Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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Preprint no.: 65 2003

GLOBAL SOLUTIONS OF NONLINEAR TRANSPORT EQUATIONS FOR CHEMOSENSITIVE MOVEMENT

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Abstract. A widespread phenomenon in moving microorganisms and cells is their ability to orient themselves in dependence of chemical signals. In this paper we discuss kinetic models for chemosensitive movement, which take into account also evaluations of gradient fields of chemical stimuli which subsequently influence the motion of the respective microbiological species. The basic type of model was discussed by Alt [1], [2] and in Othmer, Dunbar, and Alt [17]. Chalub, Markowich, Perthame and Schmeiser rigorously proved that, in three dimensions, these kind of kinetic models lead to the classical Keller-Segel model as its drift-diffusion limit when the equation for the chemo-attractant is of elliptic type [3], [4]. In [11] it was proved that the macroscopic diffusion limit exists in both two and three dimensions also when the equation of the chemo-attractant is of parabolic type. So far in the rigorous derivations only the density of the chemo-attractant was supposed to influence the motion of the chemosensitive species. Here we are concerned with the effects of evaluations of gradient fields of the chemical stimulus on the behavior of the chemosensitive species. In the macroscopic limit some effects result in a change of the classical parabolic Keller-Segel model for chemotaxis. Under suitable structure conditions global solutions for the kinetic models can be shown.

Introduction

The starting point of our considerations is the classical chemotaxis model as discussed by Keller and Segel (see [13] and [14]). This system is of advection-diffusion type and consists of two coupled parabolic equations

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (D(\rho, S)\nabla \rho - \chi(\rho, S)\rho \nabla S), \tag{1}$$

$$\tau \frac{\partial S}{\partial t} = D_0 \Delta S + \alpha \rho - \beta S, \quad \alpha, \beta, \tau \geq 0. \tag{2}$$

Here $\rho = \rho(x,t)$ denotes the density of chemotactic cells and $S = S(x,t)$ is the density of the chemo-attractant. The cells are attracted by the chemical and χ denotes their chemotactic sensitivity. The first rigorous derivation of the macroscopic chemotaxis equations from microscopic models, namely interacting stochastic many particle systems, was given in [21]. In [10] a survey about known results on existence of global solutions and finite time blowup for this type of model is given.

In [3] a kinetic model for equation (1) was discussed with a reduced version of equation (2) which is the Poisson equation without decay term

$$-\Delta S = \alpha \rho. \tag{3}$$

The following kinetic equation for the oriented cell density $f = f(x,v,t) \geq 0$ is considered in [3, page 3]

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \int_V (T[S]f' - T^*[S]f)dv', \tag{4}$$

where x, v, and t indicate position, velocity, and time, respectively. Here the abbreviations $f' = f(x,v',t)$, $T[S] = T[S](x,v,v',t)$ and $T^*[S] = T[S](x,v',v,t)$ are used. The first term on the right hand side of (4)

1991 *Mathematics Subject Classification.* 35K55, 45K05, 82C70, 92C17.

Key words and phrases. chemosensitive movement, sensing of gradient fields, nonlinear transport equations, global solutions, drift-diffusion limit, Keller-Segel model.

describes the turning into direction v and the second term the turning away from v. The cell density ρ fulfills

$$\rho(x,t) = \int_V f(x,v,t)dv,$$

where V is the set of admissible velocities which is assumed to be compactly supported.

Using stochastic models for the motion of bacteria and leukocytes Alt formally derived (1) from a transport equation similar to (4), [1, section 8], [2, section 3]. Later a general formulation of this velocity-jump process was presented and studied in [17, section 3]. In [18] and [19] Othmer and Hillen studied the formal diffusion limit of a transport equation of (4) by moment expansions, which generalizes parts of Alt's earlier works [1], [2]. A hyperbolic scaling and its formal limit are discussed in [6].

Based on [19] a rigorous proof of the macroscopic limit was given in [3]. After using diffusive scaling of time and space, the non-dimensional form of (4) leads to [3, page 4]

$$\epsilon^2 \frac{\partial f_\epsilon}{\partial t} + \epsilon v \cdot \nabla_x f_\epsilon = -\mathcal{T}_\epsilon[S_\epsilon](f_\epsilon), \qquad x \in \mathbb{R}^n,\ v \in V,\ t > 0 \tag{5}$$

where

$$\mathcal{T}_\epsilon[Z](g) = \int_V (T^*_\epsilon[Z]g - T_\epsilon[Z]g')dv'.$$

The diffusion limit $\epsilon \to 0$ was studied for initial conditions

$$f_\epsilon(x,v,0) = f_0(x,v), \qquad x \in \mathbb{R}^n,\ v \in V, \tag{6}$$

and (5) being coupled to equation (3) for the chemo-attractant. In [3] it was shown that the coupled nonlinear system (5), (6), and (3) results in Keller-Segel type equations for chemotaxis as its macroscopic drift-diffusion limit under suitable conditions turning kernel in three dimension (compare e.g. [3, Theorem 5] and [4, Theorem 2]). In [3] and [4], it was also proved that for suitable turning kernels, blow up can be prevented on the kinetic level for fixed $\epsilon > 0$.

In [11], as an extension of [3], the authors prove that such kinetic models have a macroscopic diffusion limit in both two and three dimensions also when the equation of the chemo-attractant is of parabolic type, i.e. $\tau > 0$, which is the original version of the chemotaxis model. An independent related result is announced to be given in [5].

In this article, we consider turning kernels depending not only on S but also on ∇S, like formally discussed, among others, in [22] and [19], i.e.

$$\epsilon^2 \frac{\partial f_\epsilon}{\partial t} + \epsilon v \cdot \nabla_x f_\epsilon = -\mathcal{T}_\epsilon[S_\epsilon, \nabla S_\epsilon](f_\epsilon), \qquad x \in \mathbb{R}^n,\ v \in V,\ t > 0 \tag{7}$$

with initial condition (6) coupled to

$$\tau \frac{\partial S_\epsilon}{\partial t} = \Delta S_\epsilon + \alpha \rho_\epsilon - \beta S_\epsilon,\ \ \tau \geq 0, \alpha > 0, \beta \geq 0. \tag{8}$$

where

$$\rho_\epsilon = \int_V f_\epsilon dv. \tag{9}$$

In the sequel, for notational convenience, we write $\mathcal{T}_\epsilon[S_\epsilon, \nabla S_\epsilon]$ as $\mathcal{T}_\epsilon[S_\epsilon]$, unless any confusion is to be expected Here we emphasize that the condition on the turning kernel include detection also of spatial gradients of the chemo-attractant by the chemotactic cells. This behavior results under certain conditions in a macroscopic model which varies from the classical Keller-Segel system by additional higher order terms.

Our main result is that suitable turning kernels which take into account the effects of gradient measurements of the chemical exclude blow up of the solutions in finite time on the kinetic level in two dimensions (compare Theorem 2.8 and Theorem 2.15 for elliptic and parabolic cases, respectively). The result can be extended to three dimensions under some restrictions on the turning kernels (compare Theorem 2.10 and Theorem 2.16 for the elliptic and parabolic cases, respectively). We also show the existence of a macroscopic diffusion limit of the kinetic model in two and three dimensions. More precisely, under similar assumptions on the turning kernel

$K[S]$ as given in [3], we prove that the coupled nonlinear system (6), (7), and (8) converges to Keller-Segel type equations and their variants for $\epsilon \to 0$ (compare Theorem 3.4). Our main tool is the potential estimate for S. In particular, in case the chemo-attractant equation is of elliptic type, i.e. $\tau = 0$ and in two dimensions, log-type estimates for the chemical S are used to obtain global existence for the kinetic model (similar techniques were used in [12, Lemma 4]).

The plan of this paper is as follows: In section 1, we introduce notations used in this article and briefly review derivations of the macroscopic equation presented in [3] and [11]. In section 2, we prove that the kinetic model (7)-(9) has a global solution for 'suitable' turning kernels. In section 3, we prove existence of the diffusion limit for a short time interval. In section 4 we give concrete examples on how the specific dependencies of the turning kernel result in different macroscopic equations.

1. Preliminaries

We first introduce some notations which will be used throughout this article and recall some of the observations presented in [3].

- We denote by G the Bessel potential, which is the fundamental solution of the differential operator $1 - \Delta$ in $\mathbb{R}^n$ (see [20, page 130-132])

$$G(x) = \frac{1}{4\pi} \int_0^\infty e^{-\pi \frac{|x|^2}{4s} - \frac{s}{4\pi}} s^{\frac{-n+2}{2}} \frac{ds}{s}. \tag{10}$$

- By Γ we denote the fundamental solution of the differential operator $\partial_t - \Delta_x + \beta$ in $\mathbb{R}^n \times \mathbb{R}_+$

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4t} - \beta t\right). \tag{11}$$

- For $\Omega \subset \mathbb{R}^n$ and $1 \le q \le \infty$, $L^q(\Omega)$ denotes the Banach space of measurable functions with

$$\|u\|_{L^q(\Omega)} = \left(\int_\Omega |u(x)|^q \, dx\right)^{1/q}, q < \infty \quad \text{and} \quad \|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_\Omega |u|.$$

- Let $\Omega_t = \Omega \times (0,t)$. For $1 \le q \le \infty$, $L^q(\Omega_t)$ denotes the Banach space of all measurable functions with the finite norm

$$\|u\|_{L^q(\Omega_t)} = \left(\int_0^t \int_\Omega |u(x,t)|^q \, dx \, dt\right)^{1/q}.$$

- For $1 \le q \le \infty$, $W^{k,q}(\Omega)$ denotes the usual Sobolev space; i.e., $W^{k,q}(\Omega) = \{u : D^\alpha u \in L^q(\Omega), 0 \le |\alpha| \le k\}$.
- $C^\alpha(\Omega)$ denotes the Banach space of functions that are Hölder continuous with exponent $\alpha \in (0,1)$, and $C^{k,\alpha}(\Omega)$ consists of all functions whose derivatives up to $k-$th order are Hölder continuous with exponent $\alpha \in (0,1)$.
- Let $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f is denoted by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-x\cdot\xi} dx$.
- The convolution of two functions f and g is denoted by $f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{\mathbb{R}^n} f(y)g(x-y)dy$.
- By $C = C(\alpha, \beta, \ldots)$ we denote a constant depending on the prescribed quantities $\alpha, \beta, \ldots$. The domain Ω considered in this article is $\mathbb{R}^n, n = 2, 3$.

To make this note self-contained, we review the formal derivation of the macroscopic equation from the kinetic model presented in [3] (compare the details in [3, page 5-7]). For simplicity we assume for a moment $\tau = 1, \alpha = 1$, and $\beta = 1$ (other cases can be formally derived in a similar way without any difficulty). Since the integral of $\mathcal{T}_\epsilon[S](f)$ with respect to the velocity vanishes, we obtain the macroscopic conservation equation

$$\frac{\partial \rho_\epsilon}{\partial t} + \nabla \cdot J_\epsilon = 0, \tag{12}$$

where $J_\epsilon(x,t) = \epsilon^{-1} \int_V v f_\epsilon(x,v,t) dv$ is the flux density. The turning kernel is assumed to have the following asymptotic expansion $T_\epsilon[S] = T_0[S] + \epsilon T_1[S] + O(\epsilon^2)$. Then the turning operator can be expanded in a similar way and

$$\mathcal{T}_k[S](f) = \int_V (T_k^*[S]f - T_k[S]f')dv'.$$

By asymptotic expansion of $f_\epsilon = f_0 + \epsilon f_1 + O(\epsilon^2)$ and $S_\epsilon = S_0 + \epsilon S_1 + O(\epsilon^2)$, the equation for the leading order terms can be obtained from (7):

$$\mathcal{T}_0[S_0](f_0) = 0, \quad S_0 = \rho_0 * \Gamma, \quad \rho_0 = \int_V f_0 dv. \tag{13}$$

Comparing coefficients in (7) results in

$$v \cdot \nabla_x f_0 = -\mathcal{T}_0[S_0](f_1) - \mathcal{T}_1[S_0](f_0) - \mathcal{T}_{0S}[S_0, S_1](f_0)$$

where $\mathcal{T}_{0S}[S_0, S_1]$ is part of the turning operator $\mathcal{T}$ and its kernel is the Frechet derivative of T_0 with respect to S, evaluated at S_0 in the direction S_1. Here, we recall the assumptions on the leading order terms of the turning operator and two useful lemmas presented in [3, (A0), Lemma 1, and Lemma 2, page 6-7].

Assumption 1.1. *There exists a bounded velocity distribution $F(v) > 0$, such that $T_0^*[S]F = T_0[S]F'$ and*

$$\int_V vF(v)dv = 0, \qquad \int_V F(v)dv = 1.$$

The turning rate $T_0[S]$ is bounded, and there exists a constant $\gamma = \gamma[S] > 0$ such that $T_0[S]/F \geq \gamma$ for all $(v, v') \in V \times V, x \in \mathbb{R}^n$, and $t > 0$.

Lemma 1.2. *Let $\zeta : \mathbb{R} \to \mathbb{R}, g : V \to \mathbb{R}$, and let*

$$\phi_\epsilon^S[S] = \frac{T_\epsilon[S]F' + T_\epsilon^*[S]F}{2}, \quad \phi_\epsilon^A[S] = \frac{T_\epsilon[S]F' - T_\epsilon^*[S]F}{2},$$

denote the symmetric and, respectively, antisymmetric part of $T_\epsilon[S]F'$. Then

$$\int_V \int_V \mathcal{T}_\epsilon(Fg)\zeta(g)dv = \frac{1}{2}\int_V \int_V \phi_\epsilon^S[S](g - g')(\zeta(g) - \zeta(g'))dv'dv$$
$$+\frac{1}{2}\int_V \int_V \phi_\epsilon^A[S](g + g')(\zeta(g) - \zeta(g'))dv'dv.$$

The same holds for $\mathcal{T}_k[S]$ with analogous definitions of $\phi_k^S[S]$ and $\phi_k^A[S]$.

Proof. See Lemma 1 in [3]. □

With $g = f/F$ and $\zeta = \mathrm{id}$ one obtains

Lemma 1.3. *Let Assumption 1.1 hold. Then, the entropy equality*

$$\int_V \mathcal{T}_0[S](f)\frac{f}{F}dv = \frac{1}{2}\int_V \int_V \phi_0^S[S]\left(\frac{f}{F} - \frac{f'}{F'}\right)^2 dv'dv \geq 0$$

holds. For $g \in L^2(V; dv/F)$, the equation $\mathcal{T}_0[S](f) = g$ has a unique solution $f \in L^2(V; dv/F)$ satisfying $\int_V f dv = 0$ if and only if $\int_V g dv = 0$.

Proof. See Lemma 2 in [3]. □

From the entropy equality, we deduce that

$$f_0(x, v, t) = \rho_0(x, t)F(v).$$

Since $\mathcal{T}_{0S}[S_0, S_1](f_0) = 0$, we obtain

$$\mathcal{T}_0[S](f_1) = -vF \cdot \nabla\rho_0 - \rho_0\mathcal{T}_1[S_0](F).$$

The right hand side satisfies the solvability condition from Lemma 1.3 and therefore the solution can be written as

$$f_1 = -\kappa(x, v, t) \cdot \nabla\rho_0(x, t) - \Theta(x, v, t)\rho_0(x, t) + \rho_1(x, t)F(v),$$

where $\kappa = \kappa[S_0]$ and $\Theta = \Theta[S_0]$ are the solutions of

$$\mathcal{T}_0[S_0](\kappa) = vF, \qquad \mathcal{T}_0[S_0](\Theta) = \mathcal{T}_1[S_0](F),$$

and ρ_1 is the macroscopic density of f_1, which is a new unknown. By passing to the limit $\epsilon \to 0$ in (12), the convection-diffusion equation reads

$$\partial_t \rho_0 - \nabla \cdot (D[S_0]\nabla \rho_0 - \rho_0 H[S_0]) = 0$$

where

$$H[S_0](x,t) = \int_V v \otimes \kappa[S_0](x,v,t)dv, \quad D[S_0] = -\int_V v\Theta[S_0](x,v,t)dv,$$

together with

$$\frac{\partial S_0}{\partial t} = \Delta S_0 + \rho_0 - S_0.$$

The specific form of $H[S_0]$ and $D[S_0]$ in dependence of different possible turning kernels will be discussed later.

2. Global solution of the kinetic model

In this section we show that solutions of the coupled system (6)-(9) in two and three dimensions do not blow up in finite time for fixed $\epsilon > 0$ if the turning kernel satisfies a certain structure condition. Without loss of generality we set $\epsilon = 1$ in (6) and $\alpha = 1$ in (8). We consider two problems, namely the elliptic and the parabolic equation for the chemo-attractant. First we recall some well-known facts needed for our purpose.

Theorem 2.1. ***(Young's inequality)*** *Suppose $1 \le p, q, r \le \infty$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^r(\mathbb{R}^n)$, then $f * g \in L^q(\mathbb{R}^n)$ and*

$$||f * g||_{L^q(\mathbb{R}^n)} \le ||f||_{L^p(\mathbb{R}^n)}||g||_{L^r(\mathbb{R}^n)}, \qquad \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1. \tag{14}$$

Proof. See e.g. [8, page 232-233]. □

Lemma 2.2. *Let $a(t)$ and $b(t)$ be positive functions. Let $y(t) > 0$ be differentiable in t and satisfy*

$$y' \le a(t)\, y \ln y + b(t)\, y.$$

Then

$$y(t) \le \left[y(0) \exp\left(\int_0^t b(s)\, e^{-\int_0^s a(\tau)d\tau} ds \right) \right]^{\exp\left(\int_0^t a(s)ds\right)}.$$

Proof. Dividing both sides of the inequality by y, we get $z' \le a(t)z + b(t)$ where $z = \ln y$. Using a standard Gronwall argument, we deduce the lemma. □

Lemma 2.3. ***(Gronwall's inequality)*** *Let g and h be positive functions. Suppose that f is continuous and satisfies*

$$f(t) \le g(t) + h(t) \int_0^t f(s)ds.$$

Then

$$f(t) \le g(t) + h(t) \int_0^t g(s) e^{\int_s^t h(\tau)d\tau} ds.$$

Proof. Computations are straightforward, and thus details are omitted (see e.g. [7, page 624-625]). □

The structure condition on the turning kernel $T[S]$ is assumed to be as follows.

Assumption 2.4. *There exist nonnegative constants $C_i \geq 0$, $i = 1, 2, ..., 5$ such that for all $x \in \mathbb{R}^n, n = 2, 3$, $v, v' \in V$, $t \in \mathbb{R}^+$, and $S \in W^{1,\infty}(\mathbb{R}^n)$, the turning kernel T satisfies*

$$0 \leq T_\epsilon[S](x, v, v', t) \leq C_1 + C_2 S(x + \epsilon v, t) + C_3 S(x - \epsilon v', t) + C_4|\nabla S(x + \epsilon v, t)| + C_5|\nabla S(x - \epsilon v', t)|, \tag{15}$$

$$|\nabla T_\epsilon[S](x, v, v, t)| \leq C_2|\nabla S(x + \epsilon v, t)| + C_3|\nabla S(x - \epsilon v', t)| + C_4|\nabla^2 S(x + \epsilon v, t)| + C_5|\nabla^2 S(x - \epsilon v', t)|. \tag{16}$$

This means that the cells can measure the concentration and the spatial gradient of the chemo-attractant up to a distance ϵ from their position and this may affect the movement of the cells.

Remark 2.5. *The turning kernel, as given above, describes the turning from direction v' into direction v. This means, that the actual or 'old' direction is evaluated by checking backwards, whereas the evaluation of possible new directions are checked forwards (e.g. by lamelliopodial protrusion) Checking the possible new directions also backwards if compared to the actual direction of motion is also possible to be taken into account in the following considerations. We leave this out to make the paper more readable. Nevertheless, it is important to note that a forward evaluation of the actual direction v' causes a technical problem in our approach so far.*

We first consider the case that the chemo-attractant equation is of elliptic type.

a) Elliptic case: $\tau = 0$

In this part, we consider the elliptic equation for the chemo-attractant S for two cases: $\beta > 0$ and $\beta = 0$. First, we treat the case when $\beta > 0$ and may set $\beta = 1$ without loss of generality, i.e.

$$-\Delta S = \rho - S. \tag{17}$$

Dimension $n = 2$**:**

We start with elementary properties of the Bessel potential G in two dimensions.

Lemma 2.6. *Let G be the Bessel potential in $\mathbb{R}^2$. Then $G \in L^p(\mathbb{R}^2)$ for any p with $1 \leq p < \infty$ and $\nabla G \in L^p(\mathbb{R}^2)$ for any p with $1 \leq p < 2$. Furthermore, the following estimates are satisfied:*

$$||G||_{L^p(\mathbb{R}^2)} \leq Cp, \qquad 1 \leq p < \infty, \tag{18}$$

$$||\nabla G||_{L^p(\mathbb{R}^2)} \leq C\frac{2p}{2-p}, \qquad 1 \leq p < 2. \tag{19}$$

Proof. For $n = 2$, the Bessel potential is (compare (10))

$$G(x) = \frac{1}{4\pi}\int_0^\infty e^{-\pi\frac{|x|^2}{4s} - \frac{s}{4\pi}} \frac{ds}{s}.$$

Using a change of variables, we have

$$||G||_{L^p(\mathbb{R}^2)} \leq C\int_0^\infty \frac{e^{-s}}{s}||e^{-\frac{|x|^2}{4s}}||_{L^p(\mathbb{R}^2)}ds \leq C\int_0^\infty e^{-s}s^{-1+1/p}ds \leq Cp.$$

We thus obtain (18). In a similar way, for the gradient of G, we have

$$||\nabla G||^p_{L^p(\mathbb{R}^2)} \leq C\int_0^\infty \frac{e^{-s}}{s^2}||xe^{-\frac{|x|^2}{4s}}||_{L^p(\mathbb{R}^2)}ds \leq C\int_0^\infty e^{-s}s^{-\frac{3}{2}+\frac{1}{p}}ds \leq C\frac{2p}{2-p},$$

as long as $1 \leq p < 2$. Therefore we deduce (19) for $1 \leq p < 2$. □

The next lemma shows various estimates for the chemo-attractant S.

Lemma 2.7. *Let S be a solution of (17) in $\mathbb{R}^2$. Then S satisfies the following estimates*

$$||S(t)||_{L^p(\mathbb{R}^2)} + ||\nabla S(t)||_{L^q(\mathbb{R}^2)} \leq C(p,q)||\rho_0||_{L^1(\mathbb{R}^2)}, \qquad 1 \leq p < \infty,\ 1 \leq q < 2, \tag{20}$$

$$||\nabla S(t)||_{L^2(\mathbb{R}^2)} \leq C||\rho_0||_{L^1(\mathbb{R}^2)}\left[\ln\left(||\rho(t)||^2_{L^2(\mathbb{R}^2)} + 1\right)\right]^{1/2}. \tag{21}$$

Proof. The first estimate (20) is an easy consequence of mass conservation, Lemma 2.6, and Young's inequality. Thus it suffices to show the estimate (21). From (17) we obtain the Fourier transform $\hat{S}(\xi) = \hat{\rho}(\xi)/(|\xi|^2+1)$, and thus, we have

$$||\nabla S(t)||_{L^2(\mathbb{R}^2)} = ||\xi \hat{S}(t)||_{L^2(\mathbb{R}^2)} = ||\frac{|\xi|\hat{\rho}(t)}{|\xi|^2+1}||_{L^2(\mathbb{R}^2)},$$

where Plancherel's equality is used. The above integral can be estimated by splitting $\mathbb{R}^2$ of the ξ-space into two parts:

$$\int_{\mathbb{R}^2} \frac{|\xi\hat{\rho}(t)|^2}{(|\xi|^2+1)^2} d\xi = \int_{|\xi|<R} \cdots + \int_{|\xi|>R} \cdots = I_1 + I_2,$$

where $R > 0$ will be chosen later. Using Hölder's inequality and Plancherel's equality we have

$$I_1 \le ||\hat{\rho}(t)||_{L^\infty(\mathbb{R}^2)} \left[\int_{|\xi|<R} \frac{|\xi|^2}{(|\xi|^2+1)^2} d\xi \right]^{1/2} \le C||\rho(t)||_{L^1(\mathbb{R}^2)} \left[\ln(R^2+1)\right]^{1/2},$$

$$I_2 \le ||\frac{|\xi|}{|\xi|^2+1}||_{L^\infty(|\xi|>R)} ||\hat{\rho}(t)||_{L^2(\mathbb{R}^2)} \le CR^{-1}||\rho(t)||_{L^2(\mathbb{R}^2)}.$$

Therefore we obtain

$$\begin{aligned} ||\nabla S(t)||_{L^2(\mathbb{R}^2)} &\le C||\rho(t)||_{L^1(\mathbb{R}^2)}\{\ln(R^2+1)\}^{1/2} + CR^{-1}||\rho(t)||_{L^2(\mathbb{R}^2)} \\ &\le C\left[1 + ||\rho(t)||_{L^1(\mathbb{R}^2)}\{\ln(||\rho(t)||^2_{L^2(\mathbb{R}^2)}+1)\}^{1/2}\right]. \end{aligned}$$

We optimized the above inequality by choosing $R = ||\rho(t)||_{L^2(\mathbb{R}^2)}$. Since $||\rho||_{L^1(\mathbb{R}^2)} = ||f_0||_{L^1(\mathbb{R}^2\times V)}$, we deduce (21) and our Lemma. □

The next theorem shows global existence of solutions for system (6)-(9) with $\tau = 0$.

Theorem 2.8. **($\tau = 0, \beta > 0$)** *Let Assumption 2.4 hold and $\beta > 0$. Assume that $f_0, \nabla f_0 \in (L^1 \cap L^\infty)(\mathbb{R}^2 \times V)$. Then there exist global solutions $f(\cdot,\cdot,t), \nabla f(\cdot,\cdot,t) \in (L^1 \cap L^\infty)(\mathbb{R}^2 \times V)$ and $S(\cdot,t) \in W^{1,p}(\mathbb{R}^2)$ for all $1 \le p \le +\infty$ of system* (6)-(9) *with $\epsilon > 0$ fixed but arbitrary.*

Proof. Without loss of generality, we assume $\epsilon = 1$. Mass is conserved for ρ, thus $||\rho(\cdot,t)||_{L^1(\mathbb{R}^n)} = ||f_0||_{L^1(\mathbb{R}^n\times V)}$.

$$\partial_t f(x,v,t) + v\cdot\nabla_x f(x,v,t) = \int_V T[S](x,v,v',t)f(x,v',t)dv' - \int_V T[S](x,v',v,t)f(x,v,t)dv'.$$

Using Assumption 2.4, we get

$$f(x,v,t) \le f_0(x-vt,v) + C\int_0^t \rho(x-vs,t-s)ds + Cf_1(x,v,t) + Cf_2(x,v,t)$$

where f_1 and f_2 satisfy

$$\partial_t f_1(x,v,t) + v\cdot\nabla_x f_1(x,v,t) = \int_V [S(x+v,t) + |\nabla S(x+v,t)|]f(x,v',t)dv',$$

$$\partial_t f_2(x,v,t) + v\cdot\nabla_x f_2(x,v,t) = \int_V [S(x-v',t) + |\nabla S(x-v',t)|]f(x,v',t)dv',$$

with initial conditions $f_i(x,v,0) = 0$ for $i = 1,2$. We first consider f_1. One can easily see that

$$f_1(x,v,t) = \int_0^t [S(x-vs+v,t-s) + |\nabla S(x-vs+v,t-s)|]\rho(x-vs,t-s)ds,$$

After simple calculations, we obtain the following estimates

$$||f_1(\cdot,\cdot,t)||_{L^p(\mathbb{R}^n\times V)} \le C\sup_{0\le s\le t} ||S(\cdot,s)||_{W^{1,p}(\mathbb{R}^n)} \int_0^t ||\rho(\cdot,t-s)||_{L^p(\mathbb{R}^n)}ds.$$

For the term f_2, we have

$$f_2(x,v,t) = \int_0^t \int_V [S(x-vs-v',t-s) + |\nabla S(x-vs-v',t-s)| f(x-vs,v',t-s)dv'ds.$$

Applying Young's inequality, (14)

$$||(S(\cdot,t-s) + |\nabla S(\cdot,t-s)|) * f(x-vs,\cdot,t-s)||_{L^\infty(V)} \leq \sup_{0<s<t} ||S(\cdot,s)||_{W^{1,p}(\mathbb{R}^n)} ||f(x-vs,\cdot,t-s)||_{L^{p'}(V)},$$

where p and p' are conjugate exponents. If $p \geq 2$, then $p' \leq p$ and so we have by interpolation between p and 1,

$$||f(x-vs,\cdot,t-s)||_{L^{p'}(V)} \leq C(V)||f(x-vs,\cdot,t-s)||_{L^p(V)}.$$

Hence,

$$||f_2(\cdot,\cdot,t)||_{L^p(\mathbb{R}^2\times V)} \leq \sup_{0<s<t} ||S(\cdot,s)||_{W^{1,p}(\mathbb{R}^2)} \int_0^t ||f(\cdot,\cdot,t-s)||_{L^p(\mathbb{R}^2\times V)} ds.$$

Therefore, summing up the estimates above, we obtain for $p \geq 2$

$$||f(\cdot,\cdot,t)||_{L^p(\mathbb{R}^2\times V)} \leq ||f_0(\cdot,\cdot)||_{L^p(\mathbb{R}^2\times V)} + C(1 + \sup_{0\leq s\leq t} ||S(\cdot,s)||_{W^{1,p}(\mathbb{R}^2)}) \int_0^t ||f(\cdot,\cdot,s)||_{L^p(\mathbb{R}^2\times V)}. \tag{22}$$

By Lemma 2.7, we have for $p = 2$

$$||f(\cdot,\cdot,t)||_{L^2(\mathbb{R}^2\times V)} \leq ||f_0(\cdot,\cdot)||_{L^2(\mathbb{R}^2\times V)} + C\left(1 + \sup_{0\leq s\leq t}\left[\ln\left(||f||^2_{L^2(\mathbb{R}^2\times V)} + 1\right)\right]^{1/2}\right) \int_0^t ||f(\cdot,\cdot,s)||_{L^2(\mathbb{R}^n\times V)}.$$

Then, applying Gronwall's inequality (2.2), we obtain $f \in L^2(\mathbb{R}^2 \times V)$. Now, using bootstrap arguments we obtain the $L^\infty-$estimate by applying repeatedly Lemma 2.6, Young's inequality (14), and Gronwall's inequality. Next we show $L^\infty-$estimates for the derivatives of f. For convenience let $j = 1, 2$ be arbitrary but fixed and we denote by $\tilde{f}$ and $\tilde{T}[S]$ the partial derivatives $\partial_{x_j} f$ and $\partial_{x_j} T[S]$ respectively.

$$\partial_t \tilde{f}(x,v,t) + v\cdot\nabla_x \tilde{f}(x,v,t) = \int_V \tilde{T}[S](x,v,v',t)f(x,v',t)dv' + \int_V T[S](x,v,v',t)\tilde{f}(x,v',t)dv'$$
$$- \int_V \tilde{T}[S](x,v',v,t)f(x,v,t)dv' - \int_V T[S](x,v',v,t)\tilde{f}(x,v,t)dv'.$$

Then, in a similar manner as before, we obtain

$$\tilde{f}(x,v,t) \leq \tilde{f}_0(x-vt,v) + C\tilde{f}_1(x,v,t) + C\tilde{f}_2(x,v,t) + C\tilde{f}_3(x,v,t) + C\tilde{f}_4(x,v,t),$$

where

$$\tilde{f}_1(x,v,t) = \int_0^t \int_V \tilde{T}[S](x-vs,v,v',t-s)f(x-vs,v',t-s)dv'ds,$$
$$\tilde{f}_2(x,v,t) = \int_0^t \int_V T[S](x-vs,v,v',t-s)\tilde{f}(x-vs,v',t-s)dv'ds,$$
$$\tilde{f}_3(x,v,t) = -\int_0^t \int_V \tilde{T}[S](x-vs,v',v,t-s)f(x-vs,v,t-s)dv'ds,$$
$$\tilde{f}_4(x,v,t) = -\int_0^t \int_V T[S](x-vs,v',v,t-s)\tilde{f}(x-vs,v,t-s)dv'ds.$$

We consider first $\tilde{f}_1(x,v,t)$. Here we use the fact that the L^∞ and L^p-norm of f, depending on t, are bounded, which was shown above. Therefore we have

$$|\tilde{f}_1(x,v,t)| \leq \sup_{0<s<t} ||f(\cdot,s)||_{L^\infty(\mathbb{R}^n\times V)} \int_0^t \int_V |\tilde{T}[S](x-vs,v,v',t-s)|dv'ds.$$

Using Assumption 2.4, one can easily see

$$||\tilde{f}_1(\cdot,\cdot,t)||_{L^p(\mathbb{R}^2\times V)} \leq C \sup_{0<s<t} ||f(\cdot,s)||_{L^\infty(\mathbb{R}^n\times V)} \sup_{0<s<t} ||S(\cdot,s)||_{W^{2,p}(\mathbb{R}^2)}$$

$$\leq C \sup_{0<s<t} ||f(\cdot,s)||_{L^\infty(\mathbb{R}^n\times V)} \sup_{0<s<t} ||\rho(\cdot,s)||_{L^p(\mathbb{R}^2)} \leq C = C(t,|V|),$$

where we used a standard estimate for the chemo-attractant equation. Since $\tilde{f}_3$ has the same structure as $\tilde{f}_1$, $\tilde{f}_3$ satisfies the estimates above. On the other hand, $\tilde{f}_2$ is estimated, due to Assumption 2.4, as follows:

$$|\tilde{f}_2(x,v,t)| \leq \sup_{0<s<t} ||S(\cdot,s)||_{W^{1,\infty}(\mathbb{R}^n)} \int_0^t \int_V \tilde{f}(x-vs,v',t-s)dv'ds.$$

Again, due to a standard estimate for the chemo-attractant equation, we get

$$|\tilde{f}_2(x,v,t)| \leq \sup_{0<s<t} ||\tilde{f}||_{L^q(\mathbb{R}^n)} \int_0^t \int_V \tilde{f}(x-vs,v',t-s)dv'ds,$$

where q is sufficiently large (i.e. $q>2$). Integration over $\mathbb{R}^n \times V$ yields

$$||\tilde{f}_2(\cdot,\cdot,t)||_{L^p(\mathbb{R}^2\times V)} \leq C \int_0^t ||\tilde{f}(\cdot,\cdot,t-s)||_{L^p(\mathbb{R}^2)} ds,$$

where we again used the boundedness of the L^p-norm of f and $C = C(|V|,t)$. $\tilde{f}_4$ can be treated in the same manner, thus we omit the details. To sum up, we obtain

$$||\nabla f(\cdot,\cdot,t)||_{L^p(\mathbb{R}^2\times V)} \leq C(|V|,t) + C(|V|,t) \int_0^t ||\nabla f(\cdot,\cdot,t-s)||_{L^p(\mathbb{R}^2\times V)} ds.$$

Gronwall's inequality justifies our claim. Repeating this process for higher regularity of f and S, we can easily see that this estimate is valid also in case $p=\infty$. This completes the proof. □

Next we consider more specific types of turning kernels in case $\beta = 0$. Let $C_2, C_3, C_5 = 0$ in Assumption 2.4 which gives the structure condition. In other words, the turning kernel T is supposed to satisfy

$$0 \leq T_\epsilon[S](x,v,v',t) \leq C(1+|\nabla S(x+\epsilon v,t)|), \quad |\nabla T_\epsilon[S](x,v,v',t)| \leq C|\nabla^2 S(x+\epsilon v,t)|. \tag{23}$$

Here we recall that the kernel $K = -(1/2\pi)\ln|x|$ for the Laplace operator $-\Delta$ has the property $\nabla K \in L^p_{loc}(\mathbb{R}^2)$ for any $1 \leq p < 2$. This will be used in the next theorem.

Theorem 2.9. *($\tau = 0, \beta = 0$): Let $f_0, f_{0x} \in L^1 \cap L^\infty(\mathbb{R}^2 \times V)$ and $\beta = 0$. Suppose the turning kernel satisfies the structure condition (23). Then there exist global solutions $f, \nabla f \in L^\infty\left((0,\infty); L^1_+ \cap L^\infty(\mathbb{R}^2 \times V)\right)$ and $\nabla S \in L^\infty\left((0,\infty); L^p(\mathbb{R}^2)\right)$ for all $2 < p \leq \infty$ of the system (6)-(9) with $\epsilon > 0$ fixed but arbitrary.*

Proof. Without loss of generality, we assume $\epsilon = 1$. We first decompose ∇S into two parts

$$\nabla S = \nabla S^L + \nabla S^S = \rho * \left(-\frac{x}{2\pi|x|^2}\mathbf{I}_{|x|\geq 1}\right) + \rho * \left(-\frac{x}{2\pi|x|^2}\mathbf{I}_{|x|\leq 1}\right),$$

where $\mathbf{I_A}$ denotes the characteristic function of a set A. By mass conservation and Young's inequality, we have

$$||\nabla S^L(t)||_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{2\pi}||f_0||_{L^1(\mathbb{R}^2\times V)}.$$

Hence the estimate reduces to considering ∇S^S only and we may replace ∇S by ∇S^S in the assumption on the turning kernel. In a similar way as described in the proof of Theorem 2.8, we obtain for $p \geq 1$

$$f(x,v,t) \leq f_0(x-vt,v) + C\int_0^t \rho(x-vs,t-s)ds + Cf_1(x,v,t),$$

where

$$f_1(x,v,t) = \int_0^t |\nabla S^S(x-vs+v,t-s)|\rho(x-vs,t-s)ds.$$

Simple calculations show

$$||f_1(\cdot,\cdot,t)||_{L^p(\mathbb{R}^2\times V)} \leq C \sup_{0\leq s\leq t} ||\nabla S^S(\cdot,s)||_{L^p(\mathbb{R}^2)} \int_0^t ||\rho(\cdot,t-s)||_{L^p(\mathbb{R}^2)} ds.$$

To sum up, we obtain

$$||f(\cdot,\cdot,t)||_{L^p(\mathbb{R}^2\times V)} \le C + C(1+\sup_{0\le s\le t}||\nabla S^S(\cdot,s)||_{L^p(\mathbb{R}^2)})\int_0^t ||f(\cdot,\cdot,t-s)||_{L^p(\mathbb{R}^n\times V)}ds. \tag{24}$$

Here we note that the above a priori estimate (24) holds for all $p\ge 1$. First we choose a specific p with $1<p<2$, which ensures, due to the Young's inequality, that

$$||S^S(\cdot,t)||_{L^p(\mathbb{R}^2)} \le C||f_0||_{L^1(\mathbb{R}^2\times V)}.$$

Then by Gronwall's inequality we get a bound, globally in time, for f in $L^p(\mathbb{R}^2)$ for such chosen p. By bootstrap arguments, we obtain $f\in L^\infty_{\text{loc}}([0,\infty);L^\infty(\mathbb{R}^2\times V))$. By similar procedures as given in the proof of Theorem 2.8, a L^∞-estimate for ∇f can be obtained. $\nabla S\in L^\infty\left((0,\infty);L^p(\mathbb{R}^2)\right), 2<p\le\infty$ is due to the Hardy-Littlewood-Sobolev theorem (see [20, page 119-120]). Since this is also verified by embedding arguments for general elliptic equations, we skip the details. This completes our proof. □

Dimension 3:

Next we state the three dimensional analog of our main theorem under some restrictions on the turning kernel. To be more precise, we assume that $C_3=C_5=0$ in (15) and (16), i.e.

$$0\le T[S](x,v,v',t)\le C\left(1+S(x+\epsilon v,t)+|\nabla S(x+\epsilon v,t)|\right), \tag{25}$$

$$|\nabla T[S](x,v,v',t)|\le C\left(|\nabla S(x+\epsilon v,t)|+|\nabla^2 S(x+\epsilon v,t)|\right). \tag{26}$$

In this situation, unlike in the two dimensional case in Theorem 2.8, it is not necessary to assume that $\beta\ne 0$. We briefly explain why C_3, C_5 are assumed to be zero in three dimension. Indeed, as seen in the previous calculations, we end up with the following estimate

$$||f(\cdot,\cdot,t)||_{L^p(\mathbb{R}^3\times V)} \le C + C\Big(1+\sup_{0\le s\le t}||S^S(\cdot,s)||_{W^{1,p}(\mathbb{R}^3)}\Big)\int_0^t ||f(\cdot,\cdot,t-s)||_{L^p(\mathbb{R}^3\times V)}ds. \tag{27}$$

On the other hand, in three dimension, due to behavior of the potential, we have

$$||S^S(\cdot,s)||_{W^{1,p}(\mathbb{R}^3)} \le C||\rho_0||_{L^1(\mathbb{R}^3)}, \quad \text{for} \quad 1\le p<\frac{3}{2}. \tag{28}$$

However, in case C_3 or C_5 are nonzero, one can easily show that estimate (27) is valid provided that $p\ge 2$ (compare the estimate for f_2 and f_4 on page 6), but this does not enable us to use bootstrap arguments to get higher regularity for f because of (28). Therefore we assume $C_3=C_5=0$. Since the proof of our next theorem is similar to the previous one, we just state it without proof.

Theorem 2.10. ($\tau=0,\beta\ge 0$)**:** *Let $f_0,\nabla f_0\in L^1\cap L^\infty(\mathbb{R}^3\times V)$ and let $\beta\ge 0$. Suppose the turning kernel satisfies the structure conditions* (25) *and* (26). *Then there exist global solutions $f,\nabla f\in L^\infty\left((0,\infty);L^1\cap L^\infty(\mathbb{R}^3\times V)\right)$ and*

1. *if $\beta=0$, then $S\in L^\infty\left((0,\infty);L^p(\mathbb{R}^3)\right)$, $3<p\le\infty$ and $\nabla S\in L^\infty\left((0,\infty);L^p(\mathbb{R}^3)\right)$, $3/2<p\le\infty$*
2. *if $\beta>0$, then $S\in L^\infty\left((0,\infty);W^{1,p}(\mathbb{R}^3)\right)$, $1\le p\le\infty$*

of system (6)-(9) *with $\epsilon>0$ fixed but arbitrary.*

Remark 2.11. *It is worth mentioning that Theorem 2.10 also holds when the turning kernel satisfies the following structure conditions, instead of* (25) *and* (26),

$$0\le T[S](x,v,v',t)\le C\left(1+S(x-\epsilon v',t)+|\nabla S(x-\epsilon v',t)|\right),$$
$$|\nabla T[S](x,v,v',t)|\le C\left(|\nabla S(x-\epsilon v',t)|+|\nabla^2 S(x-\epsilon v',t)|\right).$$

This is obvious from the procedure in the proof of Theorem 2.8.

We would like to thank B. Perthame for pointing this out during his visit at MPI MIS in Leipzig.

We do not know if the theorem above is valid in case the turning kernel fulfills the structure condition (15) *and* (16) *like in the two dimensional case.*

b) Parabolic case: $\tau > 0$

In this part, the parabolic equation for the chemo-attractant in (8) is considered. From now on we let $\tau = 1$ without loss of generality and, for simplicity, we also set $\alpha = 1$. Then (8) for S reads

$$\partial_t S - \Delta S = \rho - \beta S, \quad S(x.0) = S_0(x) \qquad \beta \geq 0. \tag{29}$$

To make our arguments simpler, from now on, we assume $S_0 = 0$ (see Remark 2.14 for the case $S_0 \neq 0$). In the next lemma we recall some basic properties of Γ in dimension 2.

Dimension 2:

Lemma 2.12. *Let Γ be the fundamental solution for the operator $\partial_t - \Delta_x + \beta$ in $\mathbb{R}^2$. Then $\Gamma \in L^p(\mathbb{R}^2)$ for any p with $1 \leq p < \infty$, and $\nabla\Gamma \in L^p(\mathbb{R}^2)$ for any q with $1 \leq p < 2$ satisfying:*

$$\int_0^t ||\Gamma(\cdot, s)||_{L^p(\mathbb{R}^2)} ds \leq C(\beta)p, \qquad 1 \leq p < \infty$$

$$\int_0^t ||\nabla\Gamma(\cdot, s)||_{L^p(\mathbb{R}^2)} ds \leq C(\beta)\frac{2p}{2-p}, \qquad 1 \leq p < 2.$$

Proof. The proof is similar to Lemma 2.6, thus we omit details. □

In the next lemma, we show L^p and L^2 estimates for S and ∇S, respectively.

Lemma 2.13. *Let S be a solution of (29) in $\mathbb{R}^2$ and $S_0 = 0$. Then S satisfies the following estimates*

$$||S(t)||_{L^p(\mathbb{R}^2)} + ||\nabla S(t)||_{L^q(\mathbb{R}^2)} \leq C(\beta, p, q)||\rho_0||_{L^1(\mathbb{R}^2)}, \qquad 1 \leq p < \infty,\ 1 \leq q < 2, \tag{30}$$

$$||\nabla S(t)||^2_{L^2(\mathbb{R}^2)} \leq C\left(1 + ||\rho_0||_{L^1(\mathbb{R}^2)}\left(1 + (\ln t)_+ + \sup_{0\leq\tau\leq t} \ln(||\rho(\tau)||^2_{L^2(\mathbb{R}^2)})\right)\right), \tag{31}$$

where $(f)_+$ indicates the positive part of f.

Proof. By Duhamel's principle and using the fundamental solution Γ in (11), we have

$$S(x,t) = \int_0^t \Gamma(\cdot, s) * \rho(\cdot, t-s) ds. \tag{32}$$

By using Lemma 2.12, mass conservation, and Young's inequality (14), we easily get (30). To estimate $||\nabla S||_{L^2(\mathbb{R}^2)}$, we take the Fourier transform of (32) and use Plancherel's equality to get

$$||\nabla S(t)||_{L^2(\mathbb{R}^2)} = ||\xi\hat{S}(t)||_{L^2(\mathbb{R}^2)} \leq \int_0^t \left\| |\xi|\hat{\Gamma}(\cdot, s)\hat{\rho}(\cdot, t-s)\right\|_{L^2(\mathbb{R}^2)} ds = \int_0^r \cdots + \int_r^t \cdots,$$

where $r > 0$ will be chosen appropriately later. Note that the Fourier transform of Γ is $\hat{\Gamma}(\xi, s) = \exp(-s(4\xi^2+\beta))$. For $0 < s < r$, due to the Hölder's inequality and Plancherel's equality, we have

$$\int_0^r \cdots \leq \int_0^r \left\| |\xi| \exp(-s(4\xi^2 + \beta))\right\|_{L^\infty(\mathbb{R}^2)} ||\hat{\rho}(s)||_{L^2(\mathbb{R}^2)} ds$$

$$\leq C \sup_{0\leq s\leq t} ||\rho||_{L^2(\mathbb{R}^2)} \int_0^r s^{-1/2} ds \leq Cr^{1/2} \sup_{0\leq s\leq t} ||\rho||_{L^2(\mathbb{R}^2)}.$$

For $r < s < t$, due to mass conservation and Hölder's inequality, now applied in the opposite way, we have

$$\int_r^t \cdots \leq \int_r^t \left\| |\xi| \exp(-s(4\xi^2 + \beta))\right\|_{L^2(\mathbb{R}^2)} ||\hat{\rho}(s)||_{L^\infty(\mathbb{R}^2)} ds$$

$$\leq C||\rho_0||_{L^1(\mathbb{R}^2)} \int_r^t \frac{1}{s} ds \leq C||\rho_0||_{L^1(\mathbb{R}^2)} |\ln t - \ln r|,$$

where we used $||\hat{\rho}||_{L^\infty(\mathbb{R}^2)} \leq ||\rho||_{L^1(\mathbb{R}^2)}$. Therefore we obtain

$$||\nabla S(t)||_{L^2(\mathbb{R}^2)} \leq C\left(r^{1/2} \sup_{0\leq s\leq t} ||\rho||_{L^2(\mathbb{R}^2)} + ||\rho_0||_{L^1(\mathbb{R}^2)} |\ln t - \ln r|\right).$$

We optimize the upper bound for the above inequality by choosing $r = \min\left\{ \left(\sup_{0\le s\le t} ||\rho||_{L^2(\mathbb{R}^2)}\right)^{-2}, t\right\}$. Therefore we deduce our lemma. □

Remark 2.14. *For the case $S_0 \neq 0$, which is assumed to be sufficiently smooth one has*

$$S(x,t) = \int_0^t \Gamma(\cdot, s) * \rho(\cdot, t-s)ds + \int_{\mathbb{R}^2} \Gamma(x-y,t)S_0(y)dy.$$

This gives the following variants of the estimates in the above lemma.

$$||S(t)||_{L^p(\mathbb{R}^2)} + ||\nabla S(t)||_{L^q(\mathbb{R}^2)} \le C(||S_0||_{L^p(\mathbb{R}^2)} + ||\nabla S_0||_{L^q(\mathbb{R}^2)} + ||\rho_0||_{L^1(\mathbb{R}^2)}), \quad 1 \le p < \infty,\ 1 \le q < 2,$$

$$||\nabla S(t)||^2_{L^2(\mathbb{R}^2)} \le C\left(1 + ||\nabla S_0||_{L^2(\mathbb{R}^2)} + ||\rho_0||_{L^1(\mathbb{R}^2)}\left(1 + (\ln t)_+ + \sup_{0\le\tau\le t} \ln(||\rho(\tau)||^2_{L^2(\mathbb{R}^2)})\right)\right).$$

Since computations are straightforward, we omit the details.

As in the previous elliptic case, we can establish global existence for the system (6) -(9) with $\tau = 1$.

Theorem 2.15. **$\tau > 0$:** *Let Assumption 2.4 hold and $\beta \ge 0$. Assume that $f_0, \nabla f_0 \in (L^1 \cap L^\infty)(\mathbb{R}^2 \times V)$. Then there exist global solutions $f(\cdot,\cdot,t), \nabla f(\cdot,\cdot,t) \in (L^1 \cap L^\infty)(\mathbb{R}^2 \times V)$ and $S(\cdot,t), \nabla S(\cdot,t) \in L^p(\mathbb{R}^2)$ for all $1 \le p \le +\infty$ of system (6)-(9) with $\epsilon > 0$ fixed but arbitrary.*

Proof. Once we have the essential estimate (31) for $||\nabla S||_{L^2(\mathbb{R}^2)}$, the proof is similar to the elliptic case with $\tau = 0$. Regularity of S is due to standard theory of general parabolic equations. Since the arguments are straightforward if compared to the elliptic case, we omit the details. □

Dimension 3:

If the turning kernel satisfies (25) and (26) instead of Assumption 2.4, global existence can also be proved in three dimension. The arguments are more or less the same as for $\tau = 0$ in three dimensions, so details are skipped.

Theorem 2.16. **$\tau > 0, \beta \ge 0$:** *Let (25) and (26) be satisfied. Assume that $f_0, \nabla f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times V)$. Then there exist global solutions $f(\cdot,\cdot,t), \nabla f(\cdot,\cdot,t) \in (L^1 \cap L^\infty)(\mathbb{R}^3 \times V)$ and $S(\cdot,t), \nabla S(\cdot,t) \in L^p(\mathbb{R}^3)$ for all $1 \le p \le +\infty$ of system (6)-(9) with $\epsilon > 0$ fixed but arbitrary.*

3. Diffusion limits of the kinetic model

In this section, the diffusion limit for kinetic models of type (6)-(9) is presented. First, in a lemma, we review estimates for S which satisfies an equation of elliptic type, i.e.

$$-\Delta S = \rho - \beta S \quad \beta \ge 0, \ \text{ in } \mathbb{R}^n,\ n = 2, 3.$$

We use standard arguments, which are known as potential theory. Proofs are straightforward (compare e.g. [9, Chap. 2 & 8] and [20, Chap. V]).

Lemma 3.1. *Let $I = [0,T) \subset \mathbb{R}$ and $0 < T < \infty$. Suppose $\rho \in L^\infty(I; (W^{1,1}(\mathbb{R}^n) \cap W^{1,q}(\mathbb{R}^n)))$ where $q > n$.*

- **(i):** *In the case either $n = 2, \beta > 0$ or $n = 3, \beta \ge 0$:*

$$S \in L^\infty(I; W^{2,p}(\mathbb{R}^n)) \cap L^\infty(I; \mathcal{C}^{2+\alpha}(\mathbb{R}^n)), \qquad 1 \le p < \infty,\ 0 < \alpha \le \frac{q-n}{q},$$

and S satisfies the following estimate

$$||S||_{L^\infty(I;W^{2,p}(\mathbb{R}^n))} + ||S||_{L^\infty(I;\mathcal{C}^{2+\alpha}(\mathbb{R}^n))} \le C(||\rho||_{L^\infty(I;W^{1,1}(\mathbb{R}^n))} + ||\rho||_{L^\infty(I;W^{1,q}(\mathbb{R}^n))}).$$

- **(ii):** *In the case $n = 2$ and $\beta = 0$:*

$$\nabla S \in L^\infty(I; W^{1,p}(\mathbb{R}^2)) \cap L^\infty(I; \mathcal{C}^{1+\alpha}(\mathbb{R}^2)), \qquad 1 \le p < \infty,\ 0 < \alpha \le \frac{q-2}{q},$$

and S satisfies the following estimate

$$||\nabla S||_{L^\infty(I;W^{1,p}(\mathbb{R}^2))} + ||\nabla S||_{L^\infty(I;\mathcal{C}^{1+\alpha}(\mathbb{R}^2))} \leq C\Big(||\rho||_{L^\infty(I;W^{1,1}(\mathbb{R}^2))} + ||\rho||_{L^\infty(I;W^{1,q}(\mathbb{R}^2))}\Big).$$

As in [3] we need similar assumptions on $\phi_\epsilon^S[S]$ and $\phi_\epsilon^A[S]$, which are the symmetric and antisymmetric parts of $T_\epsilon[S]$ (see Lemma 1.2).

Assumption 3.2. *There exist $\gamma > 0$ and a non-decreasing function $\Lambda \in L^\infty_{\text{loc}}$, such that*

$$\phi_\epsilon^S[S] \geq \gamma\left(1 - \epsilon\Lambda(||\nabla S||_{W^{1,\infty}(\mathbb{R}^n)})\right) FF'$$

$$\int_V \frac{\phi_\epsilon^A[S]^2}{F\phi_\epsilon^S[S]} dv' \leq \epsilon^2 \Lambda(||\nabla S||_{W^{1,\infty}(\mathbb{R}^n)}),$$

where $F \in L^\infty(V)$ is a positive velocity distribution satisfying Assumption 1.1.

Theorem 3.3. *Let Assumptions 1.1 and 3.2 hold and let $q > n$ with $n = 2, 3$. Suppose that the equation for the chemo-attractant S is of elliptic type. Let one of following conditions hold;*

- **(i):** *if $n = 2, \beta > 0$, the turning kernel satisfies Assumption 2.4.*
- **(ii):** *if $n = 2, \beta = 0$, the turning kernel satisfies* (23).
- **(iii):** *if $n = 3, \beta \geq 0$, the turning kernel satisfies* (25) *and* (26).

Assume further that

$$f_0 \in \Upsilon_q \equiv W^{1,1}(\mathbb{R}^n \times V) \cap W^{1,q}(\mathbb{R}^n \times V; \frac{dxdv}{F^{q-1}}).$$

Then there exists $t^* > 0$, independent of ϵ, such that the solutions f_ϵ, S_ϵ satisfy

$$f_\epsilon \in L^\infty((0,t^*); \Upsilon_q),$$

$$\nabla S_\epsilon \in L^\infty((0,t^*); W^{1,p}(\mathbb{R}^n) \cap \mathcal{C}^{1+\alpha}(\mathbb{R}^n)), \qquad 1 \leq p < \infty,\ \alpha = \frac{q-n}{q} \quad if\ n = 2, \beta = 0,$$

$$S_\epsilon \in L^\infty((0,t^*); W^{2,p}(\mathbb{R}^n) \cap \mathcal{C}^{2+\alpha}(\mathbb{R}^n)), \qquad 1 \leq p < \infty,\ \alpha = \frac{q-n}{q} \quad otherwise,$$

$$r_\epsilon = \frac{f_\epsilon - \rho_\epsilon F}{\epsilon} \in L^2((0,t^*); \mathbb{R}^n \times V : \frac{dxdvdt}{F}). \tag{33}$$

Proof. This can be shown by following the same procedure as given in the proof of Theorem 4 in [3], and therefore, we present a brief sketch of this proof. Simple calculations shows

$$\frac{d}{dt}\int_{\mathbb{R}^n}\int_V \frac{f_\epsilon^q}{F^{q-1}} dvdx \leq C\Lambda(||\nabla S||_{W^{1,\infty}(\mathbb{R}^n)}) \int_{\mathbb{R}^n}\int_V \frac{f_\epsilon^q}{F^{q-1}} dvdx.$$

The next step is to estimate S_ϵ:

$$||\nabla S_\epsilon(\cdot,t)||_{\mathcal{C}^{1,\alpha}(\mathbb{R}^n)} \leq C(1 + ||\nabla \rho_\epsilon(\cdot,t)||_{L^q(\mathbb{R}^n)}) \leq \tilde{C}(1 + ||\rho_\epsilon(\cdot,t)||_{L^q(\mathbb{R}^n)}),$$

here we used the estimates in Lemma 3.1.

$$\frac{d}{dt}\int_{\mathbb{R}^n}\int_V \frac{f_\epsilon^q}{F^{q-1}} dvdx \leq C[1 + (\int_{\mathbb{R}^n}\int_V \frac{f_\epsilon^q}{F^{q-1}} dvdx)^{\frac{1}{q}}] \int_{\mathbb{R}^n}\int_V \frac{f_\epsilon^q}{F^{q-1}} dvdx.$$

This shows the first two statements. The rest can be done by the same method as given in the proof of Theorem 4 in [3], and thus we omit the details. □

Now we are ready to prove the existence of the diffusion limit in a short time interval.

Theorem 3.4. *Let the assumption of Theorem 3.3 hold. Suppose that the equation for the chemo-attractant S is of elliptic type ($\tau = 0$). Assume further that for families (S_ϵ), which are uniformly bounded in $L^\infty_{\rm loc}([0,\infty);\mathcal{C}^{2+\alpha}(\mathbb{R}^n))$ for some α with $0 < \alpha \le 1$, such that $S_\epsilon, \nabla S_\epsilon$, and $\nabla^2 S_\epsilon$ converge to $S_0, \nabla S_0$, and $\nabla^2 S_0$ as $\epsilon \to 0$, respectively, in $L^p_{\rm loc}([0,\infty);\mathbb{R}^n)$ for some $p > n/(n-1)$ with $n = 2, 3$, we have the convergence*

$$T_\epsilon[S_\epsilon] \to T_0[S_0] \qquad in \;\; L^p_{\rm loc}([0,\infty);\mathbb{R}^n \times \bar V \times \bar V),$$

$$\frac{\mathcal{T}_\epsilon[S_\epsilon](F)}{\epsilon} = \frac{2}{\epsilon}\int_V \phi^A_\epsilon[S_\epsilon]dv' \to \mathcal{T}_1[S_0](F) \qquad in \;\; L^p_{\rm loc}([0,\infty);\mathbb{R}^n \times \bar V). \tag{34}$$

Then the solutions f_ϵ and S_ϵ of (6)-(9) satisfy

$$f_\epsilon \to \rho_0 F \qquad in \;\; L^\infty((0,t^*);\Upsilon_q) \; weak *,$$

$$\nabla S_\epsilon \to S_0 \qquad in \;\; W^{1,q}_{\rm loc}((0,t^*);\mathbb{R}^n), \qquad 1 \le q < \infty \quad if \;\; n = 2, \beta = 0.$$

$$S_\epsilon \to S_0 \qquad in \;\; W^{2,q}_{\rm loc}((0,t^*);\mathbb{R}^n), \qquad 1 \le q < \infty \quad otherwise.$$

Proof. Since the proof is similar to that of Theorem 5 in [3], we again present only a brief sketch of the procedure. First we note, due to (33), that

$$J_\epsilon = \frac{1}{\epsilon}\int_V v f_\epsilon dv = \int_V v r_\epsilon dv \in L^2((0,t^*);L^2(\mathbb{R}^n))$$

uniformly in ϵ. From the cell conservation equation $\partial_t \rho_\epsilon + \mathrm{div}\, \mathrm{J}_\epsilon = 0$, one can easily see that

$$\partial_t(\nabla S_\epsilon) \in L^2((0,t^*);L^2_{\rm loc}(\mathbb{R}^n))$$

by considering the gradient of the convolution of (8). The strong convergence follows combining the above estimate and the parabolic regularity for the convolutions defining S_ϵ and ∇S_ϵ from ρ_ϵ. Therefore, the kinetic equation (7) leads to

$$\epsilon\frac{\partial f_\epsilon}{\partial t} + v \cdot \nabla_x f_\epsilon = -\rho_\epsilon \frac{\mathcal{T}[S_\epsilon](F)}{\epsilon} - \mathcal{T}_\epsilon[S_\epsilon](r_\epsilon).$$

By assumption (34) and passing to the limit, we obtain

$$\mathcal{T}_0[S_0](r_0) = -vF \cdot \nabla\rho_0 - \rho_0 \mathcal{T}_1[S_0](F).$$

This equation can be solved due to Lemma 1.3. The limit of the cell conservation equation is $\partial_t \rho_0 + \nabla \cdot J_0 = 0$ with $J_0 = \int_V v r_0 dv$. This completes the proof. □

A similar result can be established if the chemo-attractant equation is of parabolic type. Since the method of proof does not change when compared to elliptic case, we present only the statements without further details.

Lemma 3.5. *Let $I = [0,T) \subset \mathbb{R}$ and $0 < T < \infty$ and $n = 2$ or 3. Suppose $\rho \in L^\infty(I;W^{1,1}(\mathbb{R}^n)) \cap L^\infty(I;W^{1,q}(\mathbb{R}^n))$ where $q > n$ and S satisfies the chemo-attractant equation of parabolic type with $\beta \ge 0$.*

$$S \in L^\infty(I;W^{2,p}(\mathbb{R}^2)) \cap L^\infty(I;\mathcal{C}^{2+\alpha}(\mathbb{R}^2)), \qquad 1 \le p < \infty, \;\; 0 < \alpha \le \frac{q-2}{q} \quad if \;\; n = 2,$$

$$S \in L^\infty(I;W^{2,p}(\mathbb{R}^3)) \cap L^\infty(I;\mathcal{C}^{2+\alpha}(\mathbb{R}^3)), \qquad 1 \le p < \infty, \;\; 0 < \alpha \le \frac{q-3}{q} \quad if \;\; n = 3,$$

and S satisfies the following estimate

$$||S||_{L^\infty(I;W^{2,p}(\mathbb{R}^n))} + ||S||_{L^\infty(I;\mathcal{C}^{2+\alpha}(\mathbb{R}^n))} \le C(||\rho||_{L^\infty(I;W^{1,1}(\mathbb{R}^n))} + ||\rho||_{L^\infty(I;W^{1,q}(\mathbb{R}^n))}). \quad n = 2, 3.$$

Proof. Proofs are standard and straightforward (see e.g. [15, Chap. 4] and [16, Chap. 4 & 6]). □

Again, as in the previous case, we have following result, which is independent of ϵ.

Theorem 3.6. *Let Assumption 1.1 and Assumption 3.2 hold and let $q > n$ with $n = 2, 3$. Suppose that the turning kernel satisfies* (25) *and* (26). *Assume further that*

$$f_0 \in \Upsilon_q = W^{1,1}(\mathbb{R}^n \times V) \cap W^{1,q}(\mathbb{R}^n \times V; \frac{dxdv}{F^{q-1}}).$$

Then there exists a $t^* > 0$, independent of ϵ, such that the solution f_ϵ, S_ϵ satisfies

$$f_\epsilon \in L^\infty((0,t^*); \Upsilon_q),$$

$$S_\epsilon \in L^\infty((0,t^*); W^{2,p}(\mathbb{R}^n) \cap \mathcal{C}^{2+\alpha}(\mathbb{R}^n)), \qquad 1 \le p < \infty, \ \ \alpha = \frac{q-n}{q},$$

$$r_\epsilon = \frac{f_\epsilon - \rho_\epsilon F}{\epsilon} \in L^2\left((0,t^*); \mathbb{R}^n \times V : \frac{dxdvdt}{F}\right).$$

Theorem 3.7. *Let the assumptions in Theorem 3.6 hold. Assume further that for families (S_ϵ), which are uniformly bounded in $L^\infty_{\mathrm{loc}}([0,\infty); \mathcal{C}^{2+\alpha}(\mathbb{R}^n))$ for some α with $0 < \alpha \le 1$, such that $S_\epsilon, \nabla S_\epsilon$, and $\nabla^2 S_\epsilon$ converge to $S_0, \nabla S_0$, and $\nabla^2 S_0$ as $\epsilon \to 0$, respectively, in $L^p_{\mathrm{loc}}([0,\infty); \mathbb{R}^n)$ for some $p > n/(n-1)$ with $n = 2, 3$, we have the convergence*

$$T_\epsilon[S_\epsilon] \to T_0[S_0] \qquad \text{in} \ \ L^p_{\mathrm{loc}}([0,\infty); \mathbb{R}^n \times \bar{V} \times \bar{V}),$$

$$\frac{\mathcal{T}_\epsilon[S_\epsilon](F)}{\epsilon} = \frac{2}{\epsilon}\int_V \phi^A_\epsilon[S_\epsilon]dv' \to \mathcal{T}_1[S_0](F) \qquad \text{in} \ \ L^p_{\mathrm{loc}}([0,\infty); \mathbb{R}^n \times \bar{V}).$$

Then the solutions f_ϵ and S_ϵ of (6)-(9) *satisfy*

$$f_\epsilon \to \rho_0 F \qquad \text{in} \ \ L^\infty((0,t^*); \chi_q) \ \text{weak} *,$$

$$S_\epsilon \to S_0 \qquad \text{in} \ \ L^q_{\mathrm{loc}}((0,t^*); W^{2,q}(\mathbb{R}^n)), \qquad 1 \le q < \infty,$$

Proof. The arguments are similar to those for Theorem 3.4. □

4. Examples

When dealing with chemosensitive movement of biological species, questions of major interest are, how do the individuals ‘measure’ the chemical signal, how is this information processed and what kind of behavior results. In the following we discuss some parts of this problem. First we give a short summary of the mathematical terminology which accounts for the possible biological behavior of the cells as suggested by Tranquillo and Alt, [22]:

Kinesis describes the dependence of movement of an individual on a scalar stimulus or other information about time and position along its path. Kinesis can be characterized as positional, which means induced by a purely positional signal, or temporal(ly differential), which describes an adapting response induced by a temporal(ly differential) signal. An increasing positional signal or a positive temporal(ly differential) signal can induce positive *orthokinesis* by increasing the (mean) speed, and positive *klinokinesis* by decreasing the turning rate and/or the absolute magnitude of turn angles.

The so-called signal for chemosensitive movement is the carrier of information which is perceived from a stimulus field and possibly stored by the individual, and which governs the individual response. In this context we are just looking at external signals. Depending on geometry and kinetics of stimulus perception, signal transduction and motor response, various types of signals carrying various components of information may be present at the same time.

Spatial - the stimulus is evaluated at (at least) two distinct locations around the individual, which are related to its direction (compare example 4.1, 4.3, and 4.4).

Temporal(ly) differential) - the stimulus is evaluated at (at least) two different times (compare example 4.1).

Positional - the stimulus is evaluated momentarily (compare example 4.4).

Directional - the stimulus is evaluated along the individual direction or its relation to a directional stimulus field, e.g. a spatial gradient at its position (compare examples 4.1, 4.3, 4.4, 4.5, and 4.6).

Taxis is the dependence of individual movement on a directional stimulus related to the movement direction. It can be induced by direct (e.g. spatial) or indirect (e.g. temporal) determination of the movement direction, or some other evaluations about the movement direction in relation to an external directional field (e.g. a stimulus gradient) The stimulus gradient can be spatial, temporal or perceived along the motion path. Here *klinotaxis* describes the dependence of the turning behavior on the current direction. Taxis with directed turning is given by a directional bias of the turn angle distribution of the turning rate. This can be induced only by a directional signal, but also by others than a spatial signal. Taxis without directed turning is not mentioned so often, but also of importance. This is a turning response depending on the locomotion direction. The directional signal might be spatial or temporal (for example, bacteria changing their turning frequency)

Discussions of possible turning rates in this context are also given in [1], [2], [17] and [18], [19].

In [18], [19] the macroscopic limit is formal. It is assumed that the turning kernel has an expansion in ϵ which is assumed to be given. Here the ϵ-expansion is related to possible evaluations of the chemo-attractant by the cells.

The first example is very general and allows also dependencies on time derivatives of the chemo-attractant. Since we did not prove regularity for S_t so far the macroscopic limit in this case has to be considered only formal. Nevertheless, from this example other rigorous examples can be extracted, which will be discussed after.

Example 4.1. *(formal for $\alpha > 0$, rigorous for $\alpha = 0$)*

The turning kernel we consider first is of the following form, which is of general type:

$$T_\epsilon[s] = \phi\big(S(x+\epsilon v,t), S(x-\epsilon v',t), S(x,t-\epsilon), \nabla S(x+\epsilon v,t), \nabla S(x-\epsilon v',t),$$

$$\partial_t S(x+\epsilon v,t), \partial_t S(x-\epsilon v',t), \partial_t S(x,t-\epsilon), v\big) + \epsilon\psi\Big(\frac{v\cdot v'}{|v||v'|}\Big), \tag{35}$$

where $\phi : \mathbb{R}^{12}\to\mathbb{R}$ and $\psi : \mathbb{R}\to\mathbb{R}$ are smooth and $\phi + \epsilon\psi$ is strictly positive (∇S contributes two entries, $\partial_{x_1} S$ and $\partial_{x_2} S$). Here S satisfies the chemo-attractant equation either of elliptic type or of parabolic type with $\alpha \geq 0$ and $\beta > 0$ in two dimensions. For $\alpha = 0$ the S-equation is completely decoupled, therefore for this case the derivation given below is rigorous. Our aim is to (formally) derive the macroscopic equation from the turning kernel given above. We do not include direct dependencies at this point, like $S(x,t), S_t(x,t), \nabla S(x,t)$. These will be discussed later in more specific examples.

For convenience, we use the following notational abbreviation:

$$\phi[S,\nabla S,\partial_t S,v] := \phi(S(x,t),S(x,t),S(x,t),\nabla S(x,t),\nabla S(x,t),\partial_t S(x,t),\partial_t S(x,t),\partial_t S(x,t),v),$$

$$\phi_i[S,\nabla S,\partial_t S,v] := \phi_i(S(x,t),S(x,t),S(x,t),\nabla S(x,t),\nabla S(x,t),\partial_t S(x,t),\partial_t S(x,t),\partial_t S(x,t),v),$$

where $\phi_i(\cdots)$ indicates the partial derivative of ϕ with respect to the ith argument with $i = 1, 2, ..., 12$. By the asymptotic expansion of $T_\epsilon = T_0 + \epsilon T_1 + O(\epsilon^2)$, one can easily see that

$$T_0 = T_0[S,v] = \phi[S,\nabla S,\partial_t S,v]$$

and

$$\begin{aligned} T_1 = T_1[S,v,v'] = {} & (\phi_1[S,\nabla S,\partial_t S,v]v - \phi_2[S,\nabla S,\partial_t S,v]v')\cdot\nabla S + \phi_3[S,\nabla S,\partial_t S,v]\partial_t S \\ & +(\phi_{3+i}[S,\nabla S,\partial_t S,v]v - \phi_{5+i}[S,\nabla S,\partial_t S,v]v')\cdot\nabla S_{x_i} \\ & +(\phi_8[S,\nabla S,\partial_t S,v]v - \phi_9[S,\nabla S,\partial_t S,v]v')\cdot\nabla S_t \\ & -\phi_{10}[S,\nabla S,\partial_t S,v]\partial_t^2 S + \psi\Big(\frac{v\cdot v'}{|v||v'|}\Big), \end{aligned}$$

where we used the summation convention, which is understood over repeated indices running from 1 to 2. From now on we use, unless any confusion is to be expected, the following notations.

$$T_0'[S,v] := T_0[S,v'], \qquad f_i'(v,x,t) := f_i(v',x,t), \qquad i = 1,2.$$

Furthermore, we define $\Phi, \tilde\Phi, \hat\Phi$, and $\bar\Phi$ as follows;

$$\Phi[S_0,\nabla S_0,\partial_t S_0] := \int_V T_0'[S_0,v]dv', \quad \tilde\Phi[S_0,\nabla S_0,\partial_t S_0,v] := \int_V T_1[S_0,v',v]dv',$$

$$\hat{\Phi}[S_0, \nabla S_0, \partial_t S_0, v] := \int_V T_1[S_0, v, v'] f_0'(v, x, t) dv',$$

$$\bar{\Phi}[S_0, \nabla S_0, \partial_t S_0, v] := \frac{1}{\Phi[S_0, \nabla S_0, \partial_t S_0]} \int_V T_0'[S_0, v] T_1[S_0, v, v'] dv'.$$

From $\mathcal{T}_0[S_0](f_0) = 0$*, we have*

$$f_0(v, x, t) = \frac{\phi[S_0, \nabla S_0, \partial_t S_0, v]\rho_0(x, t)}{\Phi[S_0, \nabla S_0, \partial_t S_0]},$$

and therefore, it is easy to see $\hat{\Phi}(v) = \bar{\Phi}(v)\rho_0$*. Due to* $\mathcal{T}_0[S_0](f_1) = -\mathcal{T}_1[S_0](f_0) - v \cdot \nabla f_0$*, we have*

$$f_1(v, x, t) = \frac{1}{\Phi[S_0, \nabla S_0, \partial_t S_0]} \Big(-v \cdot \nabla f_0(v, x, t) - \tilde{\Phi}[S_0, \nabla S_0, \partial_t S_0, v] f_0(v, x, t) + \hat{\Phi}[S_0, \nabla S_0, \partial_t S_0, v] \Big).$$

Computing $J_\epsilon = \int_V v f_1(v, x, t) dv$*, we obtain*

$$J_\epsilon = -\int_V \frac{v^i v^j \partial_{x_j} f_0}{\Phi[S_0, \nabla S_0, \partial_t S_0]} dv - \int_V \frac{v^i \tilde{\Phi}[S_0, \nabla S_0, \partial_t S_0, v] f_0}{\Phi[S_0, \nabla S_0, \partial_t S_0]} dv + \int_V \frac{v^i \hat{\Phi}[S_0, \nabla S_0, \partial_t S_0, v]\rho_0}{\Phi[S_0, \nabla S_0, \partial_t S_0]} dv. \tag{36}$$

The first integral in (36) *becomes*

$$\int_V \frac{v^i v^j \partial_{x_j} f_0}{\Phi[S_0, \nabla S_0, \partial_t S_0]} dv = \frac{\rho_0}{\Phi[S_0, \nabla S_0, \partial_t S_0]} \int_V \left(v^i v^j \partial_{x_j} \Big(\frac{\phi[S_0, \nabla S_0, \partial_t S_0, v]}{\Phi[S_0, \nabla S_0, \partial_t S_0]}\Big) \right) dv$$

$$+ \frac{\partial_{x_j}\rho_0}{\Phi^2[S_0, \nabla S_0, \partial_t S_0]} \int_V v^i v^j \phi[S_0, \nabla S_0, \partial_t S_0, v] dv = \frac{A_i}{\Phi}\rho_0 + \frac{B_{ij}}{\Phi^2}\partial_{x_j}\rho_0,$$

where

$$A_i = A_i[S_0, \nabla S_0, \partial_t S_0] = \int_V v^i v^j \partial_{x_j} \left(\frac{\phi[S_0, \nabla S_0, \partial_t S_0, v]}{\Phi[S_0, \nabla S_0, \partial_t S_0]} \right) dv, \tag{37}$$

$$B_{ij} = B_{ij}[S_0, \nabla S_0, \partial_t S_0] = \int_V v^i v^j \phi[S_0, \nabla S_0, \partial_t S_0, v] dv. \tag{38}$$

The second integral in (36) *leads to*

$$\int_V \frac{v^i \tilde{\Phi}[S_0, \nabla S_0, \partial_t S_0, v] f_0(v)}{\Phi[S_0, \nabla S_0, \partial_t S_0]} dv = \frac{C_i}{\Phi^2(S_0, \nabla S_0, \partial_t S_0)}\rho_0,$$

where

$$C_i = C_i[S_0, \nabla S_0, \partial_t S_0] = \int_V v^i \tilde{\Phi}[S_0, \nabla S_0, \partial_t S_0, v]\phi[S_0, \nabla S_0, \partial_t S_0, v] dv. \tag{39}$$

The last integral in (36) *becomes*

$$\int_V \frac{v^i \hat{\Phi}[S_0, \nabla S_0, \partial_t S_0, v]}{\Phi[S_0, \nabla S_0, \partial_t S_0]} dv = \int_V \frac{v^i \bar{\Phi}[S_0, \nabla S_0, \partial_t S_0, v]\rho_0}{\Phi[S_0, \nabla S_0, \partial_t S_0]} dv = \frac{D_i}{\Phi}\rho_0,$$

where

$$D_i = D_i[S_0, \nabla S_0, \partial_t S_0] = \int_V v^i \bar{\Phi}[S_0, \nabla S_0, \partial_t S_0, v] dv. \tag{40}$$

Summing up, we obtain the macroscopic equation

$$\partial_t \rho_0 = \partial_{x_i} \left(\frac{A_i}{\Phi}\rho_0 + \frac{B_{ij}}{\Phi^2}\partial_{x_j}\rho_0 + \frac{C_i}{\Phi^2}\rho_0 - \frac{D_i}{\Phi}\rho_0 \right), \qquad \Phi = \Phi[S_0, \nabla S_0, \partial_t S_0],$$

where A_i, B_{ij}, C_i *and* D_i *are defined in* (37)-(40). □

Remark 4.2. *In the above example, we remark that if we drop out the explicit dependence of the last argument v in the functional ϕ in* (35)*, then the term $\psi(v\cdot v'/|v||v'|)$ does not influence the resulting macroscopic equation any more. This is due to the fact that only C_i and D_i depend on ψ (A_i, B_i do not), and $C_i = D_i = 0$ in case ϕ is independent of v. This is well expected from a biological point of view since reorientations without any bias can not have a macroscop effect.*

In the following we will see how to evaluate A_i, B_{ij}, C_i and D_i in more specifically.

Example 4.3. *(rigorous for $\alpha \geq 0$)*

We present another example of turning kernels, which is more specific than previous one.

$$T_\epsilon[s] = \phi(S(x+\epsilon v,t), S(x-\epsilon v',t), \nabla S(x+\epsilon v,t), \nabla S(x-\epsilon v',t)), \tag{41}$$

where S satisfies chemo-attractant equation of elliptic type with $\beta > 0$ in two dimension. Note that $\phi : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is an even function with respect to the variable ∇S, and increasing and decreasing for the first and second argument, respectively. We also assume the structure condition of Assumption 1.1 and structure condition in Assumption 2.4, i.e.

$$|T_\epsilon[S](x,v,v',t)| \leq C(1 + S(x+\epsilon v,t) + S(x-\epsilon v',t) + |\nabla S(x+\epsilon v,t)| + |\nabla S(x-\epsilon v',t)|).$$

Using the asymptotic expansion of the turning kernel, i.e. $T_\epsilon[S] = T_0[S] + \epsilon T_1[S] + O(\epsilon^2)$, we can easily see that

$$T_0[S] = \phi(S(x,t), S(x,t), \nabla S(x,t), \nabla S(x,t)),$$

and

$$T_1[S] = (\phi_1(S,S,\nabla S,\nabla S)v - \phi_2(S,S,\nabla S,\nabla S)v')\cdot\nabla S$$
$$+ \sum_{i=1}^{2}(\phi_{2+i}(S,S,\nabla S,\nabla S)v - \phi_{4+i}(S,S,\nabla S,\nabla S)v')\cdot\nabla S_{x_i}.$$

Here $\phi_k, k = 1,2,...,6$ indicates differentiation of ϕ with respect to the $k-th$ argument. One can easily see that the symmetric $\phi^A_\epsilon[S]$ and antisymmetric part $\phi^S_\epsilon[S]$ of turning kernel satisfy

$$\phi^S_\epsilon[S] \geq \gamma\left(1 - \epsilon\Lambda(||\nabla S||_{W^{1,\infty}(\mathbb{R}^n)})\right)FF', \qquad \int_V \frac{\phi^A_\epsilon[S]^2}{F\phi^S_\epsilon[S]}dv' \leq \epsilon^2\Lambda(||\nabla S||_{W^{1,\infty}(\mathbb{R}^n)}),$$

where $\gamma > 0$ and $\Lambda \in L^\infty_{\text{loc}}$ is a non-decreasing function. By asymptotic expansion of f_ϵ and S_ϵ, the leading order equation becomes

$$f_0(x,v,t) = \frac{\rho_0(x,t)}{|V|}.$$

We remark that f_0 is independent of v. Since the ϵ-order equation is $\mathcal{T}_0[S_0](f_1) = -(v\cdot\nabla\rho_0)/|V| - \mathcal{T}_1[S_0](f_0)$, we have to calculate

$$\mathcal{T}_1[S_0](f_0) = -\rho_0(\phi_1+\phi_2)\nabla S_0\cdot v - \sum_{i=1}^{2}\rho_0(\phi_{2+i}+\phi_{4+i})\nabla S_{0,x_i}\cdot v.$$

Therefore,

$$\mathcal{T}_0[S_0](f_1) = -\frac{v\cdot\nabla\rho_0}{|V|} + \rho_0(\phi_1+\phi_2)\nabla S_0\cdot v + \sum_{i=1}^{2}\rho_0(\phi_{2+i}+\phi_{4+i})\nabla S_{0,x_i}\cdot v,$$

due to the solvability condition, and thus we get

$$f_1 = -\frac{v\cdot\nabla\rho_0}{|V|^2\phi} + \frac{\rho_0(\phi_1+\phi_2)\nabla S_0\cdot v}{|V|\phi} + \frac{\rho_0(\phi_{2+i}+\phi_{4+i})\nabla S_{0,x_i}\cdot v}{|V|\phi}.$$

Let $\mu = \int_V |v|^2dv$. Using the above results, we obtain the flux density $J_\epsilon = \int_V vf_1dv + O(\epsilon)$, where

$$J_\epsilon = -\frac{\mu}{2|V|^2}\frac{\nabla\rho_0}{\phi} + \frac{\mu}{2|V|}\frac{(\phi_1+\phi_2)\rho_0\nabla S_0}{\phi} + \sum_{i=0}^{2}\frac{\mu}{2|V|}\frac{(\phi_{2+i}+\phi_{4+i})\rho_0\nabla S_{0,x_i}}{\phi}.$$

Hence the diffusion limit is

$$\frac{\partial}{\partial t}\rho_0 = \nabla\cdot(D\nabla\rho_0 - \chi\rho_0\nabla S_0 - \sum_{i=1}^{2}\tilde{\chi}_i\rho_0\nabla S_{0,x_i}) \tag{42}$$

where

$$D = \frac{\mu}{2|V|^2\phi},\quad \chi = \frac{\mu(\phi_1+\phi_2)}{2|V|\phi},\quad \tilde{\chi}_i = \frac{\mu(\phi_{2+i}+\phi_{4+i})}{2|V|\phi},\qquad i=1,2,$$

coupled to $-\Delta S_0 = \rho_0 - \beta S_0$. *Here we remark that it is not known whether solutions for the macroscopic equation* (42) *blowup in finite time or not.*

Example 4.4. *If we choose an appropriate turning kernel, also the classical Keller-Segel model with constant coefficients can be obtained. Indeed, if the turning kernel* (41) *is replaced by*

$$T_\epsilon[s] = \phi(S(x,t), S(x+\epsilon v,t), \nabla S(x+\epsilon v,t), \nabla S(x-\epsilon v',t)),$$

then, by following similar computations as given above, we have

$$\frac{\partial}{\partial t}\rho_0 = \nabla\cdot\left(\frac{\mu}{2|V|^2\phi}\nabla\rho_0 - \frac{\mu\phi_2}{2|V|\phi}\rho_0\nabla S_0 - \sum_{i=1}^{2}\frac{\mu(\phi_{2+i}+\phi_{4+i})}{2|V|\phi}\rho_0\nabla S_{0,x_i}\right). \tag{43}$$

Now we specify ϕ *as follows*

$$\phi(x_1,x_2,x_3,x_4,x_5,x_6) = \varphi(x_2-x_1) + \varphi(x_5-x_3) + \varphi(x_6-x_4), \tag{44}$$

where

$$\varphi(x) = C_1\sqrt{1+x^2} + C_2 x,\qquad C_1 > C_2 > 0.$$

Concerning the gradient terms this example seems a bit artificial but it shows how higher order terms might cancel out. Since $\varphi(0) = C_1, \varphi'(0) = C_2$, *it is easy to see that* $\phi = C_1, \phi_2 = C_2, \phi_3 = \phi_4 = -C_2$, *and* $\phi_5 = \phi_6 = C_2$. *Therefore* (43) *leads to*

$$\frac{\partial}{\partial t}\rho_0 = \nabla\cdot\left(\frac{\mu}{2|V|^2C_1}\nabla\rho_0 - \frac{\mu C_2}{2|V|C_1}\rho_0\nabla S_0\right),$$

which is the classical version of the Keller-Segel model. The diffusion coefficient and chemotactic sensitivity, respectively, are $D = \mu/(2|V|^2C_1), \chi = (\mu C_2)/(2|V|C_1)$, *which are both constants in this case.* □

Example 4.5. *(rigorous,* $\alpha \geq 0$, $\beta > 0$*)*

The next example considers time variations of the chemical S.

$$T_\epsilon = \sigma S(x+\epsilon v,t) + h(\partial_t S(x,t), \nabla S(x,t), v) + C_2, \tag{45}$$

where $\sigma \geq 0$ *is a fixed constant and* $h:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}, n=2,3$ *is smooth and bounded, say* $-C_1 \leq h \leq C_1$ *with* $0 < C_1 < C_2$. *Note that the turning kernel satisfies the structure condition in Assumption 2.4. By asymptotic expansion, one can easily see that*

$$T_0 = \sigma S(x,t) + h(\partial_t S(x,t), \nabla S(x,t), v) + C_2,\qquad T_1 = \sigma v\cdot\nabla S(x,t).$$

We denote by H *the integration of* $h+C_2$ *with respect to* v, *i.e.* $H[S] = \int_V (h(\partial_t S, \nabla S, v) + C_2)\,dv$. *After simple computations, we obtain*

$$f_0(v,x,t) = \frac{\sigma S_0(x,t) + h(\partial_t S_0, \nabla S_0, v) + C_2}{\sigma S_0(x,t)|V| + H[S_0]}\rho_0,$$

$$f_1(v,x,t) = \frac{1}{\sigma S_0(x,t)|V| + H[S_0]}\left(-v\cdot\nabla f_0(v,x,t) + \sigma\rho_0 v\cdot\nabla S_0\right).$$

From now on, for simplicity, we skip the arguments of each variable unless confusion is to be expected. To obtain the macroscopic equation, we consider

$$J_\epsilon = \int_V v f_1 dv = \frac{1}{\sigma S_0|V| + H[S_0]}\left(-\int_V v(v\cdot\nabla f_0)dv + \sigma\int_V v(v\cdot\nabla S_0)dv\rho_0\right)$$

$$= \frac{1}{\sigma S_0|V| + H[S_0]} \left(-\int_V v(v\cdot\nabla f_0)dv + \sigma\mu\rho_0\nabla S_0 \right),$$

where $\mu = (1/n)\int_V |v|^2 dv$. *It remains to calculate* $\int_V v(v\cdot\nabla f_0)dv$. *Simple computations show*

$$\int_V v^i (v\cdot\nabla f_0)dv = (\frac{\mu(\sigma S_0 + C_2)}{\sigma S_0|V| + H[S_0]}\rho_0)_{x_i} + (A_{ij}[S_0]\rho_0)_{x_j},$$

where

$$A_{ij}[S_0] = \int_V \frac{v_i v_j h(\partial_t S, \nabla S, v)}{\sigma S_0|V| + H[S_0]} dv.$$

Therefore, the macroscopic equation becomes

$$\partial_t\rho_0 = \nabla\cdot\left(\frac{1}{\sigma S_0|V| + H[S_0]}\left[\nabla(\frac{\mu(\sigma S_0 + C_2)}{\sigma S_0|V| + H[S_0]}\rho_0) + (A_{ij}[S_0]\rho_0)_{x_j}\right] - \frac{\sigma\mu}{\sigma S_0|V| + H[S_0]}\rho_0\nabla S_0 \right). \tag{46}$$

This equation is rigorously derived with related turning kernel (45) *since it satisfies Assumption 3.2. As a specific example, we consider the case*

$$h(\partial_t S, \nabla S, v) = C_1\frac{\gamma\partial_t S + v\cdot\nabla S}{\mathcal{N}(S)}, \qquad \mathcal{N}(S) = \sqrt{1 + \gamma^2|\partial_t S|^2 + |\nabla S|^2},$$

where γ *is a fixed constant. Then one can easily see*

$$H[S_0] = \frac{C_1\gamma\partial_t S_0|B_1|}{\mathcal{N}(S_0)} + C_2|B_1|, \quad A_{ij}[S_0] = \frac{C_1\mu\gamma\partial_t S_0}{(\sigma S_0|B_1| + H[S_0])N(S_0)},$$

Therefore, the macroscopic equation (46) *can be explicitly calculated, namely for* $\gamma = 0$ *(*$H[S_0] = C_2|B_1|$ *and* $A_{ij}[S_0] = 0$*).*

$$\partial_t\rho_0 = \nabla\cdot\left(\frac{\mu}{(\sigma S_0 + C_2)|V|^2}\nabla\rho_0 - \frac{\sigma\mu}{(\sigma S_0 + C_2)|V|}\rho_0\nabla S_0 \right). \tag{47}$$

On the other hand, if $\sigma = 0$*, then the last term in* (46) *does not appear. In such case the macroscopic equation* (46) *reads*

$$\partial_t\rho_0 = \nabla\cdot\left(\frac{1}{H[S_0]}\nabla(\frac{\mu C_2}{H[S_0]}\rho_0) + (A_{ij}[S_0]\rho_0)_{x_j} \right),$$

where $(A_{ij}[S_0] = \int_V v_i v_j h(\partial_t S, \nabla S, v)/H[S_0]dv$.

We also remark that $A_{ij} = 0$ *in* (46) *in case* h *is odd with respect to the variable* v. □

In the next example we discuss the influence of non-local terms in h.

Example 4.6. *(formal for* $\alpha > 0$*, rigorous for* $\alpha = 0$*)*

Consider

$$T_\epsilon = \sigma S(x + \epsilon v, t) + h(\partial_t S(x + \epsilon v, t), v\cdot\nabla S(x + \epsilon v, t)) + C_2,$$

where $h : \mathbb{R}\times\mathbb{R}\to\mathbb{R}, n = 2, 3$ *is smooth and bounded, say* $-C_1 \le h \le C_1$ *with* $0 < C_1 < C_2$. *Note that it the structure condition in Assumption 2.4 is satisfied. Then*

$$T_0 = T_0[S, v] = \sigma S + h(\partial_t S, v\cdot\nabla S) + C_2,$$

$$T_1 = T_1[S, v] = \sigma v\cdot\nabla S + h_1(\partial_t S, v\cdot\nabla S)v\cdot\nabla S_t + h_2(\partial_t S, v\cdot\nabla S)v_i S_{x_i,x_j} v_j.$$

where h_1, h_2 *indicate partial derivatives with respect the first and second argument, respectively. We denote*

$$H[S] = \int_V (h(\partial_t S(x,t), v\cdot\nabla S(x,t)) + C_2)\, dv.$$

$$K[S] = \int_V T_1'(v)dv' = \int_V \left(h_1(\partial_t S, v'\cdot\nabla S)v'\cdot\nabla S_t + h_2(\partial_t S, v'\cdot\nabla S)v_i' S_{x_i,x_j} v_j' \right) dv',$$

where $\int_V v'\cdot\nabla S dv' = 0$. *From the asymptotic expansion, we get*

$$f_0 = \frac{\sigma S_0 + h(\partial_t S_0, v\cdot\nabla S_0) + C_2}{\sigma S_0|V| + H(S_0)}\rho_0, \quad f_1 = \frac{-v\cdot\nabla f_0 - f_0 K[S_0] + T_1[S_0,v]\rho_0}{\sigma S_0|V| + H[S_0]}.$$

Therefore, we have

$$J_\epsilon = \int_V v f_1 dv = -\frac{1}{\sigma S_0|V| + H[S_0]}(\int_V v_i v_j \partial_{x_j} f_0 dv + K[S_0]\int_V v_i f_0 dv - \rho_0\int_V v_i T_1[S_0,v]dv)$$

Thus, the macroscopic equation becomes

$$\partial_t\rho_0 = -\nabla\cdot J_\epsilon = \nabla\cdot\left(\frac{1}{\sigma S_0|V| + H[S_0]}(\int_V v_i v_j \partial_{x_j} f_0 dv + K[S_0]\int_V v_i f_0 dv - \rho_0\int_V v_i T_1[S_0,v]dv)\right).$$

Next we consider a specific example of the turning kernel above. For example, suppose h *is given as follows;*

$$h = h(\partial_t S(x+\epsilon v, t-\epsilon), v\cdot\nabla S(x+\epsilon v, t)) = \frac{C_1 v\cdot\nabla S(x+\epsilon v,t)}{\sqrt{1+(v\cdot\nabla S(x+\epsilon v,t))^2}}.$$

Then one can easily calculate

$$H[S_0] = C|V|, \qquad f_0 = G[S_0,v]\rho_0,$$

where

$$G[S_0,v] = \frac{\sigma S_0 + h(\partial_t S_0, v\cdot\nabla S_0) + C}{(\sigma S_0 + C)|V|}.$$

On the other hand,

$$K[S_0] = \int_V\left(\frac{1}{(1+(v\cdot\nabla S_0)^2)^{\frac{1}{2}}} - \frac{(v\cdot\nabla S_0)^2}{(1+(v\cdot\nabla S_0)^2)^{\frac{3}{2}}}\right)v_i v_j S_{0,x_i x_j} dv = L[S_0]\Delta S_0 - M[S_0]|\nabla S_0|^2\Delta S_0,$$

where

$$L[S_0] = \frac{1}{n}\int_V \frac{|v|^2}{\sqrt{1+(v\cdot\nabla S_0)^2}}dv, \qquad M[S_0] = \frac{1}{n^2}\int_V\frac{|v|^4}{(1+(v\cdot\nabla S_0)^2)^{\frac{3}{2}}}dv. \tag{48}$$

Hence, summing up, we obtain

$$f_1 = \frac{1}{(\sigma S_0 + C)|V|}\left(-v\cdot\nabla(G[S_0,v]\rho_0) - G[S_0,v]\rho_0(L[S_0]\Delta S_0 - M[S_0]|\nabla S_0|^2\Delta S_0) + T_1[S_0,v]\rho_0\right),$$

Using f_1 *above, we calculate*

$$J_\epsilon = \int_V v f_1 dv = -\frac{1}{(\sigma S_0 + C)|V|}\int_V v v_j\partial_{x_j}(G[S_0,v]\rho_0)$$
$$-\frac{(L[S_0]\Delta S_0 - M[S_0]|\nabla S_0|^2\Delta S_0)}{(\sigma S_0 + C)|V|}\rho_0\int_V vG[S_0,v] + \frac{1}{(\sigma S_0+C)|V|}\int_V vT_1[S_0,v]\rho_0.$$

After simple computations, we obtain $\int_V vT_1[S_0,v] = \sigma\mu\nabla S_0$, $\int_V vG[S_0,v] = L[S_0]\nabla S_0/(\sigma S_0 + C)|V|)$, *and* $\int_V v v_j\partial_{x_j}(G[S_0,v]\rho_0) = (\mu/|V|)\nabla\rho$ *where* $\mu = (1/n)\int_V |v|^2 dv$. *Therefore, the macroscopic equation reads*

$$\partial_t\rho_0 = \nabla\cdot\big(\frac{\mu}{(\sigma S_0 + C)|V|^2}\nabla\rho_0 - \frac{\sigma\mu}{(\sigma S_0+C)|V|}\rho_0\nabla S_0$$
$$+\frac{L[S_0](L[S_0]\Delta S_0 - M[S_0]|\nabla S_0|^2\Delta S_0)}{(\sigma S_0+C)^2|V|^2}\rho_0\nabla S_0\big),$$

where $L[S_0]$ *and* $M[S_0]$ *are defined in* (48). *The third term is completely due to the nonlocal dependencies of* h. *Compare* (47) *for the local formulation.* □

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