Global Synchronization Control of General Delayed Discrete-Time Networks With Stochastic Coupling and Disturbances

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***Abstract*—In this paper, the synchronization control problem is considered for two coupled discrete-time complex networks with time delays. The network under investigation is quite general to reflect the reality, where the state delays are allowed to be time varying with given lower and upper bounds, and the stochastic disturbances are assumed to be Brownian motions that affect not only the network coupling but also the overall networks. By utilizing the Lyapunov functional method combined with linear matrix inequality (LMI) techniques, we obtain several sufficient delay-dependent conditions that ensure the coupled networks to be globally exponentially synchronized in the mean square. A control law is designed to synchronize the addressed coupled complex networks in terms of certain LMIs that can be readily solved using the Matlab LMI toolbox. Two numerical examples are presented to show the validity of our theoretical analysis results.**

***Index Terms*—Control law, coupled networks, linear matrix inequality (LMI), Lyapunov functional, stochastic disturbance, synchronization, time-varying delay.**

I. Introduction

COMPLEX networks exist in our lives. For example, the brain is a neural network; the global economy is a network of national economies; computer viruses routinely spread through the Internet; food webs, ecosystems, and metabolic pathways can be represented by networks. The complexity of networks in the social, biological, engineering, and physical sciences gives rise to many challenges for scientists and engineers. In order to better understand the dynamical behaviors of different kinds of complex networks, an important and interesting phenomenon to investigate is the synchrony of all dynamical nodes. Synchronization problem have been attracting recurrent research interests for many complex networks that include, but are not limited to, large-scale and complex networks of chaotic oscillators [6], [12], [19], [27], the coupled systems exhibiting spatiotemporal chaos and autowaves [15], [26], and the array of coupled neural networks with or without delays [2], [8], [13], [14], [23]. For example, the theta rhythm related to the behavior of animals is produced by partial synchronization of neuronal activity in the hippocampal network [22], and an excessive synchronization of the neuronal activity over a wide area in the brain results in the epileptic rhythm [17]. Moreover, it has recently been revealed that, for message delivery in networks, a good synchronization can help achieve secure communication in terms of stable and high transportation rate.

It has now been well recognized that time delays may cause undesirable dynamic network behaviors such as oscillation and instability [5], [10], [16], [24]. Therefore, synchronization problem for complex networks with time delays has gained increasing research attention. For example, the synchronization criteria have been established in [7] for complex dynamical network models with coupling delays for both continuous and discrete time cases, which have further been improved in [5] by using less conservative delay-dependent techniques. A variational method has been used in [11] to deal with the synchronization problem for an array of linearly coupled identical connected neural networks with delays, whereas the similar problem has been addressed in [23] for an array of coupled nonlinear systems with delay and nonreciprocal time-varying coupling. More recently, by using Lyapunov functional method and Kronecker product technique, the global exponential synchronization has been established in [3] for arrays of coupled identical delayed neural networks with constant and delayed coupling. A notable fact is that most of the existing results have been concerned with the synchronization problem for continuous-time and deterministic complex networks with or without delays.

As pointed out in [18], it is rather challenging to understand the interaction topology of complex networks because of the discrete and random nature of network topology. On one hand, in a real complex network, the signal transfer could be perturbed randomly from the release of probabilistic causes such as neurotransmitters [20] and packet dropouts [21]. Synchronization control problem for stochastic networks has recently begun to receive initial research attention. For example, in [25], the synchronization control problem has been considered for stochastic neural networks with time-varying delays, and a novel control method has been given to estimate the controller gain. In [9], the complete synchronization has been achieved between unidirectionally coupled Chua's circuits within stochastic perturbation. Furthermore, in [4], by introducing the stochastic coupling term, the complete synchronization problem has been investigated for an array of linearly stochastically coupled neural networks with time delays. It is worth mentioning that the network coupling could occur in both a deterministic and

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a stochastic way, and the stochastic perturbations could act on both the coupling term and the overall networks. On the other hand, discrete-time networks could be more suitable to model digitally transmitted signals in a dynamical way. Note that discrete-time networks have already been applied in a wide range of areas, such as image processing, time series analysis, quadratic optimization problems, and system identification. Unfortunately, despite its importance in practice, the global synchronization problem for discrete-time networks with both stochastic coupling and stochastic disturbances with or without delays has not been fully investigated yet, which constitutes the main focus of this paper.

In this paper, we are interested in the synchronization control problem for stochastic discrete-time complex networks with time delays, where the stochastic disturbances are assumed to be Brownian motions that affect not only the network coupling but also the overall networks. Our main purpose is to establish some delay-dependent criteria to ensure that the two identical delayed networks with stochastic disturbances are globally exponentially synchronized. Based on the Lyapunov functional method and the stochastic analysis theory, we like to analyze and design appropriate feedback controllers with the hope that the derived synchronization criteria can be expressed in the form of linear matrix inequalities (LMIs). Note that the LMIs can be effectively solved and checked by the algorithms such as the interior-point method [1].

The rest of this paper is organized as follows. In Section II, some notations are introduced first, and then the coupled network model is presented. In Section III, via the Lyapunov functional method combined with the LMI technique, main results for synchronization are obtained, and the controller design is proposed. Two illustrative examples are given in Section IV to demonstrate the effectiveness of the acquired results. Finally, in Section V, we give the conclusion of this paper.

II. Notations and Problem Formulation

Throughout this paper, the notation $P > 0$ means that P is real symmetric and positive definite. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalue, respectively. In symmetric block matrices, we use an asterisk "$*$" to represent a term that is induced by symmetry. Let $\mathbb{E}\{\cdot\}$ be the mathematical expectation operator with respect to the given probability measure $\mathcal{P}$ and $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. For integers α, β with $\alpha < \beta$, $\mathbb{N}[\alpha, \beta]$ denotes the discrete interval given by $\mathbb{N}[\alpha, \beta] = \{\alpha, \alpha+1, \dots, \beta-1, \beta\}$ and $C(\mathbb{N}[\alpha, \beta], \mathbb{R}^n)$ means the set of all functions $\phi : \mathbb{N}[\alpha, \beta] \to \mathbb{R}^n$. I stands for the identity matrix with appropriate dimensions. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

To facilitate the readers, let us present the complex networks in a step-by-step way. We start with the following master network:

$$s(k+1) = As(k) + Bf\left(s(k)\right) + Cf_d\left(s\left(k-\tau(k)\right)\right) \quad (1)$$

where $k = 1, 2, \dots$ and $s(k) = (s_1(k), s_2(k), \dots, s_n(k))^{\mathrm{T}} \in \mathbb{R}^n$ is the state vector of the network; A is a constant matrix; matrices B and C are the connection weight matrix and the delayed connection weight matrix, respectively; $\tau(k)$ is a time-varying delay in the state which satisfies

$$\tau_m \leq \tau(k) \leq \tau_M \quad (2)$$

where τ_m and τ_M are known positive integers representing the lower and upper bounds of the delay; $f(s(k)) = (f_1(s_1(k)), f_2(s_2(k)), \dots, f_n(s_n(k)))^{\mathrm{T}}$ and $f_d(s(k-\tau(k))) = (f_{d1}(s_1(k-\tau(k))), f_{d2}(s_2(k-\tau(k))), \dots, f_{dn}(s_n(k-\tau(k))))^{\mathrm{T}}$ are the nonlinear functions. In system (1), $\{\phi(k) : k = -\tau_M, -\tau_M+1, \dots, 0\}$ is a given initial condition sequence in $C(\mathbb{N}[-\tau_M, 0], \mathbb{R}^n)$.

Throughout of this paper, the following assumption is always made.

Assumption 1: [10] There exist constants $l_i^-, l_i^+, k_i^-, k_i^+$ such that the following inequalities

$$l_i^- \leq \frac{f_i(u) - f_i(v)}{u - v} \leq l_i^+$$
$$k_i^- \leq \frac{f_{di}(u) - f_{di}(v)}{u - v} \leq k_i^+, \qquad (i = 1, 2, \dots, n)$$

hold for all different $u, v \in \mathbb{R}$.

As discussed in the introduction, real-world networks are usually coupled, and stochastic disturbances could enter both the network coupling and the overall networks. Therefore, in this paper, an array of linearly coupled identical networks with time-varying delay under study (the slave networks) is proposed as follows (without loss of generality, we only consider the case that two networks are coupled):

$$\begin{aligned} x_i(k+1) = {} & Ax_i(k) + Bf\left(x_i(k)\right) + Cf_d\left(x_i\left(k-\tau(k)\right)\right) + u_i(k) \\ & + H\left(k, x_i(k) - s(k), x_i\left(k-\tau(k)\right) - s\left(k-\tau(k)\right)\right) \\ & \times \omega_{i1}(k) + \sum_{j=1}^{2} G_{ij}\Gamma x_j(k)\left(d_i + \omega_{i2}(k)\right) \end{aligned} \quad (3)$$

where $i = 1, 2$ and $x_i(k) = (x_{i1}(k), x_{i2}(k), \dots, x_{in}(k))^{\mathrm{T}} \in \mathbb{R}^n$; $d_i > 0$ denotes the coupling strength; $u_i(\cdot)$ is the control input to ensure that $x_i(k) - s(k) \to 0$ as $k \to \infty$; $\omega_{i1}(\cdot)$, $\omega_{i2}(\cdot) (i = 1, 2)$ are independent scalar Wiener process (Brownian Motion) on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ satisfying

$$\begin{aligned} \mathbb{E}\left\{\omega_{ij}(k)\right\} &= 0 \\ \mathbb{E}\left\{\omega_{ij}^2(k)\right\} &= 1 \\ \mathbb{E}\left\{\omega_{ij}(s)\omega_{ij}(t)\right\} &= 0 \ (s \neq t) \end{aligned} \quad (4)$$

in which $i, j = 1, 2$. It is assumed that $H : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the diffusion coefficient vector and there exist matrices G_1, G_2 such that

$$H^{\mathrm{T}}(k, x, y)H(k, x, y) \leq \|G_1 x\|^2 + \|G_2 y\|^2, \quad \forall (k, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n. \quad (5)$$

Furthermore, $\Gamma \in \mathbb{R}^{n \times n}$ is a constant inner coupling matrix of the nodes, and $G = (G_{ij})_{2\times 2}$ is the out-coupling matrix of the network defined as follows. If there is a connection from node j to node $i(i \neq j)$, then $G_{ij} > 0$; otherwise, $G_{ij} = 0$.

Remark 1: It can be seen that matrix G reflects the topological structure of the networks and it also satisfies the diffusive coupling conditions

$$G_{ii} = -\sum_{j \neq i} G_{ij}, \qquad (i, j = 1, 2). \tag{6}$$

Moreover, (5) and (6) infer that when the synchronization is reached, the synchronization state is just a solution of an isolated node model (1).

Remark 2: It is notable that, in addition to the constant couplings in our model (3), we consider the state-dependent stochastic sequences $\omega_{i1}(k)$ on the overall network and $\omega_{i2}(k)$ on the coupling term. This represents one of the first attempts to deal with both deterministic and stochastic disturbances on the coupling as well as the overall network dynamics. In this sense, the model (3) is more natural and general than the existing ones including that introduced in [4].

In order to investigate the global synchronization for coupled networks (3), we let $e_i(k) = x_i(k) - s(k)$ be the synchronization error. Then, the error system follows immediately from (1) and (3) as follows:

$$\begin{aligned} e_i(k+1) = &\, Ae_i(k) + Bg\left(e_i(k)\right) + Cg_d\left(e_i\left(k - \tau(k)\right)\right) \\ &+ u_i(k) + H\left(k, e_i(k), e_i\left(k - \tau(k)\right)\right)\omega_{i1}(k) \\ &+ \sum_{j=1}^{2} G_{ij}\Gamma e_j(k)\left(d_i + \omega_{i2}(k)\right) \end{aligned} \tag{7}$$

where $g(e_i(k)) = f(x_i(k)) - f(s(k))$ and $g_d(e_i(k-\tau(k))) = f_d(x_i(k-\tau(k))) - f_d(s(k-\tau(k)))$.

From Assumption 1, one has

$$l_i^- \leq \frac{g_i(u)}{u} \leq l_i^+, \qquad k_i^- \leq \frac{g_{di}(u)}{u} \leq k_i^+, \qquad (u \in \mathbb{R} \setminus \{0\}, \quad i = 1, 2, \ldots, n). \tag{8}$$

For simplicity, we denote $L_1 = \mathrm{diag}\{l_1^- l_1^+, l_2^- l_2^+, \ldots, l_n^- l_n^+\}$, $L_2 = \mathrm{diag}\{-(l_1^- + l_1^+)/2, -(l_2^- + l_2^+)/2, \ldots, -(l_n^- + l_n^+)/2\}$; $K_1 = \mathrm{diag}\{k_1^- k_1^+, k_2^- k_2^+, \ldots, k_n^- k_n^+\}$, $K_2 = \mathrm{diag}\{-(k_1^- + k_1^+)/2, -(k_2^- + k_2^+)/2, \ldots, -(k_n^- + k_n^+)/2\}$.

Definition 1: The coupled network system (3) is said to be globally exponentially synchronized in the mean square if there exist two constants $\vartheta > 0$ and $\mu \in (0, 1)$ such that

$$\mathbb{E}\left\{\|x_i(k) - s(k)\|^2\right\} = \mathbb{E}\left\{\|e_i(k)\|^2\right\} \leq \vartheta \mu^k \times \max_{s \in \mathbb{N}[-\tau_M, 0]} \mathbb{E}\{\|e_i(s)\|^2\}, \qquad i = 1, 2$$

hold for all $k \geq \kappa$, where κ is a sufficiently large positive integer.

To obtain our main results, we need the following lemma known as the Schur complement:

Lemma 1: Let $Q(x) = Q^{\mathrm{T}}(x)$, $R(x) = R^{\mathrm{T}}(x)$, and $S(x)$ depend affinely on x [1]. Then, the following LMI:

$$\begin{bmatrix} Q(x) & S(x) \\ S^{\mathrm{T}}(x) & R(x) \end{bmatrix} > 0$$

holds if and only if one of the following conditions holds:

1) $R(x) > 0$, $Q(x) - S(x)R^{-1}(x)S^{\mathrm{T}}(x) > 0$;
2) $Q(x) > 0$, $R(x) - S^{\mathrm{T}}(x)Q^{-1}(x)S(x) > 0$.

III. Main Results

In this section, by utilizing the Lyapunov functional method combining with the LMI techniques, let us first derive delay-dependent stability criterion for the following unforced system of (7):

$$\begin{aligned} e_i(k+1) = &\, Ae_i(k) + Bg\left(e_i(k)\right) + Cg_d\left(e_i\left(k - \tau(k)\right)\right) \\ &+ H\left(k, e_i(k), e_i\left(k - \tau(k)\right)\right)\omega_{i1}(k) \\ &+ \sum_{j=1}^{2} G_{ij}\Gamma e_j(k)\left(d_i + \omega_{i2}(k)\right) \end{aligned} \tag{9}$$

and then design a controller $u_i(k)$ that synchronizes the coupled network (3) with stochastic disturbances.

Theorem 1: The unforced system (9) is globally exponentially stable in the mean square if there exist seven positive definite matrices P, Q_1, Q_2, R_1, R_2, $\bar{R}_1$, $\bar{R}_2$, four positive diagonal matrices $S_i = \mathrm{diag}\{s_{i1}, s_{i2}, \ldots, s_{in}\}(i = 1, 2, 3, 4)$ and a scalar $\lambda > 0$ such that the following LMIs hold:

$$P < \lambda I \qquad \Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^{\mathrm{T}} & \Xi_{22} \end{bmatrix} < 0 \tag{10}$$

where we have the expressions for Ξ_{11}, Ξ_{12}, and Ξ_{22}, shown at the bottom of the next page, with $\Pi_2 = \big(G_{11}G_{12}(1 + d_1^2) + G_{22}G_{21}(1 + d_2^2)\big)\Gamma^T P \Gamma + d_1 G_{12} A^T P \Gamma + d_2 G_{21} \Gamma^T P A$

$$\begin{aligned} \Pi_1 = &\, (\tau_M - \tau_m + 1)Q_1 + A^{\mathrm{T}}PA - P + \lambda G_1^{\mathrm{T}} G_1 - S_1 L_1 \\ &+ R_1 + \bar{R}_1 + \left((1 + d_1^2)G_{11}^2 + \left(1 + d_2^2\right)G_{21}^2\right)\Gamma^{\mathrm{T}} P \Gamma \\ &+ d_1 G_{11}(A^{\mathrm{T}} P \Gamma + \Gamma^{\mathrm{T}} P A) \\ \Pi_3 = &\, (\tau_M - \tau_m + 1)Q_2 + A^{\mathrm{T}}PA - P + \lambda G_1^{\mathrm{T}} G_1 - S_2 L_1 \\ &+ R_2 + \bar{R}_2 + \left((1 + d_1^2)G_{12}^2 + (1 + d_2^2)G_{22}^2\right)\Gamma^{\mathrm{T}} P \Gamma \\ &+ d_2 G_{22}(A^{\mathrm{T}} P \Gamma + \Gamma^{\mathrm{T}} P A). \end{aligned}$$

Proof: It follows easily from (8) that

$$\big(g_i\left(e_{1i}(k)\right) - l_i^- e_{1i}(k)\big)\big(g_i\left(e_{1i}(k)\right) - l_i^+ e_{1i}(k)\big) \leq 0, \qquad i = 1, 2, \ldots, n \tag{11}$$

which is equivalent to

$$\begin{bmatrix} e_1(k) \\ g\left(e_1(k)\right) \end{bmatrix} \begin{bmatrix} l_i^- l_i^+ \delta_i \delta_i^{\mathrm{T}} & -\frac{l_i^- + l_i^+}{2} \delta_i \delta_i^{\mathrm{T}} \\ -\frac{l_i^- + l_i^+}{2} \delta_i \delta_i^{\mathrm{T}} & \delta_i \delta_i^{\mathrm{T}} \end{bmatrix} \times \left[e_1^{\mathrm{T}}(k) \quad g^{\mathrm{T}}\left(e_1(k)\right)\right] \leq 0, \qquad i = 1, 2, \ldots, n \tag{12}$$

where δ_i is the n-dimensional unit column vector having one element on its ith row and zeros elsewhere. Multiplying both sides of (12) by s_{1i} and summing up from 1 to n with respect to i, we have

$$\begin{bmatrix} e_1(k) \\ g\left(e_1(k)\right) \end{bmatrix} \begin{bmatrix} L_1 S_1 & L_2 S_1 \\ L_2 S_1 & S_1 \end{bmatrix} \left[e_1^{\mathrm{T}}(k) \quad g^{\mathrm{T}}\left(e_1(k)\right)\right] \leq 0. \tag{13}$$

Similarly, we have the following inequalities:

$$\begin{bmatrix} e_2(k) \\ g(e_2(k)) \end{bmatrix} \begin{bmatrix} L_1S_2 & L_2S_2 \\ L_2S_2 & S_2 \end{bmatrix} \begin{bmatrix} e_2^{\mathrm{T}}(k) & g^{\mathrm{T}}(e_2(k)) \end{bmatrix} \leq 0 \quad (14)$$

$$\begin{bmatrix} e_1(k-\tau(k)) \\ g_d(e_1(k-\tau(k))) \end{bmatrix} \begin{bmatrix} K_1S_3 & K_2S_3 \\ K_2S_3 & S_3 \end{bmatrix} \times \begin{bmatrix} e_1^{\mathrm{T}}(k-\tau(k)) & g_d^{\mathrm{T}}(e_1(k-\tau(k))) \end{bmatrix} \leq 0 \quad (15)$$

$$\begin{bmatrix} e_2(k-\tau(k)) \\ g_d(e_2(k-\tau(k))) \end{bmatrix} \begin{bmatrix} K_1S_4 & K_2S_4 \\ K_2S_4 & S_4 \end{bmatrix} \times \begin{bmatrix} e_2^{\mathrm{T}}(k-\tau(k)) & g_d^{\mathrm{T}}(e_2(k-\tau(k))) \end{bmatrix} \leq 0. \quad (16)$$

Consider the following Lyapunov functional of system (9):

$$\begin{aligned} V(k) =& V_1(k)+V_2(k)+V_3(k)+V_4(k)+V_5(k) \\ =& \sum_{i=1}^{2} e_i^{\mathrm{T}}(k)Pe_i(k) + \sum_{i=1}^{2}\sum_{j=k-\tau(k)}^{k-1} e_i^{\mathrm{T}}(j)Q_ie_i(j) \\ &+\sum_{i=1}^{2}\sum_{j=k-\tau_m}^{k-1} e_i^{\mathrm{T}}(j)R_ie_i(j) + \sum_{i=1}^{2}\sum_{j=k-\tau_M}^{k-1} e_i^{\mathrm{T}}(j)\bar{R}_ie_i(j) \\ &+\sum_{i=1}^{2}\sum_{j=-\tau_M+1}^{-\tau_m}\sum_{m=k+j}^{k-1} e_i^{\mathrm{T}}(m)Q_ie_i(m). \end{aligned} \quad (17)$$

Calculating the difference of $V_1(k)$ along the solutions of (9) and taking the mathematical expectation, noting the independent properties of stochastic processes $\omega_{i1}(\cdot)$, $\omega_{i2}(\cdot)$, and (4), we obtain (18), shown at the bottom of the next page.

Similarly, we have

$$\begin{aligned} \mathbb{E}\{\Delta V_2(k)\} =& \sum_{i=1}^{2}\mathbb{E}\Bigg\{ e_i^{\mathrm{T}}(k)Q_ie_i(k) - e_i^{\mathrm{T}}(k-\tau(k)) \\ &\times Q_ie_i(k-\tau(k)) + \sum_{j=k+1-\tau(k+1)}^{k-\tau_m} e_i^{\mathrm{T}}(j)Q_ie_i(j) \\ &+ \sum_{j=k-\tau_m+1}^{k-1} e_i^{\mathrm{T}}(j)Q_ie_i(j) - \sum_{j=k-\tau(k)+1}^{k-1} e_i^{\mathrm{T}}(j)Q_ie_i(j)\Bigg\} \\ \leq& \sum_{i=1}^{2}\mathbb{E}\Bigg\{ e_i^{\mathrm{T}}(k)Q_ie_i(k) - e_i^{\mathrm{T}}(k-\tau(k)) \\ &\times Q_ie_i(k-\tau(k)) + \sum_{j=k+1-\tau_M}^{k-\tau_m} e_i^{\mathrm{T}}(j)Q_ie_i(j)\Bigg\} \end{aligned} \quad (19)$$

$$\mathbb{E}\{\Delta V_3(k)\} = \sum_{i=1}^{2}\mathbb{E}\Bigg\{ e_i^{\mathrm{T}}(k)R_ie_i(k) - e_i^{\mathrm{T}}(k-\tau_m) \times R_ie_i(k-\tau_m)\Bigg\} \quad (20)$$

$$\mathbb{E}\{\Delta V_4(k)\} = \sum_{i=1}^{2}\mathbb{E}\Bigg\{ e_i^{\mathrm{T}}(k)\bar{R}_ie_i(k) - e_i^{\mathrm{T}}(k-\tau_M) \times \bar{R}_ie_i(k-\tau_M)\Bigg\} \quad (21)$$

$$\mathbb{E}\{\Delta V_5(k)\} = \sum_{i=1}^{2}\mathbb{E}\Bigg\{ (\tau_M-\tau_m)e_i^{\mathrm{T}}(k)Q_ie_i(k) - \sum_{j=k+1-\tau_M}^{k-\tau_m} e_i^{\mathrm{T}}(j)Q_ie_i(j)\Bigg\}. \quad (22)$$

Conditions (5) and (10) indicate that

$$\begin{aligned} &H^{\mathrm{T}}(k,e_i(k),e_i(k-\tau(k)))\,PH(k,e_i(k),e_i(k-\tau(k))) \\ &\leq \lambda\left(e_i^{\mathrm{T}}(k)G_1^{\mathrm{T}}G_1e_i(k) + e_i^{\mathrm{T}}(k-\tau(k))\,G_2^{\mathrm{T}}G_2e_i(k-\tau(k))\right), \\ &i=1,2. \end{aligned} \quad (23)$$

$$\Xi_{11} = \begin{bmatrix} \Pi_1 & 0 & 0 & 0 & A^{\mathrm{T}}PB - S_1L_2 + d_1G_{11}\Gamma^{\mathrm{T}}PB & A^{\mathrm{T}}PC + d_1G_{11}\Gamma^{\mathrm{T}}PC \\ * & -R_1 & 0 & 0 & 0 & 0 \\ * & * & -Q_1 + \lambda G_2^{\mathrm{T}}G_2 - S_3K_1 & 0 & 0 & -S_3K_2 \\ * & * & * & -\bar{R}_1 & 0 & 0 \\ * & * & * & * & B^{\mathrm{T}}PB - S_1 & B^{\mathrm{T}}PC \\ * & * & * & * & * & C^{\mathrm{T}}PC - S_3 \end{bmatrix}$$

$$\Xi_{12} = \begin{bmatrix} \Pi_2 & 0 & 0 & 0 & d_2G_{21}\Gamma^{\mathrm{T}}PB & d_2G_{21}\Gamma^{\mathrm{T}}PC \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ d_1G_{12}B^{\mathrm{T}}P\Gamma & 0 & 0 & 0 & 0 & 0 \\ d_1G_{12}C^{\mathrm{T}}P\Gamma & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Xi_{22} = \begin{bmatrix} \Pi_3 & 0 & 0 & 0 & A^{\mathrm{T}}PB - S_2L_2 + d_2G_{22}\Gamma^{\mathrm{T}}PB & A^{\mathrm{T}}PC + d_2G_{22}\Gamma^{\mathrm{T}}PC \\ * & -R_2 & 0 & 0 & 0 & 0 \\ * & * & -Q_2 + \lambda G_2^{\mathrm{T}}G_2 - S_4K_1 & 0 & 0 & -S_4K_2 \\ * & * & * & -\bar{R}_2 & 0 & 0 \\ * & * & * & * & B^{\mathrm{T}}PB - S_2 & B^{\mathrm{T}}PC \\ * & * & * & * & * & C^{\mathrm{T}}PC - S_4 \end{bmatrix}$$

Denoting

$$\xi(k)=\Big(e_1^{\mathrm{T}}(k)\ e_1^{\mathrm{T}}(k-\tau_m)\ e_1^{\mathrm{T}}\left(k-\tau(k)\right)\ e_1^{\mathrm{T}}(k-\tau_M)\ g^{\mathrm{T}}\left(e_1(k)\right)\ g_d^{\mathrm{T}}\left(e_1\left(k-\tau(k)\right)\right)\ e_2^{\mathrm{T}}(k)\ e_2^{\mathrm{T}}(k-\tau_m)\ e_2^{\mathrm{T}}\left(k-\tau(k)\right)\ e_2^{\mathrm{T}}(k-\tau_M)\ g^{\mathrm{T}}\left(e_2(k)\right)\ g_d^{\mathrm{T}}\left(e_2\left(k-\tau(k)\right)\right)\Big)^{\mathrm{T}}$$

it follows from (13)–(16) and (18)–(23) that

$$\mathbb{E}\left\{\Delta V(k)\right\} \le \mathbb{E}\left\{\xi^{\mathrm{T}}(k)\Xi\xi(k)\right\} \le -\lambda_{\min}(-\Xi)\mathbb{E}\left\{\|e_1(k)\|^2+\|e_2(k)\|^2\right\}. \quad (24)$$

From the definition (17) of $V(k)$, it is easy to obtain that

$$\mathbb{E}\left\{V(k)\right\} \ge \lambda_{\min}(P)\mathbb{E}\left\{\|e_1(k)\|^2+\|e_2(k)\|^2\right\} \quad (25)$$

$$\mathbb{E}\left\{V(k)\right\} \le \lambda_{\max}(P)\mathbb{E}\left\{\|e_1(k)\|^2+\|e_2(k)\|^2\right\} + \rho\mathbb{E}\left\{\sum_{j=k-\tau_M}^{k-1}\left(\|e_1(j)\|^2+\|e_2(j)\|^2\right)\right\} \quad (26)$$

where $\rho=(\tau_M-\tau_m+1)\max\{\lambda_{\max}(Q_1),\lambda_{\max}(Q_2)\}+\max\{\lambda_{\max}(R_1),\lambda_{\max}(R_2)\}+\max\{\lambda_{\max}(\bar{R}_1),\lambda_{\max}(\bar{R}_2)\}$. For any given scalar $\varepsilon>1$, taking k to be sufficiently large, we have from (24) and (26) that

$$\begin{aligned}\mathbb{E}\left\{\varepsilon^k V(k)\right\} &= \mathbb{E}\left\{V(0)+\sum_{j=0}^{k-1}\left(\varepsilon^{j+1}\Delta V(j)+\varepsilon^j(\varepsilon-1)V(j)\right)\right\}\\ &\le \mathbb{E}\Bigg\{\lambda_{\max}(P)\left(\|e_1(0)\|^2+\|e_2(0)\|^2\right)\\ &\quad+\rho\sum_{j=-\tau_M}^{-1}\left(\|e_1(j)\|^2+\|e_2(j)\|^2\right)\\ &\quad+\left(-\varepsilon\lambda_{\min}(-\Xi)+(\varepsilon-1)\lambda_{\max}(P)\right)\\ &\quad\times\sum_{j=0}^{k-1}\varepsilon^j\left(\|e_1(j)\|^2+\|e_2(j)\|^2\right)+(\varepsilon-1)\rho\\ &\quad\times\sum_{j=0}^{k-1}\sum_{l=j-\tau_M}^{j-1}\varepsilon^j\left(\|e_1(l)\|^2+\|e_2(l)\|^2\right)\Bigg\}\end{aligned} \quad (27)$$

$$\begin{aligned}\mathbb{E}\left\{\Delta V_1(k)\right\} &= \mathbb{E}\left\{V_1(k+1)-V_1(k)\right\}\\ &= \sum_{i=1}^{2}\mathbb{E}\Bigg\{\left[Ae_i(k)+Bg\left(e_i(k)\right)+Cg_d\left(e_i\left(k-\tau(k)\right)\right)+d_i\sum_{j=1}^{2}G_{ij}\Gamma e_j(k)\right]^{\mathrm{T}}\\ &\qquad\times P\left[Ae_i(k)+Bg\left(e_i(k)\right)+Cg_d\left(e_i\left(k-\tau(k)\right)\right)+d_i\sum_{j=1}^{2}G_{ij}\Gamma e_j(k)\right]\\ &\qquad-e_i^{\mathrm{T}}(k)Pe_i(k)+H\left(k,e_i(k),e_i\left(k-\tau(k)\right)\right)PH\left(k,e_i(k),e_i\left(k-\tau(k)\right)\right)\\ &\qquad+\left(\sum_{j=1}^{2}G_{ij}\Gamma e_j(k)\right)^{\mathrm{T}}P\left(\sum_{j=1}^{2}G_{ij}\Gamma e_j(k)\right)\Bigg\}\\ &= \sum_{i=1}^{2}\mathbb{E}\Bigg\{e_i^{\mathrm{T}}(k)(A^{\mathrm{T}}PA-P)e_i(k)+g_d^{\mathrm{T}}\left(e_i\left(k-\tau(k)\right)\right)C^{\mathrm{T}}PCg_d\left(e_i\left(k-\tau(k)\right)\right)\\ &\qquad+g^{\mathrm{T}}\left(e_i(k)\right)B^{\mathrm{T}}PBg\left(e_i(k)\right)+2e_i^{\mathrm{T}}(k)A^{\mathrm{T}}P\left[Bg\left(e_i(k)\right)+Cg_d\left(e_i\left(k-\tau(k)\right)\right)\right]\\ &\qquad+2g^{\mathrm{T}}\left(e_i(k)\right)B^{\mathrm{T}}PCg_d\left(e_i\left(k-\tau(k)\right)\right)+H\left(k,e_i(k),e_i\left(k-\tau(k)\right)\right)PH\left(k,e_i(k),e_i\left(k-\tau(k)\right)\right)\Bigg\}\\ &\quad+\mathbb{E}\Bigg\{\left(\left(1+d_1^2\right)G_{11}^2+\left(1+d_2^2\right)G_{21}^2\right)e_1^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}P\Gamma e_1(k)+\left(\left(1+d_1^2\right)G_{12}^2+\left(1+d_2^2\right)G_{22}^2\right)\\ &\qquad\times e_2^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}P\Gamma e_2(k)+2\left(G_{11}G_{12}\left(1+d_1^2\right)+G_{21}G_{22}\left(1+d_2^2\right)\right)e_1^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}P\Gamma e_2(k)\\ &\qquad+2\Big[d_1G_{11}e_1^{\mathrm{T}}(k)A^{\mathrm{T}}P\Gamma e_1(k)+e_1^{\mathrm{T}}(k)\left(d_1G_{12}A^{\mathrm{T}}P\Gamma+d_2G_{21}\Gamma^{\mathrm{T}}PA\right)e_2(k)+d_2G_{22}e_2^{\mathrm{T}}(k)A^{\mathrm{T}}P\Gamma e_2(k)\Big]\\ &\qquad+2\Big[d_1G_{11}e_1^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}PBg\left(e_1(k)\right)+d_1G_{12}e_2^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}PBg\left(e_1(k)\right)\\ &\qquad\quad+d_2G_{21}e_1^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}PBg\left(e_2(k)\right)+d_2G_{22}e_2^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}PBg\left(e_2(k)\right)\Big]\\ &\qquad+2\Big[d_1G_{11}e_1^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}PCg_d\left(e_1\left(k-\tau(k)\right)\right)+d_1G_{12}e_2^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}PCg_d\left(e_1\left(k-\tau(k)\right)\right)\\ &\qquad\quad+d_2G_{21}e_1^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}PCg_d\left(e_2\left(k-\tau(k)\right)\right)+d_2G_{22}e_2^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}PCg_d\left(e_2\left(k-\tau(k)\right)\right)\Big]\Bigg\}\end{aligned} \quad (18)$$

where

$$\begin{aligned}
&\mathbb{E}\left\{\sum_{j=0}^{k-1}\sum_{l=j-\tau_M}^{j-1}\varepsilon^j\left(\|e_1(l)\|^2+\|e_2(l)\|^2\right)\right\}\\
&\leq \mathbb{E}\Bigg\{\tau_M\varepsilon^{\tau_M}\sum_{l=-\tau_M}^{-1}\left(\|e_1(l)\|^2+\|e_2(l)\|^2\right)\\
&\quad+\left(\sum_{l=0}^{k-1-\tau_M}\sum_{s=1}^{\tau_M}+\sum_{l=k-\tau_M}^{k-1}\sum_{s=1}^{k-l}\right)\\
&\quad\times\varepsilon^l\varepsilon^s\left(\|e_1(l)\|^2+\|e_2(l)\|^2\right)\Bigg\}\\
&\leq \tau_M^2\varepsilon^{\tau_M}\sup_{l\in\mathbb{N}[-\tau_M,0]}\mathbb{E}\left\{\|e_1(l)\|^2+\|e_2(l)\|^2\right\}\\
&\quad+\tau_M\varepsilon^{\tau_M}\mathbb{E}\left\{\sum_{l=0}^{k-1}\varepsilon^l\left(\|e_1(l)\|^2+\|e_2(l)\|^2\right)\right\}. \quad (28)
\end{aligned}$$

Substituting (28) into (27) gives

$$\begin{aligned}
\mathbb{E}\left\{\varepsilon^k V(k)\right\} \leq\ & p_1(\varepsilon)\sup_{l\in\mathbb{N}[-\tau_M,0]}\mathbb{E}\left\{\|e_1(l)\|^2+\|e_2(l)\|^2\right\}\\
&+p_2(\varepsilon)\mathbb{E}\left\{\sum_{l=0}^{k}\varepsilon^l\left(\|e_1(l)\|^2+\|e_2(l)\|^2\right)\right\} \quad (29)
\end{aligned}$$

where $p_1(\varepsilon)=\lambda_{\max}(P)+\rho\tau_M+\rho(\varepsilon-1)\tau_M^2\varepsilon^{\tau_M}$, $p_2(\varepsilon)=(\varepsilon-1)\lambda_{\max}(P)-\varepsilon\lambda_{\min}(-\Xi)+\rho(\varepsilon-1)\tau_M\varepsilon^{\tau_M}$. Since $p_1(\varepsilon)$ and $p_2(\varepsilon)$ are continuous functions of ε and $p_1(1)>0$, $p_2(1)<0$, there must exist a scalar $\mu>1$ such that $p_1(\mu)>0$ and $p_2(\mu)\leq 0$, which leads to the fact that

$$\begin{aligned}
\mathbb{E}\left\{\|e_1(k)\|^2+\|e_2(k)\|^2\right\} \leq\ & \frac{p_1(\mu)}{\lambda_{\min}(P)}\left(\frac{1}{\mu}\right)^k\sup_{l\in\mathbb{N}[-\tau_M,0]}\\
&\times\mathbb{E}\left\{\|e_1(l)\|^2+\|e_2(l)\|^2\right\}. \quad (30)
\end{aligned}$$

From Definition 1, (30) means that the coupled network system (9) is globally exponentially stable in the mean square, and the proof is then completed. ■

Next, we are going to design a controller $u_i(k)$ in order to make the coupled system (3) to be synchronized. For simplicity of the implementation, we adopt the following memoryless state-feedback controller:

$$u_i(k)=Ke_i(k). \quad (31)$$

Substitute (31) into (7) to give the following closed-loop system:

$$\begin{aligned}
e_i(k+1)=\ &(A+K)e_i(k)+Bg\left(e_i(k)\right)\\
&+Cg_d\left(e_i\left(k-\tau(k)\right)\right)\\
&+H\left(k,e_i(k),e_i\left(k-\tau(k)\right)\right)\omega_{i1}(k)\\
&+\sum_{j=1}^{2}G_{ij}\Gamma e_j(k)\left(d_i+\omega_{i2}(k)\right) \quad (32)
\end{aligned}$$

where $K\in\mathbb{R}^{n\times n}$ is a constant gain matrix to be determined.

Theorem 2: The coupled stochastic disturbed system (3) is globally exponentially synchronized in the mean square via the memoryless state-feedback controller (31) if there exist seven positive definite matrices P, Q_1, Q_2, R_1, R_2, $\bar{R}_1$, $\bar{R}_2$, four positive diagonal matrices $S_i (i=1,2,3,4)$, one arbitrary matrix $\widetilde{K}$, and a scalar $\lambda>0$ such that the following LMIs hold:

$$P<\lambda I \qquad \Sigma=\begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{12}^{\mathrm{T}} & \Sigma_{22}\end{bmatrix}<0 \quad (33)$$

where the expressions for Σ_{11}, Σ_{12}, Σ_{22}, Υ_2, Υ_1, and Υ_3 are shown at the bottom of the next page. Moreover, the controller gain is given by $K=P^{-1}\widetilde{K}^{\mathrm{T}}$.

Proof: From Theorem 1, one knows that system (33) is globally exponentially stable in the mean square if there exist seven matrices $P>0$, $Q_1>0$, $Q_2>0$, $R_1>0$, $R_2>0$, $\bar{R}_1>0$, $\bar{R}_2>0$, four diagonal matrices $S_i>0 (i=1,2,3,4)$, and a scalar $\lambda>0$ such that

$$P<\lambda I \qquad \widetilde{\Xi}=\begin{bmatrix}\widetilde{\Xi}_{11} & \widetilde{\Xi}_{12}\\ \widetilde{\Xi}_{12}^{\mathrm{T}} & \widetilde{\Xi}_{22}\end{bmatrix}<0 \quad (34)$$

where $\widetilde{\Xi}_{11}$, $\widetilde{\Xi}_{12}$, and $\widetilde{\Xi}_{22}$ are similar as Ξ_{11}, Ξ_{12}, and Ξ_{22} in Theorem 1 with the only difference that A is substituted by $A+K$.

On the other hand, by Lemma 1, $\widetilde{\Xi}<0$ is equivalent to

$$\widetilde{\Sigma}=\begin{bmatrix}\widetilde{\Sigma}_{11} & \widetilde{\Sigma}_{12}\\ \widetilde{\Sigma}_{12}^{\mathrm{T}} & \widetilde{\Sigma}_{22}\end{bmatrix}<0 \quad (35)$$

where $\widetilde{\Sigma}_{11}$, $\widetilde{\Sigma}_{12}$, and $\widetilde{\Sigma}_{22}$ are the same as Σ_{11}, Σ_{12}, and Σ_{22} by just noting that $K=P^{-1}\widetilde{K}^{\mathrm{T}}$. ■

In the following, one special case is discussed. The proof of the subsequent corollary is similar to that of Theorem 2 and, hence, omitted here. Consider the coupled network system (3) without stochastic terms. In this case, (3) reduces to

$$\begin{aligned}
x_i(k+1)=\ &Ax_i(k)+Bf\left(x_i(k)\right)\\
&+Cf_d\left(x_i\left(k-\tau(k)\right)\right)+u_i(k)+d_i\sum_{j=1}^{2}G_{ij}\Gamma x_j(k) \quad (36)
\end{aligned}$$

and we can obtain the following result:

Corollary 1: The discrete-time coupled system (36) is globally exponentially synchronized via the memoryless state-feedback controller (31) if there exist seven positive definite matrices P, Q_1, Q_2, R_1, R_2, $\bar{R}_1$, $\bar{R}_2$, four positive diagonal matrices $S_i (i=1,2,3,4)$ and one arbitrary matrix $\widetilde{K}$ such that the following LMI holds:

$$\Delta=\begin{bmatrix}\Delta_{11} & \Delta_{12}\\ \Delta_{12}^{\mathrm{T}} & \Delta_{22}\end{bmatrix}<0 \quad (37)$$

where the expressions for Δ_{11}, Δ_{12}, Δ_{22}, Θ_2, Θ_1, and Θ_3 are shown at the bottom of page 1080. Moreover, the controller gain $K=P^{-1}\widetilde{K}^{\mathrm{T}}$.

Remark 3: In our main results, the synchronization control problem is considered for two coupled discrete-time complex networks with time-delays, and the control law is designed to synchronize the addressed coupled complex networks in terms of certain LMIs that can be readily solved using Matlab LMI toolbox. It should be pointed out that the variables of the LMIs are essentially the parameters of the addressed complex networks. Therefore, once an adequate complex network is established, and the corresponding parameters are identified, we can analyze the exponential synchronization control problem of the complex network by simply checking the feasibility of the LMIs. Note that the verification of the solvability of LMIs can be conveniently done by utilizing the numerically efficient Matlab LMI toolbox, and no turning of parameters will be needed [1]. In the past decade, LMIs have gained much attention for their computational tractability and usefulness in system engineering (see, e.g., [1]) as the so-called interior point method has been proved to be numerically very efficient for solving the LMIs. The number of analysis and design problems that can be formulated as LMI problems is large and continues to grow.

IV. Two Numerical Examples

In this section, two simple examples are presented to justify Theorem 2 acquired in the previous section.

Example 1: Consider an isolated network (1) with parameters as follows:

$$\tau_m = 2$$

$$A = \begin{bmatrix} -0.0830 & -0.9386 & -0.6559 & -0.5539 \\ -0.6616 & -0.5905 & -0.4519 & -0.6801 \\ -0.5170 & -0.4406 & -0.8397 & -0.3672 \\ -0.1710 & -0.9419 & -0.5326 & -0.2393 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.002 & 0.01 & 0.001 & 0 \\ 0 & -0.002 & -0.013 & 0.01 \\ -0.01 & 0.001 & 0.002 & 0 \\ 0 & 0 & 0.01 & 0.012 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.03 & 0 & 0 & 0.03 \\ -0.03 & 0.03 & 0 & 0.03 \\ 0 & -0.03 & 0.003 & 0.03 \\ 0 & 0.03 & -0.03 & 0.036 \end{bmatrix}$$

and $f(s(k)) = f_d(s(k)) = (\tanh(2s_1(k)), \tanh(-4s_2(k)), \tanh(-2s_1(k)), \tanh(2s_1(k)))^{\mathrm{T}}$. It can be easily determined that $K_1 = L_1 = 0$ and $K_2 = L_2 = \mathrm{diag}\{-1, 2, 1, -1\}$.

$$\Sigma_{11} = \begin{bmatrix} \Upsilon_1 & 0 & 0 & 0 & -S_1L_2 + d_1G_{11}\Gamma^{\mathrm{T}}PB & d_1G_{11}\Gamma^{\mathrm{T}}PC & A^{\mathrm{T}}P + \widetilde{K} \\ * & -R_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & -Q_1 + \lambda G_2^{\mathrm{T}}G_2 - S_3K_1 & 0 & 0 & -S_3K_2 & 0 \\ * & * & * & -\bar{R}_1 & 0 & 0 & 0 \\ * & * & * & * & -S_1 & 0 & B^{\mathrm{T}}P \\ * & * & * & * & * & -S_3 & C^{\mathrm{T}}P \\ * & * & * & * & * & * & -P \end{bmatrix}$$

$$\Sigma_{12} = \begin{bmatrix} \Upsilon_2 & 0 & 0 & 0 & d_2G_{21}\Gamma^{\mathrm{T}}PB & d_2G_{21}\Gamma^{\mathrm{T}}PC & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_1G_{12}B^{\mathrm{T}}P\Gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ d_1G_{12}C^{\mathrm{T}}P\Gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Sigma_{22} = \begin{bmatrix} \Upsilon_3 & 0 & 0 & 0 & -S_2L_2 + d_2G_{22}\Gamma^{\mathrm{T}}PB & d_2G_{22}\Gamma^{\mathrm{T}}PC & A^{\mathrm{T}}P + \widetilde{K} \\ * & -R_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & -Q_2 + \lambda G_2^{\mathrm{T}}G_2 - S_4K_1 & 0 & 0 & -S_4K_2 & 0 \\ * & * & * & -\bar{R}_2 & 0 & 0 & 0 \\ * & * & * & * & -S_2 & 0 & B^{\mathrm{T}}P \\ * & * & * & * & * & -S_4 & C^{\mathrm{T}}P \\ * & * & * & * & * & * & -P \end{bmatrix}$$

$$\Upsilon_2 = \left(G_{11}G_{12}\left(1 + d_1^2\right) + G_{22}G_{21}\left(1 + d_2^2\right)\right)\Gamma^{\mathrm{T}}P\Gamma + d_1G_{12}(A^{\mathrm{T}}P\Gamma + \widetilde{K}\Gamma) + d_2G_{21}(\Gamma^{\mathrm{T}}PA + \Gamma^{\mathrm{T}}\widetilde{K}^{\mathrm{T}})$$

$$\Upsilon_1 = (\tau_M - \tau_m + 1)Q_1 - P + \lambda G_1^{\mathrm{T}}G_1 - S_1L_1 + R_1 + \bar{R}_1 + \left(\left(1 + d_1^2\right)G_{11}^2 + \left(1 + d_2^2\right)G_{21}^2\right)\Gamma^{\mathrm{T}}P\Gamma + d_1G_{11}\left(A^{\mathrm{T}}P\Gamma + \Gamma^{\mathrm{T}}PA + \widetilde{K}\Gamma + \Gamma^{\mathrm{T}}\widetilde{K}^{\mathrm{T}}\right)$$

$$\Upsilon_3 = (\tau_M - \tau_m + 1)Q_2 - P + \lambda G_1^{\mathrm{T}}G_1 - S_2L_1 + R_2 + \bar{R}_2 + \left(\left(1 + d_1^2\right)G_{12}^2 + \left(1 + d_2^2\right)G_{22}^2\right)\Gamma^{\mathrm{T}}P\Gamma + d_2G_{22}\left(A^{\mathrm{T}}P\Gamma + \Gamma^{\mathrm{T}}PA + \widetilde{K}\Gamma + \Gamma^{\mathrm{T}}\widetilde{K}^{\mathrm{T}}\right)$$

Now, let two identical networks in (1) be coupled and disturbed with the following parameters:

$$d_1 = 0.1, \qquad d_2 = 0.2$$
$$G = \begin{bmatrix} -0.1-a & 0.1+a \\ 0.2 & -0.2 \end{bmatrix}$$
$$\Gamma = \begin{bmatrix} -0.0579 & -0.0444 & -0.0404 & -0.0260 \\ -0.0867 & -0.0300 & -0.0390 & -0.0087 \\ -0.0407 & -0.0401 & -0.0360 & -0.0429 \\ -0.0113 & -0.0833 & -0.0140 & -0.0257 \end{bmatrix}$$
$$G_1 = \begin{bmatrix} 0.0595 & 0.0990 & 0.1570 & 0.0007 \\ 0.0850 & 0.1413 & 0.0148 & 0.0441 \\ 0.0238 & 0.0487 & 0.0788 & 0.0003 \end{bmatrix}$$
$$G_2 = \begin{bmatrix} 0.0568 & 0.0525 & 0.2703 & 0.1448 \\ 0.0427 & 0.0416 & 0.2818 & 0.1128 \\ 0.0804 & 0.1797 & 0.0664 & 0.1571 \end{bmatrix}.$$

Let $a = 0$ and τ_M vary from the value 3. By using the Matlab LMI Control Toolbox, a feasible solution to the LMIs in (33) can always be found when $\tau_M \in \mathbb{N}[3, 5]$. For example, taking $\tau_M = 4$, the corresponding solution is listed as follows:

$$P = \begin{bmatrix} 122.7081 & 5.7527 & 6.1273 & 4.8039 \\ 5.7527 & 131.0024 & 7.1094 & 6.4317 \\ 6.1273 & 7.1094 & 141.2300 & 16.6209 \\ 4.8039 & 6.4317 & 16.6209 & 124.8004 \end{bmatrix}$$
$$Q_1 = \begin{bmatrix} 18.8309 & 2.2670 & 3.8593 & 3.0423 \\ 2.2670 & 21.1439 & 4.3440 & 4.3243 \\ 3.8593 & 4.3440 & 32.4577 & 10.2498 \\ 3.0423 & 4.3243 & 10.2498 & 23.0821 \end{bmatrix}$$
$$Q_2 = \begin{bmatrix} 18.8315 & 2.2673 & 3.8595 & 3.0424 \\ 2.2673 & 21.1442 & 4.3441 & 4.3243 \\ 3.8595 & 4.3441 & 32.4578 & 10.2500 \\ 3.0424 & 4.3243 & 10.2500 & 23.0829 \end{bmatrix}$$
$$R_1 = \bar{R}_1 = \begin{bmatrix} 18.6853 & -1.4547 & -2.6639 & -1.7570 \\ -1.4547 & 16.9616 & -3.3628 & -2.6644 \\ -2.6639 & -3.3628 & 11.7259 & -5.0486 \\ -1.7570 & -2.6644 & -5.0486 & 16.4176 \end{bmatrix}$$
$$R_2 = \bar{R}_2 = \begin{bmatrix} 18.6854 & -1.4543 & -2.6635 & -1.7567 \\ -1.4543 & 16.9614 & -3.3624 & -2.6640 \\ -2.6635 & -3.3624 & 11.7260 & -5.0485 \\ -1.7567 & -2.6640 & -5.0485 & 16.4171 \end{bmatrix}$$
$$\widetilde{K} = \begin{bmatrix} 18.1433 & 91.8042 & 82.3544 & 34.6644 \\ 128.1228 & 91.4326 & 88.0246 & 133.0618 \\ 90.7563 & 70.6068 & 134.8563 & 87.5989 \\ 75.0734 & 94.9697 & 63.6401 & 41.3365 \end{bmatrix}$$
$$S_1 = \mathrm{diag}\{13.0810, 4.0177, 7.3422, 10.3274\}$$
$$S_2 = \mathrm{diag}\{13.0815, 4.0183, 7.3428, 10.3272\}$$
$$S_3 = \mathrm{diag}\{8.5475, 1.9206, 3.1295, 5.6770\}$$
$$S_4 = \mathrm{diag}\{8.5480, 1.9208, 3.1298, 5.6775\}$$
$$\lambda = 159.7163.$$

$$\Delta_{11} = \begin{bmatrix} \Theta_1 & 0 & 0 & 0 & -S_1L_2 + d_1G_{11}\Gamma^{\mathrm{T}}PB & d_1G_{11}\Gamma^{\mathrm{T}}PC & A^{\mathrm{T}}P + \widetilde{K} \\ * & -R_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & -Q_1 - S_3K_1 & 0 & 0 & -S_3K_2 & 0 \\ * & * & * & -\bar{R}_1 & 0 & 0 & 0 \\ * & * & * & * & -S_1 & 0 & B^{\mathrm{T}}P \\ * & * & * & * & * & -S_3 & C^{\mathrm{T}}P \\ * & * & * & * & * & * & -P \end{bmatrix}$$

$$\Delta_{12} = \begin{bmatrix} \Theta_2 & 0 & 0 & 0 & d_2G_{21}\Gamma^{\mathrm{T}}PB & d_2G_{21}\Gamma^{\mathrm{T}}PC & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_1G_{12}B^{\mathrm{T}}P\Gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ d_1G_{12}C^{\mathrm{T}}P\Gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Delta_{22} = \begin{bmatrix} \Theta_3 & 0 & 0 & 0 & -S_2L_2 + d_2G_{22}\Gamma^{\mathrm{T}}PB & d_2G_{22}\Gamma^{\mathrm{T}}PC & A^{\mathrm{T}}P + \widetilde{K} \\ * & -R_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & -Q_2 - S_4K_1 & 0 & 0 & -S_4K_2 & 0 \\ * & * & * & -\bar{R}_2 & 0 & 0 & 0 \\ * & * & * & * & -S_2 & 0 & B^{\mathrm{T}}P \\ * & * & * & * & * & -S_4 & C^{\mathrm{T}}P \\ * & * & * & * & * & * & -P \end{bmatrix}$$

$$\Theta_2 = \left(G_{11}G_{12}d_1^2 + G_{22}G_{21}d_2^2\right)\Gamma^{\mathrm{T}}P\Gamma + d_1G_{12}(A^{\mathrm{T}}P\Gamma + \widetilde{K}\Gamma) + d_2G_{21}(\Gamma^{\mathrm{T}}PA + \Gamma^{\mathrm{T}}\widetilde{K}^{\mathrm{T}})$$

$$\Theta_1 = (\tau_M - \tau_m + 1)Q_1 - P - S_1L_1 + R_1 + \bar{R}_1 + \left(d_1^2G_{11}^2 + d_2^2G_{21}^2\right)\Gamma^{\mathrm{T}}P\Gamma + d_1G_{11}(A^{\mathrm{T}}P\Gamma + \Gamma^{\mathrm{T}}PA + \widetilde{K}\Gamma + \Gamma^{\mathrm{T}}\widetilde{K}^{\mathrm{T}})$$

$$\Theta_3 = (\tau_M - \tau_m + 1)Q_2 - P - S_2L_1 + R_2 + \bar{R}_2 + \left(d_1^2G_{12}^2 + d_2^2G_{22}^2\right)\Gamma^{\mathrm{T}}P\Gamma + d_2G_{22}(A^{\mathrm{T}}P\Gamma + \Gamma^{\mathrm{T}}PA + \widetilde{K}\Gamma + \Gamma^{\mathrm{T}}\widetilde{K}^{\mathrm{T}})$$

Moreover, the controller gain matrix is obtained by

$$K = P^{-1}\widetilde{K}^{\mathrm{T}} = \begin{bmatrix} 0.0840 & 0.9578 & 0.6559 & 0.5533 \\ 0.6602 & 0.5858 & 0.4379 & 0.6696 \\ 0.5262 & 0.4416 & 0.8406 & 0.3662 \\ 0.1704 & 0.9403 & 0.5421 & 0.2266 \end{bmatrix}.$$

From Theorem 2, we know that the two coupled subsystems with stochastic disturbances and different initial conditions are globally exponentially synchronized with the given control law. Now, to show that the value of the parameter "a" does influence the synchronous motion, we let $\tau_M \equiv 4$, and the parameter "a" vary from -0.1. According to our main results, the LMIs in (33) are always feasible when $a \in [-0.1, 2.4853]$. In other words, the coupled network is synchronized when $a \in [-0.1, 2.4853]$.

Example 2: In order to analyze the influence of stochastic disturbances onto the dynamics of the coupled system, we now consider model (36) without stochastic disturbances and then compare the synchronous behavior with that in Example 1. The coefficients and parameters are the same as those in Example 1. By using the Matlab LMI Control Toolbox, a solution to the LMI in (37) can always be found when $\tau_M \in \mathbb{N}[3, 53]$. Comparing to the results in Example 1, we arrive at the conclusion that the random disturbances reduce the synchronous dynamics. For example, take $\tau_M = 53$, the corresponding solution is listed as follows:

$$P = \begin{bmatrix} 470.8045 & 153.3697 & -41.7123 & -106.7561 \\ 153.3697 & 823.3083 & -24.4258 & -337.0709 \\ -41.7123 & -24.4258 & 143.3909 & 59.6581 \\ -106.7561 & -337.0709 & 59.6581 & 425.6428 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 7.4625 & 2.9786 & -0.0726 & -1.5957 \\ 2.9786 & 15.3541 & -0.5364 & -6.2125 \\ -0.0726 & -0.5364 & 1.9650 & 0.7852 \\ -1.5957 & -6.2125 & 0.7852 & 7.5622 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 7.4905 & 2.9875 & -0.0617 & -1.5946 \\ 2.9875 & 15.3753 & -0.5344 & -6.2285 \\ -0.0617 & -0.5344 & 1.9718 & 0.7872 \\ -1.5946 & -6.2285 & 0.7872 & 7.5860 \end{bmatrix}$$

$$\widetilde{K} = \begin{bmatrix} 100.2649 & 485.3352 & 66.0326 & -127.1442 \\ 421.8741 & 301.5249 & 65.1491 & 126.7220 \\ 282.9604 & 257.8321 & 114.8793 & 63.3447 \\ 323.7366 & 551.0384 & 26.5753 & -166.5562 \end{bmatrix}$$

$$R_1 = \bar{R}_1 = \begin{bmatrix} 26.2741 & -0.9600 & -14.2561 & -8.7078 \\ -0.9600 & 7.9973 & 1.2198 & -5.5421 \\ -14.2561 & 1.2198 & 12.1709 & 6.8628 \\ -8.7078 & -5.5421 & 6.8628 & 10.1536 \end{bmatrix}$$

$$R_2 = \bar{R}_2 = \begin{bmatrix} 26.2832 & -0.8757 & -14.1574 & -8.6561 \\ -0.8757 & 7.7390 & 1.3061 & -5.1645 \\ -14.1574 & 1.3061 & 12.2651 & 6.8822 \\ -8.6561 & -5.1645 & 6.8822 & 9.7521 \end{bmatrix}$$

$$S_1 = \mathrm{diag}\{9.4119, 0.9675, 8.0734, 5.5794\}$$

$$S_2 = \mathrm{diag}\{9.7495, 0.9816, 7.8801, 5.4895\}$$

$$S_3 = \mathrm{diag}\{4.8229, 1.2810, 0.8603, 2.3303\}$$

$$S_4 = \mathrm{diag}\{4.8486, 1.2796, 0.8589, 2.3449\}.$$

Moreover, the controller gain matrix is obtained as

$$K = \begin{bmatrix} 0.0835 & 0.9575 & 0.6558 & 0.5532 \\ 0.6594 & 0.5857 & 0.4380 & 0.6699 \\ 0.5261 & 0.4416 & 0.8409 & 0.3661 \\ 0.1707 & 0.9398 & 0.5423 & 0.2266 \end{bmatrix}.$$

From Corollary 1, we know that the two coupled subsystems without stochastic disturbances and with different initial conditions are globally exponentially synchronized with the given control law.

V. Conclusion

In this paper, the synchronization problem has been analyzed for two identical coupled discrete-time complex networks with time-varying delay. In the complex system, both the overall networks and the network couplings are subject to stochastic disturbances. First, an easy-to-verify condition has been established under which the synchronization error dynamics is globally exponentially stable in the mean square. Second, a controller is designed to guarantee the coupled system to be synchronized by using a combination of LMI approach and the stochastic analysis tools. The LMI-based conditions obtained in this paper are dependent not only on the lower bound but also on the upper bound of the time-varying delay, which can be solved efficiently via the Matlab LMI Toolbox. Two numerical examples have been presented to show the validity of our theoretical analysis results.

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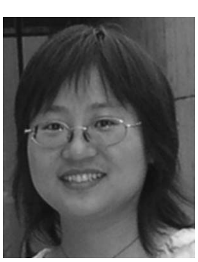

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