_t induces the standard flat metric on $\mathbf{R}^2$ and has principal curvatures $t/(1+t^2)^{1/2}$ and $(1+t^2)^{1/2}/t$, i.e. $K\equiv 0$. These surfaces are not totally umbilical and complete.

The following proposition can be proved by a similar method as Theorem 41 on p. 137 of [25].

Proposition 4.2. *There exists no complete spacelike surface in the* 3*-dimensional de Sitter space* $S_1^3(1)$ *with constant Gauss curvature* $K>1$.

Combining Proposition 4.1 with Proposition 4.2, we have the following theorem.

Theorem 4.4. *Let* M *be a complete spacelike surface in the* 3*-dimensional de Sitter space* $S_1^3(1)$ *with constant Gauss curvature* $K>0$*. Then* M *is totally umbilical.*

Remark 4.1. We conclude that Theorem 4.4 is the best possible in view of Examples 4.3 and 4.4.

Choosing $\phi_{ij}=h_{ij}=\lambda_i\delta_{ij}$ in Theorem 1.1 and noting that $R_{ijij}=c-\lambda_i\lambda_j$, we obtain the following result.

Proposition 4.3. *Let* M *be an* n*-dimensional compact spacelike hypersurface with nonnegative sectional curvature in an* $(n+1)$*-dimensional de Sitter space* $S_1^{n+1}(c)$*. Suppose that*

$$|\nabla B|^2\geq n^2|\nabla H|^2. \tag{4.22}$$

Then either M *is totally umbilical, or* M *has two different principal curvatures.*

Corollary 4.2. *Assume that* M *is an* n*-dimensional compact spacelike hypersurface with nonnegative sectional curvature in an* $(n+1)$*-dimensional de Sitter space* $S_1^{n+1}(c)$*. Suppose that one of the following conditions holds:*

(1) *the mean curvature* H *is constant,*

(2) *the normalized scalar curvature* R *is constant and not greater than* c*.*

Then M *is totally umbilical.*

Proof. It is clear that case (1) implies (4.22). Now we assume that $R-c=$ constant≤ 0. By the Gauss equation (4.3), we have

$$n^2H^2-|B|^2=n(n-1)(c-R)=\text{constant}\geq 0.$$

Thus (4.22) holds by Lemma 1.2. We conclude that there are at most two constant and distinct λ_i's by Proposition 4.3 and assumption (1) or (2). It follows that M is totally umbilical from the compactness of M. This completes the proof of Corollary 4.2.

Corollary 4.3. *Assume that M is an n-dimensional compact spacelike hypersurface with nonnegative sectional curvature in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(c)$. Suppose the normalized scalar curvature R is proportional to the mean curvature H of M, that is*

$$R = aH, \tag{4.23}$$

where a is any constant. Then M is totally umbilical.

Proof. By use of the Gauss equation (4.3) and the assumption (4.23), we have

$$|B|^2 = n^2H^2 - n(n-1)(c - aH). \tag{4.24}$$

Taking the covariant derivative of (4.24), we have for every k

$$2\sum_{i,j} h_{ij}h_{ijk} = (2n^2H + n(n-1)a)H_k.$$

It follows that

$$4|B|^2|\nabla hB|^2 \geq 4\sum_k\Big(\sum_{i,j} h_{ij}h_{ijk}\Big)^2 = (2n^2H + n(n-1)a)^2|\nabla H|^2. \tag{4.25}$$

By (4.23) and (4.24), we have

$$\begin{aligned}(2n^2H + n(n-1)a)^2 - 4n^2|B|^2 &= (4n^4H^2 + n^2(n-1)^2a^2 + 4n^3(n-1)Ha) \\ &\quad - n^2(4n^2H^2 - 4n(n-1)(c - aH)) \\ &= n^2(n-1)((n-1)a^2 + 4nc) > 0.\end{aligned} \tag{4.26}$$

Combining (4.25) with (4.26), we find that (4.22) holds. Thus we conclude that there are at most two constant and distinct λ_i's by Proposition 4.3 and the assumption (4.23). It follows that M is totally umbilical from the compactness of M. This completes the proof of Corollary 4.3.

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Haizhong Li
Department of Applied Mathematics
Tsinghua University
Beijing, 100084
People's Republic of China
email: hli@math.tsinghua.edu.cn