

# GLOBAL SOLUTIONS AND THEIR ASYMPTOTIC BEHAVIOR FOR BENJAMIN-ONO-BURGERS TYPE EQUATIONS

A.S. FOKAS AND L. LUO\*

Department of Mathematics, Imperial College, University of London  
London, SW7 2BZ, United Kingdom

(Submitted by: Jerry L. Bona)

**1. Introduction.** This paper studies an initial-value problem for the generalized Benjamin-Ono-Burgers equation (BOB)

$$u_t + P(u)_x - \nu u_{xx} - \mathbb{H}(u_{xx}) = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (1.1)$$

$$u(x, 0) = f(x), \quad x \in \mathbb{R}, \quad (1.2)$$

where  $P(u) : \mathbb{R} \rightarrow \mathbb{R}$  is a given  $C^\infty$  function satisfying certain growth conditions to be specified below, and  $\mathbb{H}$  is the Hilbert transform defined by the principal-value integral

$$\mathbb{H}u(x) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{u(y)}{x - y} dy. \quad (1.3)$$

In equation (1.1), subscripts denote partial differentiation,  $u(x, t)$  is a real-valued function, and  $\nu$  is a positive number. Using PDE techniques, the following results are obtained about the solution and its long time asymptotic behavior of the above initial value problem:

1. Let  $P(u)$  satisfy either of the following two restrictions,

$$\limsup_{u \rightarrow +\infty} \frac{|P'(u)|}{|u|^2} \leq C, \quad (1.4)$$

---

Received for publication January 1998.

\*Current address: Department of Mathematical Sciences, New Jersey Institute of Technology, Newark, NJ 07102.

AMS Subject Classifications: 35B40, 35B45, 35C20, 35Q58, 35Q72, 45G10, 76B15.

or when  $\Lambda(u) \leq 0$  and  $P'(u) \leq 0$ ,

$$|P'(u)| \leq Ce^{|u|}, \quad (1.5)$$

where  $\Lambda'(u) = P(u)$ ,  $\Lambda(0) = 0$ , prime denotes differentiation, and  $C$  is some positive number. Then, for any  $f(x)$  with finite  $H^1(\mathbb{R})$ -norm, the initial-value problem defined by equations (1.1) and (1.2) has a global solution in  $H^1(\mathbb{R})$  for any finite  $t > 0$ . (Typical examples of  $P(u)$  satisfying (1.4) and (1.5) are  $u^2$  and  $e^{-u}$ , respectively.)

**2.** Let

$$P(u) = cu^{p+1}, \quad p > 2, \quad c \text{ a constant.} \quad (1.6)$$

Assume that the initial-value problem (1.1) and (1.2) has a global solution\*. Then the long time behavior of the solution  $u$  is decomposed into two parts. One part is identical with the long time behavior of the solution  $w(x, t)$  of the corresponding linearized equation, and the other part is a higher order term which is given explicitly in terms of  $P(u)$ : If  $p > 2$ ,

$$\lim_{t \rightarrow +\infty} t^{\frac{3}{2}} (|u(\cdot, t) - w(\cdot, t)|_2)^2 = \frac{c^2}{4\nu(8\nu\pi)^{\frac{1}{2}}} \left( \int_0^{+\infty} \int_{-\infty}^{\infty} u^{p+1}(x, \tau) dx d\tau \right)^2. \quad (1.7)$$

This paper is a continuation of [6] but can be read independently. The only results of [6] used here without proof are the long time behavior of the solution  $w$  of the linear equation of (1.1), and the leading behavior of  $u$  of the equation (1.1) for long time.

We conclude this introduction with some remarks.

**1.** In the analysis of the long time asymptotics of equation (1.1) we have assumed that  $P(u)$  is given by equation (1.6). This assumption was made only in order to simplify the analytical derivations. Actually, since the solutions of equations (1.1) and (1.2) decay to zero in the  $L_2$  and in the  $L_\infty$  norms, one can choose  $T$  large enough such that  $|P(u)| \leq |c||u|^{p+1}$  for  $t \geq T$ . Hence, in general, the assumption on  $P$  will be that it vanishes at  $u = 0$  at least in the order  $p + 1$  for  $p > 2$ .

**2.** If  $\nu = 0$  and  $P(u) = \frac{1}{2}u^2$ , equation (1.1) becomes the Benjamin-Ono equation. Using the fact that this equation possesses infinitely many conservation laws, it is possible to show that its initial-value problem is globally well posed in  $H^1(\mathbb{R})$  [1, 12, 13, 16, 18]. The Benjamin-Ono equation

---

\*if  $c < 0$  and  $p$  is even, or if  $c > 0$  and  $p = 2$ , it follows from **1** above that this is the case for all  $f \in H^1(\mathbb{R})$ ; otherwise  $f$  should be sufficiently small in  $L_2$ -norm [6].

was originally derived as a model in the study of internal waves in deep, stratified fluids [3, 15]. If the dissipation effects cannot be neglected, then the term  $-\nu u_{xx}$  must be added [10], and the so-called Benjamin-Ono-Burgers equation (BOB)

$$u_t + uu_x - \nu u_{xx} - \mathbb{H}(u_{xx}) = 0 \quad (1.8)$$

is obtained. We note that the physical derivation of the BOB equation gives rise to the additional term  $u_x$ . However, for the results obtained in this paper, without loss of generality one can consider equation (1.8) and (1.1), since the relevant estimates are not affected if one uses a moving frame of reference.

**3.** The results presented here have certain similarities with the corresponding results for the generalized Korteweg-de Vries-Burgers equation

$$u_t + u_x + P(u)_x - \nu u_{xx} + u_{xxx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (1.9)$$

and the generalized regularized long-wave-Burgers equation

$$u_t + u_x + P(u)_x - \nu u_{xx} - u_{xxt} = 0, \quad x \in \mathbb{R}, \quad t > 0. \quad (1.10)$$

We recall that these equations have global solutions when the growth of  $P(u)$  is less than quintic for equation (1.9), while there is no growth restriction for equation (1.10). Furthermore, if  $P(u)$  is given by (1.6), the leading order behavior of the long time asymptotics of the solutions is the same as that of their corresponding linearized equations (see [4, 5]). The case of quadratic nonlinearity is investigated in [2].

**4.** The asymptotic results presented here are for generic initial data. If the initial data have some additional property, the decay rate of corresponding solutions will be higher than that of solutions with generic initial data. For instance, if the Fourier transform of the initial data vanishes in power  $\alpha$ , then the corresponding solutions of the equation (1.1) will decay faster than the solutions with generic initial data in the power  $\frac{\alpha}{2}$  (see [6]).

**5.** For the derivation of some of the results presented here we have used the relationship between the  $H^{\frac{1}{2}}$  and  $L_\infty$  norms given in [7].

**Notation.** The  $L_q$ -norm of a function  $f$  which is  $q$ th-power absolutely integrable on  $\mathbb{R}$  is denoted by  $|f|_q$  for  $1 \leq q < \infty$ , and similarly  $|f|_\infty = \|f\|_{L_\infty}$ . If  $m \geq 0$  is an integer,  $H^m(\mathbb{R})$  will be the Sobolev space consisting of those  $L_2(\mathbb{R})$ -functions whose first  $m$  generalized derivatives lie in  $L_2(\mathbb{R})$ , equipped with the usual norm,  $\|f\|_{H^m(\mathbb{R})} = \|f\|_m = \sum_{k=0}^m |f^{(k)}|_2$ . If  $m$  is

not integer, then the norm for  $H^m(\mathbb{R})$  will be defined by  $\|f\|_m^2 = \int_{-\infty}^{\infty} (1 + s^2)^m |\hat{f}(s)|^2 ds$ , where  $\hat{f}$  is the Fourier transform of a function  $f$  defined by  $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$ .

## 2. Global solutions.

**Theorem 2.1.** *Let  $u(x, t)$  satisfy equations (1.1) and (1.2) where  $\nu > 0$ ,  $f(x) \in H^1(\mathbb{R})$  and  $P(u) : \mathbb{R} \rightarrow \mathbb{R}$  is  $C^\infty$  and satisfies either of the two constraints given by equations (1.4) and (1.5). Then there exists a unique global solution  $u(x, t)$  in  $H^1(\mathbb{R})$  for any finite  $t > 0$ .*

**Proof.** The local solution can be easily obtained by applying the contraction mapping theorem, or semigroup theorem. The global solution exists if the  $H^1(R)$ -norm of the solution is bounded for all  $t > 0$ . Multiplying equation (1.1) by  $2u$  and integrating the result with respect to  $x$  and  $t$  over  $\mathbb{R} \times [0, t]$ , it follows that

$$(|u(\cdot, t)|_2)^2 + 2\nu \int_0^t (|u_x(\cdot, \tau)|_2)^2 d\tau = (|f|_2)^2. \quad (2.1)$$

Indeed, using

$$\begin{aligned} \int_{-\infty}^{\infty} u(P(u))_x dx &= \int_{-\infty}^{\infty} u_x P(u) dx = - \int_{-\infty}^{\infty} u_x \Lambda'(u) dx = - \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \Lambda(u) dx, \\ -2 \int_{-\infty}^{\infty} u u_{xx} dx &= 2 \int_{-\infty}^{\infty} u_x^2 dx, \\ \int_{-\infty}^{\infty} u \mathbb{H}(u_{xx}) dx &= - \int_{-\infty}^{\infty} u_x \mathbb{H}(u_x) dx = 0, \end{aligned}$$

and integrating with respect to  $x$ , it follows from equation (1.1) that

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} u^2 dx + 2\nu \int_{-\infty}^{\infty} u_x^2 dx = 0.$$

Integrating with respect to  $t$  this equation becomes (2.1).

Multiplying equation (1.1) by  $2u_{xx}$  and integrating the result over  $\mathbb{R} \times [0, t]$ , it follows that

$$(|u_x(\cdot, t)|_2)^2 + 2\nu \int_0^t (|u_{xx}(\cdot, \tau)|_2)^2 d\tau = (|f'|_2)^2 + 2 \int_0^t \int_{-\infty}^{\infty} u_{xx} P(u)_x dx d\tau. \quad (2.2)$$

Indeed, using

$$2 \int_{-\infty}^{\infty} u_{xx} u_t dx = -2 \int_{-\infty}^{\infty} u_x u_{xt} dx = -\frac{\partial}{\partial t} \int_{-\infty}^{\infty} u_x^2 dx,$$

equation (1.1) implies

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} u_x^2 dx + 2\nu \int_{-\infty}^{\infty} u_{xx}^2 dx = 2 \int_{-\infty}^{\infty} u_{xx} (P(u))_x dx,$$

which becomes equation (2.2) after integrating with respect to  $t$ .

Using Cauchy-Schwarz and Young's inequalities,

$$2 \int_{-\infty}^{\infty} ab dx \leq 2 \left( \int_{-\infty}^{\infty} a^2 dx \right)^{\frac{1}{2}} \left( \int_{-\infty}^{\infty} b^2 dx \right)^{\frac{1}{2}} \leq \alpha \int_{-\infty}^{\infty} a^2 dx + \frac{1}{\alpha} \int_{-\infty}^{\infty} b^2 dx,$$

where  $\alpha$  is an arbitrary positive number, together with  $|P(u)_x|_2 = |u_x P'(u)|_2 \leq |u_x|_2 |P'(u)|_\infty$ , one finds that

$$\int_{-\infty}^{\infty} u_{xx} (P(u))_x dx \leq \nu (|u_{xx}(\cdot, t)|_2)^2 + \frac{1}{\nu} (|u_x(\cdot, t)|_2)^2 (|P'(u(\cdot, t))|_\infty)^2.$$

Using this last estimate in equation (2.2), it follows that

$$\begin{aligned} & (|u_x(\cdot, t)|_2)^2 + \nu \int_0^t (|u_{xx}(\cdot, \tau)|_2)^2 d\tau \\ & \leq (|f'|_2)^2 + \frac{1}{\nu} \int_0^t (|u_x(\cdot, \tau)|_2)^2 (|P'(u(\cdot, \tau))|_\infty)^2 d\tau. \end{aligned} \quad (2.3)$$

Equations (2.1) and (2.3) are the main equations used for the proof of the global existence of  $u(x, t)$ . We consider the two cases (1.4) and (1.5) separately:

(i)  $|P'(u)|_\infty \leq C(|u|_\infty)^2$ . In this case, since  $(|u|_\infty)^2 \leq |u|_2 |u_x|_2$  and  $|u|_2$  is bounded (see equation (2.1)), it follows that  $|P'(u)|_\infty \leq C(|f|_2) |u_x|_2$ . Using this estimate to replace  $|P'|_\infty$  in equation (2.3), one obtains

$$(|u_x(\cdot, t)|_2)^2 + \nu \int_0^t (|u_{xx}(\cdot, \tau)|_2)^2 d\tau \leq (|f'|_2)^2 + C(\nu, |f|_2) \int_0^t (|u_x(\cdot, \tau)|_2)^4 d\tau. \quad (2.4)$$

Using Gronwall's inequality, namely if  $y(t) \leq c + \int_0^t g(\tau)y(\tau) d\tau$ , then  $y(t) \leq \widetilde{C}e^{\int_0^t g(\tau)d\tau}$ , and letting  $y = g = (|u_x(\cdot, t)|_2)^2$ , equation (2.4) yields

$$(|u_x(\cdot, t)|_2)^2 \leq C_1(|f|_2)e^{C_2(\nu, |f|_2) \int_0^t (|u_x(\cdot, \tau)|_2)^2 d\tau}. \quad (2.5)$$

However,  $\int_0^t (|u_x(\cdot, \tau)|_2)^2 d\tau$  is bounded (see equation (2.1)), thus equation (2.5) implies that  $|u_x(\cdot, t)|_2$  is bounded for all  $t > 0$ .

(ii)  $|P'(u)|_\infty \leq Ce^{|u|_\infty}$ ,  $\Lambda \leq 0$  and  $P' \leq 0$ . In this case, the following estimate is valid,

$$(\|u(\cdot, t)\|_{\frac{1}{2}})^2 + \nu \int_0^t \int_{-\infty}^{\infty} |y|^3 |\hat{u}(y, \tau)|^2 dy d\tau \leq C(\|f\|_{\frac{1}{2}})^2. \quad (2.6)$$

Indeed, multiplying equation (1.1) by  $\mathbb{H}(u_x) - P(u)$ , integrating the result over  $\mathbb{R} \times [0, t]$ , and using the facts that

$$\begin{aligned} - \int_{-\infty}^{\infty} \mathbb{H}(u_x) u_{xx} dx &= \int_{-\infty}^{\infty} u_x \mathbb{H}(u_{xx}) dx, \\ \int_{-\infty}^{\infty} P(u) u_{xx} dx &= - \int_{-\infty}^{\infty} P'(u) u_x^2 dx, \\ -P(u) u_t &= -\Lambda'(u) u_t = -\frac{\partial \Lambda}{\partial t} \\ \int_{-\infty}^{\infty} \mathbb{H}(u_x) u_t dx &= \frac{1}{2} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \mathbb{H}(u_x) u dx, \end{aligned}$$

it follows that

$$\begin{aligned} &\frac{1}{2} \int_{-\infty}^{\infty} u \mathbb{H}(u_x) dx + \nu \int_0^t \int_{-\infty}^{\infty} u_x (\mathbb{H}(u_{xx})) dx d\tau \\ &= \int_{-\infty}^{\infty} \Lambda(u) dx + \nu \int_0^t \int_{-\infty}^{\infty} P'(u) u_x^2 dx d\tau + \int_{-\infty}^{\infty} \left( \frac{1}{2} f \mathbb{H}(f_x) - \Lambda(f) \right) dx. \end{aligned}$$

If  $\Lambda(u) \leq 0$  and  $P'(u) \leq 0$ , this equation implies equation (2.6).

Applying the Brezis-Wainger inequality [7]

$$|u|_\infty \leq C \|u\|_{\frac{1}{2}} [1 + \log(1 + \|u\|_{\frac{3}{2}})]^{\frac{1}{2}}, \quad (2.7)$$

it follows that

$$|P'(u(\cdot, t))|_\infty \leq Ce^{|u(\cdot, t)|_\infty} \leq Ce^{C \|u\|_{\frac{1}{2}} [1 + \log(1 + \|u\|_{\frac{3}{2}})]^{\frac{1}{2}}} \leq C(\nu, \|f\|_{\frac{1}{2}}) \|u\|_{\frac{3}{2}}, \quad (2.8)$$

since  $|u|_{\frac{1}{2}}$  is bounded and is independent of  $t$  (see (2.6)). Using the estimate (2.8) in (2.3) to replace  $|P'(u)|_\infty$ , it follows that

$$\begin{aligned} & (|u_x(\cdot, t)|_2)^2 + \nu \int_0^t (|u_{xx}(\cdot, \tau)|_2)^2 d\tau \\ & \leq (|f'|_2)^2 + C(\nu, \|f\|_{\frac{1}{2}}) \int_0^t (|u_x(\cdot, \tau)|_2)^2 (\|u(\cdot, \tau)\|_{\frac{3}{2}})^2 d\tau. \end{aligned} \quad (2.9)$$

By using (2.1) in (2.9), one shows that

$$\begin{aligned} & (|u_x(\cdot, t)|_2)^2 + \nu \int_0^t (|u_{xx}(\cdot, \tau)|_2)^2 d\tau \\ & \leq (\|f\|_1)^2 + C(\nu, \|f\|_{\frac{1}{2}}) \int_0^t \left[ (|u_x(\cdot, \tau)|_2)^2 \int_{-\infty}^{\infty} |y^3| |\hat{u}(y, \tau)|^2 dy \right] d\tau. \end{aligned} \quad (2.10)$$

Using Gronwall's inequality, with  $y(t) = (|u_x(\cdot, t)|_2)^2$  and

$$g(t) = \int_{-\infty}^{\infty} |y^3| |\hat{u}(y, t)|^2 dy,$$

together with the boundedness of  $\int_0^t \int_{-\infty}^{\infty} |y^3| |\hat{u}(y, \tau)|^2 dy d\tau$ , because of equation (2.6), equation (2.10) implies that  $|u_x(\cdot, t)|_2$  is bounded.

**3. Asymptotic behavior of solutions.** Let  $w(x, t)$  solve the linearized version of equations (1.1) and (1.2), i.e.,

$$w_t - \nu w_{xx} - \mathbb{H}(w_{xx}) = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (3.1)$$

$$w(x, 0) = f(x), \quad x \in \mathbb{R}. \quad (3.2)$$

If  $f \in H^1(\mathbb{R})$ , these equations imply

$$w(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\nu y^2 t + i|y|yt + i y x) \hat{f}(y) dy = S(t)f(x). \quad (3.3)$$

Furthermore, if  $f \in H^1(\mathbb{R}) \cap L_1(\mathbb{R})$ , then

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2}} \int_{-\infty}^{\infty} w^2(x, t) dx = \lim_{t \rightarrow \infty} t^{\frac{1}{2}} (|S(t)f(x)|_2)^2 = (8\nu\pi)^{-\frac{1}{2}} \left( \int_{-\infty}^{\infty} f(x) dx \right)^2, \quad (3.4)$$

$$\lim_{t \rightarrow \infty} t^{\frac{3}{2}} \int_{-\infty}^{\infty} w_x^2(x, t) dx = (128\nu^3\pi)^{-\frac{1}{2}} \left( \int_{-\infty}^{\infty} f(x) dx \right)^2, \quad (3.5)$$

where  $S(t)f(x)$  is defined in equation (3.3) (see Dix [8, Theorem 2.1.1, and Corollary 2.2.7] and see also [6]).

It was shown in [6] that if  $f \in H^1(\mathbb{R}) \cap L_1(\mathbb{R})$ , the solution  $u$  of (1.1) and (1.2), with  $P(u)$  given by (1.6), satisfies

$$|u(\cdot, t)|_2 \leq C(1+t)^{-\frac{1}{4}} \quad \text{and} \quad |u_x(\cdot, t)|_2 \leq C(1+t)^{-\frac{3}{4}} \quad (3.6)$$

for  $t \geq 0$ . The same decay estimates follow from the results in [8] under closely related assumptions on the initial data.

Using equations (3.4)-(3.6) we can now derive the decay results of the difference between the solution of (1.1) and (1.2) and the solution of the corresponding linear equation (3.1) and (3.2).

**Theorem 3.1.** *Let  $f \in H^1(\mathbb{R}) \cap L_1(\mathbb{R})$ . Let  $u$  satisfy the equations (1.1) and (1.2) with  $\nu > 0$  and  $P(u) = cu^{p+1}$  for  $p > 2$ . Let  $w$  satisfy the linearized equations (3.1) and (3.2). Then the difference between  $u$  and  $w$  in  $L_2$ -norm satisfies equation (1.7).*

**Proof.** Let  $v = u - w$ . Then  $v$  solves

$$v_t - \nu v_{xx} - \mathbb{H}(v_{xx}) + cu^p u_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (3.7)$$

$$v(x, 0) = 0, \quad x \in \mathbb{R}. \quad (3.8)$$

Taking the Fourier transform of equation (3.7) with respect to the spatial variable  $x$ , and solving the resulting ordinary differential equation, it follows that  $\hat{u}$  satisfies the integral equation

$$\hat{u}(y, t) - \hat{w}(y, t) = -ci \int_0^t y \exp((- \nu y^2 + iy|y|)(t - \tau)) \widehat{u^{p+1}}(y, \tau) d\tau. \quad (3.9)$$

Using  $|u^{p+1}(\cdot, t)|_1 \leq |u(\cdot, t)|_\infty^{p-1}(|u(\cdot, t)|_2)^2$ , together with equation (3.6), one obtains

$$|u^{p+1}(\cdot, t)|_1 \leq C(1+t)^{-\frac{p}{2}}. \quad (3.10)$$

If  $p > 2$ , equation (3.10) implies that  $\int_0^\infty \int_{-\infty}^\infty u^{p+1} dx d\tau$  exists. Then the limit appearing in the left-hand side of equation (1.7) can be computed. Using Parseval's identity and equation (3.9), it follows that

$$\begin{aligned} \lim_{t \rightarrow +\infty} t^{\frac{3}{2}} (|u(\cdot, t) - w(\cdot, t)|_2)^2 &= \lim_{t \rightarrow +\infty} t^{\frac{3}{2}} (|\hat{u}(\cdot, t) - \hat{w}(\cdot, t)|_2)^2 \\ &= \lim_{t \rightarrow +\infty} t^{\frac{3}{2}} \left( \left| ci \int_0^t y \exp((- \nu y^2 + iy|y|)(t - \tau)) \widehat{u^{p+1}}(y, \tau) d\tau \right|_2 \right)^2 \\ &= \lim_{t \rightarrow +\infty} t^{\frac{3}{2}} \int_{-\infty}^\infty c^2 y^2 \left| \int_0^t e^{(- \nu y^2 + iy|y|)(t - \tau)} \widehat{u^{p+1}}(y, \tau) d\tau \right|^2 dy. \end{aligned}$$



Making the change of variable  $s = y\sqrt{t}$ , and using  $\lim_{t \rightarrow \infty} \exp[(\nu s^2 - is|s|)\frac{\tau}{t}] \rightarrow 1$  for any fixed  $s$  and  $\tau \in [0, t)$ , this limit becomes

$$\begin{aligned}
& \lim_{t \rightarrow +\infty} \int_{-\infty}^{\infty} c^2 s^2 e^{-2\nu s^2} \left| \int_0^t e^{(\nu s^2 - is|s|)\frac{\tau}{t}} \widehat{u^{p+1}}\left(\frac{s}{\sqrt{t}}, \tau\right) d\tau \right|^2 ds \\
&= c^2 \int_{-\infty}^{\infty} s^2 e^{-2\nu s^2} \left| \int_0^{+\infty} \widehat{u^{p+1}}(0, \tau) d\tau \right|^2 ds \\
&= c^2 \int_{-\infty}^{\infty} s^2 e^{-2\nu s^2} ds \left( \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \int_{-\infty}^{\infty} u^{p+1}(x, \tau) dx d\tau \right)^2 \\
&= \frac{c^2}{2\pi(2\nu)^{\frac{3}{2}}} \int_0^{+\infty} s^{\frac{1}{2}} e^{-s} ds \left( \int_0^{+\infty} \int_{-\infty}^{\infty} u^{p+1}(x, \tau) dx d\tau \right)^2 \\
&= \frac{c^2}{4\nu(8\nu\pi)^{\frac{1}{2}}} \left( \int_0^{+\infty} \int_{-\infty}^{\infty} u^{p+1}(x, \tau) dx d\tau \right)^2, \tag{3.11}
\end{aligned}$$

where we have taken the limit inside the integral since the relevant integral is finite. In fact, the use of (3.10) shows that for  $p > 2$ ,

$$\begin{aligned}
& \left| \int_0^t e^{(\nu s^2 - is|s|)\frac{\tau}{t}} \widehat{U^{p+1}}\left(\frac{s}{\sqrt{t}}, \tau\right) d\tau \right| \leq \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t e^{\nu s^2 \frac{\tau}{t}} |u^{p+1}(\cdot, \tau)|_1 d\tau \\
&\leq e^{\frac{\nu s^2}{2}} \int_0^{\frac{t}{2}} \frac{C d\tau}{(1+\tau)^{p/2}} + \int_{\frac{t}{2}}^t \frac{C e^{\nu s^2 \frac{\tau}{t}}}{(1+\tau)^{p/2}} d\tau \tag{3.12} \\
&\leq C e^{\frac{\nu s^2}{2}} + \frac{Ct}{(1+\frac{t}{2})^{p/2}} \frac{e^{\nu s^2} - e^{\frac{\nu s^2}{2}}}{\nu s^2} \leq C \left[ e^{\frac{\nu s^2}{2}} + e^{\nu s^2} \frac{1 - e^{-\frac{\nu s^2}{2}}}{s^2} \right].
\end{aligned}$$

Hence, the integrand in (3.11) is bounded as

$$\begin{aligned}
& s^2 e^{-2\nu s^2} \left| \int_0^t e^{(\nu s^2 - is|s|)\frac{\tau}{t}} \widehat{U^{p+1}}\left(\frac{s}{\sqrt{t}}, \tau\right) d\tau \right|^2 \tag{3.13} \\
&\leq C s^2 e^{-2\nu s^2} \left[ e^{\frac{\nu s^2}{2}} + e^{\nu s^2} \frac{1 - e^{-\frac{\nu s^2}{2}}}{s^2} \right]^2 \leq C \left[ s^2 e^{-\nu s^2} + \frac{(1 - e^{-\frac{\nu s^2}{2}})^2}{s^2} \right],
\end{aligned}$$

for all  $t > 0$ . Note that the left-hand side of (3.13) is a  $L_1$ -function. Hence, one can use the Dominated-Convergence Theorem to (3.11).

**Acknowledgments.** This work was partially supported by EPSRC. The authors thank Professor J. Bona for stimulating discussions. L. Luo thanks

Professor Rumei Luo for his encouragement. The authors thank the referees for their comments.

### REFERENCES

- [1] Abdelouhab, L., Bona, J.L., Felland, M. and Saut, J.-C., *Nonlocal models for nonlinear dispersive waves*, Phys. D, 40 (1989), 360-392.
- [2] Amick, C.J., Bona, J.L. and Schonbek, M.E., *Decay of solutions of some nonlinear wave equations*, J. Differential Equations, 81 (1989), 1-49.
- [3] Benjamin, T.B., *Internal waves of permanent form in fluids of great depth*, J. Fluid Mech., 29 (1967), 559-592.
- [4] Bona, J.L. and Luo, L., *Decay of the solutions to nonlinear, dispersive wave equations*, Diff. & Int. Equations, 6 (1993), 961-980.
- [5] ———, *More results on the decay of solutions to nonlinear, dispersive wave equations*, Discrete and Continuous Dynamical Systems, 1 (1995), 151-193.
- [6] ———, *Large-Time Asymptotics of the Generalized Benjamin-Ono-Burgers Equation*, (submitted) (1995).
- [7] Brezis, H. and Wainger, S., *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. P.D.E. 5 (1980), 773-789.
- [8] Dix, D.B., *Temporal asymptotic behavior of solutions of the Benjamin-Ono-Burgers equation*, J. Differential Equations, 90 (1991), 238-287.
- [9] Dix, D.B., *The dissipation of nonlinear dispersive waves: The case of asymptotically weak nonlinearity*, Comm. P.D.E, 17 (1992), 1665-1693.
- [10] Edwin, P.M. and Roberts, B., *The Benjamin-Ono-Burgers equation: An application in solar physics*, Wave Motion, 8 (1986), 151-158.
- [11] Ginibre, J. and Velo, G., *Smoothing properties and existence of solutions for the generalized Benjamin-Ono equation*, J. Differential Equations, 93 (1991), 150-212.
- [12] Iório, R.J., *On the Cauchy problem for the Benjamin-Ono equation*, Comm. P.D.E., 11 (1986), 1031-1081.
- [13] Kenig, C.E., Ponce, G. and Vega, L., *On the generalized Benjamin-Ono equation*, Trans. Amer. Math. Soc., 342 (1994), 155-172.
- [14] Naumkin, P.I. and Shishmarev, I.A., *Nonlinear Nonlocal Equations in the Theory of Waves*, Translation of Mathematical Monographs, 133 AMS (1994).
- [15] Ono, H., *Algebraic solitary waves in stratified fluids*, J. Phys. Soc. Japan, 39 (1975), 1082-1091.
- [16] Tom, M.M., *Smoothing properties of some weak solutions of the Benjamin-Ono equation*, Differential & Int. Equations, 3 (1990), 683-694.
- [17] Zhang, L., *Initial value problem for a nonlinear parabolic equation with singular integral-differential term*, Acta 8 (1992), 367-376.
- [18] Zhou, Y. and Guo, B., *Initial value problems for a nonlinear singular integral-differential equation of deep water*, Lecture notes in Mathematics, 1306 (1986), 278-290.