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Global stability of a modified Leslie-Gower model with Beddington-DeAngelis functional response

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Abstract

A predator-prey model with modified Leslie-Gower and Beddington-DeAngelis functional response is studied. The local stability of the equilibria and the permanence of the system are investigated. By applying the fluctuation lemma, qualitative analysis and Lyapunov direct method, respectively, three sufficient conditions on the global asymptotic stability of a positive equilibrium are obtained.

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Keywords: Leslie-Gower; permanence; global asymptotic stability; Lyapunov function; Dulac function; fluctuation lemma

1 Introduction

As pointed out by Berryman [1], the dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Leslie [2, 3] introduced the following famous Leslie predator-prey system:

$$\begin{aligned} \dot{x}(t) &= \big(r_1 - b_1 x(t)\big)x(t) - p\big(x(t)\big)y(t), \\ \dot{y}(t) &= \left(r_2 - a_2\frac{y(t)}{x(t)}\right)y(t), \end{aligned} \tag{1.1}$$

where $x(t)$, $y(t)$ stand for the population (the density) of the prey and the predator at time t, respectively. The parameters r_1 and r_2 are the intrinsic growth rates of the prey and the predator, respectively. b_1 measures the strength of competition among individuals of species x. The value $\frac{r_1}{b_1}$ is the carrying capacity of the prey in the absence of predation. The predator consumes the prey according to the functional response $p(x)$ and carries capacity $\frac{x}{a_2}$. The parameter a_2 is a measure of the food quantity that the prey provides converted to predator birth. The term y/x is the Leslie-Gower term which measures the loss in the predator population due to rarity (per capita y/x) of its favorite food. Leslie model is a predator-prey model where the carrying capacity of the predator is proportional to the number of prey, stressing the fact that there are upper limits to the rates of increase in both prey x and predator y, which are not recognized in the Lotka-Volterra model. These upper limits can be approached under favorable conditions: for the predators, when the number of prey per predator is large; for the prey, when the number of predators (and perhaps the number of prey also) is small [4]. For more details of the model, one can see [4–9] and the references cited therein. Holling [10] suggested three different kinds of functional response for different kinds of species to model the phenomena of predation, which made the standard Lotka-Volterra system more realistic. When $p(x) = \frac{a_1 x}{x+k_1}$, the functional response $p(x)$ is called Holling-type II.

Recently, Aziz-Alaoui and Daher Okiye [11] pointed out that in the case of severe scarcity, y can switch over to other populations but its growth will be limited by the fact that its most favorite food x is not available in abundance. To solve such a problem, they suggested to add a positive constant d to the denominator and proposed the following predator-prey model with modified Leslie-Gower and Holling-type II schemes:

$$\begin{aligned} \dot{x}(t) &= \left(r_1 - b_1 x(t) - \frac{a_1 y(t)}{x(t) + k_1}\right) x(t), \\ \dot{y}(t) &= \left(r_2 - \frac{a_2 y(t)}{x(t) + k_2}\right) y(t), \end{aligned} \tag{1.2}$$

where r_1, b_1, r_2, a_2 have the same meaning as in models (1.1). a_1 is the maximum value of the per capita reduction rate of x due to y, k_1 (respectively, k_2) measures the extent to which the environment provides protection to prey x (respectively, to the predator y). The authors studied the boundedness and global stability of positive equilibrium of system (1.2). Since then, system (1.2) and its non-autonomous versions have been studied by incorporating delay and impulses, harvesting and so on (see, for example, [12–29]). In [12], we studied the structure, linearized stability and the global asymptotic stability of equilibria of (1.2). Nindjin *et al.* [13] incorporated time delay to system (1.2) and studied the global stability and persistence of the delayed system by using the Lyapunov functional. Yafia *et al.* [14] and [15] further studied the occurrence of Hopf bifurcation at equilibrium by using the time delay as a parameter of bifurcation. Nindjin and Aziz-Alaoui [16] studied uniform persistence and global stability of three Leslie-Gower-type species food chain system. Gakkhar and Singh [17] studied the dynamic behaviors of a modified Leslie-Gower predator-prey system with seasonally varying parameters. Guo and Song [18], Song and Li [19] further considered the influence of impulsive effect. Zhu and Wang [20] obtained sufficient conditions for the existence and global attractivity of positive periodic solutions of system (1.2) with periodic coefficients. Liu and Wang [21] considered a stochastic predator-prey system with modified Leslie-Gower and Holling-type II schemes with Lévy jumps. The results showed that the Lévy jumps can change the properties of the population systems significantly. Kar and Ghorai [22] obtained local stability, global stability, influence of harvesting and bifurcation of a delayed predator-prey system in the presence of harvesting. Two stage-structured predator-prey models with modified Leslie-Gower and Holling-type II schemes were studied in [23–25]. Gupta and Chandra [26] discussed the permanence, stability and bifurcation (saddle-node bifurcation, transcritical, Hopf-Andronov and Bogdanov-Takens) of a modified Leslie-Gower prey-predator model with Michaelis-Menten type prey harvesting. Ji *et al.* [27, 28] showed there was a stationary distribution of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation and it has ergodic property. Lian and Xu [29] discussed the Hopf bifurcation of a predator-prey system with Holling type IV functional response and time delay.

As we all know, the functional response can be classified into two types: prey-dependent and predator-dependent. Prey-dependent depends on prey density only, while predator-dependent means that the functional response is a function of both the preys and the predators densities. Recently, the prey-dependent functional responses have been challenged by several ecologists. There is a growing explicit biological and physiological evidence [30–32] that in many situations, especially when the predator has to search for food (and therefore has to shave or compete for food), a more suitable general predator-prey theory should be predator-dependent. This is supported by numerous fields and laboratory experiments and observations [33, 34]. Starting from this argument and the traditional prey-dependent-only model, Arditi and Ginzburg [33] first proposed the ratio-dependent predator-prey model. Many authors have observed that the ratio-dependent models can exhibit much richer, more complicated and more reasonable or acceptable dynamics, but it has somewhat singular behavior at low densities which has been the source of controversy [35]. For the ratio-dependent predator-prey models, one can refer to [36–39].

Beddington-DeAngelis functional response $\frac{\alpha x}{a+bx+cy}$ was first proposed by Beddington and DeAngelis [40, 41]. Predator-prey model with Beddington-DeAngelis functional response has rich dynamical features, which can describe the species and the ecological systems more reasonably. Beddington-DeAngelis functional response is similar to the well-known Holling type II functional response but has an extra term cy in the denominator modeling mutual interference among predators and has some of the same qualitative features as the ratio-dependent form but avoids some of the singular behaviors of ratio-dependent models at low densities.

On the other hand, in 2001, Skalski and Gilliam [42] have presented statistical evidence from nineteen predator-prey systems that three predator-dependent functional responses (Beddington-DeAngelis, Crowley-Martin, and Hassell-Varley) can provide better description of predator feeding over a range of predator-prey abundances. In some cases, the Beddington-DeAngelis type preformed even better. Theoretical studies have shown that the dynamics of models with predator-dependent functional responses can differ considerably from those with prey-dependent functional responses. Although much progress has been seen in the study of predator-prey models with modified Leslie-Gower (see [11–29]), to the best of the authors' knowledge, seldom did scholars consider the modified Leslie-Gower model with Beddington-DeAngelis functional response. Stimulated by above reasons, in this paper we will incorporate the Beddington-DeAngelis functional response into model (1.2) and consider the following model which is the generalization of model (1.2):

$$\begin{aligned} \dot{x}(t) &= \left(r_1 - px(t) - \frac{\alpha y(t)}{a + bx(t) + cy(t)}\right)x(t), \\ \dot{y}(t) &= \left(r_2 - \frac{\beta y(t)}{x(t) + k}\right)y(t), \end{aligned} \tag{1.3}$$

with initial conditions $x(0) > 0$ and $y(0) > 0$. The parameters $r_1, p, \alpha, a, b, c, r_2, \beta$ and k are positive constants and have the same meaning as in model (1.2).

It is easy to see that both the first quadrant R^2_+ and the positive first quadrant $\operatorname{Int} R^2_+$ are invariant for system (1.3). As a result, solutions $(x(t), y(t))$ to (1.3) with $(x(0), y(0)) \in \operatorname{Int} R^2_+$ are all positive solutions.

The rest of this paper is organized as follows. In Section 2, we discuss the structure of nonnegative equilibria to (1.3) and their local stability, which motivates us to study permanence and global stability of (1.3) respectively in Section 3 and Section 4.

For more works on this direction, one could refer to [43–51] and the references cited therein.

2 Nonnegative equilibria and their local stability

The Jacobian matrix of system (1.3) is

$$J = \begin{pmatrix} r_1 - 2px - \frac{\alpha y(a+cy)}{(a+bx+cy)^2} & -\frac{\alpha x(a+bx)}{(a+bx+cy)^2} \\ \frac{\beta y^2}{(x+k)^2} & r_2 - \frac{2\beta y}{x+k} \end{pmatrix}.$$

An equilibrium E of (1.3) is (linearly) stable if the real parts of both eigenvalues of $J(E)$ are negative, and therefore a sufficient condition for stability is

$$\operatorname{tr}\bigl(J(E)\bigr) < 0 \quad \text{and} \quad \det\bigl(J(E)\bigr) > 0. \tag{2.1}$$

Obviously, (1.3) has three boundary equilibria, $E_0 = (0,0)$, $E_1 = (\frac{r_1}{p},0)$ and $E_2 = (0,\frac{r_2 k}{\beta})$, whose Jacobian matrices are

$$J(E_0) = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix},$$

$$J(E_1) = \begin{pmatrix} -r_1 & -\frac{\alpha r_1}{ap+br_1} \\ 0 & r_2 \end{pmatrix},$$

and

$$J(E_2) = \begin{pmatrix} r_1 - \frac{\alpha r_2 k}{a\beta + cr_2 k} & 0 \\ \frac{r_2^2}{\beta} & -r_2 \end{pmatrix},$$

respectively. As a direct consequence of (2.1), we have the following result.

Proposition 2.1

(i) *Both E_0 and E_1 are unstable.*

(ii) *E_2 is locally asymptotically stable if $\alpha r_2 k > ar_1\beta + r_1 r_2 ck$, while it is unstable if $\alpha r_2 k < ar_1\beta + r_1 r_2 ck$.*

Besides the three boundary equilibria, (1.3) may have (componentwise) positive equilibria. Suppose that $\hat{E} = (\hat{x},\hat{y})$ is such an equilibrium. Then

$$\begin{aligned} r_1 - p\hat{x} &= \frac{\alpha\hat{y}}{a + b\hat{x} + c\hat{y}}, \\ \hat{y} &= \frac{r_2(\hat{x}+k)}{\beta}. \end{aligned} \tag{2.2}$$

One can easily see that $\hat{x}$ satisfies

$$(r_2cp + bp\beta)\hat{x}^2 + B\hat{x} + \alpha r_2k - ar_1\beta - r_1r_2ck = 0, \tag{2.3}$$

where $B \stackrel{\triangle}{=} \alpha r_2 + pr_2ck + ap\beta - r_1r_2c - br_1\beta$. Moreover, for convenience, we denote

$$\Delta \stackrel{\triangle}{=} B^2 - 4(r_2cp + bp\beta)(\alpha r_2k - ar_1\beta - r_1r_2ck).$$

Hence, we have the following result.

Proposition 2.2 *Suppose that*

$$(\mathrm{H}_0) \quad \alpha r_2k < ar_1\beta + r_1r_2ck$$

holds, then system (1.3) *has a unique positive equilibrium* $\hat{E} = (\hat{x}, \hat{y})$, *where*

$$\hat{x} = \frac{-B + \sqrt{\Delta}}{2(r_2cp + bp\beta)} \quad \textit{and} \quad \hat{y} = \frac{r_2(\hat{x} + k)}{\beta}.$$

We now want to study the stability of the positive equilibrium $\hat{E} = (\hat{x}, \hat{y})$. It follows from the Jacobian matrix of systems (1.3) and (2.2) that

$$\begin{aligned} \operatorname{tr}\bigl(J(\hat{E})\bigr) &= \frac{(r_1b - ap)\hat{x} - 2pb\hat{x}^2 - pc\hat{x}\hat{y}}{a + b\hat{x} + c\hat{y}} - r_2 \\ &= \frac{-(2pb\beta + pcr_2)\hat{x}^2 + (r_1b\beta - ap\beta - r_2b\beta - r_2^2c)\hat{x} - r_2^2ck - pcr_2k - ar_2\beta}{a\beta + (b\beta + cr_2)\hat{x} + cr_2k}, \end{aligned}$$

and

$$\begin{aligned} \det\bigl(J(\hat{E})\bigr) &= \frac{r_2(2pb\hat{x}^2 - (r_1b - ap)\hat{x} + pc\hat{x}\hat{y})}{a + b\hat{x} + c\hat{y}} + \frac{{r_2}^2\hat{x}(a + b\hat{x})(r_1 - p\hat{x})}{\beta(a + b\hat{x} + c\hat{y})\hat{y}} \\ &= \frac{r_2\hat{x}}{a + b\hat{x} + c\hat{y}}\left(2pb\hat{x} - (r_1b - ap) + \frac{pcr_2(\hat{x} + k)}{\beta} + \frac{(a + b\hat{x})(r_1 - p\hat{x})}{(\hat{x} + k)}\right) \\ &= \frac{r_2\hat{x}}{\beta(a + b\hat{x} + c\hat{y})(\hat{x} + k)}\bigl((pb\beta + pcr_2)\hat{x}^2 + 2(pcr_2k + pbk\beta)\hat{x} \\ &\quad + pcr_2k^2 + ar_1\beta + apk\beta - r_1bk\beta\bigr). \end{aligned}$$

Thus, if $r_1b < ap$, we have

$$\operatorname{tr}\bigl(J(\hat{E})\bigr) < 0 \quad \text{and} \quad \det\bigl(J(\hat{E})\bigr) > 0.$$

Hence, the following proposition follows from (2.1).

Proposition 2.3 *Assume that*

$$(\mathrm{H}_1) \quad r_1b < ap$$

holds, then the positive equilibrium $\hat{E}$ *is locally asymptotically stable.*

Proposition 2.1 and Proposition 2.3 naturally motivate us to seek sufficient conditions on the global stability of E_2 and the unique positive equilibrium to (1.3). To achieve it, we need the bounds for positive solutions.

3 Boundedness and permanence

The following result can be proved by slightly modifying the proof of Lemma 3.2 of Chen [52] and it will play an important role in finding the bounds for positive solutions to (1.3).

Lemma 3.1 *If $a > 0$, $b > 0$ and $\dot{x} \geq x(b - ax)$, when $t \geq t_0$ and $x(t_0) > 0$, we have*

$$\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}.$$

If $a > 0$, $b > 0$ and $\dot{x} \leq x(b - ax)$, when $t \geq t_0$ and $x(t_0) > 0$, we have

$$\limsup_{t\to+\infty} x(t) \leq \frac{b}{a}.$$

Proposition 3.1 *Let $(x(t), y(t))$ be any positive solution of (1.3). Then*

$$\limsup_{t\to+\infty} x(t) \leq \frac{r_1}{p} \triangleq M_1, \tag{3.1}$$

and

$$m_2 \leq \liminf_{t\to+\infty} y(t) \leq \limsup_{t\to+\infty} y(t) \leq M_2,$$

where $m_2 \triangleq \frac{r_2 k}{\beta}$ and $M_2 \triangleq \frac{r_1 r_2 + p r_2 k}{p\beta}$.

Proof Since $(x(t), y(t))$ is a positive solution of (1.3), we have

$$\begin{aligned} &\dot{x} \leq (r_1 - px)x, \\ &\dot{y} \geq \left(r_2 - \frac{\beta y}{k}\right)y. \end{aligned} \tag{3.2}$$

Then (3.1) and $\liminf_{t\to+\infty} y(t) \geq m_2$ follow directly from Lemma 3.1. Thus, for any $\varepsilon > 0$, there exists $T > 0$ such that

$$x(t) \leq \frac{r_1}{p} + \varepsilon \quad \text{for } t \geq T.$$

This combined with the second equation of (1.3) leads to

$$\dot{y} \leq \left(r_2 - \frac{\beta y}{\frac{r_1}{p} + \varepsilon + k}\right)y \quad \text{for } t \geq T.$$

Using Lemma 3.1 again, one has

$$\limsup_{t\to+\infty} y(t) \leq \frac{r_2(\frac{r_1}{p} + \varepsilon + k)}{\beta}.$$

Thus $\limsup_{t\to\infty} y(t) \leq M_2$ by letting $\varepsilon \to 0$. □

Proposition 3.2 *Suppose that*

$$(\mathrm{H}_2) \quad ar_1p\beta > \alpha(r_1r_2 + pr_2k).$$

If $(x(t), y(t))$ *is a positive solution to system* (1.3), *then*

$$\liminf_{t\to+\infty} x(t) \geq \frac{ar_1p\beta - \alpha(r_1r_2 + pr_2k)}{ap^2\beta} \triangleq m_1.$$

Proof Denote

$$\varepsilon_0 = \frac{ar_1p\beta - \alpha(r_1r_2 + pr_2k)}{\alpha p\beta}.$$

Then, for $\varepsilon \in (0, \varepsilon_0)$, we have

$$ar_1p\beta > \alpha(r_1r_2 + pr_2k + p\beta\varepsilon).$$

According to Proposition 3.1, there exists $T > 0$ such that

$$y(t) \leq \frac{r_1r_2 + pr_2k}{p\beta} + \varepsilon \quad \text{for } t \geq T.$$

This, combined with the first equation of (1.3), produces

$$\dot{x} \geq \left(r_1 - px - \frac{\alpha(r_1r_2 + pr_2k + p\beta\varepsilon)}{ap\beta}\right)x \quad \text{for } t \geq T.$$

It follows from Lemma 3.1 that

$$\liminf_{t\to+\infty} x(t) \geq \frac{r_1 - \frac{\alpha(r_1r_2+pr_2k+p\beta\varepsilon)}{ap\beta}}{p}.$$

This gives $\liminf_{t\to+\infty} x(t) \geq m_1$ by letting $\varepsilon \to 0$. □

Combing Proposition 3.1 with Proposition 3.2 gives the permanence of (1.3).

Theorem 3.1 *Suppose that* (H_2) *holds, then* (1.3) *is permanent.*

4 Global asymptotic stability

The goal of this section is to establish sufficient conditions on the global asymptotic stability of equilibrium to (1.3). The first two results are proved by employing the fluctuation lemma, which is cited below for the convenience of the readers. See Hirsch *et al.* [53] or Tineo [54] for more details on the fluctuation lemma.

Lemma 4.1 (Fluctuation lemma) *Let* $x(t)$ *be a bounded differentiable function on* $[\alpha, \infty)$. *Then there exist sequences* $\tau_n \to \infty$ *and* $\sigma_n \to \infty$ *such that*

(i) $\dot{x}(\tau_n) \to 0$ *and* $x(\tau_n) \to \limsup_{t\to\infty} x(t) = \overline{x}$ *as* $n \to \infty$;
(ii) $\dot{x}(\sigma_n) \to 0$ *and* $x(\sigma_n) \to \liminf_{t\to\infty} x(t) = \underline{x}$ *as* $n \to \infty$.

Theorem 4.1 *Assume that*

$$(\mathrm{H}_3) \quad \alpha r_1 p\beta + br_1{}^2\beta + cr_1 r_2 kp < \alpha pr_2 k.$$

Then $E_2 = (0, \frac{r_2 k}{\beta})$ *is globally asymptotically stable for system* (1.3).

Proof Let $(x(t), y(t))$ be any positive solution of (1.3). According to (H_3), we can choose $\varepsilon \in (0, \frac{r_2 k}{\beta})$ such that

$$\alpha r_1 p\beta + br_1{}^2\beta + cr_1 r_2 kp < \alpha pr_2 k + (cr_1 - br_1 - \alpha)p\beta\varepsilon. \tag{4.1}$$

It follows from Proposition 3.1 that there exists $T > 0$ such that

$$x(t) \le \frac{r_1}{p} + \varepsilon \quad \text{and} \quad y(t) \ge \frac{r_2 k}{\beta} - \varepsilon \quad \text{for } t \ge T.$$

These inequalities combined with the first equation of (1.3) give us

$$\dot{x} \le \left(r_1 - \frac{\alpha p(r_2 k - \beta\varepsilon)}{ap\beta + br_1\beta + cr_2 kp + (b-c)p\beta\varepsilon}\right) \quad \text{for } t \ge T,$$

or

$$\frac{\dot{x}}{x} \le \left(r_1 - \frac{\alpha p(r_2 k - \beta\varepsilon)}{ap\beta + br_1\beta + cr_2 kp + (b-c)p\beta\varepsilon}\right) \quad \text{for } t \ge T. \tag{4.2}$$

Integrating both sides of (4.2) on interval $[T, t]$ leads to

$$x(t) \le x(T)\exp\left\{\left(r_1 - \frac{\alpha p(r_2 k - \beta\varepsilon)}{ap\beta + br_1\beta + cr_2 kp + (b-c)p\beta\varepsilon}\right)(t - T)\right\}.$$

Due to (4.1), one has $\limsup_{t\to+\infty} x(t) \le 0$. Hence

$$\lim_{t\to+\infty} x(t) = 0. \tag{4.3}$$

Proposition 3.1 again tells us that $y(t)$ is bounded and $\overline{y} \triangleq \limsup_{t\to+\infty} y(t) \ge \underline{y} \triangleq \liminf_{t\to+\infty} y(t) > 0$. By Lemma 4.1, there exist sequences $\tau_n \to \infty$, $\sigma_n \to \infty$ such that $\dot{y}(\tau_n) \to 0$, $\dot{y}(\sigma_n) \to 0$, $y(\tau_n) \to \overline{y}$, and $y(\sigma_n) \to \underline{y}$, as $n \to \infty$.

It follows from the second equation of (1.3) and $\lim_{t\to+\infty} x(t) = 0$ that

$$0 = \left(r_2 - \frac{\beta\overline{y}}{k}\right)\overline{y} \quad \text{or} \quad \overline{y} = \frac{r_2 k}{\beta}. \tag{4.4}$$

Similarly, one can show that

$$\underline{y} = \frac{r_2 k}{\beta}. \tag{4.5}$$

Equation (4.4) combined with (4.5) implies that

$$\lim_{t\to+\infty} y(t) = \frac{r_2 k}{\beta}. \tag{4.6}$$

It follows from (4.3) and (4.6) that $E_2 = (0, \frac{r_2 k}{\beta})$ is globally asymptotically stable. □

Theorem 4.2 *In addition to* (H_2), *further suppose that*

$$(\mathrm{H}_4)\quad r_1 b\beta + \alpha r_2 \le 2bp\beta m_1 + ap\beta + cpr_2 k + cr_1 r_2,$$

where m_1 is defined in Proposition 3.2. *Then system* (1.3) *has a unique positive equilibrium which is globally asymptotically stable.*

Proof Note that (H_2) implies (H_0), thus (1.3) has a unique positive equilibrium according to Proposition 2.2. Let $(x(t), y(t))$ be any positive solution of (1.3). By the results in Section 3, $\overline{x} \triangleq \limsup_{t\to\infty} x(t) \ge \underline{x} \triangleq \liminf_{t\to\infty} x(t) \ge m_1$ (> 0) and $\overline{y} \triangleq \limsup_{t\to\infty} y(t) \ge \underline{y} \triangleq \liminf_{t\to\infty} y(t) > 0$.

We claim that $\overline{x} = \underline{x}$. Suppose that $\overline{x} > \underline{x}$. According to Lemma 4.1, there exist sequences $\xi_n \to \infty$, $\eta_n \to \infty$, $\tau_n \to \infty$ and $\sigma_n \to \infty$, such that $\dot{x}(\xi_n) \to 0$, $\dot{x}(\eta_n) \to 0$, $x(\xi_n) \to \overline{x}$ and $x(\eta_n) \to \underline{x}$, $\dot{y}(\tau_n) \to 0$, $\dot{y}(\sigma_n) \to 0$, $y(\tau_n) \to \overline{y}$ and $y(\sigma_n) \to \underline{y}$, as $n \to \infty$. First, it follows from the second equation of (1.3) that

$$\dot{y}(\tau_n) \le \left(r_2 - \frac{\beta y(\tau_n)}{\sup_{t\ge\tau_n} x(\tau_n) + k}\right) y(\tau_n)$$

and

$$\dot{y}(\tau_n) \ge \left(r_2 - \frac{\beta y(\tau_n)}{\inf_{t\ge\tau_n} x(\tau_n) + k}\right) y(\tau_n).$$

Letting $n \to \infty$ gives us

$$0 \le \left(r_2 - \frac{\beta \overline{y}}{\overline{x} + k}\right)\overline{y}$$

and

$$0 \ge \left(r_2 - \frac{\beta \overline{y}}{\underline{x} + k}\right)\overline{y}.$$

Hence

$$\frac{r_2(\underline{x} + k)}{\beta} \le \overline{y} \le \frac{r_2(\overline{x} + k)}{\beta}. \tag{4.7}$$

Similar arguments as above also produce

$$\frac{r_2(\underline{x} + k)}{\beta} \le \underline{y} \le \frac{r_2(\overline{x} + k)}{\beta}. \tag{4.8}$$

Second, from the first equation of (1.3), we have

$$\dot{x}(\xi_n) = \left(r_1 - px(\xi_n) - \frac{\alpha y(\xi_n)}{a + bx(\xi_n) + cy(\xi_n)}\right) x(\xi_n). \tag{4.9}$$

Equation (4.9) implies

$$\dot{x}(\xi_n) \le \left(r_1 - px(\xi_n) - \frac{\alpha \inf_{t\ge\xi_n} y(\xi_n)}{a + bx(\xi_n) + c\inf_{t\ge\xi_n} y(\xi_n)}\right) x(\xi_n).$$

Taking limit as $n \to \infty$, one obtains

$$0 \le \left(r_1 - p\overline{x} - \frac{\alpha \underline{y}}{a + b\overline{x} + c\underline{y}}\right)\overline{x}.$$

This, combined with (4.8), gives us

$$0 \le \left(r_1 - p\overline{x} - \frac{\alpha r_2(\underline{x} + k)}{a\beta + b\beta\overline{x} + cr_2(\underline{x} + k)}\right)\overline{x}.$$

It follows that

$$(r_1 b\beta - ap\beta)\overline{x} - bp\beta\overline{x}^2 - cpr_2\overline{x}(\underline{x} + k) + cr_1r_2\underline{x} + ar_1\beta + cr_1r_2k \ge \alpha r_2(\underline{x} + k). \tag{4.10}$$

Similarly, one can show that

$$(r_1 b\beta - ap\beta)\underline{x} - bp\beta\underline{x}^2 - cpr_2\underline{x}(\overline{x} + k) + cr_1r_2\overline{x} + ar_1\beta + cr_1r_2k \le \alpha r_2(\overline{x} + k). \tag{4.11}$$

Multiplying (4.10) by –1 and adding it to (4.11), we have

$$bp\beta\left(\overline{x}^2 - \underline{x}^2\right) + (ap\beta - r_1b\beta + cpr_2k + cr_1r_2 - \alpha r_2)(\overline{x} - \underline{x}) \le 0.$$

Due to $\overline{x} > \underline{x}$, one gets

$$bp\beta(\overline{x} + \underline{x}) + (ap\beta + cpr_2k + cr_1r_2 - r_1b\beta - \alpha r_2) \le 0. \tag{4.12}$$

On the other hand,

$$\begin{aligned} &bp\beta(\overline{x} + \underline{x}) + (ap\beta + cpr_2k + cr_1r_2 - r_1b\beta - \alpha r_2) \\ &\quad > 2bp\beta m_1 + (ap\beta + cpr_2k + cr_1r_2 - r_1b\beta - \alpha r_2) \\ &\quad \ge 0 \quad \text{by } (\mathrm{H}_4), \end{aligned}$$

which contradicts with (4.12). Therefore, $\overline{x} = \underline{x}$ and the claim is proved. The claim implies that $\lim_{t\to+\infty} x(t)$ exists and we denote it by x^*. Then it follows from (4.7) and (4.8) that $\lim_{t\to+\infty} y(t)$ exists and $\lim_{t\to+\infty} y(t) \stackrel{\triangle}{=} y^* = \frac{r_2(x^*+k)}{\beta} > 0$. Letting $n \to \infty$ in (4.9) gives us

$$(r_2c\beta + bp\beta){x^*}^2 + (\alpha r_2 + pr_2ck + ap\beta - r_1r_2c - br_1\beta)x^* + \alpha r_2k - ar_1\beta - r_1r_2ck = 0.$$

Thus (x^*, y^*) satisfies

$$\begin{aligned} r_1 - px^* &= \frac{\alpha y^*}{a + bx^* + cy^*}, \\ y^* &= \frac{r_2(x^* + k)}{\beta}. \end{aligned}$$

That is, (x^*, y^*) is a positive equilibrium of (1.3). This completes the proof as the positive equilibrium is unique and so the unique equilibrium point is globally asymptotically stable. □

The following results in this section are proved by qualitative method and applying the Lyapunov direct method with the Lyapunov function.

Theorem 4.3 *Assume that* (H_0) *and* (H_1) *hold, then* (1.3) *has a unique positive equilibrium which is globally asymptotically stable.*

Proof According to Proposition 2.2, (1.3) has a unique positive equilibrium $\hat{E} = (\hat{x}, \hat{y})$. Taking Dulac function $D(x,y) = x^{-1}(a + bx + cy)y^{-2}$, we obtain

$$\frac{\partial(DP)}{\partial x} + \frac{\partial(DQ)}{\partial y} = \frac{(r_1 b - ap) - 2pbx - pcy}{y^2} - \frac{r_2(a + bx)}{xy^2} - \frac{\beta c}{x(x + k)},$$

where (P,Q) is the vector field of (1.3). By the positivity of x, y, it is easy to obtain that $\frac{\partial(DP)}{\partial x} + \frac{\partial(DQ)}{\partial y} < 0$ if (H_1) holds. Then, by the Dulac criteria, (1.3) admits no limit cycles or separatrix cycles. Proposition 2.3 shows that $\hat{E} = (\hat{x}, \hat{y})$ is locally asymptotically stable when (H_1) holds. On the other hand, (1.3) admits only four equilibria E_i $(i = 0, 1, 2)$ and $\hat{E}$. Also, Proposition 2.1 shows that E_i $(i = 0, 1, 2)$ are all unstable when $\alpha r_2 k < ar_1\beta + r_1 r_2 ck$ holds. So, according to Proposition 3.1 and the Poincaré-Bendixson theorem, $\hat{E}$ is globally asymptotically stable. □

Theorem 4.4 *Suppose that* (H_0) *holds, further assume that*

$$(\mathrm{H}_5)\quad \frac{\alpha b\hat{y}}{a} + \frac{\alpha M_2}{2k} + \frac{\alpha(a + b\hat{x})}{2a} < p(a + b\hat{x} + c\hat{y}) \quad \textit{and} \quad M_2 a < k(a - b\hat{x})$$

hold, then (1.3) *has a unique positive equilibrium* $\hat{E} = (\hat{x}, \hat{y})$ *which is globally asymptotically stable.*

Proof Let $(x(t), y(t))$ be any positive solution of (1.3). According to Proposition 2.2, (1.3) has a unique positive equilibrium $\hat{E} = (\hat{x}, \hat{y})$. From (H_5), we can choose an $\varepsilon > 0$ such that

$$\frac{\alpha b\hat{y}}{a} + \frac{\alpha(M_2 + \varepsilon)}{2k} + \frac{\alpha(a + b\hat{x})}{2a} < p(a + b\hat{x} + c\hat{y}) \quad \text{and} \quad \frac{M_2 + \varepsilon}{2k} + \frac{a + b\hat{x}}{2a} < 1. \tag{4.13}$$

Moreover, it follows from Proposition 3.1 that there exists $T > 0$ such that

$$0 < y(t) \le M_2 + \varepsilon \quad \text{for } t \ge T. \tag{4.14}$$

Let $V(x,y) = V_1(x,y) + V_2(x,y)$, where $V_1(x,y) = (a + b\hat{x} + c\hat{y})(x - \hat{x} - \hat{x}\ln(\frac{x}{\hat{x}}))$ and $V_2(x,y) = \frac{\alpha(\hat{x}+k)}{\beta}(y - \hat{y} - \hat{y}\ln(\frac{y}{\hat{y}}))$. Calculating the derivative of V along the solution of system (1.3), we have

$$\begin{aligned}
\dot{V}(x,y) &= (a + b\hat{x} + c\hat{y})(x - \hat{x})\left(r_1 - px - \frac{\alpha y}{a + bx + cy}\right) + \frac{\alpha(\hat{x} + k)}{\beta}(y - \hat{y})\left(r_2 - \frac{\beta y}{x + k}\right) \\
&= (a + b\hat{x} + c\hat{y})(x - \hat{x})\left(-p(x - \hat{x}) + \frac{\alpha\hat{y}}{a + b\hat{x} + c\hat{y}} - \frac{\alpha y}{a + bx + cy}\right) \\
&\quad + \frac{\alpha(\hat{x} + k)}{\beta}(y - \hat{y})\left(\frac{\beta\hat{y}}{\hat{x} + k} - \frac{\beta y}{x + k}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\alpha b\hat{y}}{a+bx+cy} - p(a+b\hat{x}+c\hat{y})\right)(x-\hat{x})^2 - \alpha(y-\hat{y})^2 \\
&\quad + \left(\frac{\alpha y}{x+k} - \frac{\alpha(a+b\hat{x})}{a+bx+cy}\right)(x-\hat{x})(y-\hat{y}) \\
&\le \left(\frac{\alpha b\hat{y}}{a+bx+cy} - p(a+b\hat{x}+c\hat{y})\right)(x-\hat{x})^2 - \alpha(y-\hat{y})^2 \\
&\quad + \left(\frac{\alpha y}{x+k} + \frac{\alpha(a+b\hat{x})}{a+bx+cy}\right)\frac{(x-\hat{x})^2+(y-\hat{x})^2}{2} \\
&\le \left(\frac{\alpha b\hat{y}}{a} - p(a+b\hat{x}+c\hat{y})\right)(x-\hat{x})^2 - \alpha(y-\hat{y})^2 \\
&\quad + \left(\frac{\alpha y}{k} + \frac{\alpha(a+b\hat{x})}{a}\right)\frac{(x-\hat{x})^2+(y-\hat{x})^2}{2} \\
&= \left(\frac{\alpha b\hat{y}}{a} - p(a+b\hat{x}+c\hat{y}) + \frac{\alpha y}{2k} + \frac{\alpha(a+b\hat{x})}{2a}\right)(x-\hat{x})^2 \\
&\quad + \left(\frac{\alpha y}{2k} + \frac{\alpha(a+b\hat{x})}{2a} - \alpha\right)(y-\hat{y})^2 \\
&\le \left(\frac{\alpha b\hat{y}}{a} - p(a+b\hat{x}+c\hat{y}) + \frac{\alpha(M_2+\varepsilon)}{2k} + \frac{\alpha(a+b\hat{x})}{2a}\right)(x-\hat{x})^2 \\
&\quad + \alpha\left(\frac{M_2+\varepsilon}{2k} + \frac{a+b\hat{x}}{2a} - 1\right)(y-\hat{y})^2.
\end{aligned}
$$

According to (4.13) and (4.14), $\dot{V}(x,y) < 0$ strictly for all $x, y > 0$ except the positive equilibrium $\hat{E} = (\hat{x},\hat{y})$, where $\dot{V}(x,y) = 0$. Thus, $V(x,y)$ satisfies Lyapunov's asymptotic stability theorem, and the positive equilibrium $\hat{E}$ of system (1.3) is globally asymptotically stable. This ends the proof of Theorem 4.4. □

Conclusion

In this paper, we consider a predator-prey model with modified Leslie-Gower and Beddington-DeAngelis functional response. We discuss the structure of nonnegative equilibria and their local stability. Also, the permanence of the system is investigated. By applying the fluctuation lemma, qualitative analysis and Lyapunov direct method, respectively, three sufficient conditions on the global asymptotic stability of a positive equilibrium are obtained. Compare Theorem 4.2 with Theorem 4.3. Since (H_2) contains (H_0), what will happen when (H_0) and (H_4) hold? This is a further problem, which can be studied in the future.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author wrote the manuscript carefully, read and approved the final manuscript.

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