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Global Spectral Characterization of Chaotic Dynamics

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We propose a new global characterization of chaotic dynamics. It gives a unified description of some aspects of chaos (e.g., the relationships among dimensions of invariant measures, Lyapunov exponents, and entropies; strange attractor vs repeller; and the variational principle). We apply this formalism to a few examples employing a practical algorithm.

Recent efforts to characterize complex objects, such as strange attractor of nonlinear dynamical systems or fractal sets arising in many fields of science, reached a concept of spectrum of singularities through the study of fluctuation of scaling indices.[1] Employing a similar formalism we obtain a new global spectral characterization of dynamical behavior of chaos, which describes a range of Lyapunov exponents and of entropy. It is closely connected with some important aspects of chaos, i.e., with the relationships between dimensions of invariant measures, Lyapunov exponents and entropies,[2] with variational principle of chaos,[3] and with strange repeller and chaotic transient phenomena.[4),5]

Significance of fluctuation of metric entropy or Lyapunov exponents have been noticed and stressed by several authors.[6),7] Suppose that d-dimensional phase space is partitioned to boxes of size ε. Trajectory of the dynamical system $x(t)$ is measured at interval of time τ. Let $P(i_1, \cdots, i_n)$ be the joint probability that $x(t=\tau)$ is in box i_1, $\cdots$, $x(t=n\tau)$ is in box i_n. We define a partition function $\Gamma(q, n)$,

$$\Gamma(q, n)=\sum^{N(n)} P(i_1, \cdots, i_n)^q , \tag{1}$$

where the sum of Eq. (1) is taken over all possible sequences $i_1, \cdots, i_n$. Where $N(n)$ is the number of possible sequences $i_1, \cdots, i_n$ with length n. If one knows Jacobian of map or linearized flow, the joint probability is obtained. For simplicity, let us start from one-dimensional map $x_{n+1}=f(x_n)$. A small interval i_1 of size ε expands by mapping to an interval of size $|f'(x_0)|\varepsilon$ ($x_0\in$box i_1), then the joint probability tends to

$$P(i_1, \cdots, i_n)\sim|(f^n)'(x_0)|^{-1}\sim e^{-\gamma n} \tag{2}$$

as $\varepsilon\to 0$. Where $\gamma=(1/n)\ln|(f^n)'(x_0)|$. An extension to high-dimensional systems is

$$\Gamma(q, n)=\sum^{N(n)}|\det T_x{}^{n+}|^{-q} , \tag{3}$$

where $T_x{}^{n+}$ is a matrix restricted in the expanding subspace of the Jacobian, $T_x{}^n=$matrix $[\partial(f^n)_i/\partial x_j]$. (Extension to flow is also straightforward.) The order-q Renyi entropy[6] is defined with the use of the partition function

$$K_q=-\frac{1}{q-1}\lim_{n\to\infty}\frac{1}{n}\ln\Gamma(q, n) . \tag{4}$$

Almost the same expression has been defined by Takahashi and Oono,[3] and independently by Fujisaka.[7]

A spectral representation of the fluctuation of entropy or the sum of the Lyapunov exponents is obtained directly with the same formalism of Halsey et al.,[1] however, implication of the results is not trivial as to be described later. The joint probability $p(i_1, \cdots, i_n)$ decreases exponentially as n increases,

$$P(i_1, \cdots, i_n)\sim e^{-\gamma n\tau} \tag{5}$$

(for the case of map, one sets $\tau=1$). γ is an expansion rate for the orbit and it approaches the sum of the positive Lyapunov exponents as n goes to infinity. On the other hand, the number of sequences with length n increases exponentially as n increases. Therefore the number of orbits which has the expansion rate γ between γ' and $\gamma'+d\gamma'$ will be the form

$$N(\gamma')d\gamma'\sim d\gamma' e^{h(\gamma')n\tau} . \tag{6}$$

Substituting Eqs. (5) and (6) into Eq. (1), and adopting an approximation that $\Gamma(q, n)$ is dominated by $\gamma(q)$ which minimize the term $[q\gamma'-h(\gamma')]$,[1] as $n\to\infty$ thus $\exp(-n\tau)\to 0$, we obtain

$$\Psi(q)\equiv-\lim_{n\to\infty}\frac{1}{n\tau}\ln\Gamma(q, n)$$
$$=q\gamma(q)-h(\gamma(q)) . \tag{7}$$

If one knows $\Psi(q)$ or K_q, $\gamma(q)$ and $h(\gamma(q))$ are obtained as

$$\gamma(q)=\frac{d}{dq}\Psi(q)\,, \tag{8a}$$

$$h(\gamma(q))=q\gamma(q)-\Psi(q)\,. \tag{8b}$$

We assumed that $h(\gamma)$ is a continuous function of γ. The function $h(\gamma)$ has some peculiar properties such that $(d/d\gamma)h(\gamma)=q$ and $(d^2/d\gamma^2)h(\gamma)<0$, and from Eqs. (7) and (1)

$$h(\gamma(0))=\lim_{n\to\infty}\frac{1}{n\tau}\ln N(n)$$

$$=h_{\rm topol}\,, \tag{9}$$

where $h_{\rm topol}$ denotes topological entropy. γ describes the expansion rate, and, thus, is the sum of positive Lyapunov exponents; and h is an exponent representing how fast the number of such orbits grows, thus, is the dynamical entropy. It is evident, however, that the expansion rate γ converges to a unique value in the limit as n goes to infinity due to the ergodicity. This implies that one invariant measure gives one point in the curve $h(\gamma)$. It is a well-known fact that for a given dynamical system there exist infinitely many invariant measures in general. Some (probably one) of them are visible (observable) and the rest of them are invisible (unobservable). Those invisible invariant measures have their own entropies, Lyapunov exponents and dimensions, and can be measured by seeing transient phenomena[4),8)] or by inverse mapping.[2)] We conjecture that: When a set of infinite number of invariant measures with positive entropy is taken into account, the spectrum $h(\gamma)$ becomes a continuous function.

This conjecture is justified for the exactly soluble model. Let us consider a generalized tent map

$$f(x)=\begin{cases}x/a & x\in[0,\,a/(a+b)]\,,\\(1-x)/b & x\in[a/(a+b),\,1]\,.\end{cases} \tag{10}$$

The partition function $\Gamma(q,\,n)$ and its Legendre transform are[3)]

$$\Gamma(q,\,n)=\sum_{m=0}^{n}{}_nC_m a^{q(n-m)}b^{qm}$$

$$=(a^q+b^q)^n\,, \tag{11a}$$

$$\gamma(q)=\frac{\partial\Psi(q)}{\partial q}$$

$$=\frac{-a^q\ln a-b^q\ln b}{a^q+b^q}\,, \tag{11b}$$

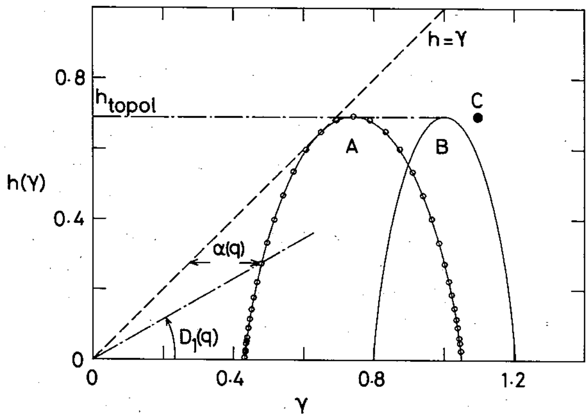

Fig. 1. The plot of h vs γ for the map, Eq. (10). Curve A corresponds to the case for $a=0.35$, $b=1-a$. Curve B and dot C correspond to the case for $a=0.3$, $b=0.45$ and for $a=b=1/3$, respectively ($a+b<1$ means the strange repeller). The $h(\gamma)$ estimated for case A by response measurement are marked by the circles. Implications of $h(\gamma)$ are schematically shown.

$$h(\gamma(q))=\ln(a^q+b^q)-q\frac{a^q\ln a+b^q\ln b}{a^q+b^q}\,. \tag{11c}$$

For simplicity, where $\Gamma(q,\,n)$ was calculated by using the generating partition $\{[0,\,a),\,(1-b,\,1]\}$ and by counting all possible symbolic sequences. One sequence corresponds to one invariant measure as $n\to\infty$. If one take equipartition with ε intervals, the coefficients ${}_nC_m$ must be modified by $A\times{}_nC_m$ (A does not depend on n). Nevertheless, $\Psi(q)$, $\gamma(q)$ and $h(q)$ are unchanged, because n is very large. In Fig. 1 we illustrate $h(\gamma)$ from Eq. (11) for a few sets of a and b. Range of the Lyapunov exponent is evidently $\gamma_{\rm min}={\rm Min}(-\ln a,\,-\ln b)$, $\gamma_{\rm max}={\rm Max}(-\ln a,\,-\ln b)$, and $h(\gamma(0))=\ln 2=h_{\rm topol}$. In the case $a+b=1$, the equality of Lyapunov exponent and entropy holds for $q=1$, $\gamma(1)=h(\gamma(1))=-a\ln a-b\ln b$. This point corresponds to an observable invariant measure. The curve $h(\gamma)$ is tangent to the line $h=\gamma$ at the point $(\gamma(1),\,h(\gamma(1))$ from Eqs. (11) and (8). For $a+b<1$ the system has only the strange repeller and the curve $h(\gamma)$ is no more tangent to the line $h=\gamma$. Criterion of observability is related to Eq. (1). In general, the probability to observe an invariant measure which has an expansion rate γ is

$$(e^{-n\tau})^{\gamma-h(\gamma)}\,, \tag{12}$$

since the orbits will diffuse by expansion as

$\exp(-\gamma n\tau)$ while the number of those orbits increase as $\exp(h(\gamma)n\tau)$. As the probability, Eq. (12), must be less than or equal to 1, then

$$\gamma \geq h(\gamma) \tag{13}$$

must hold when $\gamma > 0$. Equation (13) means that the sum of the positive Lyapunov exponents is greater than the metric entropy.

Now, implications of the spectrum $h(\gamma)$ are clear from the above discussions as schematically shown in Fig. 1.

1) As n goes to infinity, a chaotic invariant measure is observable only when the equality of Eq. (13) holds (i.e., $\gamma = h(\gamma)$ at $q=1$). It corresponds to a tangential point of the curve $h(\gamma)$ and the line $h=\gamma$.

2) For the other chaotic invariant measures, since the relation $\gamma > h(\gamma)$ holds they are invisible in equilibrium state ($n\to\infty$) but might be observable in transient behavior. Distribution of length of the transients is in the form $\exp(-\alpha n\tau)$ by Eq. (12). α is the escape rate[9] from an invisible invariant measure. Thus,

$$\text{Escape Rate: } \alpha(q) = \gamma(q) - h(\gamma(q))\,. \tag{14}$$

With varying q, one can see different invariant measures $\mu(q)$'s with smaller probabilities.

3) Generalizing the conjecture by Grassberger on dimension of invariant measure,[2),4)] we conjecture that the information dimension $D_1(\mu(q))$ of an invariant measure $\mu(q)$ of one-dimensional map which has Lyapunov exponent $\gamma(q)$ and entropy $h(q)$ for arbitrary q is

$$D_1(\mu(q)) = \frac{h(q)}{\gamma(q)}\,. \tag{15}$$

It follows from 1)~3) that for the one-dimensional map the information dimension of observable invariant measure is 1 while for invisible invariant measure $D_1(\mu)$ is less than 1.

Response of nonlinear dynamical systems and $h(\gamma)$ spectrum

As described above, if the sum of Eq. (1) is taken for an experimentally obtained time series (it is a realization of an observable invariant measure), we will obtain only one point $h(\gamma)=\gamma$ for sufficiently large n. In order to obtain the global structure of $h(\gamma)$, it is necessary to see all of invisible invariant measures. Is it possible in experiments? Recently, Kantz and Grassberger[4)] successfully showed that invisible chaos (repeller) can be observed by measuring transient orbits.[5)] We extend this idea to observe $h(\gamma)$ in numerical experiments. Namely, the sum of Eq. (1) is taken for many transient orbits with different random initial conditions. Observable invariant measure is an equilibrium distribution of most probable state. Since invisible invariant measures corresponding to $q\neq 1$ are non-equilibrium states, they can be seen when responses (or transients) of the system are measured. For practical application we replace the sum of Eq. (1) by ensemble average $\langle\ \rangle$

$$\Gamma(q,n) = \sum^{N(n)} P(i_1,\cdots,i_n)^q = \langle P(i_1,\cdots,i_n)^{q-1}\rangle\,, \tag{16}$$

where the ensemble average $\langle\ \rangle$ is taken for random initial conditions. Application of this method to the map of Eq. (10) gives correct results as shown in Fig. 1 by circles. It was also confirmed that the curve $h(\gamma)$ obtained from Eqs. (16) and (8) for the logistic map is in good agreement with the other numerical estimate from a quite different point of view.[11)]

Finally we apply the method to the case of intermittent chaos near the period 3 window of the logistic map

$$x_{n+1} = a x_n(1-x_n)\,. \tag{17}$$

Figure 2 shows the estimated spectrum $h(\gamma)$ by response measurement for the case $a=3.82842$ (cf.

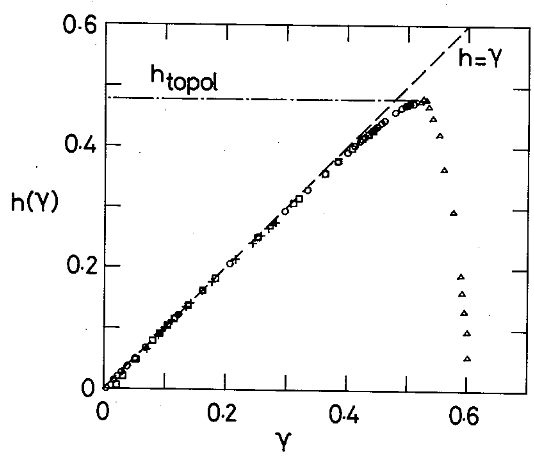

Fig. 2. The $h(\gamma)$ spectrum estimated for intermittent chaos of the logistic map. Where the transient length n is varied as, $n=200$ (○), 500(□) and 1000(+). The right portion of the curve (△) was drawn for a guide, which was calculated for the orbits obtained by the inverse mapping.[2)]

$a_c = 1 + \sqrt{8} = 3.8284271\cdots$). Topological entropy h_{topol} for this parameter is numerically calculated by utilizing the kneading theory.[10] We obtain $h_{\text{topol}} = 0.481212\cdots$. The estimated value of $h(\gamma(0)) = 0.48$ is in good agreement with h_{topol}. Furthermore, since the parameter is very close to the critical value a_c, the Lyapunov number of observable measure is very small ($\lambda = 0.061\cdots$). It follows that $\gamma_{\min}$ and $h(1) = \gamma(1)$ have to lie in the vicinity of the origin in spite of $h(\gamma(0)) = h_{\text{topol}}$ being apart from zero. Moreover, for intermittent chaos the Lyapunov exponent grows rapidly from zero near the onset point $a = a_c$. This implies the curve $h(\gamma)$ must be very close to the line $h = \gamma$. The estimated spectrum of Fig. 2 shows these properties very well. A distinctive feature of intermittent chaos is that: There exist infinitely many invisible chaotic invariant measures with very small escape rate and they cause large fluctuations to the Lyapunov exponent.

Measurement of the spectrum of expansion rate in real experiments should give a new knowledge on global metric structure of nonlinear dynamical system. We thank Professor Takeuchi, Professor Mori and Dr. Matsushita for many stimulating discussions.

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Note added: In preparation of this letter we were informed of the work of Eckmann and Procaccia (Phys. Rev. **A34** (1986), 659). They have obtained the same expression with Eq. (8), however, in implementation their results are far from ours. Our discussion on the physical meaning of the spectrum $h(\gamma)$ and the practical algorithm is quite new.