COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL XIX, 95–105 (1966)

# Global Solutions of the Cauchy Problem for Quasi-Linear First-Order Equations in Several Space Variables*

EDWARD CONWAY AND JOEL SMOLLER
*University of California at San Diego* *University of Michigan*

## 1. Introduction

In recent years, there has been some interest in quasi-linear differential equations of conservation type, and much progress has been made in the case of a single space variable (see [1] for a survey of the literature). The results obtained here extend part of this theory to several space variables. To be more precise, we are concerned with solutions, which are defined in the region $t \geqq 0$, of the Cauchy problem

$$u_t + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} f_j(u) = 0\,, \qquad u(0, x) = u_0(x)\,, \tag{1.1}$$

where $x = (x_1, \cdots, x_n)$, $u = u(t, x)$, and the $f_j$, $j = 1, 2, \cdots, n$, are continuously differentiable functions of a single real variable. If some of the $f_j$ are nonlinear, it is well known that one cannot, in general, obtain continuous solutions, and that shocks (jump discontinuities) develop in a finite time, [1]. The solutions we obtain will therefore be weak solutions in the sense that

$$\int_{t \geq 0} \left[ u\phi_t + \sum_{j=1}^{n} f_j(u)\phi_{x_j} \right] dx\, dt + \int_{t=0} u_0(x)\phi(0, x)\, dx = 0\,, \tag{1.2}$$

for every continuously differentiable test function $\phi = \phi(t, x)$ having compact support.

Our main hypotheses concern the initial function $u_0(x)$. In addition to requiring that $u_0(x)$ be bounded, we also require that $u_0(x)$ be of bounded variation in the sense of Tonelli-Cesari on any compact set. A function $f$ is said to have bounded variation in the sense of Tonelli-Cesari over a compact set $Q$ if there exists a set $Z$ of measure zero in $Q$ such that the functions

$$V_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) = \operatorname*{Var}_{Q-Z} f(x_1, \cdots, x_{i-1}, \cdot, x_{i+1}, \cdots, x_n)\,,$$
$$i = 1, 2, \cdots, n\,,$$

* This paper represents results obtained under the Visiting Membership program at the Courant Institute of Mathematical Sciences, New York University, with the sponsorship of the National Science Foundation, Grant No. NSF-GP-3465. 

are measurable and summable. This is equivalent to the statement that the gradient of $f$, in the sense of the theory of distributions, is a measure whose total variation is finite over the compact set $Q$, [2]. If we let $\mathfrak{F}$ be the class of bounded functions in $R_n$ having locally bounded variation in the sense of Tonelli-Cesari, then we can state our main theorem in the following way.

THEOREM 1. *Let $f_i$, $i = 1, 2, \cdots, n$, be continuously differentiable functions of a single real variable. If $u_0(x) \in \mathfrak{F}$, then there exists a function $u(t, x)$ which is a weak solution of (1.1) in the region $t > 0$ having $u_0(x)$ as initial value. Moreover, for each fixed $t$, $u(t, x) \in \mathfrak{F}$, and $u(t, x)$ has the same upper and lower bounds as $u_0(x)$.*

In order to prove this theorem, we shall make use of a finite difference scheme proposed by Lax in [3]. In the case of a single space variable, Oleinik [4] used this scheme in order to prove existence of a weak solution; our methods are related to hers.

As a consequence of the method of proof, we shall also prove the following theorem.

THEOREM 2. *Monotonicity in one or more of the space variables is persistent in time; i.e., if $u_0(x)$ is monotonic in one or more variables, then for each fixed $t$, the solution $u(t, x)$ constructed in Theorem 1 is monotonic in the same sense in the same variables.*

The plan of the paper is the following: In Section 2 we introduce the difference scheme and derive the basic estimates for the solution of these equations. In Section 3 we construct a sequence of functions from the solution of the difference equations, and, using a compactness criterion due to de Giorgi [5] and Fleming [6], extract a subsequence which converges in the topology of $L_1$ convergence on compacta. The limit of this subsequence is then shown to be a solution of our problem. The last section consists of some concluding remarks.

## 2. Estimates for the Difference Equations

In order to facilitate the presentation, we shall avoid cumbersome notation by restricting ourselves to the case $n = 2$. It will be quite clear that everything which we do in this case can be extended at once to the more general case. Thus we shall consider the initial value problem

$$u_t + \frac{\partial}{\partial x} f(u) + \frac{\partial}{\partial y} g(u) = 0\,, \qquad u(0, x, y) = u_0(x, y)\,, \tag{2.1}$$

where $f$ and $g$ are functions in the class $C^1$ and $u_0 \in \mathfrak{F}$. By a solution of the problem (2.1), we shall mean that

$$\iiint_{t>0} [u\phi_t + f(u)\phi_x + g(u)\phi_y]\, dx\, dy\, dt + \iint_{t=0} u_0(x, y)\phi(0, x, y)\, dx\, dy = 0 \tag{2.2}$$

for all $C^1$ functions $\phi = \phi(t, x, y)$ having compact support.

Let the half-space $t \geqq 0$ be covered by a grid defined by the planes

$$t = kh\,, \qquad x = nq\,, \qquad y = mp\,,$$

where $h$, $q$, and $p$ are fixed positive real numbers, $k$ runs through the non-negative integers, and $n$ and $m$ assume all integral values. In the region $t > 0$, we consider the finite difference scheme defined by

$$\tag{2.3} \begin{aligned} &\frac{u^{k+1}_{n,m} - \frac{1}{4}(u^k_{n+1,m} + u^k_{n-1,m} + u^k_{n,m+1} + u^k_{n,m-1})}{h} \\ &\qquad + \frac{f(u^k_{n+1,m}) - f(u^k_{n-1,m})}{2q} + \frac{g(u^k_{n,m+1}) - g({}^k_{n,m-1})}{2p} = 0\,, \end{aligned}$$

where we are using the notation

$$u^\alpha_{\beta,\gamma} = u(\alpha h, \beta q, \gamma p)\,.$$

LEMMA 1. *Let $M_0 \leqq u^0_{n,m} \leqq M_1$ for all $n$ and $m$, and let $A$ and $B$ be defined by*

$$A = \max_{M_0 \leqq u \leqq M_1} |f'(u)|\,, \qquad B = \max_{M_0 \leqq u \leqq M_1} |g'(u)|\,.$$

*Then if the stability requirements $Ah/q < \frac{1}{2}$ and $Bh/p < \frac{1}{2}$ are fulfilled, we have $M_0 \leqq u^k_{n,m} \leqq M_1$ for all values of $n$, $m$ and $k$.*

Proof: By using the mean value theorem, we can write (2.3) in the form

$$\begin{aligned} u^{k+1}_{n,m} &= \tfrac{1}{4}(u^k_{n+1,m} + u^k_{n-1,m} + u^k_{n,m+1} + u^k_{n,m-1}) \\ &\quad - \frac{h}{2q} f'(\alpha_{n,m})(u^k_{n+1,m} - u^k_{n-1,m}) \\ &\quad - \frac{h}{2p} g'(\beta_{n,m})(u^k_{n,m+1} - u^k_{n,m-1})\,, \end{aligned}$$

where $\alpha_{n,m}$ is some intermediate value between $u^k_{n+1,m}$ and $u^k_{n-1,m}$, and $\beta_{n,m}$ is some intermediate value between $u^k_{n,m+1}$ and $u^k_{n,m-1}$. This last equation can then be written as

$$\tag{2.4} \begin{aligned} u^{k+1}_{n,m} &= u^k_{n+1,m}\left[\frac{1}{4} - \frac{h}{2q} f'(\alpha_{n,m})\right] + u^k_{n-1,m}\left[\frac{1}{4} + \frac{h}{2q} f'(\alpha_{n,m})\right] \\ &\quad + u^k_{n,m+1}\left[\frac{1}{4} - \frac{h}{2p} g'(\beta_{n,m})\right] + u^k_{n,m-1}\left[\frac{1}{4} + \frac{h}{2p} g'(\beta_{n,m})\right]. \end{aligned}$$

Now the coefficients of the $u^k_{i,j}$ appearing in (2.4) are non-negative and add up to one so that if we inductively assume that $M_0 \leqq u^k_{n,m} \leqq M_1$ for all $n$ and $m$, we see at once that $M_0 \leqq u^{k+1}_{n,m} \leqq M_1$ for all $n$ and $m$.

In what follows, we shall always assume that the stability conditions $Bh/p < \frac{1}{2}$ and $Ah/q < \frac{1}{2}$ are satisfied.

LEMMA 2. *If $p$, $q$ and $h$ satisfy*

$$q \leqq \delta h , \qquad p \leqq \delta h , \tag{2.5}$$

*for some fixed $\delta > 0$, then for arbitrary $X$ we have the estimate*

$$\begin{aligned} &\sum_X (|u^k_{n+1,m} - u^k_{n,m}| \, p + |u^k_{n,m+1} - u^k_{n,m}| \, q) \\ &\qquad \leqq \sum_{X+\delta(kh)} (|u^0_{n+1,m} - u^0_{n,m}| \, p + |u^0_{n,m+1} - u^0_{n,m}| \, q) , \end{aligned} \tag{2.6}$$

*where by $\sum_L$ we mean summation over all $n$, $m$ satisfying $q\,|n| \leqq L$, $p\,|m| \leqq L$.*

Proof: We let $w^k_{n,m} = u^k_{n+1,m} - u^k_{n,m}$ and $v^k_{n,m} = u^k_{n,m+1} - u^k_{n,m}$. From (2.3) we obtain, after applying the mean value theorem,

$$\begin{aligned} w^k_{n,m} = {} & w^{k-1}_{n+1,m} \left[ \frac{1}{4} - \frac{h}{2q} f'(\theta_{n+1,m}) \right] + w^{k-1}_{n-1,m} \left[ \frac{1}{4} + \frac{h}{2q} f'(\theta_{n-1,m}) \right] \\ & + w^{k-1}_{n,m+1} \left[ \frac{1}{4} - \frac{h}{2p} g'(\xi_{n,m+1}) \right] + w^{k-1}_{n,m-1} \left[ \frac{1}{4} + \frac{h}{2p} g'(\xi_{n,m-1}) \right] , \end{aligned} \tag{2.7}$$

where $\theta_{i,j}$ and $\xi_{i,j}$ are intermediate values for $u^{k-1}_{i+1,j}$ and $u^{k-1}_{i,j}$. Because of the stability requirement on $h$, $p$, and $q$, the four coefficients of the $w^{k-1}_{i,j}$ in (2.7) are non-negative. Therefore,

$$\sum_{\substack{q|n| \leq X \\ p|m| \leq X}} |w^k_{n,m}| \leqq \sum_{\substack{q|n| \leq X+q \\ p|m| \leq X}} \tfrac{1}{2} |w^{k-1}_{n,m}| + \sum_{\substack{q|n| \leq X \\ p|m| \leq X+p}} \tfrac{1}{2} |w^{k-1}_{n,m}| \leqq \sum_{\substack{q|n| \leq X+q \\ p|m| \leq X+p}} |w^{k-1}_{n,m}| \, .$$

In the same way we can obtain an identical estimate for $\Sigma\, |v^k_{n,m}|$. If we make use of (2.5), we can express these estimates as

$$\sum_X (|w^k_{n,m}| \, p + |v^k_{n,m}| \, q) \leqq \sum_{X+\delta h} (|w^{k-1}_{n,m}| \, p + |v^{k-1}_{n,m}| \, q) \, .$$

If we now apply this estimate to the right side of this last inequality, and continue in this way $k$ times, we obtain the desired estimate (2.6).

LEMMA 3. *If $w^0_{n,m} \leqq 0$ ($\geqq 0$) for all $n$ and $m$, then $w^k_{n,m} \leqq 0$ ($\geqq 0$) for all $k$, $n$, and $m$. The same statement is true for $v^k_{n,m}$.*

Proof: Because of the stability restriction on $h$, $p$, and $q$, all four coefficients in (2.7) are non-negative. Therefore, if $w^{k-1}_{i,j} \leqq 0$ ($\geqq 0$) for all $i$ and $j$, then $w^k_{n,m}$ will also be $\leqq 0$ ($\geqq 0$).

LEMMA 4. *Let $\delta > 0$ satisfy* (2.5). *Then if $\alpha > \beta$,*

$$\sum_X |u^\alpha_{n,m} - u^\beta_{n,m}| \, qp \leqq c(\alpha - \beta) h , \tag{2.8}$$

*where the constant $c$ is defined by*

$$c = \delta \sum_{X+\delta h\alpha} (p \, |u^0_{n+1,m} - u^0_{n,m}| + q \, |u^0_{n,m+1} - u^0_{n,m}|) \, . \tag{2.9}$$

Proof: We first consider the transition from two adjacent time intervals. If we use (2.4), we can write, for $\beta < k + 1 \leqq \alpha$,

$$\begin{aligned} u^{k+1}_{n,m} - u^k_{n,m} = {} & (u^k_{n+1,m} - u^k_{n,m}) \left(\frac{1}{4} - \frac{h}{2q} f'(\alpha_{n,m})\right) \\ & + (u^k_{n-1,m} - u^k_{n,m}) \left(\frac{1}{4} + \frac{h}{2q} f'(\alpha_{n,m})\right) \\ & + (u^k_{n,m+1} - u^k_{n,m}) \left(\frac{1}{4} - \frac{h}{2p} g'(\beta_{n,m})\right) \\ & + (u^k_{n,m-1} - u^k_{n,m}) \left(\frac{1}{4} + \frac{h}{2p} g'(\beta_{n,m})\right) . \end{aligned}$$

Therefore, since the coefficients of the differences of the two $u^k_{i,j}$ are non-negative, and since the sum of the first two as well as the last two coefficients is $\frac{1}{2}$, we can write

$$\begin{aligned} \sum_{\substack{q|n|\leqq X \\ p|m|\leqq X}} |u^{k+1}_{n,m} - u^k_{n,m}|\, qp \leqq {} & qp \sum_{p|m|\leqq X} \sum_{q|n|\leqq X+q} |u^k_{n+1,m} - u^k_{n,m}| \\ & + qp \sum_{q|n|\leqq X} \sum_{p|m|\leqq X+p} |u^k_{n,m+1} - u^k_{n,m}| \\ \leqq {} & \delta h \sum_{\substack{q|n|\leqq X+q \\ p|m|\leqq X+p}} (p\, |u^k_{n+1,m} - u^k_{n,m}| + q\, |u^k_{n,m+1} - u^k_{n,m}|) . \end{aligned}$$

But then if we use Lemma 2, we get

$$\begin{aligned} & \sum_X |u^{k+1}_{n,m} - u^k_{n,m}|\, qp \\ & \leqq \delta h \sum_{X+\delta(k+1)h} (p\, |u^0_{n+1,m} - u^0_{n,m}| + q\, |u^0_{n,m+1} - u^0_{n,m}|) \\ & \leqq \delta h \sum_{X+\delta(\alpha h)} (p\, |u^0_{n+1,m} - u^0_{n,m}| + q\, |u^0_{n,m+1} - u^0_{n,m}|) \\ & = ch . \end{aligned}$$

It follows from the triangle inequality that

$$\sum_X |u^\alpha_{n,m} - u^\beta_{n,m}|\, qp \leqq \sum_{k=\beta}^{\alpha-1} \sum_X |u^{k+1}_{n,m} - u^k_{n,m}|\, qp \leqq \sum_{k=\beta}^{\alpha-1} ch = c(\alpha - \beta)h , \tag{2.10}$$

which is the desired result.

## 3. Proof of the Main Theorem

In this section we shall prove the theorem stated in the introduction for the case $n = 2$; i.e., we shall prove

THEOREM 1′. *If $u_0(x,y)$ belongs to the class $\mathfrak{F}$, then there exists a weak solution $u(t, x, y)$ of the Cauchy problem (2.1) for all $t$, $t \geqq 0$. For each fixed $t$, the function $u$ considered as a function of $x$ and $y$ belongs to the class $\mathfrak{F}$ and has the same upper and lower bounds as $u_0$.*

We shall obtain $u(t, x, y)$ as the limit of a sequence of solutions of the difference equation (2.3). However, rather than consider solutions of the difference equation to be defined only on grid points $t = kh$, $x = nq$ and $y = mp$ as was done in Section 2, we wish to consider them as functions defined throughout $t, x, y$-space. We accomplish this by defining

$$U(t, x, y) = u^k_{n,m}$$

for $kh \leqq t < (k+1)h$, $nq \leqq x < (n+1)q$, $mp \leqq y < (m+1)p$. We refer to these functions as "grid functions". The first step in the proof of the above theorem is

LEMMA 5. *There exists a sequence of grid functions $U^i_0$, having the same upper and lower bounds as $u_0$, such that on any compact set $Q \subset R^2$, $U^i_0$ converges to $u_0$ in $L_1(Q)$ and $V(U^i_0\,; Q)$ is uniformly bounded for all $i$. $V(U^i_0\,; Q)$ is defined as the total variation in the sense of Tonelli-Cesari of the function $U^i_0$ over the set $Q$.*

Proof: Let $Q_i$, $i \geqq 1$, be a sequence of squares such that $Q_i \subset Q_{i+1}$ and the union of the $Q_i$ is $R^2$. Let $w_\lambda$ be the gaussian averaging kernel having the circle of radius $\lambda$ as its support and let $u_\lambda = u_0 * w_\lambda$. It is a classical result that

$$\|u_\lambda - u_0\|_{L_1(Q)} \to 0 \qquad \text{as} \qquad \lambda \to 0$$

for any compact set $Q$. De Giorgi [7] has show that $V(u_\lambda\,; Q)$ is a non-increasing function of $\lambda$ and that

$$\lim_{\lambda \to 0} V(u_\lambda\,; Q) = V(u; Q)\,. \tag{3.1}$$

Let $\{\lambda_i\}$, $i \geqq 1$, be a sequence of positive numbers converging to zero. Let $U^i_0$ be a grid function which coincides with $u_{\lambda_i}$ at the grid points, the mesh size being so small that

$$\|u_{\lambda_i} - U^i_0\|_{L_1(Q_i)} \leqq \lambda_i\,.$$

Now if $Q$ is any compact set, then $Q \subseteq Q_k$ for some $k$. Therefore, if $i \geqq k$, we have

$$\begin{aligned}\|u_0 - U^i_0\|_{L_1(Q)} &\leqq \|u_0 - u_{\lambda_i}\|_{L_1(Q)} + \|u_{\lambda_i} - U^i_0\|_{L_1(Q_i)} \\ &\leqq \|u_0 - u_{\lambda_i}\|_{L_1(Q)} + \lambda_i\,,\end{aligned}$$

and we see that $U^i_0$ converges to $u_0$ in $L_1(Q)$ as $i \to \infty$. Since $U^i_0$ coincides with $u_{\lambda_i}$ at grid points, it is clear that

$$V(U^i_0\,; Q) \leqq V(u_{\lambda_i}\,; Q)\,,$$

so that using (3.1) we see that $V(U^i_0\,; Q)$ is bounded independently of $i$. That the $U^i_0$ have the same upper and lower bounds as $u_0$ follows at once from the positivity of $w_\lambda$ and the fact that $\int w_\lambda = 1$. This completes the proof of the lemma.

We now let $U^i(t, x, y)$ be the solution of the difference equation (2.3) having $U^i_0(x, y)$ as initial values. We assume that $h_i$ has been chosen so that

$$\frac{1}{\delta} q_i \leqq h_i \leqq \frac{1}{2A} q_i \qquad \text{and} \qquad \frac{1}{\delta} p_i \leqq h_i \leqq \frac{1}{2B} p_i\,.$$

Lemmas 1 through 4 are then valid. But grid functions $U$ are functions whose gradients are measures having their mass concentrated along the grid lines, so that we have for any square $Q \subset R^2$

$$\sum_{n,m} (p\,|u^k_{n+1,m} - u^k_{n,m}| + q\,|u^k_{n,m+1} - u^k_{n,m}|) = V(U; Q)\,,$$

where $u^k_{n,m} = U(kh, nq, mp)$ and the summation is over all values $n$, $m$ such that $(nq, mp) \in Q$. We see therefore that Lemmas 2 and 4 are equivalent to the following lemmas, where $Q(L)$ indicates the square $|x| \leqq L$ and $|y| \leqq L$.

LEMMA 2′. *For any fixed* $t \geqq 0$,

$$V(U^i(t, x, y); Q(X)) \leqq V(U^i_0(x, y); Q(X + \delta t)) .$$

LEMMA 4′. *For* $t > s$,

$$\iint_{Q(X)} |U^i(t, x, y) - U^i(s, x, y)|\, dx\, dy \leqq \delta(t - s) V(U^i_0 ; Q(X + \delta t)) .$$

We are now prepared to prove

LEMMA 6. *From the sequence of functions* $\{U^i\}$, *a subsequence (also denoted by* $\{U^i\}$) *can be selected which converges to a function* $u(t, x, y)$ *for each fixed* $t$ *in the sense that*

$$\iint_Q |U^i(t, x, y) - u(t, x, y)|\, dx\, dy$$

*converges to zero as* $i \to \infty$ *for any square* $Q$.

Proof: From Lemma 2′ and Lemma 5 it follows that the functions $U^i(t, x, y)$, for any fixed value of $t$, have uniformly bounded total variation (in the sense of Tonelli-Cesari) over any fixed square $Q$ in $R^2$. But then, according to a theorem of de Giorgi [5] and Fleming [6], the functions $U^i$ are compact in $L_1$ over this square. We can, therefore, extract a subsequence that converges in $L_1$ on this square. By a diagonalization procedure we can select a subsequence of $\{U^i\}$ that converges in $L_1$ over any square. By a further diagonalization process we can obtain a subsequence, denoted by $\{U^i(t, x, y)\}$, which for $t = t_j$, $j = 1, 2, \cdots$, converges in $L_1$ on compact sets, the sequence $\{t_j\}$ being dense on the positive $t$-axis; i.e., if $Q$ is any square, then

$$\iint_Q |U^i(t_j, x, y) - U^l(t_j, x, y)|\, dx\, dy$$

converges to zero as $i \to \infty$ and $l \to \infty$. But now for any value of $t$ we have

$$\begin{aligned} &\iint_Q |U^i(t, x, y) - U^l(t, x, y)|\, dx\, dy \\ &\quad \leqq \iint_Q |U^i(t, x, y) - U^i(t_j, x, y)|\, dx\, dy \\ &\qquad + \iint_Q |U^l(t, x, y) - U^l(t_j, x, y)|\, dx\, dy \\ &\qquad + \iint_Q |U^i(t_j, x, y) - U^l(t_j, x, y)|\, dx\, dy \end{aligned} \tag{3.2}$$

for every $t_j$. From Lemma 4′ and Lemma 5, we see that the right-hand side of (3.2) is majorized by

$$\iint_Q |U^i(t_j, x, y) - U^l(t_j, x, y)| \, dx \, dy + c\,|t - t_j| , \tag{3.3}$$

where $c$ is independent of $i$ and $l$. It is clear that this quantity can be made as small as we wish by first choosing $t_j$ close to $t$ and then choosing $i$ and $l$ sufficiently large. This completes the proof of the lemma.

We can improve the result of the last lemma by showing that the convergence in question is uniform in $t$, $0 \leqq t \leqq T$, for any $T > 0$. To see this, let $\varepsilon$ be any positive number. Since $\{t_j\}$ is dense, we can find a finite number of the $t_j$ such that for any $t$, $0 \leqq t \leqq T$, we have $c|t - t_j| < \varepsilon/2$ for some one of a finite number of the $t_j$. Here $c$ is the constant appearing in (3.3), but for $t = T$. This majorizes the corresponding constant for any other value of $t$, $0 \leqq t \leqq T$ (c.f. Lemma 4′). We then choose $i$ and $l$ so large that

$$\iint_Q |U^i(t_j, x, y) - U^l(t_j, x, y)| \, dx \, dy < \frac{\varepsilon}{2}$$

for each of the finite number of the $t_j$. From (3.2) and (3.3), it follows that for such $i$ and $l$ values we have

$$\iint_Q |U^i(t, x, y) - U^l(t, x, y)| \, dx \, dy < \varepsilon$$

for all $t$, $0 \leqq t \leqq T$.

Because of this uniformity of convergence we see that

$$\int_0^T \iint_Q |U^i(t, x, y) - U^l(t, x, y)| \, dx \, dy \, dt$$

converges to zero as $i, l \to \infty$. Therefore we can conclude that the limit function $u(t, x, y)$ is measurable. Furthermore, since $L_1$ convergence implies convergence almost everywhere of a subsequence, we see that $u(t, x, y)$ has the same upper and lower bounds as do the $U^i(t, x, y)$ and these in turn have the same upper and lower bounds as $u_0(x, y)$. Finally, to see that $u(t, x, y) \in \mathfrak{F}$ for each fixed $t$, we recall that if $U^i$ converges to $u$ in $L_1(Q)$, then for each fixed $\lambda$

$$\frac{\partial U^i_\lambda}{\partial x} \to \frac{\partial u_\lambda}{\partial x} \quad \text{and} \quad \frac{\partial U^i_\lambda}{\partial y} \to \frac{\partial u_\lambda}{\partial y}$$

in $L_1(Q)$. (Here we are employing the gaussian averaging kernels as in the proof of Lemma 5.) From this it follows that, for each fixed $\lambda$,

$$\left\| \frac{\partial U^i_\lambda}{\partial x} \right\|_{L_1(Q)} + \left\| \frac{\partial U^i_\lambda}{\partial y} \right\|_{L_1(Q)} = V(U^i_\lambda ; Q) \to V(u_\lambda ; Q) .$$

Using (3.1), we see that $V(u; Q)$ has the same bound as $V(U^i; Q)$.

The last step in the proof of Theorem 1′ is given by

LEMMA 7. *The function $u(t, x, y)$ satisfies equation (2.2) for any $C^3$ function $\phi = \phi(t, x, y)$ which has compact support.*

Proof: To prove this lemma we employ a device used by Oleinik in [4]. By means of our difference scheme (2.3), we can write

$$\frac{u_{n,m}^{k+1} - u_{n,m}^k}{h} - \frac{u_{n+1,m}^k - 2u_{n,m}^k + u_{n-1,m}^k}{2q^2} \cdot \frac{q^2}{h} + \frac{2f(u_{n+1,m}^k)}{2q} - \frac{2f(u_{n-1,m}^k)}{2q}$$
$$+ \frac{u_{n,m}^{k+1} - u_{n,m}^k}{h} - \frac{u_{n,m+1}^k - 2u_{n,m}^k + u_{n,m-1}^k}{2p^2} \cdot \frac{p^2}{h}$$
$$+ \frac{2g(u_{n,m+1}^k)}{2p} - \frac{2g(u_{n,m-1}^k)}{2p} = 0\,.$$

If we multiply this equation by $\phi_{n,m}^k$, we have

$$\frac{\phi_{n+1,m}^{k+1} u_{n+1,m}^{k+1} - \phi_{n,m}^k u_{n,m}^k}{h} - u_{n,m}^{k+1} \frac{(\phi_{n,m}^{k+1} - \phi_{n,m}^k)}{h}$$
$$+ \frac{q^2}{2h} \cdot \frac{2\phi_{n,m}^k - \phi_{n+1,m}^k - \phi_{n-1,m}^k}{q^2} u_{n,m}^k + \frac{\phi_{n+1,m}^k u_{n,m}^k - \phi_{n,m}^k u_{n-1,m}^k}{2h}$$
$$+ \frac{\phi_{n-1,m}^k u_{n,m}^k - \phi_{n,m}^k u_{n+1,m}^k}{2h} + \frac{\phi_{n+1,m}^k f(u_{n+1,m}^k) - \phi_{n-1,m}^k f(u_{n-1,m}^k)}{p}$$
$$- f(u_{n+1,m}^k) \frac{(\phi_{n+1,m}^k - \phi_{n,m}^k)}{p} - f(u_{n-1,m}^k) \frac{(\phi_{n,m}^k - \phi_{n-1,m}^k)}{p}$$
$$+ \frac{\phi_{n,m+1}^{k+1} u_{n,m+1}^{k+1} - \phi_{n,m}^k u_{n,m}^k}{h} - u_{n,m}^{k+1} \frac{(\phi_{n,m}^{k+1} - \phi_{n,m}^k)}{h}$$
$$+ \frac{p^2}{2h} \cdot \frac{2\phi_{n,m}^k - \phi_{n,m+1}^k - \phi_{n,m-1}^k}{p^2} u_{n,m}^k + \frac{\phi_{n,m+1}^k u_{n,m}^k - \phi_{n,m-1}^k}{2h}$$
$$+ \frac{\phi_{n,m-1}^k u_{n,m}^k - \phi_{n,m}^k u_{n,m+1}^k}{2h} + \frac{\phi_{n,m+1}^k g(u_{n,m+1}^k) - \phi_{n,m}^k g(u_{n,m-1}^k)}{q}$$
$$- g(u_{n,m+1}^k) \frac{(\phi_{n,m+1}^k - \phi_{n,m}^k)}{q} - g(u_{n,m-1}^k) \frac{(\phi_{n,m}^k - \phi_{n,m-1}^k)}{q} = 0\,.$$

We sum this equation over all integers $n$ and $m$ and all non-negative integers $k$. Then the desired equality (2.2) follows from Lemma 6. This procedure is carried out in detail by Oleinik in Lemma 7 of [4].

We can now complete the proof of Theorem 1. Since the function $u(t, x, y)$ is bounded and the set of $C^3$ functions having compact support is dense in the set of $C^1$ functions having compact support, the equality (2.2) is valid for any $\phi = \phi(t, x, y)$ which belong to this larger class. Thus the theorem is proved.

THEOREM 2'. *Monotonicity in $x$ or $y$ is persistent in time; i.e., if $u_0(x, y)$ is monotonic in $x$ or $y$, then for each fixed $t$, the solution constructed in Theorem 1' is monotonic in the same sense in the same variables.*

Proof: From Lemma 3, we see that monotonicity in $x$ or $y$ is persistent in the solution of the difference equations. Therefore, if we start out with monotonic initial data, it follows that the simple functions $U^i$, for a fixed $t$, will be monotonic in the same variables in the same sense. Hence the same is true of the limit function $u(t, x, y)$ since $L_1$ convergence on compacta implies convergence almost everywhere of a subsequence.

## 4. Concluding Remarks

1. Our main theorem shows that, for each $t$, $u(t, x)$ belongs to the class $\mathfrak{F}$. Therefore, from a theorem of Krickeberg [2], the surface formed from the solution, for each fixed $t$, has finite Lebesgue area. However, at the present time we are unable to analyze the nature of the discontinuities of our solution. In particular, it would be of interest to know whether discontinuities other than shocks can occur.

2. It is known [1], that weak solutions of (1.1) are not uniquely determined by their initial data. In the case $n = 1$, Oleinik [4] has shown that weak solutions satisfying an additional condition ("entropy condition") are uniquely determined by their initial data. In the general case, $n > 1$, the question of uniqueness is still unresolved.

## Addendum[1]

Shortly after this paper was accepted for publication we obtained an additional result concerning the regularity of the solution constructed in Theorem 1. In the proof of that theorem we showed that the solution $u$ was, for each fixed $t$, a function of locally bounded variation (in the sense of Tonelli-Cesari) in the space variables. In other words, the derivatives (in the sense of the theory of distributions) with respect to the space variables, are locally finite measures in $R_n$ for each fixed $t$. However, we have made no statement concerning the regularity of the solution with respect to $t$. In this note we shall show that the solution is in fact a function of locally bounded variation (Tonelli-Cesari) in both the space and time variables; i.e., in the sense of the theory of distributions, the derivatives $u_t$ and $u_{x_i}$ are locally finite measures in the subset of $R_{n+1}$ defined by $t \geqq 0$.

To prove this statement (we again give the details only for the case $n = 2$), we observe that the functions $U^i(t, x, y)$ defined in Section 3 are clearly functions of locally bounded variation in all three variables. The variation over the closed set $K$ defined by $0 \leqq t_1 \leqq t \leqq t_2$ and $(x, y) \in X$, is given by

$$\operatorname*{Var}_K (U^i) = \iiint_K (|U_t^i| + |U_x^i| + |U_y^i|)\, dt\, dx\, dy$$

$$= \sum_K \{pq\, |u_{n,m}^{k+1} - u_{n,m}^k| + ph\, |u_{n+1,m}^k - u_{n,m}^k| + qh\, |u_{n,m+1}^k - u_{n,m}^k|\},$$

[1] This section was added in August, 1965.

where $\sum_K$ means summation over all indices $k, n, m$ such that the point $(kh, nq, mp)$ is in $K$. (Of course, the derivatives are taken in the sense of the theory of distributions.) Now from (2.10), we have

$$\sum_K pq\,|u_{n,m}^{k+1} - u_{m,n}^{k}| = \sum_{t_1 \leq kh \leq t_2} \sum_X pq\,|u_{n,m}^{k+1} - u_{n,m}^{k}| \leqq c(t_2 - t_1)\,,$$

while from Lemmas 2 and 5 we have

$$\begin{aligned}\sum_K &\{ph\,|u_{n+1,m}^{k} - u_{n,m}^{k}| + qh\,|u_{n,m+1}^{k} - u_{n,m}^{k}|\} \\ &= \sum_{t_1 \leq kh \leq t_2} h \sum_X \{p\,|u_{n+1,m}^{k} - u_{n,m}^{k}| + q\,|u_{n,m+1} - u_{n,m}^{k}|\} \\ &\leqq \sum_{t_1 \leq kh \leq t_2} hc' = c'(t_2 - t_1)\,.\end{aligned}$$

Here $c$ and $c'$ are independent of the mesh size and depend only on the initial data. We thus see that the total variation of the functions $U^i$ over the set $K$ is uniformly bounded in $i$. Therefore by the De Giorgi-Fleming criterion [6], it follows that the sequence $\{U^i\}$ is compact in the topology of $L_1$ convergence on compact subsets of $t \geqq 0$. Consequently the limit function obtained in Theorem 1 is seen to be of locally bounded variation in the half-space $t \geqq 0$.

In view of these remarks, Theorem 1 can be amended so as to read as follows:

THEOREM. *Let $f_i$, $i = 1, 2, \cdots, n$, be continuously differentiable functions of a single real variable and let $u_0(x_1, \cdots, x_n) \in \mathfrak{F}$. Then there exists a weak solution of the initial value problem* (1.1). *This solution is of locally bounded variation* (*in the sense of Tonelli-Cesari*) *in the half-space $t \geqq 0$, and is in $\mathfrak{F}$ for each fixed $t \geqq 0$. Moreover the solution has the same upper and lower bounds as $u_0$.*

*Acknowledgement.* We would like to thank Casper Goffman for bringing the works of Fleming, de Giorgi, and Krickeberg to our attention. We would also like to thank George Minty for a helpful technical suggestion.

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Received May, 1965.