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Global Smooth Solutions to the n-Dimensional Damped Models of Incompressible Fluid Mechanics with Small Initial Datum

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Abstract In this paper, we consider the n-dimensional ($n \geq 2$) damped models of incompressible fluid mechanics in Besov spaces and establish the global (in time) regularity of classical solutions provided that the initial data are suitable small.

Keywords Boussinesq equations · Surface quasi-geostrophic equation · Magnetohydrodynamics equations · Damping · Global regularity · Small initial datum

Mathematics Subject Classification 35Q35 · 35B35 · 35B65 · 76D03

1 Introduction

The n-dimensional incompressible damped Boussinesq system concerned here can be represented in the form

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$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nu u + \nabla P = \theta e_n, \\ \partial_t \theta + (u \cdot \nabla)\theta + \eta\theta = 0, \\ \nabla \cdot u = 0 \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \end{cases} \tag{1.1}$$

where $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is a vector field denoting the velocity, $\theta : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is a scalar function denoting the temperature in the content of thermal convection and the density in the modeling of geophysical fluids, P is the scalar pressure, e_n is the unit vector in the x_n direction, and $\nu \geq 0$ and $\eta \geq 0$ are real parameters. In the case when $\nu = \eta = 0$, the system (1.1) reduces to the standard inviscid n-dimensional Boussinesq system. Also the system (1.1) becomes the standard viscous Boussinesq equations when νu and $\eta\theta$ are replaced by $-\nu\Delta u$ and $-\eta\Delta\theta$, respectively.

The Boussinesq system is extensively used in the atmospheric sciences and oceanographic turbulence in which rotation and stratification are important (see for example Pedlosky (1987) and references therein). Thus, over the past few years, the Boussinesq system has been studied extensively theoretically, see Abidi and HmidI (2007), Cao and Wu (2013), Chae (2006), Chae and Wu (2012), Cao and Titi (2007), Chae and Nam (1997), Constantin and Vicol (2012), Danchin and Paicu (2009, 2011), Shu (1994), Hmidi et al. (2010, 2011), Hou and Li (2005), Larios et al. (2010), Miao and Xue (2011), Xu (2010), Xu and Ye (2013), Ye (2014) and references therein. Mathematically the 2D Boussinesq equations serve as a lower dimensional model of the 3D hydrodynamics equations. In fact, the Boussinesq equations retain some key features of the 3D Navier-Stokes and the Euler equations such as the vortex-stretching mechanism. As pointed out in Majda and Bertozzi (2002), the inviscid Boussinesq equations can be identified with the 3D Euler equations for axisymmetric flows. Constructing global unique solutions for some nonconstant θ_0 is a challenging open problem (even in the two-dimensional case) which has many similarities with the global existence problem for the three-dimensional incompressible Euler equations. The global well-posedness of completely inviscid Boussinesq system is still an outstanding open problem.

In this paper, we are also concerned with the following Euler equations for the homogeneous incompressible fluid flows with damping in $\mathbb{R}^n$:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \mu u + \nabla P = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x). \end{cases} \tag{1.2}$$

Here, $\mu \geq 0$ is real parameter. $u = (u_1, u_2, \cdots, u_n)$, $u_j = u_j(x, t)$, $j = 1, 2, \cdots, n$, is the velocity of the flow, and $P = P(x, t)$ is the scalar pressure. It is a obvious fact that in the case $\mu = 0$, the Eq. (1.2) reduce to the standard incompressible n-dimension Euler system. The local in time well-posedness for the n-dimension Euler equations in the standard Sobolev space $H^m(\mathbb{R}^n)$, $m > \frac{n+2}{2}$, was done by Kato (1972), and the problem of finite time singularity for the local smooth solution is still an outstanding open problem. In this direction, there is a celebrated result on the blow-up criterion Beale et al. (1984) and its refinements (Chae 2006; Constantin et al. 1996) taking into account geometric considerations on the vorticity directions. In Chae (2004), Chae proved local existence and uniqueness of solutions to n-dimensional Euler in critical Besov space (for velocity) $B^{\frac{n}{p}+1}_{p,1}(\mathbb{R}^n)$ with $p \in (1, \infty)$. The local well-posedness in $B^1_{\infty,1}(\mathbb{R}^n)$ was settled by authors Pak and Park (2004). We refer the readers to the interesting works on damped compressible Euler equations and damped Navier-Stokes equations (for instance Sideris et al. 2003; Constantin et al. 2014). The finite time blow-up problem of the local classical solution is known as one of the most important and difficult problems in partial differential equations (see e.g., Constantin 1994, 2007; Kozono and Taniuchi 2000; Majda and Bertozzi 2002).

Finally, the n-dimensional surface quasi-geostrophic equation (SQG) is considered here.

$$\begin{cases} \partial_t\theta + (u\cdot\nabla)\theta + \gamma\theta = 0, \\ u_j = \mathcal{R}_j\theta, \quad \nabla\cdot u = 0, \\ \theta(x,0) = \theta_0(x). \end{cases} \tag{1.3}$$

Here, $\gamma \geq 0$ is a parameter. The function $\theta : \mathbb{R}^{\mathrm{n}} \times \mathbb{R}_+ \to \mathbb{R}$ represents the potential temperature, and the fluid velocity $u : \mathbb{R}^{\mathrm{n}} \times \mathbb{R}_+ \to \mathbb{R}^{\mathrm{n}}$ is determined by $\theta(x,t)$ via the formula

$$u_j = \pm\mathcal{R}_{\pi(j)}\theta, \quad \pi(j) \text{ is a permutation of } j, \quad j = 1, 2, \ldots, n$$

where u_j may take either a plus or a minus sign, and $R_j = \frac{\partial_j}{\sqrt{-\Delta}}$ are Riesz transforms. In the particular case $n = 2$, we have

$$u = \mathcal{R}^\perp\theta = \left(\frac{\partial_{x_2}}{(-\Delta)^{\frac{1}{2}}}\theta, \ \frac{-\partial_{x_1}}{(-\Delta)^{\frac{1}{2}}}\theta\right) = (\mathcal{R}_2\theta, \ -\mathcal{R}_1\theta),$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the 2-D Riesz transforms.

Recently, the SQG has been intensively investigated because of both its mathematical importance and its potential for applications in meteorology and oceanography (Pedlosky 1987). The global regularity problem for this case remains open. This is probably the simplest active scalar equation for which the global regularity is unknown. It shares parallel properties with the 3D Euler equations. Also the system (1.3) becomes the standard the dissipative SQG when $\gamma\theta$ is replaced $\gamma(-\Delta)^\alpha\theta$. The global regularity issue concerning the SQG has recently been studied very extensively, and important progress has been made. Here, we only mention some works on the SQG (see e.g., Caffarelli and Vasseur 2010; Chae and Constantin 2012; Chae et al. 2011; Cao and Titi 2007; Constantin and Vicol 2012; Constantin 2001; Constantin and Wu 2008; Constantin and Wu 2009; Constantin and Wu 1999; Córdoba and Córdoba 2004; Dong and Li 2008; Gancedo 2008; Hmidi et al. 2007; Kiselev 2011; Kiselev et al. 2007; Li and Rodrigo 2009; Miao and Xue 2011; Wu 2002). In particular, the global regularity for the critical case ($\gamma\theta$ replaced by $\gamma(-\Delta)^{\frac{1}{2}}\theta$) has been successfully established by De Giorgi techniques in Caffarelli and Vasseur (2010) and a non local maximum principle verified by the modulus of continuity at time 0 in Kiselev et al. (2007) for the case $n = 2$. The situation in the supercritical case ($\gamma\theta$ replaced by $\gamma(-\Delta)^\varrho\theta$ with $\varrho < \frac{1}{2}$) is only partially understood at the time of writing both the cases $n = 2$

and $n > 2$. Small data global existence results for the dissipative SQG have been obtained in various functional settings, see Chae and Lee (2003), Chen et al. (2007), Córdoba and Córdoba (2004), Hmidi et al. (2007), Miura (2006), Wu (2005a, b) for more details.

The last fluid mechanic model concerned in this paper is the n-dimensional incompressible magnetohydrodynamics system (MHD) with damping which can be written as follows:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \kappa u = -\nabla P + (b \cdot \nabla) b, \\ \partial_t b + (u \cdot \nabla) b + \lambda b = (b \cdot \nabla) u, \\ \nabla \cdot u = 0, \ \nabla \cdot b = 0, \\ u(x, 0) = u_0(x), \ b(x, 0) = b_0(x), \end{cases} \tag{1.4}$$

where $\kappa \geq 0$ and $\lambda \geq 0$ are real parameters. Here, $u = u(x, t) \in \mathbb{R}^n$, $P = P(x, t) \in \mathbb{R}$, and $b = b(x, t) \in \mathbb{R}^n$ denote the velocity vector, scalar pressure, and the magnetic field of the fluid, respectively. In the case of zero magnetic field, the system (1.4) reduces to the incompressible damped Euler system (1.2). The system (1.4) becomes the completely inviscid MHD when $\kappa = \lambda = 0$. Meanwhile, replacing κu and λb by $-\kappa \Delta u$ and $-\lambda \Delta b$, respectively, the system (1.4) reduces to the viscous MHD.

The MHD system models the complex phenomena in fluid mechanics, such as the magnetic reconnection in astrophysics and geomagnetic dynamo in geophysics, plasmas, liquid metals, and salt water, etc. Besides its important physical applications, the MHD system is also mathematically significant. Furthermore, an important feature of the MHD system is the induction effect, which brings about the strong coupling of the magnetic field and the velocity field. As a result of the strong coupling, the MHD system is considerably more complicated than the system of ordinary hydrodynamics. The question of whether a solution to the completely inviscid MHD system can develop a finite time singularity from smooth initial data with finite energy both in 2-dimension and high dimension is still a challenging open problem. Therefore, it is natural to examine the MHD system with damped term. The study of MHD has attracted the interest of many mathematicians (see, e.g., Adhikar et al. 2014; Caflisch et al. 1997; Cao et al. 2013; Cao and Wu 2011; Cao et al. 2014; Chen et al. 2008; Chen et al. 2010; Fan et al. 2014; Sermange and Temam 1983; Tran et al. 2013; Wu 2003; Wu 2011; Wu 2008; Ye and Xu 2014; Yamazaki 2014a, b; Zhou 2007 and the references therein).

The global regularity problem for the inviscid incompressible fluid mechanics appears to be out of reach in spite of the progress on the local well-posedness and regularity criteria. This work is partially aimed at understanding this difficult problem by examining how damping affects the regularity of solutions to incompressible fluid mechanics. Therefore, in the absence of a global well-posedness theory for large initial data, it is interesting to consider incompressible fluid mechanics with damping. In this paper, we aim at establishing the global (in time) regularity of classical solutions provided that the initial data are suitable small.

Finally, the rest of this paper is organized as follows. In Sect. 2, we introduce the notations, recall materials related to Besov spaces, provide useful lemmas, and state the main results. The proof of the first main result is given in Sect. 3. Sections 4 and 5 present the proofs of Theorems 2.14 and 2.16, respectively. For the sake of completeness, we provide the local existence and uniqueness to the system (1.4) in the appendix. Also the proofs of two commutator (see Lemmas 2.6 and 2.7 below) estimates are provided in Appendix.

2 Preliminaries and Main Results

Throughout the paper, C stands for some real positive constant which may be different in each occurrence and independent on the initial data. We shall sometimes use the notation $A \lesssim B$ which stands for $A \leq CB$. Before we state the main results, we first explain the notations and conventions used throughout this paper. Here, $w \triangleq \nabla u - (\nabla u)^{\mathrm{T}}$ and $J \triangleq \nabla b - (\nabla b)^{\mathrm{T}}$ (where A^{T} denotes the transpose of A) stand for the vorticity and the current, respectively. In dimension $n = 2$, it can be identified with the scalar function $\omega = \partial_1 u_2 - \partial_2 u_1$, while for $n = 3$ with the vector field $\omega = \nabla \times u \equiv (\partial_2 u_3 - \partial_3 u_2,\ \partial_3 u_1 - \partial_1 u_3,\ \partial_1 u_2 - \partial_2 u_1)^{\mathrm{T}}$. For all Banach space X and an interval I of $\mathbb{R}$, we denote by $C(I; X)$, the set of continuous functions on I with values in X.

In this preparatory section, we recall the Littlewood-Paley operators and their elementary properties which allow us to define the Besov spaces. Related materials can be found in several books and many papers (see for example Bahour et al. 2011; Chemin 2009; Miao et al. 2012; Triebel 1992)

Let (χ, φ) be a couple of smooth functions with values in [0, 1] such that $\chi \in C_0^\infty(\mathbb{R}^n)$ is supported in the ball $\mathcal{B} \triangleq \{\xi \in \mathbb{R}^n, |\xi| \leq \frac{4}{3}\}$, $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ is supported in the annulus $\mathcal{C} \triangleq \{\xi \in \mathbb{R}^n, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, and they satisfy

$$\chi(\xi) + \sum_{j\in\mathbb{N}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1,\ \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

For every $u \in S'$ (tempered distributions), we define the nonhomogeneous Littlewood-Paley operators as follows:

$$\Delta_j u = 0,\ \text{for } j \leq -2; \quad \Delta_{-1} u = \chi(D)u; \quad \Delta_j u = \varphi(2^{-j}D)u, \ \text{ for } j \in \mathbb{N}.$$

We shall also use the following low-frequency cutoff:

$$S_j u = \sum_{-1 \leq k \leq j-1} \Delta_k u.$$

Meanwhile, we define the homogeneous dyadic blocks as

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u \quad \forall j \in \mathbb{Z}, \quad \text{and} \quad \dot{S}_j u = \sum_{k \leq j-1} \dot{\Delta}_k u.$$

Let us recall the definition of homogeneous and inhomogeneous Besov spaces through the dyadic decomposition.

Definition 2.1 Let $s \in \mathbb{R}$, $(p, r) \in [1, +\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as a space of $f \in S'(\mathbb{R}^n)$ such that

$$\dot{B}^s_{p,r} = \{f \in S'(\mathbb{R}^n); \|f\|_{\dot{B}^s_{p,r}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jrs}\|\dot{\Delta}_j f\|^r_{L^p}\right)^{\frac{1}{r}}, & \forall\, r < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j f\|_{L^p}, & \forall\, r = \infty. \end{cases}$$

Definition 2.2 Let $s \in \mathbb{R}$, $(p, r) \in [1, +\infty]^2$. The inhomogeneous Besov space $B^s_{p,r}$ is defined as a space of $f \in S'(\mathbb{R}^n)$ such that

$$B^s_{p,r} = \{f \in S'(\mathbb{R}^n); \|f\|_{B^s_{p,r}} < \infty\},$$

where

$$\|f\|_{B^s_{p,r}} = \begin{cases} \left(\sum_{j\geq -1} 2^{jrs}\|\Delta_j f\|^r_{L^p}\right)^{\frac{1}{r}}, & \forall\, r < \infty, \\ \sup_{j\geq -1} 2^{js}\|\Delta_j f\|_{L^p}, & \forall\, r = \infty. \end{cases}$$

For $s > 0$, $(p, r) \in [1, +\infty]^2$, we define the inhomogeneous Besov space norm $B^s_{p,r}$ as

$$\|f\|_{B^s_{p,r}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,r}}. \tag{2.1}$$

When $p = r = 2$, we have $B^s_{2,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$, where $H^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n); \|f\|_{H^s(\mathbb{R}^n)} < \infty\}$ is the potential Banach space with the norm

$$\|f\|_{H^s(\mathbb{R}^n)} \triangleq \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2\, d\xi\right)^{\frac{1}{2}}.$$

For more details about function spaces, we refer to Triebel (1992).

Bernstein inequalities are fundamental in the analysis involving Besov spaces, and these inequalities trade integrability for derivatives (see for instance Bahour et al. 2011; Chemin 2009).

Lemma 2.3 *(Bernstein inequality) Let $k \in \mathbb{N} \cup \{0\}$, $1 \leq a \leq b \leq \infty$. Assume that*

$$supp\, \widehat{f} \subset \left\{\xi \in \mathbb{R}^n :\ |\xi| \lesssim 2^j\right\},$$

for some integer j, then there exists a constant C_1 such that

$$\|\nabla^\alpha f\|_{L^b} \leq C_1\, 2^{jk+jn(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a}, \quad k = |\alpha|.$$

If f satisfies

$$\text{supp}\,\widehat{f} \subset \left\{\xi \in \mathbb{R}^n : \ |\xi| \approx 2^j\right\}$$

for some integer j, then

$$C_1\, 2^{jk}\|f\|_{L^b} \le \|\nabla^\alpha f\|_{L^b} \le C_2\, 2^{jk+jn(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a}, \quad k = |\alpha|,$$

where C_1 and C_2 are constants depending on α, p, and q only.

The following laws of product can be found in the reference Bahour et al. (2011); thus, we omit its proof here.

Lemma 2.4 *Suppose that $s > 0$, $(p, r) \in [1, +\infty]^2$, then the followings hold true*

$$\|uv\|_{B^s_{p,r}} \le \frac{C^{s+1}}{s}(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,r}}),$$
$$\|uv\|_{\dot{B}^s_{p,r}} \le \frac{C^{s+1}}{s}(\|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{\dot{B}^s_{p,r}}).$$

Next, we recall the well-known Calderon-Zygmund operators, which will be used to get the control between the gradient of velocity and the vorticity (see reference Chemin 2009).

Lemma 2.5 (Biot-Savart law) *Let u be a smooth divergence-free vector field, then there exists a universally positive constant C such that for any $1 < p < \infty$*

$$\|\nabla u\|_{L^p} \le C\frac{p^2}{p-1}\|\omega\|_{L^p},$$

where ω is the vorticity, namely $\omega \triangleq \nabla \times u$.

In this paper, we need some simple commutator estimates as follows, the detailed proofs stated in many references, see for example Bahour et al. (2011), Danchin (1993), Miao et al. (2012). For completeness, we give a little simple proof in the Appendix.

Lemma 2.6 *Let $s > -1$, $(p, r) \in [1, +\infty]^2$, and u is a divergence-free vector field, namely $\nabla \cdot u = 0$; we have the following inequality:*

$$\|2^{js}\|[\Delta_j, u\cdot\nabla]v\|_{L^p}\|_{l^r_j} \le C(\|\nabla u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|\nabla v\|_{L^\infty}\|\nabla u\|_{B^{s-1}_{p,r}}), \quad (2.2)$$

where

$$[\Delta_j, u\cdot\nabla]v = \Delta_j(u\cdot\nabla v) - u\cdot\nabla(\Delta_j v).$$

If we set $v = \omega \triangleq \nabla \times u$, then (2.2) reduces to

$$\|2^{js}\|[\Delta_j, u\cdot\nabla]\omega\|_{L^p}\|_{l^r_j} \le C\|\nabla u\|_{L^\infty}\|\omega\|_{B^s_{p,r}}, \forall s > -1. \quad (2.3)$$

The following commutator is very useful to handle the field v such that v is in L^∞, but its gradient is not.

Lemma 2.7 *Let $j \in Z$ be an integer, $s > -1$, $1 \le p \le \infty$, and u is a divergence-free vector field, namely $\nabla \cdot u = 0$. Then*

$$\left\|2^{js}\|[\dot{\Delta}_j, u\cdot\nabla]v\|_{L^p}\right\|_{l_j^r} \le C\|\nabla u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}} + C\|v\|_{L^\infty}\|u\|_{\dot{B}^{s+1}_{p,r}}, \tag{2.4}$$

$$\left\|2^{js}\|[\dot{\Delta}_j, u\cdot\nabla]v\|_{L^p}\right\|_{l_j^r} \le C\|\nabla u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}} + C\|v\|_{L^\infty}\|w\|_{\dot{B}^s_{p,r}}, \quad (\omega \triangleq \nabla\times u). \tag{2.5}$$

Finally, let us now state our first main global well-posedness result with suitable small initial datum. More precisely, we have the following theorem:

Theorem 2.8 *Assume $\nu > 0$ and $\eta > 0$. Suppose that $(\omega_0, \nabla\theta_0) \in B^s_{p,r} \times B^s_{p,r}$ for every $1 < p < \infty$ and all s with $s = \frac{n}{p}$ if $r = 1$ and $s > \frac{n}{p}$ if $r \in (1, \infty]$ obeys one of the smallness conditions*

$$\begin{cases} \|\omega_0\|_{B^s_{p,r}} < \frac{\eta}{2C} \triangleq B \quad and \quad \|\nabla\theta_0\|_{B^s_{p,r}} \le \dfrac{e\eta(B - \|\omega_0\|_{B^s_{p,r}})}{2C}, & \nu = \eta, \\ \|\omega_0\|_{B^s_{p,r}} < \min\left\{\frac{\nu}{2C}, \frac{\eta}{2C}\right\} \triangleq B \quad and \quad \|\nabla\theta_0\|_{B^s_{p,r}} \le \dfrac{(B - \|\omega_0\|_{B^s_{p,r}})}{2C}\left(\dfrac{\nu^\nu}{\eta^\eta}\right)^{\frac{1}{\nu-\eta}} & \nu \ne \eta, \end{cases} \tag{2.6}$$

where C is independent of ν, η, and initial datum. Then the system (1.1) has a unique global solution (u, θ) satisfying

$$(u, \theta) \in L^\infty([0,\infty); L^2(\mathbb{R}^n)), \quad (\omega, \nabla\theta) \in L^\infty([0,\infty); B^s_{p,r}(\mathbb{R}^n)).$$

Moreover, for any $T \ge 0$,

$$\|\nabla\theta(.,T)\|_{B^s_{p,r}} \le \|\nabla\theta_0\|_{B^s_{p,r}} e^{-\frac{\eta}{2}T} < \infty, \quad \|\omega(.,T)\|_{B^s_{p,r}} < B.$$

Remark 2.9 Due to the Besov imbedding, $B^s_{p,r}(\mathbb{R}^n) \hookrightarrow L^\infty$ for any $s = \frac{n}{p}$ if $r = 1$ and $s > \frac{n}{p}$ if $r \in (1, \infty]$. For any $T > 0$, we thus have $\int_0^T \|\nabla\theta(\,.\,,\tau)\|_{L^\infty} d\tau < \infty$. Applying the well-known blow-up criterion (Chae and Nam 1997; Chae 2004) to the damped system (1.1), we can obtain that the system (1.1) has a unique global smooth classical solution (u, θ). Due to the fact that the Calderon–Zygmund operator is not bounded $L^p(\mathbb{R}^n)$ for $p = 1$ or ∞, Theorem 2.8 fails for the limit cases $p = 1$ and $p = \infty$. However, if we replace the $B^s_{p,r}$ norm by $B^s_{p,r} \cap L^q$ norm, Theorem 2.8 is also true. In fact, it is easy for us to apply the standard L^q-estimate to the vorticity and temperature equations to obtain the following two elementary estimates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^q} + \nu\|\omega\|_{L^q} \le C\|\omega\|_{L^q}\|\nabla u\|_{L^\infty} + C\|\nabla\theta\|_{L^q},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{L^q} + \eta\|\nabla\theta\|_{L^q} \le C\|\nabla u\|_{L^\infty}.$$

Moreover, we can deduce from the proof of Theorem 2.8 that Theorem 2.8 is also true in the homogeneous Besov space $\dot{B}^{\frac{n}{p}}_{p,1}$ for any $1 \leq p \leq \infty$ (in fact we just need the imbedding $\dot{B}^{\frac{n}{p}}_{p,1} \hookrightarrow L^\infty$ and the boundedness of Calderon-Zygmund operators between $\dot{B}^{\frac{n}{p}}_{p,1}$, namely $\|\nabla u\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \leq C\|\nabla \times u\|_{\dot{B}^{\frac{n}{p}}_{p,1}}$. However, the limit case $\dot{B}^0_{\infty,1}$ cannot handle the same as the case $\dot{B}^{\frac{n}{p}}_{p,1}$ for any $1 \leq p < \infty$, see Remark 3.1 below for more details). These Besov spaces, especially the limit case $\dot{B}^0_{\infty,1}$, seems to be some very natural functional settings to guarantee the uniqueness of the solutions.

Remark 2.10 It is a simple observation that

$$\lim_{\nu\to\eta}\left(\frac{\nu^\nu}{\eta^\eta}\right)^{\frac{1}{\nu-\eta}} = e\eta$$

which means that (2.6) makes sense.

Remark 2.11 Inspired by the paper of Adhikar et al. (2014) which states that sufficiently small $\|\nabla u_0\|_{\dot{B}^0_{\infty,1}}$ and $\|\nabla\theta_0\|_{\dot{B}^0_{\infty,1}}$ for case $n = 2$ can guarantee the uniqueness and global existence, we impose the initial datum $\|\nabla \times u_0\|_{\dot{B}^0_{\infty,1}}$ and $\|\nabla\theta_0\|_{\dot{B}^0_{\infty,1}}$ being suitable small both in dimension $n = 2$ and $n \geq 3$; thus, our results are very similar to ones in Adhikar et al. (2014) and are even some further improvements. Indeed, by the boundedness of Calderon-Zygmund operators between homogeneous Besov spaces, one can show that $\frac{1}{C}\|\nabla u_0\|_{\dot{B}^0_{\infty,1}} \leq \|\nabla \times u_0\|_{\dot{B}^0_{\infty,1}} \leq \|\nabla u_0\|_{\dot{B}^0_{\infty,1}}$ for some $C \geq 1$. Moreover, we also have obtained similar result in another suitable functional spaces.

For the damped Euler equations (1.2), we are interested in establishing the following theorem.

Theorem 2.12 *Assume that* $\mu > 0$. *Suppose that initial vorticity* $\omega_0 \in B^s_{p,r}$ *with* $1 < p < \infty$ *and* $s = \frac{n}{p}$ *if* $r = 1$ *and* $s > \frac{n}{p}$ *if* $r \in (1, \infty]$ *obeys the smallness condition*

$$\|\omega_0\|_{B^s_{p,r}} < \frac{\mu}{2C},$$

for a constant C independent of μ. *Then (1.2) has a unique global solution u satisfying*

$$u \in L^\infty([0,\infty); L^2(\mathbb{R}^n)), \quad \omega \in L^\infty([0,\infty); B^s_{p,r}(\mathbb{R}^n)).$$

Moreover, for any $T \geq 0$

$$\|\omega(.,T)\|_{B^s_{p,r}} < \frac{\mu}{2C}.$$

Remark 2.13 One can follow the same arguments used in the proof of Theorem 2.8 to conclude the desired result of Theorem 2.12. Moreover, Theorem 2.12 is a direct consequence of Theorem 2.8 provided that we set the temperature $\theta = 0$. For these reasons, we will not give a separate proof for Theorem 2.12. Meanwhile, Theorem 2.12 holds true for the case $p = 1$ or $p = \infty$ if we replace the $B^s_{p,r}$ norm by $B^s_{p,r} \cap L^q$ norm and in the homogeneous Besov space $\dot{B}^{\frac{n}{p}}_{p,1}$ for any $1 \le p \le \infty$.

The next theorem concerns the n-dimensional damped SQG equation.

Theorem 2.14 *Assume $\gamma > 0$. Suppose that $\nabla\theta_0 \in B^s_{p,r}$ with $1 < p < \infty$ and $s = \frac{n}{p}$ if $r = 1$ and $s > \frac{n}{p}$ if $r \in (1, \infty]$* (resp. $\|\nabla\theta_0\| \in \dot{B}^0_{\infty,1}$) obeys the smallness condition

$$\|\nabla\theta_0\|_{B^s_{p,r}} < \frac{\gamma}{2C}, \quad (\text{resp. } \|\nabla\theta\|_{\dot{B}^0_{\infty,1}} < \frac{\gamma}{2C})$$

for a constant C independent of γ. Then the system (1.3) has a unique global solution u satisfying

$$\theta \in L^\infty([0,\infty); L^2(\mathbb{R}^n)), \ \nabla\theta \in L^\infty([0,\infty); B^s_{p,r}(\mathbb{R}^n))$$
$$\Big(resp.\ \nabla\theta \in L^\infty([0,\infty); \dot{B}^0_{\infty,1}(\mathbb{R}^n))\Big).$$

Moreover, for any $T \ge 0$

$$\|\nabla\theta(.,T)\|_{B^s_{p,r}} < \frac{\gamma}{2C}, \quad \left(\text{resp. } \|\nabla\theta(.,T)\|_{\dot{B}^0_{\infty,1}} < \frac{\gamma}{2C}\right). \tag{2.7}$$

Remark 2.15 By the Besov imbedding, we immediately have $\|\nabla\theta(.,t)\|_{L^\infty} < \frac{\gamma}{2C}$ for any $t \ge 0$ which is enough for high regularity as shown in Wu (2002). Consequently, (2.7) actually implies the smoothness of the solution.

The final result of this paper states that the system (1.4) with sufficiently small initial datum always possesses a unique global solution. More precisely, we will state the following theorem to end this section.

Theorem 2.16 *Assume $\kappa > 0$ and $\lambda > 0$. Let $s > 1 + \frac{n}{2}$ and assume that $(u_0, b_0) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ obeys the smallness condition*

$$\|u_0\|_{H^s(\mathbb{R}^n)} + \|b_0\|_{H^s(\mathbb{R}^n)} < \frac{\min\{\kappa,\ \lambda\}}{2C}, \tag{2.8}$$

where C independent of κ, λ, and initial datum. Then the system (1.4) admits a unique global solution $(u,\ b)$ satisfying

$$(u,\ b) \in L^\infty([0,\infty); H^s(\mathbb{R}^n)).$$

Furthermore, for any $T \ge 0$,

$$\|u(\cdot,T)\|_{H^s(\mathbb{R}^n)} + \|b(\cdot,T)\|_{H^s(\mathbb{R}^n)} < \frac{\min\{\kappa,\ \lambda\}}{2C}.$$

Remark 2.17 (A remark of conclusion) To conclude, we note that all the system above is a coupling between transport equations. Hence, preserving the initial regularity requires the velocity field to be at least locally Lipschitz with respect to the space variable. As a matter of fact, the classical transport equations have been shown to be well-posed in any Besov space $B^s_{p,r}$ embedded in $C^{0,1}$ a property which holds provided that $(s, p, r) \in \mathbb{R} \times [1, \infty] \times [1, \infty]$ satisfies

$$s > 1 + \frac{n}{p} \quad \text{or} \quad s = 1 + \frac{n}{p} \text{ and } r = 1.$$

see Bahour et al. (2011); Chemin (2009) for more details.

3 The Proof of Theorem 2.8

Proof of Theorem 2.8 The existence of local smooth solutions can be obtained without difficulty, see for example Adhikar et al. (2014), Chae and Nam (1997), Chae (2004), Majda and Bertozzi (2002). Thus, in order to complete the proof of Theorem 2.8, it is sufficient to establish some a priori estimates.

Taking the L^2 inner product of the equations $(1.1)_1$, $(1.1)_2$ with u and θ, respectively, and some calculations, we can obtain

$$\|\theta\|_{L^2} \leq \|\theta_0\|_{L^2} e^{-\eta t},$$
$$\begin{cases} \|u\|_{L^2} \leq \|u_0\|_{L^2} e^{-\nu t} + \|\theta_0\|_{L^2} t e^{-\nu t}, & \nu = \eta, \\ \|u\|_{L^2} \leq \|u_0\|_{L^2} e^{-\nu t} + \|\theta_0\|_{L^2} \frac{e^{-\eta t} - e^{-\nu t}}{\nu - \eta}, & \nu \neq \eta. \end{cases}$$

The following inequalities will be used frequently, which are some easy consequences of Besov space imbedding properties and Lemma 2.5.

$$\|\nabla u\|_{L^\infty} \leq C \|\nabla u\|_{B^s_{p,r}} \leq C \|\omega\|_{B^s_{p,r}}, \tag{3.1}$$

where index (p, r, s) satisfies the assumption in Theorem 2.8.

We differentiate the temperature equation of system (1.1) in x; then we get

$$\partial_t \nabla\theta + \eta \nabla\theta + u \cdot \nabla(\nabla\theta) + (\nabla u) \cdot \nabla\theta = 0. \tag{3.2}$$

Applying homogeneous blocks $\dot{\Delta}_j$ operator to above equality, we have

$$\partial_t \dot{\Delta}_j \nabla\theta + \eta \dot{\Delta}_j \nabla\theta + u \cdot \dot{\Delta}_j \nabla(\nabla\theta) = -\dot{\Delta}_j (\nabla u \cdot \nabla\theta) - [\dot{\Delta}_j, u \cdot \nabla]\nabla\theta,$$

which, together with L^p-norm in space variable, Hölder inequality and incompressible condition, namely $\nabla \cdot\ u = 0$, directly give (C is independent of p)

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\dot{\Delta}_j \nabla\theta\|_{L^p} + \eta \|\dot{\Delta}_j \nabla\theta\|_{L^p} \leq C \|\dot{\Delta}_j (\nabla u \cdot \nabla\theta)\|_{L^p} + C \|[\dot{\Delta}_j, u \cdot \nabla]\nabla\theta\|_{L^p}. \tag{3.3}$$

For $r \geq 1$, multiply above inequality by $2^{jsr}\|\dot{\Delta}_j\nabla\theta\|_{L^p}^{r-1}$ and then sum over j from $-\infty$ to ∞ to obtain

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{\dot{B}^s_{p,r}}^r+\eta\|\nabla\theta\|_{\dot{B}^s_{p,r}}^r \leq C\Big(\|\nabla u\cdot\nabla\theta\|_{\dot{B}^s_{p,r}}+\big\|2^{js}\|[\dot{\Delta}_j, u\cdot\nabla]\nabla\theta\|_{L^p}\big\|_{l^r_j}\Big)\|\nabla\theta\|_{\dot{B}^s_{p,r}}^{r-1}.$$

Taking advantage of Lemmas 2.4 and 2.7, embedding inequality (3.1) and some computations, we can get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{\dot{B}^s_{p,r}}+\eta\|\nabla\theta\|_{\dot{B}^s_{p,r}} &\leq C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{\dot{B}^s_{p,r}}+C\|\nabla\theta\|_{L^\infty}\|u\|_{\dot{B}^{s+1}_{p,r}}\\ &\quad+C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{\dot{B}^s_{p,r}}+C\|\nabla\theta\|_{L^\infty}\|\nabla u\|_{\dot{B}^s_{p,r}}\\ &\leq C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{\dot{B}^s_{p,r}}+C\|\nabla\theta\|_{L^\infty}\|\nabla u\|_{\dot{B}^s_{p,r}}\\ &\leq C\|\omega\|_{B^s_{p,r}}\|\nabla\theta\|_{B^s_{p,r}},\end{aligned}\tag{3.4}$$

where the following facts are used: $\|u\|_{\dot{B}^{s+1}_{p,r}} \approx \|\nabla u\|_{\dot{B}^s_{p,r}}$ and $\|f\|_{\dot{B}^s_{p,r}} \leq C\|f\|_{B^s_{p,r}}$ for any $s > 0$.

Multiplying (3.2) by $|\nabla\theta|^{p-2}\nabla\theta$ and integrating it over $\mathbb{R}^n$, we immediately derive

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{L^p}+\eta\|\nabla\theta\|_{L^p} &\leq C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^p}\\ &\leq C\|\nabla u\|_{B^s_{p,r}}\|\nabla\theta\|_{L^p}.\end{aligned}\tag{3.5}$$

Combining (3.5) with (3.4), we can get by (2.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{B^s_{p,r}}+\eta\|\nabla\theta\|_{B^s_{p,r}} \leq C\|\omega\|_{B^s_{p,r}}\|\nabla\theta\|_{B^s_{p,r}}.\tag{3.6}$$

Next we move our attention to estimate the velocity field. To do it, we firstly resort to the vorticity of the fluid which is defined as the skew-symmetric matrix

$$\omega \triangleq \nabla u-(\nabla u)^{\mathrm{T}},$$

and it satisfies the equation

$$\partial_t\omega+\nu\omega+u\cdot\nabla\omega=-\omega\cdot\nabla u-(\nabla u)^{\mathrm{T}}\cdot\omega+\nabla(\theta e_n)-[\nabla(\theta e_n)]^{\mathrm{T}}$$

For the sake of writing, we set $F(\theta)\triangleq\nabla(\theta e_n)-[\nabla(\theta e_n)]^{\mathrm{T}}$.

Applying inhomogeneous blocks Δ_j operator to the vorticity equation, we can get

$$\begin{aligned}\partial_t\Delta_j\omega+\nu\Delta_j\omega+u\cdot\nabla\Delta_j\omega=-\Delta_j(\omega\cdot\nabla u)-\Delta_j((\nabla u)^{\mathrm{T}}\cdot\omega)+\Delta_j(F(\theta))\\ -[\Delta_j, u\cdot\nabla]\omega.\end{aligned}$$

Taking L^p-norm and using the Hölder inequality, we can conclude the following differential inequality:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_j\omega\|_{L^p}+\nu\|\Delta_j\omega\|_{L^p} &\leq C\|\Delta_j(\omega\cdot\nabla u)\|_{L^p}+C\|\Delta_j((\nabla u)^{\mathrm{T}}\cdot\omega)\|_{L^p}+\|\Delta_j F(\theta)\|_{L^p}\\ &\quad + C\|[\Delta_j, u\cdot\nabla]\omega\|_{L^p}\\ &\leq C\|\Delta_j(\omega\cdot\nabla u)\|_{L^p}+C\|\Delta_j(\nabla\theta)\|_{L^p}\\ &\quad + C\|[\Delta_j, u\cdot\nabla]\omega\|_{L^p}. \end{aligned} \tag{3.7}$$

Multiplying above inequality by $2^{jsr}\|\Delta_j\omega\|_{L^p}^{r-1}$ and then summing over j from -1 to ∞, one can show that

$$\begin{aligned}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^r_{B^s_{p,r}}+\nu\|\omega\|^r_{B^s_{p,r}} &\leq C\Big(\|\omega\cdot\nabla u\|_{B^s_{p,r}}+\|F(\theta)\|_{B^s_{p,r}}\\ &\quad+\big\|2^{js}\|[\Delta_j, u\cdot\nabla]\omega\|_{L^p}\big\|_{l^r_j}\Big)\|\omega\|^{r-1}_{B^s_{p,r}}.\end{aligned}$$

Apply Lemma 2.4, Lemma 2.6, and inequality (3.1) to differential inequality (3.7) to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{B^s_{p,r}}+\nu\|\omega\|_{B^s_{p,r}} &\leq C\|\omega.\nabla u\|_{B^s_{p,r}}+C\|\nabla\theta\|_{B^s_{p,r}}+C\|\nabla u\|_{L^\infty}\|\omega\|_{B^s_{p,r}}\\ &\leq C\|\nabla u\|_{L^\infty}\|\omega\|_{B^s_{p,r}}+C\|\nabla\theta\|_{B^s_{p,r}}+C\|\omega\|_{L^\infty}\|\nabla u\|_{B^s_{p,r}}\\ &\leq C\|\omega\|^2_{B^s_{p,r}}+C\|\nabla\theta\|_{B^s_{p,r}}. \end{aligned}\tag{3.8}$$

For the sake of clarity of presentation, we denote

$$X(t)\triangleq\|\nabla\theta\|_{B^s_{p,r}},\quad Y(t)\triangleq\|\omega\|_{B^s_{p,r}}.$$

Now we can deduce from differential inequalities (3.6) and (3.8) that

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t)+\eta X(t)\leq CX(t)Y(t), \tag{3.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t)+\nu Y(t)\leq CY(t)^2+CX(t). \tag{3.10}$$

Next we claim that the above two differential inequalities obey the following global bounds:

$$X(t)\leq X(0), \tag{3.11}$$

$$Y(t)<B, \tag{3.12}$$

provided that the smallness conditions (2.6) hold true.

Now suppose (3.12) is not true, and T_0 is the first time such that (3.12) is violated, namely,

$$Y(T_0) = B; \tag{3.13}$$

moreover,

$$Y(t) < B, \quad \forall 0 \le t < T_0.$$

We can deduce from (3.9) that for any $0 \le t \le T_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) + \frac{\eta}{2}X(t) \le 0.$$

Therefore, by standard differential equation theorem, we have

$$X(t) \le X(0)e^{-\frac{\eta}{2}t} \le X(0), \quad \forall\, 0 \le t \le T_0.$$

With this bound together with (3.10), we can derive the following differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) + \frac{\nu}{2}Y(t) \le CX(0)e^{-\frac{\eta}{2}t}, \quad \forall\, 0 \le t \le T_0.$$

Thus, using standard differential equation theory, we arrive at for any $0 \le t \le T_0$

$$Y(t) \le Y(0)e^{-\frac{\nu}{2}t} + CX(0)te^{-\frac{\nu}{2}t}, \quad \nu = \eta, \tag{3.14}$$

$$Y(t) \le Y(0)e^{-\frac{\nu}{2}t} + 2CX(0)\frac{e^{-\frac{\eta}{2}t} - e^{-\frac{\nu}{2}t}}{\nu - \eta}, \quad \nu \ne \eta. \tag{3.15}$$

Now we define two functions as follows:

$$F(t) \triangleq te^{-\frac{\nu}{2}t} \quad \text{and} \quad G(t) \triangleq \frac{e^{-\frac{\eta}{2}t} - e^{-\frac{\nu}{2}t}}{\nu - \eta}, \quad \nu \ne \eta.$$

By the standard theory of calculus, it is not difficult to show that for any $t \ge 0$

$$F_{\max}(t) = F(\frac{2}{\eta}) = \frac{2}{\eta e},$$
$$G_{\max}(t) = G\Big(\frac{2}{\nu - \eta}\ln(\frac{\nu}{\eta})\Big) = \Big(\frac{\nu^{\nu}}{\eta^{\eta}}\Big)^{\frac{1}{\nu-\eta}}.$$

Therefore, combining the above two facts with (3.14) and (3.15), we can immediately show that

$$
\begin{aligned}
Y(T_0) &\le Y(0)e^{-\frac{\nu}{2}T_0} + CX(0)T_0e^{-\frac{\nu}{2}T_0} < Y(0) + \frac{2CX(0)}{\eta e} \le B,\\
Y(T_0) &\le Y(0)e^{-\frac{\nu}{2}T_0} + 2CX(0)\frac{e^{-\frac{\eta}{2}T_0} - e^{-\frac{\nu}{2}T_0}}{\nu-\eta} < Y(0) + 2CX(0)\Big(\frac{\nu^\nu}{\eta^\eta}\Big)^{\frac{1}{\nu-\eta}} \le B.
\end{aligned}
$$

Hence, we can get

$$
Y(T_0) < B
$$

which contradict with (3.13). This contradiction implies that

$$
\|\nabla\theta(.,T)\|_{B^s_{p,r}} \le \|\nabla\theta_0\|_{B^s_{p,r}} e^{-\frac{\eta}{2}T} < \infty, \quad \|\omega(.,T)\|_{B^s_{p,r}} < B,
$$

for any $T \ge 0$.

With the obtained global bounds on the solutions at disposal, we are ready to show the uniqueness. In fact, the uniqueness is easy to be proved, since the velocity and the temperature are both in Lipschitz spaces. Let (u^1, θ^1, P^1) and (u^2, θ^2, P^2) be two solutions both satisfying the system (1.1) with the same initial datum. It is easy to check that the difference $(\widetilde{u}, \widetilde{\theta}, \widetilde{P})$ with

$$
\widetilde{u} = u^1 - u^2, \quad \widetilde{\theta} = \theta^1 - \theta^2, \quad \widetilde{P} = P^1 - P^2
$$

satisfies

$$
\begin{cases}
\widetilde{u}_t + (u^1\cdot\nabla)\widetilde{u} + \nu\widetilde{u} + (\widetilde{u}\cdot\nabla)u^2 + \nabla\widetilde{P} = \widetilde{\theta}e_n,\\
\widetilde{\theta}_t + (u^1\cdot\nabla)\widetilde{\theta} + \eta\widetilde{\theta} + (\widetilde{u}\cdot\nabla)\theta^2 = 0,\\
\widetilde{u}(x,0) = 0, \quad \widetilde{\theta}(x,0) = 0.
\end{cases}
\tag{3.16}
$$

It follows from taking the L^2 inner product of the equations $(3.16)_1$, $(3.16)_2$ with $\widetilde{u}$ and $\widetilde{\theta}$, respectively, and adding them up yields

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{u}(t)\|^2_{L^2} + \|\widetilde{\theta}(t)\|^2_{L^2}) + \nu\|\widetilde{u}(t)\|^2_{L^2} + \eta\|\widetilde{\theta}(t)\|^2_{L^2}\\
&\quad\le C(1 + \|\nabla u^2\|_{L^\infty} + \|\nabla\theta^2\|_{L^\infty})(\|\widetilde{u}(t)\|^2_{L^2} + \|\widetilde{\theta}(t)\|^2_{L^2})\\
&\quad\le C(1 + \|\nabla u^2\|_{B^s_{p,r}} + \|\nabla\theta^2\|_{B^s_{p,r}})(\|\widetilde{u}(t)\|^2_{L^2} + \|\widetilde{\theta}(t)\|^2_{L^2}).
\end{aligned}
\tag{3.17}
$$

Above, Hölder's inequality and imbedding $B^s_{p,r} \hookrightarrow L^\infty$ are applied with index (p, r, s), satisfying the assumption in Theorem 2.8. Then Gronwall inequality implies that $\widetilde{u} \equiv 0$ and $\widetilde{\theta} \equiv 0$. Thus, we have completed the proof of Theorem 2.8. □

Remark 3.1 Let us state the case $\dot{B}^0_{\infty,1}$. In fact, this case is much involved as Lemma 2.4 does not hold true any more for the case $\dot{B}^0_{\infty,1}$ which leads to that we cannot get the estimates (3.4) and (3.8) as above directly. Now we state the detailed proofs for the case $\dot{B}^0_{\infty,1}$.

It follows from the estimate (3.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_j\nabla\theta\|_{L^p}+\eta\|\dot{\Delta}_j\nabla\theta\|_{L^p}\le C\|\dot{\Delta}_j(\nabla u\cdot\nabla\theta)\|_{L^p}+C\|[\dot{\Delta}_j,u\cdot\nabla]\nabla\theta\|_{L^p}.$$

We note that C is independent of p. Thus, setting $p=\infty$ in the above inequality, we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_j\nabla\theta\|_{L^\infty}+\eta\|\dot{\Delta}_j\nabla\theta\|_{L^\infty}\le C\|\dot{\Delta}_j(\nabla u\cdot\nabla\theta)\|_{L^\infty}+C\|[\dot{\Delta}_j,u\cdot\nabla]\nabla\theta\|_{L^\infty}. \tag{3.18}$$

We will handle the first term of the right-hand side of (3.18).

$$\begin{aligned}
&\|\dot{\Delta}_j(\nabla u\cdot\nabla\theta)\|_{L^\infty}\\
&\quad\le C\sum_{|k-j|\le 2}\|\dot{\Delta}_j(\dot{S}_{k-1}\nabla u\cdot\nabla\dot{\Delta}_k\theta)\|_{L^\infty}+C\sum_{|k-j|\le 2}\|\dot{\Delta}_j(\dot{\Delta}_k\nabla u\cdot\nabla\dot{S}_{k-1}\theta)\|_{L^\infty}\\
&\qquad+C\sum_{k+2\ge j}\|\dot{\Delta}_j\partial_l(\dot{\Delta}_k\nabla u_l\cdot\widetilde{\dot{\Delta}}_k\theta)\|_{L^\infty}\\
&\quad\le C\|\nabla u\|_{L^\infty}\|\dot{\Delta}_j\nabla\theta\|_{L^\infty}+C\|\nabla\theta\|_{L^\infty}\|\dot{\Delta}_j\nabla u\|_{L^\infty}\\
&\qquad+C\sum_{k+2\ge j}2^j\|\dot{\Delta}_j(\dot{\Delta}_k\nabla u_l\cdot\widetilde{\dot{\Delta}}_k\theta)\|_{L^\infty}\\
&\quad\le C\|\nabla u\|_{L^\infty}\|\dot{\Delta}_j\nabla\theta\|_{L^\infty}+C\|\nabla\theta\|_{L^\infty}\|\dot{\Delta}_j\nabla u\|_{L^\infty}\\
&\qquad+C\sum_{k+2\ge j}2^j\|\nabla u\|_{L^\infty}\|\dot{\Delta}_k\theta\|_{L^\infty}\\
&\quad\le C\|\nabla u\|_{L^\infty}\|\dot{\Delta}_j\nabla\theta\|_{L^\infty}+C\|\nabla\theta\|_{L^\infty}\|\dot{\Delta}_j\nabla u\|_{L^\infty}\\
&\qquad+C\|\nabla u\|_{L^\infty}\sum_{k+2\ge j}2^{j-k}\|\dot{\Delta}_k\nabla\theta\|_{L^\infty}.
\end{aligned} \tag{3.19}$$

By plugging (3.19) into (3.18), summing over all integer j, and applying Young inequality for series convolution, one has

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{\dot{B}^0_{\infty,1}}+\eta\|\nabla\theta\|_{\dot{B}^0_{\infty,1}}&\le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{\dot{B}^0_{\infty,1}}+C\|\nabla\theta\|_{L^\infty}\|\nabla u\|_{\dot{B}^0_{\infty,1}}\\
&\quad+C\big\|\|[\dot{\Delta}_j,u\cdot\nabla]\nabla\theta\|_{L^\infty}\big\|_{l^1_j}\\
&\le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{\dot{B}^0_{\infty,1}}+C\|\nabla\theta\|_{L^\infty}\|\nabla u\|_{\dot{B}^0_{\infty,1}}\\
&\le C\|\omega\|_{\dot{B}^0_{\infty,1}}\|\nabla\theta\|_{\dot{B}^0_{\infty,1}},
\end{aligned} \tag{3.20}$$

where commutator estimate (2.4), imbedding inequalities $\|\nabla\theta\|_{L^\infty}\le C\|\nabla\theta\|_{\dot{B}^0_{\infty,1}}$, and $\|\nabla u\|_{L^\infty}\le C\|\nabla u\|_{\dot{B}^0_{\infty,1}}\le C\|\omega\|_{\dot{B}^0_{\infty,1}}$ are used.

For the vorticity equations, we set $p = \infty$ in (3.7) and replace Δ_j by $\dot{\Delta}_j$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_j\omega\|_{L^\infty} + \nu\|\dot{\Delta}_j\omega\|_{L^\infty} \le C\|\dot{\Delta}_j\big[(\omega\cdot\nabla u) + (\nabla u)^{\mathrm{T}}\cdot\omega)\big]\|_{L^\infty} + \|\dot{\Delta}_j F(\theta)\|_{L^\infty} + C\|[\dot{\Delta}_j, u\cdot\nabla]\omega\|_{L^\infty}. \tag{3.21}$$

Here, we show only how to bound the term $\|\dot{\Delta}_j[(\omega\cdot\nabla u) + (\nabla u)^{\mathrm{T}}\cdot\omega)]\|_{L^\infty}$ as others can be bounded directly as in the proof of Theorem 2.8. It is remarkable to point out that when dimension $n = 2$ and $n = 3$, this term becomes simple. Thus, we only treat the case $n \ge 4$. The (i, j) component of matrix $\omega\cdot\nabla u + (\nabla u)^{\mathrm{T}}\cdot\omega$ can be rewritten as

$$\begin{aligned}\big\{\omega\cdot\nabla u + (\nabla u)^{\mathrm{T}}\cdot\omega\big\}_{i,j} &= \sum_{k=1}^{n}(\omega_{i,k}\partial_k u_j + \partial_k u_i\omega_{k,j})\\ &= \sum_{k=1}^{n}\big((\partial_i u_k - \partial_k u_i)\partial_k u_j + \partial_k u_i(\partial_k u_j - \partial_j u_k)\big)\\ &= \sum_{k=1}^{n}\big(\partial_i u_k\partial_k u_j - \partial_k u_i\partial_j u_k\big)\\ &= \sum_{k=1}^{n}\partial_k\big(u_j\partial_i u_k - u_i\partial_j u_k\big), \quad (\nabla\cdot u = 0).\end{aligned} \tag{3.22}$$

Therefore,

$$\begin{aligned}&\|\dot{\Delta}_j(\omega\cdot\nabla u + (\nabla u)^{\mathrm{T}}\cdot\omega)\|_{L^\infty}\\ &\le C\sum_{|k-j|\le 2}\|\dot{\Delta}_j(\dot{S}_{k-1}\nabla u\cdot\nabla\dot{\Delta}_k u)\|_{L^\infty} + C\sum_{|k-j|\le 2}\|\dot{\Delta}_j(\dot{\Delta}_k\nabla u\cdot\nabla\dot{S}_{k-1}u)\|_{L^\infty}\\ &\quad + C\sum_{k+2\ge j}\sum_{l=1}^{n}\|\dot{\Delta}_j\partial_l(\dot{\Delta}_k u_j\partial_i\widetilde{\dot{\Delta}_k}u_l - \dot{\Delta}_k u_i\partial_j\widetilde{\dot{\Delta}_k}u_l)\|_{L^\infty}\\ &\le C\|\nabla u\|_{L^\infty}\|\dot{\Delta}_j\nabla u\|_{L^\infty} + \sum_{k+2\ge j}2^j\|\dot{\Delta}_j(\widetilde{\dot{\Delta}_k}\nabla u\cdot\dot{\Delta}_k u)\|_{L^\infty}\\ &\le C\|\nabla u\|_{L^\infty}\|\dot{\Delta}_j\nabla u\|_{L^\infty} + \sum_{k+2\ge j}2^j\|\nabla u\|_{L^\infty}\|\dot{\Delta}_k u\|_{L^\infty}\\ &\le C\|\nabla u\|_{L^\infty}\|\dot{\Delta}_j\nabla u\|_{L^\infty} + C\|\nabla u\|_{L^\infty}\sum_{k+2\ge j}2^{j-k}\|\dot{\Delta}_k\nabla u\|_{L^\infty}.\end{aligned} \tag{3.23}$$

Inserting (3.23) into (3.21), summing over all integer j from $-\infty$ to ∞ and applying Young inequality for series convolution, we can show

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{\dot{B}^0_{\infty,1}} + \nu\|\omega\|_{\dot{B}^0_{\infty,1}} \le C\|\nabla u\|_{L^\infty}\|\nabla u\|_{\dot{B}^0_{\infty,1}} + C\|\nabla\theta\|_{\dot{B}^0_{\infty,1}} + C\big\|\|[\dot{\Delta}_j, u\cdot\nabla]\omega\|_{L^\infty}\big\|_{l^1_j}$$

$$
\begin{aligned}
&\le C\|\nabla u\|_{L^\infty}\|\nabla u\|_{\dot B^0_{\infty,1}} + C\|\nabla\theta\|_{\dot B^0_{\infty,1}}\\
&\le C\|\omega\|^2_{\dot B^0_{\infty,1}} + C\|\nabla\theta\|_{\dot B^0_{\infty,1}}, \qquad (3.24)
\end{aligned}
$$

where the following imbedding inequalities are used

$$
\begin{aligned}
&\|\nabla u\|_{L^\infty} \le C\|\nabla u\|_{\dot B^0_{\infty,1}} \le C\|\omega\|_{\dot B^0_{\infty,1}},\\
&\big\| \|[\dot\Delta_j, u\cdot\nabla]\omega\|_{L^\infty}\big\|_{l^1_j} \le C\|\nabla u\|_{L^\infty}\|\omega\|_{\dot B^0_{\infty,1}} + C\|\omega\|_{L^\infty}\|\nabla u\|_{\dot B^0_{\infty,1}},
\end{aligned}
$$

[see (2.4) and (2.5)].

Therefore, we get (3.20) and (3.24) which are corresponding to (3.6) and (3.8), respectively. The desired results can be obtained the same as the proof of Theorem 2.8.

4 The Proof of Theorem 2.14

Proof of Theorem 2.14 The following inequalities will be used frequently, which are some easy consequences of Besov space imbedding properties and Boundedness of Calderon-Zygmund operators on Besov spaces,

$$
\|\nabla u\|_{L^\infty} \le C\|\nabla u\|_{B^s_{p,r}} \le C\|\nabla\theta\|_{B^s_{p,r}} \ \ (1<p<\infty), \qquad (4.1)
$$

$$
\|\nabla u\|_{L^\infty} \le C\|\nabla u\|_{\dot B^0_{\infty,1}} \le C\|\nabla\theta\|_{\dot B^0_{\infty,1}}, \qquad (4.2)
$$

where index (p, r, s) satisfies the assumption in Theorem 2.14.

Differentiating the temperature equation of system (1.3), we can derive

$$
\partial_t\nabla\theta + \gamma\nabla\theta + u\cdot\nabla(\nabla\theta) + (\nabla u)\cdot\nabla\theta = 0. \qquad (4.3)
$$

Applying homogeneous blocks $\dot\Delta_j$ operator to above equality, we have

$$
\partial_t\dot\Delta_j\nabla\theta + \gamma\dot\Delta_j\nabla\theta + u\cdot\dot\Delta_j\nabla(\nabla\theta) = -\dot\Delta_j(\nabla u\cdot\nabla\theta) - [\dot\Delta_j, u\cdot\nabla]\nabla\theta,
$$

which, together with L^p-norm in space variable, Hölder inequality, and incompressible condition, directly leads

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\Delta_j\nabla\theta\|_{L^p} + \gamma\|\dot\Delta_j\nabla\theta\|_{L^p} \le C\|\dot\Delta_j(\nabla u\cdot\nabla\theta)\|_{L^p} + C\|[\dot\Delta_j, u\cdot\nabla]\nabla\theta\|_{L^p}.
$$

For $r \ge 1$, multiply above inequality by $2^{jsr}\|\dot\Delta_j\nabla\theta\|^{r-1}_{L^p}$ and then sum over j from $-\infty$ to ∞ to obtain

$$
\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|^r_{\dot B^s_{p,r}} + \gamma\|\nabla\theta\|^r_{\dot B^s_{p,r}} \le C\Big(\|\nabla u\cdot\nabla\theta\|_{\dot B^s_{p,r}} + \big\|2^{js}\|[\dot\Delta_j, u\cdot\nabla]\nabla\theta\|_{L^p}\big\|_{l^r_j}\Big)\|\nabla\theta\|^{r-1}_{\dot B^s_{p,r}}.
$$

Taking advantage of Lemmas 2.4, 2.7, and (4.1) and some calculations give rise to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{\dot{B}^s_{p,r}}+\gamma\|\nabla\theta\|_{\dot{B}^s_{p,r}} &\le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{\dot{B}^s_{p,r}}+C\|\nabla\theta\|_{L^\infty}\|u\|_{\dot{B}^{s+1}_{p,r}}\\
&\quad+C\|\nabla\theta.\nabla u\|_{\dot{B}^s_{p,r}}\\
&\le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{\dot{B}^s_{p,r}}+C\|\nabla\theta\|_{L^\infty}\|u\|_{\dot{B}^{s+1}_{p,r}}\\
&\quad+C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{\dot{B}^s_{p,r}}+C\|\nabla\theta\|_{L^\infty}\|\nabla u\|_{\dot{B}^s_{p,r}}\\
&\le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{\dot{B}^s_{p,r}}+C\|\nabla\theta\|_{L^\infty}\|\nabla u\|_{\dot{B}^s_{p,r}}\\
&\le C\|\nabla\theta\|_{B^s_{p,r}}\|\nabla\theta\|_{\dot{B}^s_{p,r}}+C\|\nabla\theta\|_{B^s_{p,r}}\|\nabla\theta\|_{\dot{B}^s_{p,r}}\\
&\le C\|\nabla\theta\|^2_{B^s_{p,r}}.
\end{aligned} \tag{4.4}$$

Here, inequality (4.1), the bound $\|\nabla u\|_{\dot{B}^s_{p,r}}\le C\|\nabla\theta\|_{\dot{B}^s_{p,r}}$, and $\|f\|_{\dot{B}^s_{p,r}}\le C\|f\|_{B^s_{p,r}}$, $\forall s>0$ have been used.

Multiplying (4.3) by $|\nabla\theta|^{p-2}\nabla\theta$ and integrating over $\mathbb{R}^n$, we immediately derive

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{L^p}+\gamma\|\nabla\theta\|_{L^p} &\le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^p}\\
&\le C\|\nabla u\|_{B^s_{p,r}}\|\nabla\theta\|_{L^p}\\
&\le C\|\nabla\theta\|^2_{B^s_{p,r}},
\end{aligned} \tag{4.5}$$

where (2.1) and (4.1) are applied.

Combining (4.4) with (4.5), we can get the inhomogeneous- type differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{B^s_{p,r}}+\gamma\|\nabla\theta\|_{B^s_{p,r}}\le C\|\nabla\theta\|^2_{B^s_{p,r}}. \tag{4.6}$$

Let us move to the case $\dot{B}^0_{\infty,1}$. In fact, this case is much involved as Lemma 2.4 does not hold true any more for the case $\dot{B}^0_{\infty,1}$ so that we cannot obtain estimate (4.5) directly. Now we state the detailed proofs for the case $\dot{B}^0_{\infty,1}$.

From the temperature equation of system (1.3), we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_j\nabla\theta\|_{L^\infty}+\gamma\|\dot{\Delta}_j\nabla\theta\|_{L^\infty}\le C\|\dot{\Delta}_j(\nabla u\cdot\nabla\theta)\|_{L^\infty}+C\|[\dot{\Delta}_j,u\cdot\nabla]\nabla\theta\|_{L^\infty}. \tag{4.7}$$

We just handle the first term of the right-hand side of the above inequality differently.

$$\begin{aligned}
&\|\dot{\Delta}_j(\nabla u\cdot\nabla\theta)\|_{L^\infty}\\
&\quad\le C\sum_{|k-j|\le 2}\|\dot{\Delta}_j(\dot{S}_{k-1}\nabla u\cdot\nabla\dot{\Delta}_k\theta)\|_{L^\infty}+C\sum_{|k-j|\le 2}\|\dot{\Delta}_j(\dot{\Delta}_k\nabla u\cdot\nabla\dot{S}_{k-1}\theta)\|_{L^\infty}
\end{aligned}$$

$$
\begin{aligned}
&+ C \sum_{k+2\geq j} \|\dot{\Delta}_j \partial_l (\dot{\Delta}_k \nabla u_l \cdot \widetilde{\dot{\Delta}_k}\theta)\|_{L^\infty} \\
&\leq C\|\nabla u\|_{L^\infty}\|\dot{\Delta}_j \nabla\theta\|_{L^\infty} + C\|\nabla\theta\|_{L^\infty}\|\dot{\Delta}_j \nabla u\|_{L^\infty} \\
&\quad + C \sum_{k+2\geq j} 2^j \|\dot{\Delta}_j (\dot{\Delta}_k \nabla u_l \cdot \widetilde{\dot{\Delta}_k}\theta)\|_{L^\infty} \\
&\leq C\|\nabla u\|_{L^\infty}\|\dot{\Delta}_j \nabla\theta\|_{L^\infty} + C\|\nabla\theta\|_{L^\infty}\|\dot{\Delta}_j \nabla u\|_{L^\infty} \\
&\quad + C \sum_{k+2\geq j} 2^j \|\nabla u\|_{L^\infty}\|\dot{\Delta}_k \theta\|_{L^\infty} \\
&\leq C\|\nabla u\|_{L^\infty}\|\dot{\Delta}_j \nabla\theta\|_{L^\infty} + C\|\nabla\theta\|_{L^\infty}\|\dot{\Delta}_j \nabla u\|_{L^\infty} \\
&\quad + C\|\nabla u\|_{L^\infty} \sum_{k+2\geq j} 2^{j-k} \|\dot{\Delta}_k \nabla\theta\|_{L^\infty}. \qquad (4.8)
\end{aligned}
$$

By plugging (4.8) into (4.7), summing over all integer j and applying Young inequality for series convolution, one has

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{\dot{B}^0_{\infty,1}} + \gamma\|\nabla\theta\|_{\dot{B}^0_{\infty,1}} &\leq C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{\dot{B}^0_{\infty,1}} + C\|\nabla\theta\|_{L^\infty}\|\nabla u\|_{\dot{B}^0_{\infty,1}} \\
&\quad + C\Big\| \|[\dot{\Delta}_j, u\cdot\nabla]\nabla\theta\|_{L^\infty}\Big\|_{l^1_j} \\
&\leq C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{\dot{B}^0_{\infty,1}} + C\|\nabla\theta\|_{L^\infty}\|\nabla u\|_{\dot{B}^0_{\infty,1}} \\
&\leq C\|\nabla\theta\|^2_{\dot{B}^0_{\infty,1}}. \qquad (4.9)
\end{aligned}
$$

For the sake of clarity of presentation, we denote

$$
X(t) \triangleq \|\nabla\theta\|_{B^s_{p,r}} \quad \left(X(t) \triangleq \|\nabla\theta\|_{\dot{B}^0_{\infty,1}}\right).
$$

Now we can deduce from differential inequality (3.6) or (3.20) that

$$
\frac{\mathrm{d}}{\mathrm{d}t}X(t) + \gamma X(t) \leq CX^2(t). \qquad (4.10)
$$

Next we claim that the above differential inequality obeys the following global bounds:

$$
X(t) < \frac{\gamma}{2C}. \qquad (4.11)
$$

Now suppose (4.11) is not true and T_0 is the first time such that (4.11) is violated, namely,

$$
X(T_0) = \frac{\gamma}{2C}; \qquad (4.12)
$$

moreover,

$$
X(t) < \frac{\gamma}{2C}, \quad \forall 0 \leq t < T_0.
$$

We can deduce from (4.10) that for any $0 \le t \le T_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) + \frac{\gamma}{2}X(t) \le 0.$$

Therefore,

$$X(T_0) \le X(0)e^{-\frac{\gamma}{2}T_0} < X(0) < \frac{\gamma}{2C}.$$

This is a contradiction. Thus, we get the global bound. Now the uniqueness is easy to prove since the function θ is in Lipschitz spaces. Thus, we have completed the proof of Theorem 2.14. □

5 The Proof of the Theorem 2.16

The goal of this section is to present the proof of Theorem 2.16. The existence and uniqueness of local smooth solutions can be done without any difficulty as in the case of Euler or Navier–Stokes system, see for example Majda and Bertozzi (2002), Sermange and Temam (1983); for the sake of completeness, the local theory will be provided in Appendix. Thus, our efforts are focused on proving global a priori bounds for (u, b) in the initial functional setting $H^s(\mathbb{R}^n)$ with $s > 1 + \frac{n}{2}$.

Proof of Theorem 2.14 Testing the equations $(1.4)_1$ and $(1.4)_2$ by u and b, respectively, and adding them up, we can find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \kappa\|u\|_{L^2}^2 + \lambda\|b\|_{L^2}^2 = 0, \tag{5.1}$$

where we have used the facts

$$\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot u\,\mathrm{d}x = \int_{\mathbb{R}^3}(u\cdot\nabla b)\cdot b\,\mathrm{d}x = 0$$

and

$$\int_{\mathbb{R}^3}(b\cdot\nabla b)\cdot u\,\mathrm{d}x + \int_{\mathbb{R}^3}(b\cdot\nabla u)\cdot b\,\mathrm{d}x = 0. \tag{5.2}$$

To obtain the higher regularity of u and b, we apply the operator Λ^s with $s > 1 + \frac{n}{2}$ to both sides of equations (1.4)

$$\begin{cases} \partial_t\Lambda^s u + (u\cdot\nabla)\Lambda^s u + \kappa\Lambda^s u = -\nabla\Lambda^s P - [\Lambda^s,\, u\cdot\nabla]u + [\Lambda^s,\, b\cdot\nabla]b + (b\cdot\nabla)\Lambda^s b,\\ \partial_t\Lambda^s b + (u\cdot\nabla)\Lambda^s b + \lambda\Lambda^s b = (b\cdot\nabla)\Lambda^s u + [\Lambda^s,\, b\cdot\nabla]u - [\Lambda^s,\, u\cdot\nabla]b, \end{cases} \tag{5.3}$$

Taking the L^2 inner product of above equations $(5.3)_1$ and $(5.3)_2$ with $\Lambda^s u$ and $\Lambda^s b$, respectively, adding them up, we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s b(t)\|_{L^2}^2) + \kappa\|\Lambda^s u\|_{L^2}^2 + \lambda\|\Lambda^s b\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^n} [\Lambda^s, u\cdot\nabla]u\cdot\Lambda^s u\, dx + \int_{\mathbb{R}^n} [\Lambda^s, b\cdot\nabla]b\cdot\Lambda^s u\, dx \\
&\quad -\int_{\mathbb{R}^n} [\Lambda^s, u\cdot\nabla]b\cdot\Lambda^s b\, \mathrm{d}x + \int_{\mathbb{R}^n} [\Lambda^s, b\cdot\nabla]u\cdot\Lambda^s b\, \mathrm{d}x,
\end{aligned} \tag{5.4}$$

where we have used the facts

$$\int_{\mathbb{R}^n} (u\cdot\nabla\Lambda^s u)\cdot\Lambda^s u\, \mathrm{d}x = \int_{\mathbb{R}^n} (u\cdot\nabla\Lambda^s b)\cdot\Lambda^s b\, \mathrm{d}x = 0$$

and

$$\int_{\mathbb{R}^n} (b\cdot\nabla\Lambda^s b)\cdot\Lambda^s u\, dx + \int_{\mathbb{R}^n} (b\cdot\nabla\Lambda^s u)\cdot\Lambda^s b\, dx = 0 \tag{5.5}$$

which can be deduced from the $\nabla\cdot u = \nabla\cdot b = 0$ and integrating by parts.

To estimate the four terms in the right-hand side of (5.4), we need the commutator estimate (see, e.g., Kato 1990)

$$\|[\Lambda^s, f]g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}),$$

where $s > 0, p_2, p_3 \in (1,\infty)$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Therefore, by above commutator estimate, we can deduce that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} [\Lambda^s, u\cdot\nabla]u\cdot\Lambda^s u\, dx\right| \le C\|\nabla u\|_{L^\infty}\|\Lambda^s u(t)\|_{L^2}^2, \\
&\left|\int_{\mathbb{R}^n} [\Lambda^s, b\cdot\nabla]b\cdot\Lambda^s u\, dx\right| \le C\|\nabla b\|_{L^\infty}\|\Lambda^s b\|_{L^2}^2, \\
&\left|\int_{\mathbb{R}^n} [\Lambda^s, u\cdot\nabla]b\cdot\Lambda^s b\, dx\right| \le C(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty})(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2), \\
&\left|\int_{\mathbb{R}^n} [\Lambda^s, b\cdot\nabla]u\cdot\Lambda^s b\, dx\right| \le C(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty})(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s b(t)\|_{L^2}^2) + \kappa\|\Lambda^s u\|_{L^2}^2 + \lambda\|\Lambda^s b\|_{L^2}^2 \\
&\quad \le C(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty})(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2).
\end{aligned} \tag{5.6}$$

Combining above estimates (5.1) and (5.6), using the equivalent definition (2.1), we can show

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{H^s}^2+\|b(t)\|_{H^s}^2)+\kappa\|u\|_{H^s}^2+\lambda\|b\|_{H^s}^2\\&\quad\le C(\|\nabla u\|_{L^\infty}+\|\nabla b\|_{L^\infty})(\|u(t)\|_{H^s}^2+\|b\|_{H^s}^2)\\&\quad\le C(\|u(t)\|_{H^s}+\|b(t)\|_{H^s})(\|u(t)\|_{H^s}^2+\|b\|_{H^s}^2),\end{aligned}\tag{5.7}$$

where in the last inequality, we have used the fact $H^s(\mathbb{R}^n)\hookrightarrow L^\infty(\mathbb{R}^n)$ due to $s>1+\frac{n}{2}$.

As a consequence, one has

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{H^s}^2+\|b(t)\|_{H^s}^2)+\kappa\|u\|_{H^s}^2+\lambda\|b\|_{H^s}^2\\&\quad\le C(\|u(t)\|_{H^s}+\|b(t)\|_{H^s})(\|u(t)\|_{H^s}^2+\|b\|_{H^s}^2),\end{aligned}\tag{5.8}$$

For notational convenience, we set

$$H(t)\triangleq\sqrt{\|u(t)\|_{H^s}^2+\|b(t)\|_{H^s}^2}.$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t)+\min\{\kappa,\ \lambda\}H(t)\le CH(t)^2.$$

An argument similar to that used in the proof of Theorem 2.14 yields the desired global bounds. Hence, we have completed the proof of Theorem 2.16. □

Remark 5.1 We do not know whether Theorem 2.16 still holds true if we replace the space H^s by more general space $B^s_{p,r}$. The key reason is that the a priori estimates heavily rely on the L^2 cancelation relations (5.2) and (5.5). If one considers the vorticity $w\triangleq\nabla u-(\nabla u)^{\mathrm{T}}$ and the current $J\triangleq\nabla b-(\nabla b)^{\mathrm{T}}$ equations, then the difficulty comes from the nonlinear terms $b\cdot\nabla J$ and $b\cdot\nabla\omega$. In fact, the following two cancelations are not true for $q\ne 2$:

$$\int_{\mathbb{R}^3}(b\cdot\nabla)J\cdot(|\omega|^{q-2}\omega)\,\mathrm{d}x+\int_{\mathbb{R}^3}(b\cdot\nabla)\omega\cdot(|J|^{q-2}J)\,\mathrm{d}x=0,$$

$$\int_{\mathbb{R}^3}b\cdot\nabla\dot{\Delta}_jJ\cdot(|\dot{\Delta}_j\omega|^{q-2}\dot{\Delta}_j\omega)\,\mathrm{d}x+\int_{\mathbb{R}^3}b\cdot\nabla\dot{\Delta}_j\omega\cdot(|\dot{\Delta}_jJ|^{q-2}\dot{\Delta}_jJ)\,\mathrm{d}x=0.$$

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6 Appendix: Local Existence and Uniqueness Theory to Damped MHD

For sake of completeness, we provide the local existence and uniqueness to the system (1.4) with initial data $(u_0, b_0) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ with $s > 1 + \frac{n}{2}$. The local existence and uniqueness results to the system (1.1), (1.2), and (1.3) can be obtained by the method similar to Chapter 3 in Majda and Bertozzi (2002). More precisely, we state the following local result.

Proposition 6.1 *Let initial datum $(u_0, b_0) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ with $s > 1 + \frac{n}{2}$. Assume that $\nabla \cdot u_0 = 0,\ \nabla \cdot b_0 = 0$. There exists a positive time T depending on $\|u_0\|_{H^s}$ and $\|b_0\|_{H^s}$ such that the system (1.4) admits a unique solution (u, b) in $C([0, T]; H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n))$.*

To prove Proposition 6.1, the main step is to modify the system (1.4) in order to easily produce a family of global smooth solutions. In order to do this, we may, for instance, make use of the Friedrichs method. Now we define the spectral cutoff as follows:

$$\widehat{\mathcal{J}_N f}(\xi) = \chi_{B(0,N)}(\xi)\widehat{f}(\xi),$$

where $N > 0,\ B(0, N) = \{\xi \in \mathbb{R}^n \,|\, |\xi| \le N\}$, $\chi_{B(0,N)}$ is the characteristic function on $B(0, N)$. Also we define

$$L^2_N \triangleq \left\{ f \in L^2(\mathbb{R}^n) |\ \text{supp}\ \widehat{f} \subset B(0, N) \right\}.$$

It is easy for us to show the following properties (here the proof will be omitted) which will be used frequently later.

Lemma 6.2 *Let $N > 0$ and for any $f \in H^s(\mathbb{R}^n)$, the followings hold true*

$$\|\mathcal{J}_N f\|_{L^2(\mathbb{R}^n)} \le \|f\|_{L^2(\mathbb{R}^n)}, \quad \|\nabla^k \mathcal{J}_N f\|_{H^s(\mathbb{R}^n)} \le CN^k \|f\|_{H^s(\mathbb{R}^n)}, \tag{6.1}$$

$$\|\mathcal{J}_N f - f\|_{H^s(\mathbb{R}^n)} \to 0, \ \textit{as}\ \ N \to \infty, \tag{6.2}$$

$$\|\nabla^k \mathcal{J}_N f\|_{L^\infty(\mathbb{R}^n)} \le CN^{k+\frac{n}{2}} \|f\|_{L^2(\mathbb{R}^n)}, \tag{6.3}$$

$$\int_{\mathbb{R}^n} \mathcal{J}_N f g \mathrm{d}x = \int_{\mathbb{R}^n} f \mathcal{J}_N g \mathrm{d}x, \quad \int \mathcal{P} f \cdot g \,\mathrm{d}x = \int f \cdot \mathcal{P} g \,\mathrm{d}x, \tag{6.4}$$

$$\|\mathcal{J}_N f\|_{H^s(\mathbb{R}^n)} \le CN^s \|f\|_{L^2(\mathbb{R}^n)}, \quad \|\mathcal{P} f\|_{H^s(\mathbb{R}^n)} \le \|f\|_{H^s(\mathbb{R}^n)} \tag{6.5}$$

where $\mathcal{P}$ denotes the Leray projection onto divergence-free vector fields.

Proof of Proposition 6.1 The first step is to establish a smooth solution (u^N, b^N) in space L^2_N satisfying

$$\begin{cases} \partial_t u^N + \mathcal{P}\mathcal{J}_N((\mathcal{P}\mathcal{J}_N u^N \cdot \nabla)\mathcal{P}\mathcal{J}_N u^N) + \kappa \mathcal{P}\mathcal{J}_N u^N = \mathcal{P}\mathcal{J}_N((\mathcal{J}_N b^N \cdot \nabla)\mathcal{J}_N b^N), \\ \partial_t b^N + \mathcal{J}_N((\mathcal{P}\mathcal{J}_N u^N \cdot \nabla)\mathcal{J}_N b^N) + \lambda \mathcal{J}_N b^N = \mathcal{J}_N((\mathcal{J}_N b^N \cdot \nabla)\mathcal{P}\mathcal{J}_N u^N), \\ \nabla \cdot u^N = 0,\ \nabla \cdot b^N = 0, \\ u^N(x, 0) = \mathcal{J}_N u_0(x),\ b^N(x, 0) = \mathcal{J}_N b_0(x). \end{cases} \tag{6.6}$$

Claim 1: For any fixed $N > 0$, approximate system (6.6) has a unique global (in time) smooth solution (u^N, b^N) satisfying

$$(u^N, b^N) \in C([0, \infty); H^{\bar{s}}(\mathbb{R}^n)), \quad \text{for any } \bar{s} \geq 0.$$

Now we give the outline to prove the above claim. First, applying the L^2 estimate to (6.6) tells us that for any $0 \leq t \leq T$ with any $T \geq 0$

$$\begin{aligned} &\|u^N(.,t)\|_{L^2}^2 + \|b^N(.,t)\|_{L^2}^2 + 2\int_0^T (\kappa\|\mathcal{J}_N u^N(.,s)\|_{L^2}^2 + \lambda\|\mathcal{J}_N b^N(.,s)\|_{L^2}^2)\,\mathrm{d}t \\ &\quad \leq \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \end{aligned} \tag{6.7}$$

We write

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u^N \\ b^N \end{pmatrix} \\ &\quad = \begin{bmatrix} -\mathcal{P}\mathcal{J}_N((\mathcal{P}\mathcal{J}_N u^N \cdot \nabla)\mathcal{P}\mathcal{J}_N u^N) - \kappa\mathcal{P}\mathcal{J}_N u^N + \mathcal{P}\mathcal{J}_N((\mathcal{J}_N b^N \cdot \nabla)\mathcal{J}_N b^N) \\ -\mathcal{J}_N((\mathcal{P}\mathcal{J}_N u^N \cdot \nabla)\mathcal{J}_N b^N) - \lambda\mathcal{J}_N b^N + \mathcal{J}_N((\mathcal{J}_N b^N \cdot \nabla)\mathcal{P}\mathcal{J}_N u^N) \end{bmatrix}. \end{aligned}$$

For convenience of notation, we denote the right-hand side of the above differential equations as $F(u^N, b^N)$.

It is not difficult to show that F satisfies the local Lipschitz condition for any fixed N. That is, the difference $\|F(u^N, b^N) - F(\widetilde{u}^N, \widetilde{b}^N)\|_{H^{\bar{s}}}$ satisfies

$$\|F(u^N, b^N) - F(\widetilde{u}^N, \widetilde{b}^N)\|_{H^{\bar{s}}} \leq \widetilde{C}\|(u^N, b^N) - (\widetilde{u}^N, \widetilde{b}^N)\|_{H^{\bar{s}}},$$

where $\widetilde{C} = C\max(N^{\bar{s}+1+\frac{n}{2}}, \kappa, \lambda, \|u_0\|_{L^2} + \|b_0\|_{L^2})$.

Taking advantage of the Cauchy-Lipschitz theorem (Picard's Theorem, see Majda and Bertozzi 2002), we can find that for any fixed N, there exists the unique solution (u^N, b^N) in in $C([0, T_N); H^{\bar{s}}(\mathbb{R}^n) \times H^{\bar{s}}(\mathbb{R}^n))$ with $T_N = T(N, u_0, b_0)$. In fact, it is not hard to extend the local solution to the global solution based on the above estimates. In fact, we just set $\widetilde{u}^N = \widetilde{b}^N = 0$, then we can obtain immediately that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u^N\|_{H^{\bar{s}}} + \|b^N\|_{H^{\bar{s}}}) \leq C\max(N^{\bar{s}+1+\frac{n}{2}}, \kappa, \lambda, \|u_0\|_{L^2} + \|b_0\|_{L^2})(\|u^N\|_{H^{\bar{s}}} + \|b^N\|_{H^{\bar{s}}}).$$

Gronwall inequality yields for any $T \geq 0$

$$\|u^N(.,T)\|_{H^{\bar{s}}} + \|b^N(.,T)\|_{H^{\bar{s}}} \leq e^{C\max(N^{\bar{s}+1+\frac{n}{2}}, \kappa, \lambda, \|u_0\|_{L^2} + \|b_0\|_{L^2})T}.$$

Thus, we have proved **Claim 1**.

Due to $\mathcal{J}_N^2 = \mathcal{J}_N$, $\mathcal{P}^2 = \mathcal{P}$, and $\mathcal{P}\mathcal{J}_N = \mathcal{J}_N\mathcal{P}$, we can discover that $(\mathcal{P}u^N, b^N)$ and $(\mathcal{J}_N u^N, \mathcal{J}_N b^N)$ are also solutions to approximate system (6.6) with the same initial datum. Thanks to the uniqueness, we thus find

$$\mathcal{P}u^N = u^N, \; \mathcal{J}_N u^N = u^N \; \textit{and} \; \mathcal{J}_N b^N = b^N.$$

Consequently, approximate system (6.6) reduces to

$$\begin{cases} \partial_t u^N + \mathcal{P}\mathcal{J}_N((u^N\cdot\nabla)u^N) + \kappa u^N = \mathcal{P}\mathcal{J}_N(b^N\cdot\nabla)b^N, \\ \partial_t b^N + \mathcal{J}_N((u^N\cdot\nabla)b^N) + \lambda b^N = \mathcal{J}_N((b^N\cdot\nabla)u^N), \\ \nabla\cdot u^N = 0,\ \nabla\cdot b^N = 0, \\ u^N(x,0) = \mathcal{J}_N u_0(x),\ b^N(x,0) = \mathcal{J}_N b_0(x). \end{cases} \tag{6.8}$$

The same argument to that used in obtaining (5.8) together with the fact (6.4), we can also get the nonhomogeneous H^s bound as follows:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u^N(t)\|_{H^s}^2 + \|b^N(t)\|_{H^s}^2) + \kappa\|u^N\|_{H^s}^2 + \lambda\|b^N\|_{H^s}^2 \\ &\quad \le C(\|\nabla u^N\|_{L^\infty} + \|\nabla b^N\|_{L^\infty})(\|u^N(t)\|_{H^s}^2 + \|b^N\|_{H^s}^2) \\ &\quad \le C(\|u^N\|_{H^s} + \|b^N\|_{H^s})(\|u^N(t)\|_{H^s}^2 + \|b^N\|_{H^s}^2). \end{aligned} \tag{6.9}$$

For the convenience of notation, we also denote

$$X(t) = \sqrt{\|u^N(t)\|_{H^s}^2 + \|b^N\|_{H^s}^2}.$$

Consequently, (6.9) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) + \min\{\kappa,\ \lambda\}X(t) \le CX(t)^2.$$

Standard calculations show that for all N

$$\sup_{0\le t\le T}\left(\sqrt{\|u^N(t)\|_{H^s}^2 + \|b^N\|_{H^s}^2}\right) \le \frac{\sqrt{\|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2}}{1 - CT\sqrt{\|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2}}. \tag{6.10}$$

Thus, the family $(u^N,\ b^N)$ is uniformly bounded in $C([0,T];\ H^s)$ with $s > 1 + \frac{n}{2}$, provided that $T < \left(C(\|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2)\right)^{-\frac{1}{2}}$.

Therefore, one can conclude from (6.7) and (6.10) that

- $(u^N,\ b^N)_{N\in\mathbb{N}}$ is bounded in $L^\infty([0,T];\ L^2(\mathbb{R}^n))$,
- $(u^N,\ b^N)_{N\in\mathbb{N}}$ is bounded in $L^\infty([0,T];\ H^s(\mathbb{R}^n))$ for some $s > 1 + \frac{n}{2}$.

This is enough to pass to the limit (up to extraction) in (6.8). In fact, we have

$$\begin{aligned} &\|\mathcal{P}\mathcal{J}_N((u^N u^N)\|_{L^2} \le C\|u^N\|_{L^4}^2 \le C, \quad \|\mathcal{P}\mathcal{J}_N((b^N b^N)\|_{L^2} \le C\|b^N\|_{L^4}^2 \le C, \\ &\|\mathcal{P}\mathcal{J}_N((u^N b^N)\|_{L^2} \le C\|u^N\|_{L^4}\|b^N\|_{L^4} \le C, \end{aligned}$$

where we have used the following interpolation:

$$\|f\|_{L^4} \le C\|f\|_{L^2}^{\frac{4s-n}{4s}}\|f\|_{H^s}^{\frac{n}{4s}}, \quad s \ge \frac{n}{4}.$$

Note that

$$\begin{aligned}\partial_t u^N &= -\mathcal{P}\mathcal{J}_N((u^N\cdot\nabla)u^N) - \kappa u^N + \mathcal{P}\mathcal{J}_N(b^N\cdot\nabla)b^N,\\ \partial_t b^N &= -\mathcal{J}_N((u^N\cdot\nabla)b^N) - \lambda b^N + \mathcal{J}_N((b^N\cdot\nabla)u^N).\end{aligned}$$

Thus, it is not hard to see that

$$\partial_t u^N,\ \partial_t b^N \in L_t^\infty([0,T]);\ H_x^{-\sigma}(\mathbb{R}^n) \quad \text{for any } \sigma \ge 1.$$

Consequently, we assume that

$$\partial_t u^N,\ \partial_t b^N \in L^4_{Loc}([0,T]);\ H_x^{-2}(\mathbb{R}^n).$$

Since the embedding $L^2 \hookrightarrow H^{-2}$ is locally compact, the well-known Aubin-Lions argument (see e.g., Constantin and Foias 1988; Temam 2002) allows us to conclude that, up to extraction, subsequence $(u^N, b^N)_{N\in\mathbb{N}}$ satisfies

$$\|u^N - u^{N'}\|_{L^2}, \quad \|b^N - b^{N'}\|_{L^2} \to 0, \quad as \quad N,\ N' \to \infty.$$

By the interpolation ($\|u\|_{H^{s'}} \le C\|u\|_{L^2}^{1-\frac{s'}{s}}\|u\|_{H^s}^{\frac{s'}{s}}$ for any $s' < s$), we can show that

$$\|u^N - u^{N'}\|_{H^{s'}}, \quad \|b^N - b^{N'}\|_{H^{s'}} \to 0, \quad as \quad N,\ N' \to \infty, \ \text{for any } s' < s,$$

which imply that we have strong convergence limit (u, b) in $C([0,T]; H^{s'})$ (see Claim 2 below for detailed proof) with any $s' < s$. Therefore, it is enough for us to show that up to extraction, sequence $(u^N, b^N)_{N\in\mathbb{N}}$ has a limit (u, b) satisfying

$$\begin{cases}\partial_t u + \mathcal{P}((u\cdot\nabla)u) + \kappa u = \mathcal{P}((b\cdot\nabla)b),\\ \partial_t b + (u\cdot\nabla)b + \lambda b = (b\cdot\nabla)u,\\ \nabla\cdot u = 0,\ \nabla\cdot b = 0,\\ u(x,0) = u_0(x),\ b(x,0) = b_0(x).\end{cases} \tag{6.11}$$

Claim 2: $(u, b) \in L^\infty([0,T]; H^s(\mathbb{R}^n)\times H^s(\mathbb{R}^n))$ and $(u, b) \in$Lip$([0,T]; H^{s-1}(\mathbb{R}^n)\times H^{s-1}(\mathbb{R}^n))$. Moreover, $(u, b) \in C([0,T]; H^s(\mathbb{R}^n)\times H^s(\mathbb{R}^n))$.

From the above argument, it is easy to show that

$$\sup_{0\le t\le T}\|(u^N, b^N)\|_{H^s} \le M_1 < \infty, \tag{6.12}$$

$$\sup_{0\le t\le T}\|(\partial_t u^N, \partial_t b^N)\|_{H^{s-1}} \le M_2 < \infty. \tag{6.13}$$

Therefore, (u^N, b^N) is uniformly bounded in the Hilbert space $L^2([0, T]; H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n))$ such that there exists a subsequence that converges weakly to

$$(u, b) \in L^2([0, T]; H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)). \tag{6.14}$$

For the fixed $t \in [0, T]$, the sequence $(u^N(., t), b^N(., t))$ is uniformly bounded in $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$, so that it also has a subsequence that converges weakly to $(u(t), b(t)) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$. Consequently, $\|(u, b)\|_{H^s}$ is bounded for any $t \in [0, T]$ which together with (6.14) implies that $(u, b) \in L^\infty([0, T]; H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n))$. Applying the same arguments and (6.13), we can show that $(u, b) \in \mathrm{Lip}([0, T]; H^{s-1}(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n))$.

In fact, we can get from the local existence theorem that

$$(u^N, b^N) \in C^1([0, T]; H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)), \tag{6.15}$$

and

$$(u^N, b^N) \to (u, b) \in L^\infty([0, T]; H^{s'}(\mathbb{R}^n) \times H^{s'}(\mathbb{R}^n)) \quad \text{for any } s' \leq s. \tag{6.16}$$

Now we will show that (u, b) is strongly continuous in $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ in time. It suffices to consider $u \in C([0, T]; H^s(\mathbb{R}^n)$ as the same fashion can be applied to b to obtain the desired result.

By the equivalent norm, it yields

$$\|u(t_1) - u(t_2)\|_{H^s} = \left\{\left(\sum_{j<N} + \sum_{j\geq N}\right)(2^{js}\|\Delta_j u(t_1) - \Delta_j u(t_2)\|_{L^2})^2\right\}^{\frac{1}{2}}. \tag{6.17}$$

Let $\varepsilon > 0$ be arbitrarily small. Due to $u \in L^\infty([0, T]; H^s(\mathbb{R}^n))$, there exists a integer $N > 0$ such that

$$\left\{\sum_{j\geq N}(2^{js}\|\Delta_j u(t_1) - \Delta_j u(t_2)\|_{L^2})^2\right\}^{\frac{1}{2}} < \frac{\varepsilon}{2}. \tag{6.18}$$

Recalling the system $(6.11)_1$, we obtain

$$\begin{aligned}\Delta_j u(t_1) - \Delta_j u(t_2) &= \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}\tau}\Delta_j u(\tau)\,\mathrm{d}\tau \\ &= \int_{t_1}^{t_2} \Delta_j \mathcal{P}[(b\cdot\nabla)b - (u\cdot\nabla)u - \kappa u](\tau)\,\mathrm{d}\tau. \end{aligned}\tag{6.19}$$

Therefore, we can get

$$\sum_{j<N} 2^{2js}\|\Delta_j u(t_1) - \Delta_j u(t_2)\|_{L^2}^2$$

$$
\begin{aligned}
&= \sum_{j<N} 2^{2js}\Big(\Big\| \int_{t_1}^{t_2} \Delta_j \mathcal{P}[(b\cdot\nabla)b - (u\cdot\nabla)u - \kappa u](\tau)\,\mathrm{d}\tau \Big\|_{L^2}\Big)^2 \\
&\le \sum_{j<N} 2^{2js}\Big(\int_{t_1}^{t_2} \|\Delta_j[(b\cdot\nabla)b - (u\cdot\nabla)u - \kappa u]\|_{L^2}(\tau)\,\mathrm{d}\tau\Big)^2 \\
&\le \sum_{j<N} 2^{2js}\Big(\int_{t_1}^{t_2} [\|\Delta_j(b\cdot\nabla)b\|_{L^2} + \|(u\cdot\nabla)u\|_{L^2} + \kappa\|u\|_{L^2}](\tau)\,\mathrm{d}\tau\Big)^2 \\
&= \sum_{j<N} 2^{2j}\Big(\int_{t_1}^{t_2} [2^{j(s-1)}\|\Delta_j(b\cdot\nabla)b\|_{L^2} + 2^{j(s-1)}\|(u\cdot\nabla)u\|_{L^2} + \kappa 2^{j(s-1)}\|u\|_{L^2}](\tau)\,\mathrm{d}\tau\Big)^2 \\
&\le C\sum_{j<N} 2^{2j}\Big(\|(b\cdot\nabla)b\|^2_{H^{s-1}}|t_1-t_2| + \|(u\cdot\nabla)u\|^2_{H^{s-1}}|t_1-t_2| + \kappa\|u\|^2_{H^{s-1}}|t_1-t_2|\Big) \\
&\le C\sum_{j<N} 2^{2j}|t_1-t_2|\Big(\|b\|^2_{L^\infty}\|\nabla b\|^2_{H^{s-1}} + \|\nabla b\|^2_{L^\infty}\|b\|^2_{H^{s-1}} + \|u\|^2_{L^\infty}\|\nabla u\|^2_{H^{s-1}} \\
&\quad + \|\nabla u\|^2_{L^\infty}\|u\|^2_{H^{s-1}} + \kappa\|u\|^2_{H^{s-1}}\Big) \\
&\le C2^{2N}|t_1-t_2|\Big(\|b\|^2_{H^s}\|b\|^2_{H^s} + \|u\|^2_{H^s}\|u\|^2_{H^s} + \kappa\|u\|^2_{H^s}\Big),
\end{aligned} \tag{6.20}
$$

where the Sobolev imbeddings $H^s(\mathbb{R}^n) \hookrightarrow H^{s-1}(\mathbb{R}^n)$ and $H^{s-1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ with $s > 1 + \frac{n}{2}$ are used several times in the last inequality.
Thus, the following holds true

$$
\Big\{\sum_{j<N} (2^{js}\|\Delta_j u(t_1) - \Delta_j u(t_2)\|_{L^2})^2\Big\}^{\frac{1}{2}} < \frac{\varepsilon}{2} \tag{6.21}
$$

provided $|t_1 - t_2|$ small enough.
Combining (6.18) with (6.21) implies $u \in C([0,T]; H^s(\mathbb{R}^n))$. Consequently, the **Claim 2** holds true. The uniqueness can be easily obtained as the velocity field and magnetic field are both in Lipschitz space. Therefore, the proof of Proposition 6.1 is completed. □

Now we will state the following fundamental commutator estimates which have been used repeatedly in the proofs of Lemmas 2.6 and 2.7.

Lemma 6.3 (See Bahour et al. 2011) *Let θ be a C^1 function on $\mathbb{R}^n$ such that $|x|\check{\theta}(x) \in L^1$. There exists a constant C such that for any Lipschitz function a with gradient in L^p and any function b in L^q, we have, for any positive λ,*

$$
\|[\theta(\lambda^{-1}D), a]b\|_{L^r} \le C\lambda^{-1}\|\nabla a\|_{L^p}\|b\|_{L^q} \;\; with \;\; \frac{1}{p} + \frac{1}{q} = \frac{1}{r},\;\; (p,q) \in [1,\infty]^2.
$$

With the aid of above Lemma, we will prove the commutator (2.3) and Lemma 2.7.

Proof of Commutator 2.3 The proof of (2.2) follows the general line of presentation in Bahour et al. (2011), Miao et al. (2012) and is standard; thus, we omit it. Now our efforts focus on the commutator (2.3). Using the notion of para-products, we write

$$[\Delta_j, u\cdot\nabla]\omega = K_1 + K_2 + K_3$$

where

$$K_1 = \sum_{|k-j|\le 2}[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\omega, \quad K_2 = \sum_{|k-j|\le 2}[\Delta_j, \Delta_k u\cdot\nabla]S_{k-1}\omega,$$
$$K_3 = \sum_{k+1\ge j}[\Delta_j, \Delta_k u\cdot\nabla]\widetilde{\Delta_k}\omega \quad \text{and} \quad \widetilde{\Delta_k} = \Delta_{k-1} + \Delta_k + \Delta_{k+1}.$$

By using Bernstein inequality and Lemma 6.3 above, we arrive at

$$\begin{aligned}
\|K_1\|_{L^p} &\le C\sum_{i=1}^n\sum_{|k-j|\le 2} 2^{-j}\|\nabla S_{k-1}u_i\|_{L^\infty}\|\partial_i\Delta_k\omega\|_{L^p}\\
&\le C\sum_{|k-j|\le 2} 2^{k-j}\|\nabla S_{k-1}u\|_{L^\infty}\|\Delta_k\omega\|_{L^p}\\
&\le C\|\nabla u\|_{L^\infty}\sum_{|k-j|\le 2} 2^{k-j}\|\Delta_k\omega\|_{L^p}.
\end{aligned}$$

We can bound K_2 ($k\ge 1$ otherwise $K_2\equiv 0$) similar to K_1 as follows:

$$\begin{aligned}
\|K_2\|_{L^p} &\le C\sum_{i=1}^n\sum_{|k-j|\le 2} 2^{-j}\|\nabla\Delta_k u_i\|_{L^p}\|\partial_i S_{k-1}\omega\|_{L^\infty}\\
&\le C\sum_{|k-j|\le 2} 2^{-j}\|\nabla\Delta_k u\|_{L^p}2^k\| \, S_{k-1}\omega\|_{L^\infty}\\
&\le C\sum_{|k-j|\le 2} 2^{-j}\|\Delta_k\nabla u\|_{L^p}2^k\|S_{k-1}\omega\|_{L^\infty}\\
&\le C\sum_{|k-j|\le 2} 2^{k-j}\|\Delta_k\nabla u\|_{L^p}\|\omega\|_{L^\infty}\\
&\le C\|\nabla u\|_{L^\infty}\sum_{|k-j|\le 2} 2^{k-j}\|\Delta_k\omega\|_{L^p},
\end{aligned}$$

where $\|\omega\|_{L^\infty}\le C\|\nabla u\|_{L^\infty}$ and $\|\Delta_k\nabla u\|_{L^p}\le C\|\Delta_k\omega\|_{L^p}$ for any $k\ge 0$ are applied (in fact, $k\ge 0$ is only needed when $p=1$ or $p=\infty$).

We decompose K_3 into the following two parts:

$$K_3 = \sum_{j-1\le k\le j}[\Delta_j, \Delta_k u\cdot\nabla]\widetilde{\Delta_k}\omega + \sum_{k>j}[\Delta_j, \Delta_k u\cdot\nabla]\widetilde{\Delta_k}\omega \triangleq K_3^1 + K_3^2.$$

For K_3^1, making use of Lemma 6.3 above, we have

$$\begin{aligned}\|K_3^1\|_{L^p} &\le C \sum_{j-1\le k\le j} \|[\Delta_j, \Delta_k u\cdot\nabla]\widetilde{\Delta_k}\omega\|_{L^p}\\ &\le C \sum_{j-1\le k\le j} 2^{-j}\|\nabla\Delta_k u\|_{L^\infty}\|\nabla\widetilde{\Delta_k}\omega\|_{L^p}\\ &\le C\|\nabla u\|_{L^\infty} \sum_{j-1\le k\le j} 2^{k-j}\|\widetilde{\Delta_k}\omega\|_{L^p}\\ &\le C\|\nabla u\|_{L^\infty} \sum_{j-1\le k\le j} 2^{k-j}\|\Delta_k\omega\|_{L^p}.\end{aligned}$$

For the term K_3^2, we do not need to use the structure of the commutator. Thanks to divergence-free condition, we can rewrite K_3^2 as follows:

$$K_3^2 = \sum_{k>j}\Delta_j\partial_l(\Delta_k u_l\widetilde{\Delta_k}\omega) + \sum_{|k-j|\le 3, k>j}\Delta_k u_l\Delta_j\partial_l\widetilde{\Delta_k}\omega.$$

By using Bernstein inequality, we can obtain (bearing in mind $k>j \Rightarrow k\ge 0$)

$$\begin{aligned}\|K_3^2\|_{L^p} &\le C\sum_{k>j}\|\Delta_j\partial_l(\Delta_k u_l\widetilde{\Delta_k}\omega)\|_{L^p} + C\sum_{|k-j|\le 3, k>j}\|\Delta_k u_l\Delta_j\partial_l\widetilde{\Delta_k}\omega\|_{L^p}\\ &\le C\sum_{k>j}2^{j-k}\|\nabla\Delta_k u\|_{L^\infty}\|\widetilde{\Delta_k}\omega\|_{L^p} + C\sum_{|k-j|\le 3, k>j}2^{j-k}\|\nabla\Delta_k u\|_{L^\infty}\|\widetilde{\Delta_k}\omega\|_{L^p}\\ &\le C\|\nabla u\|_{L^\infty}\sum_{k>j}2^{j-k}\|\Delta_k\omega\|_{L^p}.\end{aligned}$$

Plugging all the obtained estimates together, we have

$$\begin{aligned}\|[\Delta_j, u\cdot\nabla]\omega\|_{L^p} \le{}& C\|\nabla u\|_{L^\infty}\sum_{|k-j|\le 2}2^{k-j}\|\Delta_k\omega\|_{L^p} + C\|\nabla u\|_{L^\infty}\sum_{j-1\le k\le j}2^{k-j}\|\Delta_k\omega\|_{L^p}\\ &+ C\|\nabla u\|_{L^\infty}\sum_{k>j}2^{j-k}\|\Delta_k\omega\|_{L^p}.\end{aligned}$$

Multiplying above inequality by 2^{js}, taking l_j^r then applying the discrete Young inequality, we can show that

$$\begin{aligned}\left\|2^{js}\|[\Delta_j, u\cdot\nabla]w\|_{L^p}\right\|_{l_j^r} \le{}& C\|\nabla u\|_{L^\infty}\Big\|\sum_{|k-j|\le 2}2^{(j-k)(s-1)}2^{ks}\|\Delta_k\omega\|_{L^p}\Big\|_{l_j^r}\\ &+ C\|\nabla u\|_{L^\infty}\Big\|\sum_{j-1\le k\le j}2^{(j-k)(s-1)}2^{ks}\|\Delta_k\omega\|_{L^p}\Big\|_{l_j^r}\\ &+ C\|\nabla u\|_{L^\infty}\Big\|\sum_{k>j}2^{(j-k)(s+1)}2^{ks}\|\Delta_k\omega\|_{L^p}\Big\|_{l_j^r}\end{aligned}$$

$$\le C\|\nabla u\|_{L^\infty}\|\omega\|_{B^s_{p,r}},$$

where $s+1>0$ can guarantee that

$$\Big\|\sum_{k>j}2^{(j-k)(s+1)}2^{ks}\|\Delta_k\omega\|_{L^p}\Big\|_{l^r_j}\le C\|\omega\|_{B^s_{p,r}}.$$

Therefore, the commutator (2.3) is then proved. □

Proof of Lemma 2.7 Using the notion of para-products, we write

$$[\dot\Delta_j, u\cdot\nabla]v = J_1+J_2+J_3$$

where

$$J_1=\sum_{|k-j|\le 2}[\dot\Delta_j,\dot S_{k-1}u\cdot\nabla]\dot\Delta_k v,\quad J_2=\sum_{|k-j|\le 2}[\dot\Delta_j,\dot\Delta_k u\cdot\nabla]\dot S_{k-1}v,$$
$$J_3=\sum_{k+1\ge j}[\dot\Delta_j,\dot\Delta_k u\cdot\nabla]\widetilde{\dot\Delta}_k v\quad\text{and}\quad \widetilde{\dot\Delta}_k=\dot\Delta_{k-1}+\dot\Delta_k+\dot\Delta_{k+1}.$$

Since the summation above is for k satisfying $|j-k|\le 2$ and can be replaced by a constant multiple of the representative term with $k=j$, then it follows from Bernstein inequality and Lemma 6.3 above that

$$\begin{aligned}\|J_1\|_{L^p}&\le C\sum_{i=1}^n\sum_{|k-j|\le 2}2^{-j}\|\nabla\dot S_{k-1}u_i\|_{L^\infty}\|\nabla\dot\Delta_k v\|_{L^p}\\&\le C\sum_{|k-j|\le 2}2^{k-j}\|\nabla\dot S_{k-1}u\|_{L^\infty}\|\dot\Delta_k v\|_{L^p}\\&\le C\|\nabla u\|_{L^\infty}\|\dot\Delta_j v\|_{L^p}.\end{aligned}$$

Similarly, we can deal with J_2 as follows

$$\begin{aligned}\|J_2\|_{L^p}&\le C\sum_{i=1}^n\sum_{|k-j|\le 2}2^{-j}\|\nabla\dot\Delta_k u_i\|_{L^p}\|\nabla\dot S_{k-1}v\|_{L^\infty}\\&\le C\sum_{|k-j|\le 2}2^{-j}\|\nabla\dot\Delta_k u\|_{L^p}\|\nabla\dot S_{k-1}v\|_{L^\infty}\\&\le C\sum_{|k-j|\le 2}2^{-j}2^k\|\dot\Delta_k u\|_{L^p}2^k\|\dot S_{k-1}v\|_{L^\infty}\\&\le C\sum_{|k-j|\le 2}2^{2k-2j}\|\dot\Delta_k u\|_{L^p}2^j\|v\|_{L^\infty}\\&\le C2^j\|v\|_{L^\infty}\|\dot\Delta_j u\|_{L^p}.\end{aligned}$$

It is much more involved to handle the remainder term J_3. We split it into two terms: high frequencies and low frequencies:

$$J_3 = \sum_{k\geq j-1} \dot{\Delta}_j[\dot{\Delta}_k u \cdot \nabla \widetilde{\dot{\Delta}_k} v] + \sum_{|k-j|\leq 3} \dot{\Delta}_k u \cdot \nabla \dot{\Delta}_j \widetilde{\dot{\Delta}_k} v \triangleq J_3^1 + J_3^2.$$

For the first term we do not need to use the structure of the commutator. We estimate separately each term of the commutator by using Bernstein inequalities: Thanks to divergence-free condition, we can rewrite J_3^1 as follows:

$$J_3^1 = \sum_{k\geq j-1} \sum_{l=1}^{n} \dot{\Delta}_j \partial_l [\dot{\Delta}_k u_l \widetilde{\dot{\Delta}_k} v].$$

Hence,

$$\begin{aligned}
\|J_3^1\|_{L^p} &\leq C \sum_{k\geq j-1} 2^j \|\dot{\Delta}_k u \widetilde{\dot{\Delta}_k} v\|_{L^p} \\
&\leq C \sum_{k\geq j-1} 2^{j-k} \|\dot{\Delta}_k \nabla u\|_{L^\infty} \|\widetilde{\dot{\Delta}_k} v\|_{L^p} \\
&\leq C \sum_{k\geq j-1} 2^{j-k} \|\nabla u\|_{L^\infty} \|\dot{\Delta}_k v\|_{L^p} \\
&\leq C \|\nabla u\|_{L^\infty} \sum_{k\geq j-1} 2^{j-k} \|\dot{\Delta}_k v\|_{L^p}.
\end{aligned}$$

For the second term, we use Bernstein inequalities to obtain

$$\begin{aligned}
\|J_3^2\|_{L^p} &\leq C \sum_{|k-j|\leq 3} \|\dot{\Delta}_k u \cdot \nabla \dot{\Delta}_j \widetilde{\dot{\Delta}_k} v\|_{L^p} \\
&\leq C 2^j \|v\|_{L^\infty} \|\dot{\Delta}_j u\|_{L^p}.
\end{aligned}$$

Putting all the above estimates together, we have

$$\begin{aligned}
\|[\dot{\Delta}_j, u \cdot \nabla] v\|_{L^p} \leq\ & C\|\nabla u\|_{L^\infty} \|\dot{\Delta}_j v\|_{L^p} + C 2^j \|v\|_{L^\infty} \|\dot{\Delta}_j u\|_{L^p} \\
& + C\|\nabla u\|_{L^\infty} \sum_{k\geq j-1} 2^{j-k} \|\dot{\Delta}_k v\|_{L^p}.
\end{aligned}$$

Multiplying above inequality by 2^{js} then taking l_j^r yields

$$\begin{aligned}
\left\| 2^{js} \|[\dot{\Delta}_j, u \cdot \nabla] v\|_{L^p} \right\|_{l_j^r} \leq\ & C\|\nabla u\|_{L^\infty} \|v\|_{\dot{B}^s_{p,r}} + C\|v\|_{L^\infty} \|u\|_{\dot{B}^{s+1}_{p,r}} \\
& + C\|\nabla u\|_{L^\infty} \Big\| \sum_{k\geq j-1} 2^{(j-k)(s+1)} 2^{ks} \|\dot{\Delta}_k v\|_{L^p} \Big\|_{l_j^r}
\end{aligned}$$

$$\leq C\|\nabla u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}} + C\|v\|_{L^\infty}\|u\|_{\dot{B}^{s+1}_{p,r}} + \|C_k \star D_m(j)\|_{l^r_j}$$
$$\leq C\|\nabla u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}} + C\|v\|_{L^\infty}\|u\|_{\dot{B}^{s+1}_{p,r}},$$

where $C_k = \chi_{[k\leq 1]}2^{(s+1)k}$ and $D_m = 2^{ms}\|\dot{\Delta}_m v\|_{L^p}$. Here, the discrete Young inequality has been applied. Note the following fact

$$\|u\|_{\dot{B}^{s+1}_{p,r}} \approx \|\nabla u\|_{\dot{B}^s_{p,r}} \approx \|\omega\|_{\dot{B}^s_{p,r}};$$

thus, the desired inequalities can be obtained immediately. Therefore, we have completed the proof Lemma 2.7. □

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