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GLOBAL STRUCTURAL STABILITY OF A SADDLE NODE BIFURACTION

BY

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ABSTRACT. S. Newhouse, J. Palis, and F. Takens have recently proved the global structural stability of a one parameter unfolding of a saddle node when the nonwandering set is finite and transversality conditions are satisfied. (The diffeomorphism is Morse-Smale except for the saddle node.) Using their local unfolding of a saddle node and our method of compatible families of unstable disks (instead of the more restrictive method of compatible systems of unstable tubular families), we are able to extend one of their results to the case where the nonwandering set is infinite. We assume that a saddle node is introduced away from the rest of the nonwandering set, which is hyperbolic (Axiom A), and that a (strong) transversality condition is satisfied.

1. Statement of the theorem. We consider M a compact manifold without boundary, and $I = [-1, 1]$. We consider C^r *one parameter families of diffeomorphisms*, i.e. a C^r function $f: I \times M \to M$ such that for each $\mu \in I$, $f_\mu(\cdot) = f(\mu, \cdot): M \to M$ is a diffeomorphism. We denote the set of such C^r one parameter families of diffeomorphisms with the C^r by $\mathfrak{D}^r$. We let $\mathrm{Diff}^r(M)$ be the C^r diffeomorphisms on M. For $f, g \in \mathfrak{D}^r$, we say that g *is semiconjugate to* f *near* $\mu = 0$, if there is an $\alpha > 0$ and continuous functions $h: [-\alpha, \alpha] \times M \to M$ and injective $k: [-\alpha, \alpha] \to R$ with $k(0) = 0$ such that $h_\mu f_\mu(x) = g_{k(\mu)} h_\mu(x)$ for all $\mu \in [-\alpha, \alpha]$ and $x \in M$. If h_μ is one to one for each $\mu \in [-\alpha, \alpha]$, we say that g *is conjugate to* f near $\mu = 0$. An $f \in \mathfrak{D}^r$ is called *structurally stable* near $\mu = 0$ if there is a neighborhood $\mathfrak{N}$ of f such that, for $g \in \mathfrak{N}$, g is conjugate to f near $\mu = 0$ (and the α is independent of g).

A point $x \in M$ is a *periodic point* of $f \in \mathrm{Diff}^r(M)$ if $f^n(x) = x$ for some $n \geq 1$. For $f \in \mathrm{Diff}^r(M)$, let $\Omega(f) \subset M$ be the *nonwandering set* of f, i.e., $x \in \Omega(f)$ if for every neighborhood U of x we have $U \cap \bigcup\{f^n(U): n \geq 1\} \neq \emptyset$. For $U \subset M$, let $\mathcal{O}(U, f) = \bigcup\{f^n(U): n \in Z\}$.

For $f \in \mathrm{Diff}^r(M)$, we say a periodic point p of period n is a *saddle node* if 1 is an eigenvalue of $Df^n(p)$ of multiplicity one and all the other eigenvalues

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have absolute value different from 1. Then there is a splitting of the tangent space at p, $T_pM = E_p^u \oplus E_p^s \oplus E_p^c$ and a $\lambda < 1$ and $C > 0$ such that for $i > 0$

$$Tf^{ni}v = v \quad \text{for } v \in E_p^c,$$
$$|Tf^{ni}v| \leq C\lambda^i|v| \quad \text{for } v \in E_p^s,$$
$$|Tf^{-ni}v| \leq C\lambda^i|v| \quad \text{for } v \in E_p^u.$$

Here Tf^{ni} is the induced map on tangent vectors (derivative). By changing the metric we can take $C = 1$. The *strong stable manifold* of p are the points that go to p at an exponential rate,

$$W^{ss}(p,f) = \Big\{ y \in M: d\big(f^{ni}(p), f^{ni}(y)\big) \leq d(p,y)\big[(\lambda + 1)/2\big]^i \quad \text{for } i \text{ sufficiently large} \Big\}.$$

$W^{ss}(p,f)$ has dimension equal to $\dim E_p^s$. We can extend $W^{ss}(p,f)$ to a foliation of a neighborhood of p. See §4 where we use methods related to those of [2, Theorem 6]. We call this the *strong stable foliation* in a neighborhood of p. Similarly we have the strong unstable manifold, $W^{uu}(p,f)$, and the strong unstable foliation. There is also an invariant manifold $W^c_{\text{loc}}(p,f)$ tangent to E_p^c called the center manifold.

For $f \in \mathfrak{D}^r$, we say *f adds a saddle node p* at $\mu = 0$ if p is a saddle node of f_0 and there is a neighborhood U of p in M and $\alpha > 0$ such that, for $\mu \in [-\alpha, 0)$, f_μ has no periodic points in U and, for $\mu \in (0, \alpha]$, f_μ has two hyperbolic periodic points in U. We are assuming this is a generic bifurcation as given in [1], [14], or [4].

An $f \in \text{Diff}^r(M)$ has a *hyperbolic structure* on $\Lambda \subset M$ (satisfies Axiom A) if (Ab) the periodic points of f are dense in Λ and (Aa) there are continuous subbundles E^u and E^s of $TM|\Lambda$ and constants $0 < \lambda < 1$ and $C > 0$ such that $TM|\Lambda = E^u \oplus E^s$ and for $n > 0$

$$|Tf^n v| \leq C\lambda^n|v| \quad \text{for } v \in E^s, \qquad |Tf^{-n}v| \leq C\lambda^n|v| \quad \text{for } v \in E^u.$$

It follows that E^u and E^s are invariant under Tf. A Riemannian metric on M is called *adapted* if we can take $C = 1$ above. They always exist [7]. We will always be using a C^∞ adapted metric below. Let d be the associated distance on M.

The *stable manifold* of $x \in M$ is the set of points

$$W^s(x,f) = \{ y \in M: d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty \}.$$

The *unstable manifold* of x is $W^u(x,f) = W^s(x, f^{-1})$. We write

$$W_r^s(x,f) = \{ y \in W^s(x,f): d\big(f^n(x), f^n(y)\big) \leq r \text{ for } n \geq 0 \}.$$

Similarly $W_r^u(x,f)$. For $\Lambda \subset M$ and $\sigma = u, s$ we let $W^\sigma(\Lambda) = \bigcup\{W^\sigma(x,f): x \in \Lambda\}$. If $\mathrm{Diff}^r(M)$ has a hyperbolic structure on $\Omega(f)$, then $M = W^s(\Omega(f)) = W^u(\Omega(f))$. Also $W^s(x,f)$ and $W^u(x,f)$ are then injective by immersed submanifolds for all points $x \in M$. See [7] or [13]. If $f \in \mathrm{Diff}^r(M)$ has a saddle node p and $\Omega(f) - \mathcal{O}(p,f)$ has a hyperbolic structure then it is still true that $M = W^s(\Omega(f))$ but $W^s(p,f)$ is a manifold with boundary $W^{ss}(p,f)$, the strong stable manifold.

Assume f adds a saddle node p at $\mu = 0$ and $\Omega(f_0) - \mathcal{O}(p,f_0)$ has a hyperbolic structure. Then there is a decomposition of the nonwandering set $\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_k$ where the Ω_i are pairwise disjoint and each Ω_i is cloaed, invariant by f_0, and transitive. We say f_0 has the *no cycle property* if it is possible to number the Ω_i so that if $W^u(\Omega_j,f_0) \cap W^s(\Omega_i,f_0) \neq \varnothing$ then $j < i$.

A diffeomorphism $f \in \mathrm{Diff}^r(M)$ such that $\Omega(f)$ has a hyperbolic structure is said to satisfy the strong *transversality condition* if $W^s(x,f)$ and $W^u(x,f)$ intersect transversally at x for all points $x \in M$. A diffeomorphism $f \in \mathrm{Diff}^r(M)$ with a saddle node p and with $\Omega(f) - \mathcal{O}(p,f)$ having a hyperbolic structure is said (in this paper) to satisfy the *strong transversality condition* if $W^s(x,f)$ and $W^u(x,f)$ intersect transversally at x for all $x \in M$ and further that $W^s(x,f)$ are transverse to the strong unstable foliation near p and $W^u(x,f)$ are transverse to the strong stable foliation near p. If $f \in \mathfrak{D}^r$ adds a saddle node at $\mu = 0$ and f_0 satisfies the strong transversality condition and no cycle property then for μ near 0, $\Omega(f_\mu)$ has a hyperbolic structure and f_μ satisfies the strong transversality condition. This fact is implicit in our proof below. Also see [5].

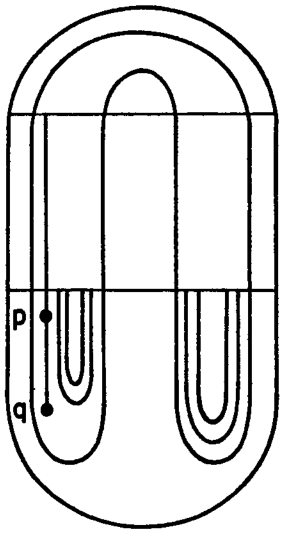
For $f \in \mathfrak{D}^r$, we use the induced parameterized metric (distance) on M which for $\mu \in I$ is given by

$$d_{f\mu}(x,y) = \sup\left\{d\left(f_\mu^n(x), f_\mu^n(y)\right): n \in Z\right\}.$$

This is similar to the d_f metric introduced by Robbin and used in [11].

THEOREM. *Assume $f \in \mathfrak{D}^\infty$ adds a saddle node p at $\mu = 0$. Also assume $\Omega(f_0) - \mathcal{O}(p,f_0)$ has a hyperbolic structure, and f_0 satisfies the strong transversality condition and no cycle condition. Then f is structurally stable near $\mu = 0$.*

An example where this theorem applies but that of [6] does not can be constructed by adding a saddle node to the horseshoe on the two spheres [13]. The original diffeomorphism has a sink, a source and a horseshoe. The saddle node p is added near the sink in such a manner that the "hooks" from the horseshoe still go to the sink q.



For a more complete introduction to global dynamical systems see [13], [7], or [10]. For a more complete discussion of the various types of stability for one parameter families of diffeomorphisms see [6]. For an introduction to the various types of global bifurcation see [4] and [5]. In §2, we sketch the proof of structural stability using compatible families of unstable disks indicating the changes necessary. For more details see [11]. For an introduction to this method see [12].

The main technique from analysis that we use is that of strong unstable manifolds. We need a result of the following type. Let $V \subset M$ be an invariant compact C^1 manifold for f and have a splitting $TM|V = E^u \oplus TV \oplus E^c$ such that E^c may have a weak expansion but E^u is much more expanding than other directions. Then there is a C^1 manifold through V tangent to E^u called the strong unstable, $W^{uu}(V, F)$. See [2, Theorem 6.1]. Moreover if g is C^1 near f and if we assume g has an invariant manifold V' that is C^1 near V, then $W^{uu}(V', g)$ is C^1 near $W^{uu}(V, f)$. We do not know a reference for this last part of the theorem. Actually we need the theorem in the Lipschitz category with uniformities instead of compactness. Therefore we prove these results in §4. Certainly the theory of stable and unstable manifolds is very old, going back to Hadamard and Perron. The reader might check [15, Chapter 7] for a more complete history.

2. Global aspects of the proof. In this section we sketch the proof of structural stability using compatible families of unstable disks given in [11], and indicate the changes necessary. See [11] for more details or [12] for an introduction to this method.

We are given $f \in \mathfrak{D}^r$ such that f adds a saddle node, p, at $\mu = 0$,

$\Omega(f_0) - \mathcal{O}(p, f_0)$ has a hyperbolic structure, and f_0 satisfies the strong transversality condition. Also assume f_0 satisfies the no cycle property, so $\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_K$ where the Ω_i are pairwise disjoint and each Ω_i is closed, invariant by f_0, and transitive (f_0 has a dense orbit.) Each Ω_i is called a *basic set*. Since f_0 satisfies the strong transversality condition, it is possible to index the Ω_i so that if $W^u(\Omega_j, f_0) \cap W^s(\Omega_i, f_0) \neq \varnothing$ then $j < i$. The saddle node is one of the basic sets, $\Omega_q = \mathcal{O}(p, f_0)$.

For $g \in \mathfrak{D}^r$ near f, we are given by [6] a conjugacy on the local center manifold of p for f to that of p' for g, h_1: $W^c_{\text{loc}}(0, p, f) \to W^c_{\text{loc}}(a, p', g)$ and k: $[-\alpha, \alpha] \to [-\alpha', \alpha']$ such that $h_1(\mu, f_\mu(x)) = g(k(\mu), h_{1\mu}(x))$. Here $W^c_{\text{loc}}(\ , p, f)$ is two dimensional including the parameter direction. We reparameterize g so we can take $k(\mu) = \mu$ and $a = 0$ in the rest of the proof.

By *compatible families of unstable disks* we mean that there are neighborhoods U_i of Ω_i and families $\{D_i^u(\mu, x, g)$: $x \in \mathcal{O}(U_i, f_\mu)\}$ for $1 < i < K$ such that

(0) for f and $x \in W^u(\Omega_i, f_\mu)$ we have $D_i^u(\mu, x, f) \subset W^u(x, f_\mu)$;

(1) $D_i^u(\mu, x, g)$ is a C^1 disk near x with dimension equal dim E^u_y for $y \in \Omega_i$ and the disk depends continuously on x and μ in the C^1 topology;

(2) (invariance) $g_\mu D_i^u(\mu, x, g) \supset D_i^u(\mu, f_\mu(x), g)$;

(3) (compatibility) if $i < j$ and $x \in \mathcal{O}(U_i, f_\mu) \cap \mathcal{O}(U_j, f_\mu)$ then $D_i^u(\mu, x, g) \supseteq D_j^u(\mu, x, g)$;

(4) the family $\{D_i^u(\mu, x, g)$: $x \in U_i\}$ is $d_{f\mu}$ Lipschitz with uniform Lipschitz constant over U_i (as explained in §§3–4);

(5) the family for g, $\{D_i^u(\mu, x, g)$: $x \in U_i\}$, is both C^0 and $d_{f\mu}$ Lipschitz near the family for f, $\{D_i^u(\mu, x, f)$: $x \in U_i\}$, for g near f;

(6) the d_f Lipschitz jet of the family for f varies uniformly continuously along fibers. (This is a technical point explained in [11] to make the induction work.)

We construct the families of unstable disks for g near f and μ near 0 as in [11, §5]. We use induction on k, proving that conditions (1)–(6) are satisfied for $1 < i, j < k$. When $k = 1$, Ω_1 is a repellor (unstable manifolds form a neighborhood but Ω_i is not necessarily a point). We can construct the disks on a neighborhood U_1 by the generalized unstable manifold theorem [11, Theorem 3.1 and 3.2]. Assuming (1)–(6) are satisfied up to $k - 1$, we constant over U_i (as explained in §§3–4);

We take a neighborhood U_k of Ω_k and differentiable subbundles of $TM|U_k$, E_k^{ud} and E_k^{sd}, such that $E_k^{ud}|\Omega_k$ approximates $E^u|\Omega_k$ and $E_k^{sd}|\Omega_k$ approximates $E^s|\Omega_k$. By taking U_k small enough f_μ satisfies hyperbolic estimates with respect to the splitting $TM|U_k = E_k^{ud} \oplus E_k^{sd}$. (Actually we need to take continuous extensions and then approximate them by differentiable subbundles later in the construction. The reader can consult

[11, §5].) We take a fundamental domain F_k^s of $W^s(\Omega_k)$,

$$F_k^s = \text{closure}\{ W_\delta^s(\Omega_k) - fW_\delta^s(\Omega_k)\}.$$

We take V_k^s a neighborhood of F_k^s that is disjoint from Ω_k (a fundamental neighborhood). Using a procedure introduced by Palis [8] or [9] we can construct disks $D_k^u(\mu, x, g)$ for $x \in V_k^s$ that satisfy conditions (1)–(6), [11, Lemma 5.3]. The generalized unstable manifold theorem [11, Theorems 3.1 and 3.2] says that these disks extend to a neighborhood U_k of Ω_k and satisfy (1)–(6).

We continue by induction on k until $k = q$ and we are at the saddle node, $\Omega_q = \mathcal{O}(p, f_0)$. We want to construct disks whose dimension equals the dimension of the strong unstable manifold of p. We cannot just construct unstable disks on a fundamental neighborhood V_q^s and extend these to a neighborhood of Ω_q because the strong unstable manifold theorem does not give permanence under a small perturbation. In the next section we show how to use the conjugacy on the local center manifold constructed in [6] to construct a point through which the (strong) unstable disk passes. Picking this point correctly gives permanence for g near f. Specifying this point is like specifying the component of the unstable disks in the center direction. The assumption that the unstable manifolds of Ω_i for $i < q$ are transverse to the strong stable foliation in a neighborhood of Ω_q implies we are free to specify this component in the center direction and still get disks that are compatible with the earlier families.

For $k > q$, the proof of the induction step is as in [11]. The assumption, that the stable manifolds of Ω_i for $i > q$ are transverse to the strong unstable foliation in a neighborhood of Ω_q, implies that the unstable disks $D_q^u(\mu, x, f)$ are transverse to the stable manifolds of Ω_i.

Once we have constructed all the unstable disks, we reverse the process and look at the map $f_\mu^{-1} \times g_\mu^{-1}$ on the unstable disks,

$$\begin{array}{ccc} \bigcup\{x\} \times D_i^u(\mu, x, g) & \xrightarrow{f_\mu^{-1}\times g_\mu^{-1}} & \bigcup\{x\} \times D_i^u(\mu, x, g) \\ \downarrow & & \downarrow \\ U_i & \xrightarrow{f_\mu^{-1}} & f_\mu^{-1}U_i \end{array}$$

The map is a contraction on fibers even for $i = q$. Therefore we can use [11, Theorems 3.1 and 3.2] to get an invariant section exactly as in [11, §6]. We start constructing an invariant section on U_K and continue back by induction. We get h_μ for $\mu \in [-\alpha, \alpha]$ and $\alpha > 0$ small enough such that $h_\mu f_\mu = g_\mu h_\mu$ and h_μ is $d_{f\mu}$ Lipschitz near the identity. Then [11, Lemma 6.2] proves h_μ is one-to-one. Therefore h_μ is a conjugacy.

3. Geometric aspects of the construction near the saddle node. From the results of [6], we know that there is a conjugacy, h_1, C^0 near the identity from the local center manifold of f to the local center manifold of g,

$$h_1\colon W^c_{\mathrm{loc}}(0,p,f) \to W^c_{\mathrm{loc}}(0,p',g)$$

with $h_1(\mu, x) = (\mu, h_{1\mu}(x))$. Here p' is the saddle node for g. Note, W^c_{loc} is two dimensional because it includes the parameter direction. Also remember we have already adjusted the parameterization of g in §2, so h_1 preserves μ.

We want to use h_1 on $W^c_{\mathrm{loc}}(p,f)$ to construct a function h_3 defined on a neighborhood U_q of $\mathcal{O}(p,f_0)$. We first construct a strong unstable foliation for both f and g (tubular families). We use these foliations to extend h_1 to a conjugacy

$$h_2\colon W^{cu}_{\mathrm{loc}}(0,p,f) \to W^{cu}_{\mathrm{loc}}(0,p',g).$$

Next we construct a strong stable foliation for f and g in a neighborhood of p. Then we let $h_{3\mu}(x)$ be the point where the appropriate leaf (using h_2) of the strong stable foliation of g intersects the unstable disk $D^u_i(\mu, x, g)$ if $i < q$ and $x \in \mathcal{O}(U_i, f_\mu)$. Then we construct compatible (strong) unstable disks for g such that

$$h_{3\mu}(x) \in D^u_q(\mu, x, g) \quad \text{and} \quad D^u_q(\mu, x, g) \subset D^u_i(\mu, x, g)$$

if $i < q$ and $x \in \mathcal{O}(U_i, f_\mu)$. The condition that $h_{3\mu}(x) \in D^u_q(\mu, x, q)$ replaces the usual condition in constructing strong unstable manifolds that the leaf goes through x, e.g., $x \in D^u_q(\mu, x, f)$ for the unperturbed f. Notice that the construction of h_3 is similar to methods in [6] using tubular families. However, we need to prove we can make everything $d_{f\mu}$ Lipschitz.

We now proceed to fill in more details of the above construction, but we leave the necessary analysis to prove h_3 and D^u_q are $d_{f\mu}$ Lipschitz until §4. We let U_q be a neighborhood of p in M such that we have the estimates on U_q for f_μ used in §4.

Using the standard methods of the strong unstable manifold theory, we can construct continuous families of C^1 disks

$$\{B^{uu}(\mu, x, f)\colon (\mu, x) \in W^c_{\mathrm{loc}}(0,p,f)\}$$

and

$$\{B^{uu}(\mu, y, g)\colon (\mu, y) \in W^c(0,p',g)\}$$

where $y \in B^{uu}(\mu, y, g)$. These continuous foliations of the center unstable manifolds $W^{cu}_{\mathrm{loc}}(0,p,f)$ and $W^{cu}_{\mathrm{loc}}(0,p',g)$ are called tubular families in the terminology of Palis [8] or [9]. These disks are not necessarily compatible with the earlier unstable disk families. Let $\pi^u\colon W^{cu}_{\mathrm{loc}}(0,p,f) \to W^c_{\mathrm{loc}}(0,p,f)$ be the projection along the fibers $B^{uu}(\mu, x, f)$. Let $D^{uu}(\mu, x, g) \subset B^{uu}(h_1\pi^u(\mu, x), g)$ be a disk near x. For $g = f$ we get $x \in D^{uu}(\mu, x, f)$. By construction the disks $D^{uu}(\mu, x, g)$ are C^0 near the disks $D^{uu}(\mu, x, f)$.

Let

$$\rho_{f\mu}(x, y) = \sup\left\{d\left(f_\mu^n(x), f_\mu^n(y)\right): f_\mu^i(x), f_\mu^i(y) \in U_q \text{ for } i \text{ between } 0 \text{ and } n\right\}$$

and

$$\rho_{g\mu}(x, y) = \sup\left\{d\left(g_\mu^n(x), g_\mu^n(y)\right): g_\mu^i(x), g_\mu^i(y) \in U_q \text{ for } i \text{ between } 0 \text{ and } n\right\}.$$

We claim the disk family $\{D^{uu}(\mu, x, g)\}$ is $\rho_{f\mu}$ Lipschitz near the family $\{D^{uu}(\mu, x, f)\}$. In §4 we prove that the family $\{B^{uu}(h_1(\mu, x), g): (\mu, x) \in W^c_{\text{loc}}(0, p, f)\}$ is $\rho_{f\mu}$ Lipschitz near the family $\{B^{uu}(\mu, x, f): (\mu, x) \in W^c_{\text{loc}}(0, p, f)\}$ using the fact that h_1 is $\rho_{f\mu}$ Lipschitz near the identity. This means that there are functions $w_f, w_g: W^c_{\text{loc}}(0, p, f) \times E^u_p \to E^c_p \times E^s_p$ such that $w_f - w_g$ is Lipschitz small with the usual norm on $E^c_p \times E^s_p$ and $\max\{\rho_{f\mu}(x, y), |v - v'|\}$ on the domain where $(\mu, x, v), (\mu, y, v') \in W^c_{\text{loc}}(0, p, f) \times E^u_p$. We also prove in Lemma 7 of §4 that there is a uniform bound given by

$$\max\left\{\rho_{f\mu}(x, y), |v - v'|\right\} < C\rho_{f\mu}\left(x + v + w_f(x, \sigma), y + v' + w_f(y, v')\right).$$

Letting $z = x + v + w_p(x, v)$ and $z' = y + v' + w_f(y, v')$, we have $\pi^u z = x$, $\pi^u z' = y$ and $\max\{\rho_{f\mu}(\pi^u z, \pi^u z'), |v - v'|\} < C\rho_{f\mu}(z, z')$. This proves the family $\{D^{uu}(\mu, z, g)\}$ is $\rho_{f\mu}$ Lipschitz near the family $\{D^{uu}(\mu, z, f)\}$.

Next we want an invariant section of $\bigcup \{x\} \times D^{uu}(\mu, x, g)$ under the map $f_\mu^{-1} \times g_\mu^{-1}$. Since f^{-1} is not overflowing on $W^{cu}_{\text{loc}}(0, p, f)$, we first construct a section for

$$x \in F^{cu} = \text{closure}\left[W^{cu}_{\text{loc}}(0, p, f) - f^{-1}W^{cu}_{\text{loc}}(0, p, f)\right]$$

and then extend this section to all of $W^{cu}_{\text{loc}}(0, p, f)$ by the generalized stable manifold theorem. As in [11] we get a section h_2 that is $\rho_{f\mu}$ Lipschitz near the identity.

We then construct strong stable foliations

$$\left\{B^{ss}(\mu, x, f): (\mu, x) \in W^{cu}_{\text{loc}}(0, p, f)\right\}$$

and

$$\left\{B^{ss}(\mu, x, g): (\mu, x) \in W^{cu}_{\text{loc}}(0, p', g)\right\}.$$

Let $\pi^s: U_q \to W^{cu}(0, p, f)$ be projection along the fibers $B^{ss}(\mu, x, f)$. Let $D^{ss}(\mu, x, g) \subset B^{ss}(h_2\pi^s(\mu, x), g)$ be a disk near (μ, x). As above the family $\{D^{ss}(\mu, x, g)\}$ is $\rho_{f\mu}$ Lipschitz near the family $\{D^{ss}(\mu, x, f)\}$.

For (μ, x) in a fundamental neighborhood F^s we can pick $h_{30}(\mu, x) \in D^{ss}(\mu, x, g)$ and also so $h_{30}(\mu, x) \in D^u_i(\mu, x, g)$ if $(\mu, x) \in \mathcal{O}(U_i, f_\mu)$ and $i < q$. This h_{30} is $\rho_{f\mu}$ Lipschitz near the identity. Using the map

$$f \times g: \bigcup\,(\mu, x) \times D^{ss}(\mu, x, g) \to \bigcup\,(\mu, x) \times D^{ss}(\mu, x, g),$$

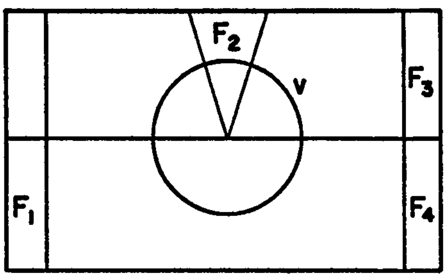
we can extend h_{30} to a section h_3 with $h_3(\mu, x) \in D^{ss}(\mu, x, g)$ for $(\mu, x) \in U_q$. This section is $\rho_{f\mu}$ Lipschitz near the identity. See [11].

We then construct the compatible family of (strong) unstable disks, $\{D_q^u(\mu, x, g)\}$, such that $h_3(\mu, x) \in D_q^u(\mu, x, g)$, and these disks are compatible with the families $\{D_i^u(\mu, x, g)\}$ for $i < q$. We first do this over the fundamental neighborhood F^s and then extend to U_q using the methods of §4. The family $\{D_q^u(\mu, x, g)\}$ is $\rho_{f\mu}$ Lipschitz and so $d_{f\mu}$ Lipschitz near the family $\{D_q^u(\mu, x, f)\}$. This completes the construction of the unstable disk family near the saddle node.

4. Analytic lemmas.

LEMMA 1. *The conjugacy* h_1: $W^c_{\text{loc}}(0, p, f) \to W^c_{\text{loc}}(0, p', g)$ *is* $\rho_{f\mu}$ *Lipschitz near the identity.*

PROOF. Take local coordinates at $(0, p)$. Adjust the diffeomorphism g so that $(0, p') = (0, p)$. This can be done by a small translation. Let V be a neighborhood in R^2 of $(0, p)$.



The conjugacy h_1 is constructed so it is differentiable on $F_1 = \text{closure}\{fW^c_{\text{loc}}(0, p, f) - W^c_{\text{loc}}(0, p, f)\}$, a wedge F_2 for $\mu > 0$, and $F_3 = \text{closure}\{f^{-1}W^c_{\text{loc}}(0, p, f) - W^c_{\text{loc}}(0, p, f)\} \cap \{\mu > 0\}$. Let $F_4 = \{f^{-1}W^c_{\text{loc}}(0, p, f) - W^c_{\text{loc}}(0, p, f)\} \cap \{\mu < 0\}$. Then [6] prove that if there are constants $0 < C_1 < C_2 < \infty$ such that for $x, y \in F_1$ and $f^n(x), f^n(y) \in F_4$ then $C_1 < |f^n(x) - f^n(y)|/|x - y| < C_2$ and similarly for g. This gives

$$|h_1 f^n(x) - h_1 f^n(y)| = |g^n h_1(x) - g^n h_1(y)| < C_2|h_1(x) - h_1(y)|$$
$$< C_2 C|x - y| < C_2 C C_1^{-1}|f^n(x) - f^n(y)|$$

where C is given by the differentiability on F_1. Therefore h_1 is Lipschitz on F_4. There is a bound on the number of iterates from any point in $W^c_{\text{loc}}(0, p, f) - V$ to F_1, F_2, F_3, or F_4. Therefore h_1 is Lipschitz outside of V. For $x, y \in V$ there is an η such that if $x, y \in F_1, f^n(x), f^n(y) \in V$ then $|g^n h_1(x) - g^n h_1(y)| < \eta|h_1(x) - h_1(y)|$. Then

$$|h_1 f^n(x) - h_1 f^n(y)| = |g^n h_1(x) - g^n h_1(t)| < \eta |h_1(x) - h_1(y)| < \eta C|x-y| < \eta C \rho_f(f^n(x), f^n(y)).$$

Therefore h_1 is ρ Lipschitz for $\mu < 0$. A similar argument applies to other regions of V. Q.E.D.

We treat the case when p is a fixed point. A periodic point is an easy generalization. We also omit writing the parameter μ although this adds no real complication. We take local coordinates at p, defined on U. We define $F\colon U \times D^n \to f(U) \times R^n$ by $F(x,y) = (f(x), f(x+v) - f(x))$. We will construct strong unstable disks when there is a ρ Lipschitz conjugacy h_1: $W^c_{\text{loc}}(p,f) \to W^c_{\text{loc}}(p', g)$. We define G by

$$G(x,v) = \big(f(x), g(h_1(x) + v) - h_1 f(x)\big).$$

Remember $h_1 f(x) = g h_1(x)$ so G preserves the zero section. (This is a different definition of G than in [11]. It is used because of the special nature of strong unstable manifolds.)

Using the local coordinates we let $\pi^u\colon TU \to E^u_p$ be projection onto the unstable component (along $E^c_x + E^s_x$ to E^u_x and then translate to E^u_p). We let $\pi^c\colon TU \to E^c_p \times E^s_p$ be the other projection. We use the letter c to emphasize this direction is not necessarily a contracting direction. We have $\|(Df(x)|E^u_x)^{-1}\| < \lambda_u < 1$ and $\|Df(x)|E^c_x \times E^s_x\| < \lambda_c$ where $\lambda_c < \lambda_u^{-1}$.

As in [11] we construct differentiable disks for points $x \in F_c =$ closure$\{fW^c_{\text{loc}}(p,f) - W^c_{\text{loc}}(p,f)\}$. These can be given by a section $w_0\colon E^u(r)|F_c \to TU$, where $E^u(r)$ is the disk bundle of radius r. We then consider trial sections (or trial disks) $w\colon E^u(r)|U \to TU$ such that $w = w_0$ on the domain of w_0. More specifically, we let $\Sigma = \{w\colon E^u(r) \to TU(r)| w = w_0$ on domain w_0, $L_{\text{fib}}(w) < L_0$, $L_{\text{hor}}(w; v_x) < L_0|v_x|$, $w(0) = 0\}$. Here

$$L_{\text{fib}}(w) = \sup\{|\pi^c w(x,v) - \pi^c w(x,y)|/|v-y| \colon x \in U\}$$

and

$$L_{\text{hor}}(w; v_k) = \limsup\{|\pi^c w(x,v) - \pi^c w(y,v)|/|x-y| \colon y \to x\}.$$

We use the norm on Σ given by

$$\|w\| = \sup\{|\pi^c w(x,v)|/|v| \colon v \neq 0, (x,v) \in E^u(r)|U\}.$$

This makes Σ a complete metric space. We look at the graph transform by F on Σ, $F_\#$. Working in the differentiable category, Fenichel in [2, Theorem 6] shows $F_\#$ is a contraction. We repeat the analysis here in Lemmas 2, and 3, because we rely heavily on Lemma 2 later to show the disks are ρ Lipschitz close.

LEMMA 2. *Let $w \in \Sigma$ and $H = (F_u w)^{-1}\colon E^u \to E^u$. Then*

$$|\pi^u H(x,v) - \pi^u H(y,v)| < \varepsilon' |\pi^u H(x,u)| \rho_f(x,y)$$

where ε' is small. The same estimate holds if $H = (G_u w)^{-1}$. (Here F_u: $TU \to E^u$. For notation see [11].)

PROOF. Let $H(x, v) = (m, z)$ and $H(y, v) = (q, t)$. Then

$$\begin{aligned}
0 &= \pi^u F_u w(m, z) - \pi^u F_u w(q, t) \\
&= \{\pi^u A_m^{uu} z - \pi^u A_m^{uu} t\} - \{(\pi^u A_u - \pi^u F_u) w(m, z) \\
&\quad - (\pi^u A_u - \pi^u F_u)(m, t, \pi^c w(q, t))\} \\
&\quad - \{\pi^u F_u w(q, t) - \pi^u F_u(m, t, \pi^c w(q, t))\}.
\end{aligned}$$

So

$$\begin{aligned}
0 &\geq |\pi^u A_m^{uu} z - \pi^u A_m^{uu} t| \\
&\quad - |(\pi^u A_u - \pi^u F_u) w(m, z) - (\pi^u A_u - \pi^u F_u)(m, t, \pi^c w(q, t))| \\
&\quad - |\pi^u F_u w(q, t) - \pi^u F_u(m, t, \pi^c w(q, t))| \\
&\geq L_{\mathrm{fib}}(\pi^u A_m^{uu})|z - t| \\
&\quad - L_{\mathrm{fib}}(\pi^u A_u - \pi^u F_u)\{|z - t| + |\pi^c w(m, z) - \pi^c w(q, z)| \\
&\qquad\qquad + |\pi^c w(q, z) - \pi^c w(q, t)|\} \\
&\quad - \sup\{\|D_1 D_{\mathrm{fib}}(\pi^u F_u)\|\} d(q, m)\{|t| + |\pi^c w(q, t)|\} \\
&\geq \lambda_u^{-1}|z - t| - \varepsilon\{|z - t| + L_0|z| d(m, q) + L_0|z - t|\} \\
&\quad - \sup\{\|D_1 D_{\mathrm{fib}}(\pi^u F_u)\|\} d(q, m)\{|t| + L_0|t|\} \\
&\geq |z - t|\{\lambda_u^{-1} - \varepsilon - \varepsilon L_0\} - d(m, q)|z|\{\varepsilon L_0 - \varepsilon(1 + L_0)|t|/|z|\}.
\end{aligned}$$

Here D_1 is the derivative with respect to the coordinate in U and D_{fib} is the fiber derivative along $T_m U$. We also made a change of scale of the norm on the fibers so that $\sup\{\|D_1 D_{\mathrm{fib}} \pi^u F_u\|\} < \varepsilon$. Using the fact that $z = \pi^u H(x, v)$, $t = \pi^u H(y, v)$, $m = f^{-1}(x)$, and $q = f^{-1}(y)$, and solving for $|z - t| = |\pi^u h(x, v) - \pi^u h(y, v)|$ we get

$$\begin{aligned}
&|\pi^u H(x, v) - \pi^u H(y, v)| \\
&\quad \leq d(f^{-1}(x), f^{-1}(y))|\pi^u H(x, v)| \\
&\qquad \cdot \{\lambda_u^{-1} - \varepsilon - \varepsilon L_0\}^{-1}\{\varepsilon - \varepsilon(1 + L_0)|t|\,|z|^{-1}\} \\
&\quad \leq \rho(x, y)|\pi^u H(x, v)|\varepsilon'.
\end{aligned}$$

We can make ε' small by making ε small.

To check for $H = (G_u w)^{-1}$ the only difference is the term

$$\left|\pi^u G_u w(q,t) - \pi^u G_u w(m,t,\pi^c w(q,t))\right|$$
$$\leq \left|g(h_1(q)+a) - gh_1(q) - g(h_1(m)+a) + gh_1(m)\right|$$

where $a = (t, \pi^c w(q,t))$. But this is

$$\leq |a| \sup_{|b| \leq |a|} \left\|Dg(h_1(q)+b) - Dg(h_1(m)+b)\right\|$$
$$\leq |a| \left|h_1(q) - h_1(m)\right| \sup_{|b| \leq |a|} \left\|D^2 g(x+b)\right\|.$$

The $|a| = |(t, \pi^c w(q,t))| \leq |t| + |\pi^c w(q,t)|$, and

$$\begin{aligned} \left|h_1(q) - h_1(m)\right| &\leq \left|h_1(a) - q - h_1(m) + m\right| + |q-m| \\ &\leq \varepsilon\rho_f(q,m) + d(q,m) \\ &\leq (1+\varepsilon)\rho_f(q,m). \end{aligned}$$

LEMMA 3. *The graph transform preserves* Σ.

PROOF. From the estimates in [11, §3], we only need to show that

$$\left|\pi^s(F_\# w)(x,v) - \pi^s(F_\# w)(y,v)\right| \leq d(x,y)|v|L_0.$$

$$\begin{aligned} &\left|F_s wH(x,v) - F_s wH(y,v)\right| \\ &\quad\leq \left|F_s w\left(f^{-1}(x), \pi^u H(x,v)\right) - F_s w\left(f^{-1}(y), \pi^u H(x,v)\right)\right| \\ &\qquad + \left|F_s w\left(f^{-1}(y), \pi^u H(x,v)\right) - F_s w\left(f^{-1}(y)\right), \pi^u H(y,v)\right| \\ &\quad\leq L(F_s)L_0\left|\pi^u H(x,v)\right| d\left(f^{-1}(x), f^{-1}(y)\right) \\ &\qquad + L_{\mathrm{fib}}(F_s)L_{\mathrm{fib}}(w)\left|\pi^u H(x,v) - \pi^u H(y,v)\right| \\ &\quad\leq (\lambda_c+\varepsilon)L_0\left|\pi^u H(x,v)\right| d\left(f^{-1}(x), f^{-1}(y)\right) \\ &\qquad + (\lambda_c+\varepsilon)L_0\varepsilon'\left|\pi^u(x,v)\right|\rho(x,y) \\ &\quad\leq (\lambda_c+\varepsilon)L_0\lambda_u(1-\lambda_u\varepsilon L_0)^{-1}|v|\rho(x,y)(1+\varepsilon') \\ &\quad\leq (L_c+\varepsilon)\lambda_u\left[(1-\lambda_u\varepsilon L_0)^{-1}(1+\varepsilon')\right]L_0|v|d(x,y). \end{aligned}$$

We know $(\lambda_c+\varepsilon)\lambda_u < 1$ and if ε and ε' are small enough then the term $[\,] < 1$. Therefore F preserves the sections of this Lipschitz type. Q.E.D.

This part of the proof is fairly standard. We have shown there are strong unstable disks for F through x and strong unstable disks for G through $h_1(x)$. What we need now to show is that these disks are ρ Lipschitz close. We do this using Lipschitz jets exactly as we did in [11]. (This is based on the methods in [3].) We use the notation of [11]. We need only show $(\mathcal{G}F)_\#$ and $(\mathcal{G}G)_\#$ are contractions (Lemma 4), the invariant section of $(\mathcal{G}F)_\#$, σ^F, is uniformly continuous on fibers (Lemma 5), and that $\|(\mathcal{G}F)_{\#}\sigma^F - (\mathcal{G}G)_{\#}\sigma^F\|$ is small with the new norm we use here (Lemma 6).

LEMMA 4. *Let $(\mathcal{G}F)_{\#}$ and $(\mathcal{G}G)_{\#}$ be the graph transforms on Lipschitz jets as defined in* [11]. *Then both are contractions with respect to the norm that divides the horizontal Lipschitz constant by $|v|$.*

PROOF. We look at the case for $(\mathcal{G}F)_{\#}$. It is enough to look at the horizontal direction because the vertical direction is contained in [11, Theorem 3.1]. Let w_i be the representatives of σ_i and h_i the right inverse of $F_u \circ w_i$.

$$\begin{aligned}
&\left|\{(\mathcal{G}F)_{\#}\sigma_1\}(x, v) - \{(\mathcal{G}F)_{\#}\sigma_2\}(x, v)\right|_{\text{hor}} \\
&\quad = \limsup_{y\to x} \frac{|F_c w_1 H_1(y, v) - F_c w_2 H_2(y, v) - F_c w_1 H_1(x, v) + F_c w_2 H_2(x, v)|}{|v|\rho(x, y)} \\
&\quad = \limsup_{y\to x} \frac{|F_c w_1 H_1(y, v) - F_c w_2 H_2(y, v)|}{|v|\rho(x, y)} \\
&\quad < L_{\text{fib}}(F_c) \limsup_{y\to x} \frac{|w_1 H_1(y, v) - w_2 H_2(y, v)|}{|v|\rho(x, y)} \\
&\quad < L_{\text{fib}}(F_c) \limsup_{y\to x} \frac{|w_1 H_1(y, v) - w_2 H_1(y, v)|}{|v|\rho(x, y)} + \frac{|w_2 H_1(y, v) - w_2 H_2(y, v)|}{|v|\rho(x, y)}.
\end{aligned}$$

First,

$$\begin{aligned}
&\limsup_{y\to x} \frac{|w_1 H_1(y, v) - w_2 H_1(y, v)|}{|v|\rho(x, y)} \\
&\quad = \limsup \frac{|w_1 H_1(y, v) - w_2 H_1(y, v) - w_1 H_1(x, v) + w_2 H_1(x, v)|}{|v|\rho(x, y)} \\
&\quad < \limsup \frac{|(w_1 - w_2)(f^{-1}y, \pi^u H_1(y, v)) - (w_1 - w_2)(f^{-1}x, \pi^u H_1(y, v))|}{|v|\rho(x, y)} \\
&\qquad + \frac{|(w_1 - w_2)(f^{-1}x, \pi^u H_1(y, v)) - (w_1 - w_2)(f^{-1}x, \pi^u H_1(x, v))|}{|v|\rho(x, y)} \\
&\quad < \limsup \frac{L_{\text{hor}}(w_1 - w_2; \pi^u H_1(y, v)) d\left(f^{-1}(y), f^{-1}(x)\right)}{|v|\rho(x, y)} \\
&\qquad + \frac{L_{\text{fib}}(w_1 - w_2)|\pi^u H_1(y, v) - \pi^u H_1(x, v)|}{|v|\rho(x, y)} \\
&\quad < \limsup \frac{|\sigma_1 - \sigma_2|_{\text{hor}} |\pi^u H_1(y, v)| d\left(f^{-1}(y), f^{-1}(x)\right)}{|v|\rho(x, y)} \\
&\qquad + \frac{|\sigma_1 - \sigma_2|_{\text{fib}} |\pi^u H_1(x, v)| \rho(x, y)\varepsilon'}{|v|\rho(x, y)} \\
&\quad < |\sigma_1 - \sigma_2|_{\text{hor}}(\lambda_u + \varepsilon) + |\sigma_1 - \sigma_2|_{\text{fib}}(\lambda_u + \varepsilon)\varepsilon'.
\end{aligned}$$

Next,

$$\limsup_{y\to x}\frac{|w_2H_1(y,v)-w_2H_2(y,v)|}{|v|\rho(x,y)}$$
$$< L_{\text{fib}}(w_2)\limsup\frac{|\pi^uH_1(y,v)-\pi^uH_2(y,v)|}{|v|\rho(x,y)}$$
$$< L_{\text{fib}}(w_2)\limsup\frac{|\pi^uH_1F_uw_2h_2(y,v)-\pi_uh_1F_uw_1H_2(y,v)|}{|v|\rho(x,y)}$$
$$< L_{\text{fib}}(w_2)L_{\text{fib}}(H_1)$$
$$\cdot\limsup\frac{|\pi^u(F_u-A_u)w_2H_2(y,v)-\pi^u(F_u-A_u)w_1H_2(y,v)|}{|v|\rho(x,y)}$$
$$< L_{\text{fib}}(w_2)L_{\text{fib}}(H_1)L_{\text{fib}}(F_u-A_u)$$
$$\cdot\limsup\frac{|\pi^uw_2H_2(y,v)-\pi^uw_1H_2(y,v)|}{|v|\rho(x,y)}$$
$$< L_{\text{fib}}(w_2)L_{\text{fib}}(H_1)L_{\text{fib}}(F_u-A_u)$$
$$\cdot\limsup\frac{|\pi^u(w_2-w_1)H_2(y,v)-\pi^u(w_2-w_1)H_2(x,v)|}{|v|\rho(x,y)}$$
$$< L_{\text{fib}}(w_2)L_{\text{fib}}(H_1)L_{\text{fib}}(F_u-A_u)$$
$$\cdot\left\{\limsup\frac{|\pi^u(w_2-w_1)H_2(y,v)-\pi^u(w_2-w_1)(f^{-1}x,\pi^uH_2(y,v))|}{|v|\rho(x,y)}\right.$$
$$\left.+\frac{\pi^u(w_2-w_1)(f^{-1}x,\pi^uH_2(y,v))-\pi^u(w_2-w_1)H_2(x,v)|}{|v|\rho(x,y)}\right\}$$
$$< L_{\text{fib}}(w_2)L_{\text{fib}}(H_1)L_{\text{fib}}(F_u-A_u)$$
$$\cdot\limsup\left\{\frac{|\sigma_2-\sigma_1|_{\text{hor}}|\pi^uH_2(y,v)|\rho(f^{-1}y,f^{-1}x)}{|v|\rho(x,y)}\right.$$
$$\left.+\frac{L_{\text{fib}}(w_2-w_1)|\pi^uH_2(y,v)-\pi^uH_2(x,v)|}{|v|\rho(x,y)}\right\}$$
$$< L_{\text{fib}}(w_2)L_{\text{fib}}(H_1)L_{\text{fib}}(F_u-A_u)|\sigma_2-\sigma_1|\{(\lambda+\varepsilon)+(\lambda_u+\varepsilon)\varepsilon'\}$$
$$< L_0\lambda(1-\varepsilon\lambda L_0)^{-1}\varepsilon|\sigma_2-\sigma_1|(\lambda_u+\varepsilon)(1+\varepsilon').$$

Therefore

$$|\{(\mathcal{G}F)_{\#}\sigma_1\}(x,v) - \{(\mathcal{G}F)_{\#}\sigma_2\}(x,v)|_{\text{hor}}$$
$$< |\sigma_2 - \sigma_1|(\lambda_u + \varepsilon)\{1 + \varepsilon' + \varepsilon L_0\lambda(1 - \varepsilon L_0)^{-1}(1 + \varepsilon')\}.$$

The last term is about one and $(\lambda_u + \varepsilon) < 1$ so we have a contraction.

LEMMA 5. *The Lipschitz jet σ^F is uniformly continuous on fibers.*

PROOF. As in [11], it is enough to show that $\mathcal{G}F$ is uniformly continuous on fibers. The calculations are much like those, but we need to use the new norm on the horizontal directions as we have done above. We leave the estimates to the reader.

LEMMA 6. $|\{(\mathcal{G}F)_{\#}\sigma^F\}(x,y) - \{(\mathcal{G}G)_{\#}\sigma^F\}(x,y)|_{\text{hor}}$ *goes to zero as g goes to f.*

PROOF. Let $v_1 = (F_u w^F)^{-1}(x,v)$, $v_2 = (G_u w^G)^{-1}(x,y)$, $\sigma^F(v_1) = J_{v_1}(w_1(\cdot) - \pi^c w^F(v_1))$, $\sigma^F(v_2) = J_{v_2}(w_2(\cdot) - \pi^c w^G(v_2))$, $H_1 = (F_u w_1)^{-1}$, $H_2 = (G_u w_2)^{-1}$, $p_1 = w^F(v_1)$, and $p_2 = w^G(v_2)$. Then

$$|\{(\mathcal{G}F)_{\#}\sigma^F\}(x,y) - \{(\mathcal{G}G)_{\#}\sigma^F\}(x,y)|_{\text{hor}}$$
$$= \limsup_{x'\to x} \frac{|F_c w_1 H_1(x',y) - F_c w_1 H_1(x,y) - G_c w_2 H_2(x',y) + G_c w_2 H_2(x,y)|}{|y|\rho(x',x)}$$
$$< \limsup\left\{ \frac{|F_c w_1 H_1(x',y) - F_c w_1[H_2(x',y) - v_2 + v_1]|}{|y|\rho(x',x)}\right.$$
$$+ \frac{|F_c w_1[H_2(x',y) - v_2 + v_1] - F_c[w_2 H_2(x',y) - p_2 + p_1]|}{|y|\rho(x',x)}$$
$$+ \frac{|F_c[w_2 H_2(x',y) - p_2 + p_1] - F_c w_2 H_2(x',y) - F_c(p_1) + F_c(p_2)|}{|y|\rho(x',x).}$$
$$\left. + \frac{|F_c w_2 H_2(x',y) - F_c(p_2) - G_c w_2 H_2(x',y) + G_c(p_2)|}{|y|\rho(x',x)}\right\}.$$

The first term is less than

$$L_{\text{fib}}(F_c)L_{\text{fib}}(w_1)\limsup \frac{|H_1(x',y) - H_2(x',y) - H_1(x,y) + H_2(x,y)|}{|y|\rho(x',x)}.$$

This term goes to zero using calculations like those in Lemma 4 above and in [11].

The second term is less than

$$L_{\text{fib}}(F_c)\limsup \frac{|w_1[H_2(x',y) - v_2 + v_1] - w_2 H_2(x',y) - w_1 H_1(x,y) + w_2 H_2(x,y)|}{|y|\rho(x',x)}.$$

This is bounded by terms involving the horizontal term

$$|\sigma^F(v_1) - \sigma^F(v_2)|_{\text{hor}}|\pi^u H_2(x,y)|\rho(f^{-1}x', f^{-1}x)$$

and the fiber term

$$L_{\text{fib}}(w_1 - w_2)|\pi^u H_2(x', y) - \pi^u H_2(x, y)|$$

in the numerator. Then $|\sigma^F(v_1) - \sigma^F(v_2)|$ goes to zero uniformly.

The third term is less than

$$L(F_c(\cdot + p_1 - p_2) - F_c(\cdot))\limsup \frac{|w_2 H_2(x', y) - w_2 H_2(x, y)|}{|y|\rho(x', x)}.$$

The first factor goes to zero and the second factor is bounded.

The fourth and last term is less than

$$L(F_c - G_c)\limsup\left\{ \frac{|w_2 H_2(x', y) - w_2(f^{-1}x', \pi^u H_2(x, y))|}{|y|\rho(x', x)} \right.$$
$$\left. + \frac{|w_2(f^{-1}x', \pi^u H_2(x, y)) - w_2 H_2(x, y)|}{|y|\rho(x', x)} \right\}$$
$$< L(F_c - G_c)L_{\text{fib}}(w_2)\limsup \frac{|\pi^u H_2(x', y) - \pi^u H_2(x, y)|}{|y|\rho(x', x)}$$
$$+ L(F_c - G_2)|\sigma^F(v_2)|_{\text{hor}}\limsup \frac{|\pi^u H_2(x, y)|\rho(f^{-1}x', f^{-1}x)}{|y|\rho(x', x)}.$$

The factor $L(F_c - G_c)$ goes to zero and the other factors are bounded.

LEMMA 7. *There is a constant C such that*

$$\max\{\rho_{f\mu}(x, x'), |v - v'|\} < C\rho_{f\mu}(x + v + w_f(x, v), x' + y' + w_f(x', v'))$$

for all x, x', v, v'.

PROOF. If $\pi: W^{cu}_{\text{loc}}(0, p, f) \to W^c_{\text{loc}}(0, p, f)$ is a trial projection (associated with a trial disk family w), then we can form the transform $(f_\# \pi)z = f\pi f^{-1}(z)$. If $\rho_{f\mu}(\pi z, \pi z') < \Lambda\rho_{f\mu}(z, z')$ for all z, z', then

$$\rho_{f\mu}((f_\#\pi)z, (f_\#\pi)z') = \rho_{f\mu}(f\pi f^{-1}(z), f\pi f^{-1}(z')) = \rho_{f\mu}(\pi f^{-1}(z), \pi f^{-1}(z'))$$
$$< \Lambda\rho_{f\mu}(f^{-1}(z), f^{-1}(z')) = \Lambda\rho_{f\mu}(z, z').$$

Therefore $f_\#$ preserves projections of bounded $\rho_{f\mu}$ Lipschitz size. Therefore the fixed point π_0 is of bounded $\rho_{f\mu}$ Lipschitz size. By setting $z = x + v + w_f(x, v)$ and $z' = x' + v' + w_f(x', v')$ we get that

$$\rho_{f\mu}(x, x') < \Lambda\rho_{f\mu}(x + v + w_f(x, v), x' + v' + w_f(x', v')).$$

Also

$$\begin{aligned}|v - v'| &< |x - x'| + |v - v'| + |w_f(x, v) - w_f(x', v')| \\ &< 3^{1/2} d\big(x + v + w_f(x, v), x' + v' + w_f(x', v')\big) \\ &< 3^{1/2} \Lambda \rho_{f\mu}\big(x + v + w_f(x, v), x' + v' + w_f(x', v')\big).\end{aligned}$$

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