Global Solutions to the Coupled Chemotaxis-Fluid Equations

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1. Introduction

1.1 Consider the coupled chemotaxis-Navier-Stokes eqns:

$$\begin{cases} \partial_t n + u\cdot\nabla n = \delta\Delta n - \nabla\cdot(\chi(c)n\nabla c), \\ \partial_t c + u\cdot\nabla c = \mu\Delta c - k(c)n, \\ \partial_t u + u\cdot\nabla u + \nabla P = \nu\Delta u - n\nabla\phi, \\ \nabla\cdot u = 0, \quad t>0, x\in\mathbb{R}^3, \end{cases} \qquad (cNS)$$

with initial data

$$(n,c,u)|_{t=0} = (n_0(x), c_0(x), u_0(x)), \quad x\in\mathbb{R}^3,$$

where

- **unknowns:**

chemotaxis variables — $n = n(t,x) \geq 0$ **(cell density)**
$c = c(t,x) \geq 0$ **(substrate concentration)**

fluid variables — $u = u(t,x) \in \mathbb{R}^3$ **(velocity)**
$P = P(t,x) \in \mathbb{R}$ **(pressure)**

- **given:**

constant coefficients — $\delta > 0$ **(cells diffusion)**
$\mu > 0$ **(substrate diffusion)**
$\nu > 0$ **(viscosity)**

variable coefficients — $\chi(c)$ **(chemotactic sensitivity)**
$k(c)$ **(consumption rate)**

potential function — $\phi = \phi(t, x)$ **(external forcing)**

- **basic assumptions (A1):**

$$(i)\ n_0(x) \geq 0,\ \ c_0(x) \geq 0,\ \ \nabla \cdot u_0(x) = 0;$$
$$(ii)\ k(0) = 0,\ \ k'(c) \geq 0.$$

Remarks:

a) $-n\nabla\phi$ **is the force exerted on the fluid by cells.**

b) **(ii) implies that** $k(c) \geq 0$ **holds for** $c \geq 0$**, that is the case of consumption of chemical substrates.**

1.2 Our interest lies in

- the existence of the free energy functional;
- the global well-posedness of the Cauchy problem on (cNS);
- the large-time behavior of solutions and convergence rates.

So far, we can answer that

- any constant steady state $(n, c, u) \equiv (n_\infty \geq 0, 0, 0)$ is asymptotically stable under small perturbations, and the rate of trend to equilibrium can be obtained;
- $\exists$ temporal *free* energy functionals $\mathcal{E}(n(t), c(t), u(t))$ and corresponding dissipation rate $\mathcal{D}(n(t), c(t), u(t))$ s.t.

$$\frac{d}{dt}\mathcal{E}(n(t), c(t), u(t)) = -\mathcal{D}(n(t), c(t), u(t)) \leq 0,$$

  provided that
  - the potential forcing is weak, or
  - the substrate concentration is small.

1.3 Background of the model system:

- Bacteria live in thin fluid layers near solid-air-water contact lines
- Chemotactic Boycott effect in sedimentation:
  - Bacteria swim up to the free surface between water and air (chemotaxis), and slide down the bottom;
  - high concentrations of Bacteria are produced at two contact lines, and the oxygen in water is consumed;
  - Bacteria at upper contact line slide down due to gravitational forcing;
  - a vortex is formed in the water due to incompressibility
- This mathematical model consists of
  - diffusion, chemotaxis and transport for bacteria;
  - diffusion, consumption and transport for Oxygen;
  - viscosity and incompressibility for fluid

1.4 Formulation of the boundary conditions: Let Ω be a bounded domain with smooth bdries. Then, on $\partial\Omega$,

$$\left(\frac{\partial n}{\partial \nu} - \chi(x) n \frac{\partial c}{\partial \nu}\right)\bigg|_{\partial\Omega} = 0 \text{ (\textbf{no-flux on} } n),$$

$$c|_{\partial\Omega} = 0 \text{ (\textbf{Dirichlet}) \textbf{or} } \frac{\partial c}{\partial \nu}\bigg|_{\partial\Omega} = 0 \text{ (\textbf{Neumann})},$$

$$u|_{\partial\Omega} = 0 \text{ (\textbf{Dirichlet})}.$$

For the semi-dimension problem:

$$\nabla\phi = g\mathbf{e}_d,$$

c **takes the mixed boundary conditions:**

$$c|_{\Gamma_+} = c_+ > 0 \text{ \textbf{Dirichlet on the upper bdy}},$$

$$\frac{\partial c}{\partial \nu}\bigg|_{\Gamma_-} = 0 \text{ \textbf{no-flux on the lower bdy}},$$

where

$$\Gamma_+ = \{x \in \partial\Omega : \mathbf{e}_d \cdot \nu(x) > 0\},$$

$$\Gamma_- = \{x \in \partial\Omega : \mathbf{e}_d \cdot \nu(x) < 0\}.$$

1.5 Related results:

- **Chemotaxis for the angiogenesis system:**

$$\partial_t n = \Delta n - \nabla \cdot (\chi n \nabla c),$$
$$\partial_t c = -c^m n, \quad t > 0, x \in \Omega,$$
$$(n, c)(0, x) = (n_0, c_0)(x), \; x \in \Omega \subseteq \mathbb{R}^d.$$

Rascle, Fontelos-Friedman-Hu, Guarguaglini-Natalini, Corrias-Perthame-Zagg, ...

- **Kinetic-fluid-coupled model:**

kinetic equation: Vlasov-type

+

fluid dynamic equations: NS or Euler (C or IC)

Caflish-Papanicolaou, Hamdache, Jabin, Goudon, Carrillo-Goudon, Mellet-Vasseur, ...

1.5 Related results (cont.):

- **Keller-Segel model (substrate is also produced by cells):**

$$\partial_t n = \Delta n - \nabla \cdot (\chi n \nabla c),$$
$$\partial_t c = \Delta c - c + n.$$

(recent progress only)

- **Chalub-Markowich-Perthame-Schmeiser:** the model was justified as a diffusion limit of a kinetic model
- **Blanchet-Dolbeault-Perthame, Blanchet-Carrillo-Masmoudi:** Parabolic-elliptic in $\mathbb{R}^2$
- **Calvez-Corrias:** Parabolic-parabolic in $\mathbb{R}^2$
- ...

2. Free energy functionals

2.1 To expose the idea, consider

$$\begin{cases} \partial_t n = \delta \Delta n - \nabla \cdot (\chi(c) n \nabla c), \\ \partial_t c = \mu \Delta c - k(c) n, \quad t > 0, x \in \mathbb{R}^d, \end{cases}$$

where the fluid component was ignored. Define

$$\mathcal{E}(n(t), c(t)) = \int_{\mathbb{R}^d} n \ln n \, dx + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \Psi(c)|^2 dx,$$

with

$$\Psi(c) = \int_0^c \left(\frac{\chi(s)}{k(s)} \right)^{1/2} ds.$$

Then, one has

Proposition (identity I)

$$\frac{d}{dt} \mathcal{E}(n(t), c(t)) = -\mathcal{D}(n(t), c(t)),$$

where the dissipation rate $\mathcal{D}(n(t), c(t))$ is given by

$$\mathcal{D}(n(t),c(t)) = \delta \int_{\mathbb{R}^d} \frac{|\nabla n|^2}{n} dx + \int_{\mathbb{R}^d} \frac{\chi'(c)k(c) + \chi(c)k'(c)}{2\chi(c)} n|\nabla \Psi|^2 dx$$

$$+\mu \int_{\mathbb{R}^d} \left| \nabla^2 \Psi - \frac{d}{dc}\sqrt{\frac{k(c)}{\chi(c)}} \nabla \Psi \otimes \nabla \Psi \right|^2 dx$$

$$-\frac{\mu}{2} \int_{\mathbb{R}^d} \frac{d^2}{dc^2}\left(\frac{k(c)}{\chi(c)}\right) |\nabla \Psi|^4 dx.$$

Moreover,

$$\mathcal{D}(n(t),c(t)) \geq 0$$

holds provided that

$$\chi(c) > 0, \ \ \frac{d}{dc}(\chi(c)k(c)) > 0, \ \ \frac{d^2}{dc^2}\left(\frac{k(c)}{\chi(c)}\right) < 0. \qquad \textbf{(A2)}$$

Proof of Proposition: it follows from the direct calculations.■

Remarks:

a) Identity I is inspired by Tupchiev-Fomina (CMMP '04) for the study of the two-dimensional case, where some inequalities were derived.

b) A typical example for $\chi(c)$ and $k(c)$ satisfying the above condition is

$$\chi(c) = \chi_0 c^{-\alpha}, \quad k(c) = k_0 c^m$$

with constants $\chi_0 > 0$, $k_0 > 0$ and

$$0 < m < 1, \quad 0 \leq \alpha < \min\{m, 1 - m\}.$$

c) When the transportation occurs, i.e.,

$$\begin{cases} \partial_t n + u \cdot \nabla n = \delta \Delta n - \nabla \cdot (\chi(c) n \nabla c), \\ \partial_t c + u \cdot \nabla c = \mu \Delta c - k(c) n, \quad t > 0, x \in \mathbb{R}^d, \end{cases}$$

one has

$$\frac{d}{dt}\mathcal{E}(n(t), c(t)) + \mathcal{D}(n(t), c(t)) = -\sum_{ij} \int_{\mathbb{R}^d} \partial_i u_j \partial_i \Psi \partial_j \Psi dx.$$

2.2 Consider the (cNS), that is the coupled chemotaxis-Navier-Stokes, with

$$\phi = \phi(x)$$

independent of time t. Then one has

Proposition (identity II)

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left(n\phi + \frac{1}{2}|u|^2\right)dx + \nu\int_{\mathbb{R}^d}|\nabla u|^2 dx$$
$$= \delta\int_{\mathbb{R}^d} n\Delta\phi dx + \int_{\mathbb{R}^d}\sqrt{k(c)\chi(c)}\, n\nabla\Psi\cdot\nabla\phi dx.$$

Proof of Proposition: it follows from the integration by parts and replacing $\phi u\cdot\nabla n$ by the eqn of n. ■

Remark. The r.h.s terms of identities I and II can be controlled provided that

- **ϕ is small in some sense, or**
- **ϕ is bounded in some sense and c is small in L^∞.**

3. Global existence of weak solutions

3.1 Consider the simplified model system of the coupled chemotaxis-Stokes equations:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \delta \Delta n - \nabla \cdot (\chi(c) n \nabla c), \\ \partial_t c + u \cdot \nabla c = \mu \Delta c - k(c) n, \\ \partial_t u + \nabla P = \nu \Delta u - n \nabla \phi, \\ \nabla \cdot u = 0, \quad t > 0, x \in \mathbb{R}^3, \end{cases} \tag{cS}$$

with

$$(n, c, u)|_{t=0} = (n_0(x), c_0(x), u_0(x)), \quad x \in \mathbb{R}^3.$$

Remark. The nonlinear convective term $u \cdot \nabla u$ may produce the new difficulty for regularity of u.

Theorem I *Let assumptions* **A1**- **A2** *hold and let* $\phi = \phi(x) \geq 0$ *be independent of t with* $\nabla\phi \in L^\infty(\mathbb{R}^3)$*. Suppose that*

$$n_0(|\ln n_0| + \langle x\rangle + \phi(x)) \in L^1(\mathbb{R}^3),$$
$$c_0 \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3), \quad \nabla\Psi(c_0) \in L^2(\mathbb{R}^3),$$
$$u_0 \in L^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$$

where $\langle x\rangle := \sqrt{1+x^2}$*. Then,* $\exists\, \epsilon_\phi > 0$*, depending only on* δ*,* μ*,* ν*,* $\|c_0\|_{L^\infty}$*, s.t. if*

$$\sup_x |x||\nabla\phi(x)| + \sup_x |x|^2|\Delta\phi(x)| \leq \epsilon_\phi,$$

the Cauchy problem of (cS) has a global-in-time weak solution (n, c, u)*, satisfying that*

$$n(t,x) \geq 0, \quad \sup_{t\geq 0} \|n(t)\|_{L^1} \leq \|n_0\|_{L^1},$$
$$c(t,x) \geq 0, \quad \sup_{t\geq 0} \|c(t)\|_{L^p} \leq \|c_0\|_{L^p}, \quad \textit{for any } 1 \leq p \leq \infty,$$

and

$$\mathcal{E}_1(t) + \int_0^t \mathcal{D}_1(s)ds \leq \mathcal{E}_1(0), \;\; \textit{for any } t \geq 0,$$

where the free energy $\mathcal{E}_1(t)$ *and its dissipation rate* $\mathcal{D}_1(t)$ *are given by*

$$\mathcal{E}_1(t) = \int_{\mathbb{R}^3} \left(n \ln n + \frac{1}{2}|\nabla \Psi(c)|^2 + \frac{1}{\lambda_1 \mu \nu} n\phi + \frac{1}{2\lambda_1 \mu \nu}|u|^2 \right) dx,$$

$$\mathcal{D}_1(t) = \frac{\delta}{2}\int_{\mathbb{R}^3} \frac{|\nabla n|^2}{n} dx + \frac{\lambda_0}{2}\int_{\mathbb{R}^3} n|\nabla \Psi|^2 dx + \frac{\lambda_1 \mu}{2}\int_{\mathbb{R}^3} |\nabla \Psi|^4 dx$$
$$+ \frac{1}{2\lambda_1 \mu}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \mu \sum_{ij} \int_{\mathbb{R}^3} \left| \partial_i \partial_j \Psi - \frac{d}{dc}\sqrt{\frac{k(c)}{\chi(c)}} \partial_i \Psi \partial_j \Psi \right|^2 dx,$$

for constants λ_0 *and* λ_1 *depending only* $\|c_0\|_\infty$*, and moreover, for any* $T > 0$*,*

$$n(|\ln n| + \langle x \rangle) \in L^\infty([0,T], L^1(\mathbb{R}^3)), \;\; u \in L^\infty([0,T] \times \mathbb{R}^3).$$

Theorem II *Let the assumption* **A1** *and also* $k'(c) > 0$ *hold, and let* $\phi = \phi(x) \geq 0$ *be independent of* t *with*

$$\sup_x (1 + |x|)|\nabla\phi(x)| + \sup_x |x|^2|\Delta\phi(x)| < \infty.$$

Suppose that

$$n_0(|\ln n_0| + \langle x\rangle + \phi(x)) \in L^1(\mathbb{R}^3),$$
$$c_0 \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3), \;\; u_0 \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3).$$

Then, $\exists\, c_* > 0$*, depending only on* δ*,* μ*,* ν *and* ϕ*, s.t. if*

$$\|c_0\|_{L^\infty} \leq c_*,$$

the Cauchy problem of (cS) has a global-in-time weak solution (n, c, u)*, satisfying that*

$$n(t,x) \geq 0, \;\; \sup_{t\geq 0} \|n(t)\|_{L^1} \leq \|n_0\|_{L^1},$$
$$c(t,x) \geq 0, \;\; \sup_{t\geq 0} \|c(t)\|_{L^p} \leq \|c_0\|_{L^p}, \;\; \textit{for any } 2 \leq p \leq \infty,$$

and

$$\mathcal{E}_2(t) + \lambda \int_0^t \mathcal{D}_2(s)ds \le C(\|n_0 \ln n_0\|_{L^1} + \|c_0\|_{H^1}^2 + \|u_0\|_{L^2}^2),$$

for any $t \ge 0$*, where the free energy* $\mathcal{E}_2(t)$ *and its dissipation rate* $\mathcal{D}_2(t)$ *are given by*

$$\mathcal{E}_2(t) = \int_{\mathbb{R}^3} n(\ln n + \lambda\phi)dx + \lambda(\|c\|_{H^1}^2 + \|u\|^2)$$
$$\mathcal{D}_2(t) = \|\nabla\sqrt{n}\|^2 + \|\nabla c\|_{H^1}^2 + \|\sqrt{n}c\|^2 + \|\sqrt{n}\nabla c\|^2 + \|\nabla u\|^2,$$

and $\lambda > 0$ *is a small constant, and moreover, for any* $T > 0$*,*

$$n(|\ln n| + \langle x\rangle) \in L^\infty([0,T], L^1(\mathbb{R}^3)), \;\; u \in L^\infty([0,T]\times\mathbb{R}^3).$$

3.2 Proof of Theorem I: (uniform a priori estimates)

a) From

$$\partial_t n + \nabla \cdot (\delta \nabla n + n(u + \chi(c)\nabla c)) = 0,$$
$$\partial_t c + u \cdot \nabla c = \mu \Delta c - k'(\xi) n c,$$

where $\xi = \xi(t,x)$ **is between** 0 **and** $c(t,x)$**, by the assumption (A1), the maximum principle implies**

$$n(t,x) \geq 0, \quad 0 \leq c(t,x) \leq \|c\|_{L^\infty} = c_M$$

for any $0 \leq t \leq T$**,** $x \in \mathbb{R}^3$**.**

b) Proof of the energy inequality: Denote $c_M = \|c_0\|_{L^\infty}$**, and define**

$$\lambda_0 = \min_{0 \leq c \leq c_M} \frac{\chi'(c)k(c) + \chi(c)k'(c)}{2\chi(c)} > 0,$$
$$\lambda_1 = \min_{0 \leq c \leq c_M} -\frac{1}{2}\frac{d^2}{dc^2}\left(\frac{k(c)}{\chi(c)}\right) > 0,$$

by assumptions A1 and A2.

The r.h.s. of identity I is bounded by

$$\frac{1}{2\lambda_1\mu}\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{\lambda_1\mu}{2}\int_{\mathbb{R}^3}|\nabla\Psi|^4dx.$$

The r.h.s. of identity II is bounded by

$$\delta\epsilon\int_{\mathbb{R}^3}\left|\frac{\sqrt{n}}{|x|}\right|^2dx+\epsilon\left(\sup_{0\le c\le c_M}k(c)\chi(c)\right)^{1/2}\int_{\mathbb{R}^3}\frac{\sqrt{n}}{|x|}\cdot\sqrt{n}|\nabla\Psi|dx$$
$$\le\left(\delta\epsilon+\frac{\sup_{0\le c\le c_M}k(c)\chi(c)}{2\lambda_0\lambda_1\mu\nu}\epsilon^2\right)\int_{\mathbb{R}^3}\frac{|\nabla n|^2}{n}dx+\frac{\lambda_0\lambda_1\mu\nu}{2}\int_{\mathbb{R}^3}n|\nabla\Psi|^2dx,$$

where one used the Hardy inequality

$$\int_{\mathbb{R}^3}\left|\frac{\sqrt{n}}{|x|}\right|^2dx\le 4\int_{\mathbb{R}^3}|\nabla\sqrt{n}|^2dx=\int_{\mathbb{R}^3}\frac{|\nabla n|^2}{n}dx,$$

and

$$\sup_x|x||\nabla\phi(x)|+\sup_x|x|^2|\Delta\phi(x)|\le\epsilon.$$

Then, $\epsilon>0$ is small $\Rightarrow\cdots$

c) Estimates on moments and $\|u\|_{L^\infty}$: Eqn of n $\Rightarrow$

$$\int_{\mathbb{R}^3} \langle x \rangle n(t,x)dx \le C\delta \|n_0\|_{L^1} T + C\|n_0\|_{L^1} T \sup_{0\le t\le T} \|u(t)\|_{L^\infty} + \frac{1}{\lambda_0}\left(\sup_{0\le c\le c_M} \chi(c)\right)\int_0^T \|k(c)n\|_{L^1} ds + \frac{\lambda_0}{4}\int_0^T \|\sqrt{n}\nabla\Psi\|^2 ds,$$

$+$ **Eqn of c $\Rightarrow$**

$$\|c(t)\|_{L^1} + \int_0^t \|k(c)n\|_{L^1} ds \le \|c_0\|_{L^1},$$

$+$ **Eqn of u $\Rightarrow$**

$$\|u(t)\|_{L^\infty} \le C\|u_0\|_{L^\infty} + C\|\nabla\phi\|_{L^\infty}\int_0^t \sqrt{t-s}\|\nabla\sqrt{n}\|^2 ds.$$

Then,

$$\int_{\mathbb{R}^3} \langle x\rangle n(t,x)dx \leq \frac{\|c_0\|_{L^1}}{\lambda_0} \sup_{0\leq c\leq c_M} \chi(c) + C(\delta + \|u_0\|_{L^\infty})\|n_0\|_{L^1} T$$
$$+C\|n_0\|_{L^1} T^{3/2} \|\nabla\phi\|_{L^\infty} \int_0^T \|\nabla\sqrt{n}\|^2 ds$$
$$+\frac{\lambda_0}{4}\int_0^T \|\sqrt{n}\nabla\Psi\|^2 ds.$$

Take the linear combination with the energy inequality $\Rightarrow$

$$\sup_{0\leq t\leq T_0} \mathcal{E}_1^+(t) + \frac{1}{2}\int_0^{T_0} D_1(s)ds$$
$$\leq \mathcal{E}_1(0) + C + \frac{\|c_0\|_{L^1}}{\lambda_0} \sup_{0\leq c\leq c_M} \chi(c) + C(\delta + \|u_0\|_{L^\infty})\|n_0\|_{L^1} T_0,$$

for some small $T_0 > 0$**, where**

$$\mathcal{E}_1^+(t) = \int_{\mathbb{R}^3} n\ln n \chi_{n\geq 1} dx + \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla\Psi(c)|^2 + \frac{1}{\lambda_1\nu} n\phi + \frac{1}{2\lambda_1\nu}|u|^2\right) dx.$$

Apply to intervals $[0,T_0],[T_0,2T_0],\cdots,[mT_0,T] \Rightarrow \cdots$ ■

3.3 Proof of Theorem II: (uniform a priori estimates)

a) c_M is small. The direct energy estimates $\Rightarrow$

$$\frac{d}{dt}\int_{\mathbb{R}^3}(n\ln n+\lambda_2|\nabla c|^2+|c|^2)dx+\mu\min\{\lambda_2,2\}\int_{\mathbb{R}^3}(|\nabla c|^2+|\nabla^2 c|^2)dx$$
$$+\min\{1,2\min_{0\le c\le c_M}k'(c)\}\int_{\mathbb{R}^3}n(|c|^2+|\nabla c|^2)dx+\frac{\delta}{4}\int_{\mathbb{R}^3}\frac{|\nabla n|^2}{n}dx$$
$$\le\frac{\lambda_2 c_M^2}{\mu}\int_{\mathbb{R}^3}|\nabla u|^2dx.$$

b) Identity II gives

$$\frac{d}{dt}\int_{\mathbb{R}^3}\left(n\phi+\frac{1}{2}|u|^2\right)dx+\nu\int_{\mathbb{R}^3}|\nabla u|^2dx$$
$$\le\delta\sup_x|x|^2|\Delta\phi(x)|\int_{\mathbb{R}^3}\frac{|\nabla n|^2}{n}dx$$
$$+\frac{1}{2}\sup_x|x||\nabla\phi(x)|\sup_{0\le c\le c_M}|\chi(c)|\left(\int_{\mathbb{R}^3}\frac{|\nabla n|^2}{n}dx+\int_{\mathbb{R}^3}n|\nabla c|^2dx\right).$$

c) Smallness of c_M + linear combination of a) and b) $\Rightarrow\cdots$ ■

3.4 Remarks:

a) It is unknown that Theorems I and II could still hold for one of the following three cases:

- **the smallness of both ϕ and $\|c_0\|_{L^\infty}$ is removed;**
- **the nonlinear convective term $\nabla \cdot (u \otimes u)$ is added;**
- **both $\chi(c)$ and $k(c)$ take the more general forms.**

b) Similar results hold for the case of

- **the space dimension $d \geq 2$, or**
- **the bounded domain with homogeneous boundary conditions**

$$\frac{\partial n}{\partial \nu}\bigg|_{\partial\Omega} = \frac{\partial c}{\partial \nu}\bigg|_{\partial\Omega} = 0, \quad u|_{\partial\Omega} = 0.$$

However, it is not clear for the general biological non-homogeneous bdry conditions.

4. Classical solutions near constant states

4.1 Consider

$$\begin{cases} \partial_t n + u \cdot \nabla n = \delta \Delta n - \nabla \cdot (\chi(c) n \nabla c), \\ \partial_t c + u \cdot \nabla c = \mu \Delta c - k(c) n, \\ \partial_t u + u \cdot \nabla u + \nabla P = \nu \Delta u - n \nabla \phi, \\ \nabla \cdot u = 0, \quad t > 0, x \in \mathbb{R}^3, \end{cases} \tag{cNS}$$

with initial data

$$(n, c, u)|_{t=0} = (n_0(x), c_0(x), u_0(x)), \quad x \in \mathbb{R}^3.$$

Suppose

$$(n_0(x), c_0(x), u_0(x)) \to (n_\infty \geq 0, 0, 0) \text{ as } |x| \to \infty.$$

Our goal is to prove

> **the constant steady state $(n_\infty, 0, 0)$ is asymptotically stable under small smooth perturbations.**

4.2 Reformulation of the Cauchy problem: Let

$$n = \sigma + n_\infty, \quad \bar{P} = P + n_\infty \phi.$$

Then,

$$\begin{cases} \partial_t \sigma + u \cdot \nabla \sigma - \delta \Delta \sigma = -\nabla \cdot (\chi(c)\sigma \nabla c) - n_\infty \nabla \cdot (\chi(c)\nabla c), \\ \partial_t c + u \cdot \nabla c - \mu \Delta c + k'(0)(\sigma + n_\infty)c = -(k(c) - k'(0)c)(\sigma + n_\infty), \\ \partial_t u + u \cdot \nabla u + \nabla \bar{P} - \nu \Delta u = -\sigma \nabla \phi, \\ \nabla \cdot u = 0, \quad t > 0, x \in \mathbb{R}^3, \end{cases}$$

with

$$(\sigma, c, u)|_{t=0} = (\sigma_0(x), c_0(x), u_0(x)) \to (0,0,0) \text{ as } |x| \to \infty,$$

where $\sigma_0 = n_0 - n_\infty$**.**

Theorem III. *Let $n_\infty \geq 0$, and the assumption* **(A1)** *hold with $n_0(x) \equiv \sigma_0(x) + n_\infty \geq 0$ for $x \in \mathbb{R}^3$, and $\phi = \phi(t,x)$ satisfy*

$$\sup_{t,x}(1+|x|)|\phi(t,x)| + \sum_{1\leq|\alpha|\leq 3} \sup_{t,x} |\partial_x^\alpha \phi(t,x)| < \infty.$$

Furthermore, suppose that $\|(\sigma_0, c_0, u_0)\|_{H^3}$ is sufficiently small. Then the Cauchy problem of (cNS) admits a unique classical solution (σ, c, u) with

$$n(t,x) \equiv \sigma(t,x) + n_\infty \geq 0, \; c(t,x) \geq 0$$

for $t \geq 0$, $x \in \mathbb{R}^3$, such that

$$\|(\sigma,c,u)(t)\|_{H^3}^2 + \lambda \int_0^t \int_{\mathbb{R}^3} (\sigma + n_\infty) \left[k(c)c + k'(0) \sum_{1\leq|\alpha|\leq 3} |\partial_x^\alpha c(s)|^2 \right] dxds$$

$$+\lambda \int_0^t \|\nabla(\sigma,c,u)(s)\|_{H^3}^2 ds \leq C\|(n_0,c_0,u_0)\|_{H^3}^2$$

hold for some constants $\lambda > 0$, C and for any $t \geq 0$.

4.3 Proof of Theorem III: (uniform a priori estimates)

a) The maximum principle $\Rightarrow$

$$\sigma + n_\infty = n(t,x) \geq 0, \quad 0 \leq c(t,x) \leq \|c\|_{L^\infty}.$$

b) Energy estimates under smallness: (Zero-order)

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \left(|u|^2 + d_2\sigma^2 + d_1 d_2 c^2\right) dx + \frac{\nu}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{d_2\delta}{4}\int_{\mathbb{R}^3} |\nabla \sigma|^2 dx$$
$$+ \frac{d_1 d_2 \mu}{2}\int_{\mathbb{R}^3} |\nabla c|^2 dx + d_1 d_2 \int_{\mathbb{R}^3} k(c)c(\sigma + n_\infty) dx \leq 0,$$

+ (high-order)

$$\frac{1}{2}\frac{d}{dt}\sum_{1\leq|\alpha|\leq 3} C_\alpha \int_{\mathbb{R}^3} |\partial_x^\alpha(\sigma, c, u)|^2 dx + \lambda \sum_{2\leq|\alpha|\leq 4}\int_{\mathbb{R}^3} |\partial_x^\alpha(\sigma, c, u)|^2 dx$$
$$+\lambda k'(0)\sum_{1\leq|\alpha|\leq 3}\int_{\mathbb{R}^3}(\sigma + n_\infty)|\partial_x^\alpha c|^2 dx \leq C\int_{\mathbb{R}^3} |\nabla(\sigma, c, u)|^2 dx,$$

(linear combination) $\Rightarrow$ uniform a priori estimates. ■

4.4 Convergence rates: There are three cases:

$$n_\infty = 0;\ n_\infty > 0, k'(0) = 0;\ n_\infty k'(0) > 0.$$

Theorem IV. *Let $n_\infty = 0$, and all conditions in Theorem III hold.*

(i) Assume $\sigma_0, c_0 \in L^1(\mathbb{R}^3)$. Then, for any $1 \le p < \infty$,

$$\|\sigma(t)\|_{L^p} \le C\|\sigma_0\|_{L^1\cap L^p}(1+t)^{-\frac{3}{2}(1-\frac{1}{p})},$$
$$\|c(t)\|\|_{L^p} \le C\|c_0\|_{L^1\cap L^p}(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}.$$

(ii) Furthermore, assume that $u_0 \in L^q(\mathbb{R}^3)$ and

$$\phi \in L^\infty(\mathbb{R}^+; L^{2q/(2-q)}(\mathbb{R}^3))$$

for $1 < q < 6/5$. Then,

$$\|u(t)\| \le C(\|u_0\|_{L^q\cap H^3} + K_0)(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})},$$

for any $t \ge 0$, where K_0 is defined by

$$K_0 = \|(\sigma_0, c_0)\|_{L^1\cap H^3} + \|\sigma_0\|_{L^1\cap L^2}\|c_0\|_{L^1\cap L^2}.$$

4.5 Proof of Theorem IV:

Energy-spectrum method (D.-Ukai-Yang '09)

a) Time-decay of c and n:

$$\frac{d}{dt}\int_{\mathbb{R}^3} c^p dx + \frac{4\mu(p-1)}{p}\int_{\mathbb{R}^3} |\nabla c^{p/2}|^2 dx \leq 0,$$

+

$$\|c(t)\|_{L^1} \leq \|c_0\|_{L^1}$$

(standard argument: interpolation inequality

$$\|f\|_{L^p} \leq C\|\nabla |f|^{p/2}\|^{\frac{2\gamma_{p,q}}{1+p\gamma_{p,q}}} \|f\|_{L^q}^{\frac{1}{1+p\gamma_{p,q}}}$$

with

$$\gamma_{p,q} = \frac{3}{2}\left(\frac{1}{q} - \frac{1}{p}\right)$$

+ Young inequality) $\Rightarrow$ time-decay of c, and similarly for n.

b) Time-decay of u: (Three steps)

Step 1. **Time-decay of high-order derivatives of (σ, c): Use the high-order energy inequality**

$$\frac{d}{dt}\|\nabla(\sigma, c)\|_{H^2}^2 + \lambda\|\nabla(\sigma, c)\|_{H^3}^2 \leq C\|\nabla(\sigma, c)\|^2,$$

+ Use the mild forms of σ, c to obtain

$$\begin{aligned}
\|\nabla\sigma(t)\| &\leq C\|\sigma_0\|_{L^p\cap H^1}(1+t)^{-\gamma_{2,p}-1/2} \\
&\quad + C\epsilon\int_0^t (1+t-s)^{-5/4}\|\nabla(\sigma, c)(s)\|_{H^1}ds, \\
\|\nabla c(t)\| &\leq C\|c_0\|_{L^1\cap H^1}(1+t)^{-\gamma_{2,p}-1/2} \\
&\quad + C\epsilon\int_0^t (1+t-s)^{-5/4}\|\nabla c(s)\|_{H^1}ds \\
&\quad + C(1+t)^{-5/4}\|c_0\|_{L^1\cap L^2}\|\sigma_0\|_{L^1\cap L^2}
\end{aligned}$$

(Gronwall inequality) $\Rightarrow$

$$\|\nabla(\sigma, c)(t)\|_{H^2} \leq C(\|\nabla(\sigma_0, c_0)\|_{H^2} + K_p)(1+t)^{-\gamma_{2,p}-1/2},$$

for any $1 \leq p \leq 2$.

Step 2. **Time-decay of high-order derivatives of u: Use the mild form**

$$u(t) = e^{\nu\Delta t}u_0 + \int_0^t e^{\nu\Delta(t-s)}(-\mathbf{P}(u\cdot\nabla u) + \mathbf{P}(\phi\nabla\sigma))ds,$$

with

$$\mathbf{P}(\sigma\nabla\phi) = -\mathbf{P}(\phi\nabla\sigma).$$

(Energy-spectrum method again + Riesz inequality) $\Rightarrow$

$$\|\nabla(\sigma, c, u)(t)\|_{H^2} \le C(\|u_0\|_{L^p\cap H^3} + K_0)(1+t)^{-\gamma_{2,p}-1/2},$$

for $1 < p \le 2$**.**

Step 3. **Time-decay of $\|u\|$: Again use the mild form and time-decay of high-order derivatives of u to get**

$$\|u(t)\| \le C(\|u_0\|_{L^q\cap H^3} + K_0)(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})},$$

for $1 < q < 6/5$**, where** $\gamma_{2,q} + 1/2 > 1$ **was used.** $\blacksquare$

5. Final remarks

Consider the more realistic mathematical model:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \delta \Delta n - \nabla \cdot [\chi(c) n (\nabla c + \nabla \phi)], \\ \partial_t c + u \cdot \nabla c = \mu \Delta c - k(c) n, \\ \partial_t u + u \cdot \nabla u + \nabla P = \nu \Delta u - n[\nabla \phi + \nabla k(c)], \\ \nabla \cdot u = 0, \quad t > 0, x \in \mathbb{R}^3. \end{cases}$$

Here

- $\nabla \phi$ **exhibit the effect of gravity on cells, and**
- $\nabla k(c)$ **exhibit the effect of the chemotactic force in the fluid equation.**

Claim: *Theorem II still holds if the smallness of both* ϕ *and* $\|c_0\|_{L^\infty}$ *is supposed. Theorems III and IV also hold.*

Remark. It is the on-going work to extend the current results to the above realistic model.

Thanks for your attention!