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Global synchronization of stochastic delayed complex networks

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Abstract This paper is concerned with the delay-dependent synchronization criterion for stochastic complex networks with time delays. Firstly, expectations of stochastic cross terms containing the Itô integral are investigated by utilizing stochastic analysis techniques. In fact, in order to obtain less conservative delay-dependent conditions for stochastic delay systems including stochastic complex (or neural) networks with time delays, how to deal with expectations of these stochastic cross terms is an important problem, and expectations of these stochastic terms were not dealt with properly in many existing results. Then, based on the investigation of expectations of stochastic cross terms, this paper proposes a novel delay-dependent synchronization criterion for stochastic delayed complex networks. In the derivation process, the mathematical development avoids bounding stochastic cross terms. Thus, the method leads to a simple criterion and shows less conservatism. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed approach.

Keywords Complex networks · Stochastic systems · Delay · Synchronization · Delay-dependent criterion

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1 Introduction

As is known to all, complex dynamical networks (CDNs) widely exist in the real world, including food-webs, ecosystems, metabolic pathways, the Internet, the world wide web, social networks, and global economic markets [1, 2]. Since the discoveries of the small-world feature [3] and the scale-free feature [4] of complex networks, the analysis and the control of the dynamical behaviors in complex networks have attracted a great deal of attention. As a significant collective behavior, the studies on synchronization phenomena of complex dynamical networks have been extensively investigated in [5–11]. Recently, it has now been well realized that in spreading information through complex networks, there always exist time delays caused by the finite speed of information transmission and the limit of bandwidth, which may decrease the quality of the system and even lead to oscillation, divergence, and instability. Accordingly, the synchronization problems for many delayed complex dynamic networks have received many researchers' interests; see, e.g., [12–17].

In real-time systems, complex networks are often subject to stochastic disturbances. For example, the signal transfer in a real complex network could be perturbed randomly from the release of probabilistic causes such as neurotransmitters and packet dropouts [21]. Hence, in order to reflect more realistic dynamical behaviors, many researchers recently began to study complex networks with stochastic disturbances, and many results for the synchronization problems of stochastic complex networks with or without time delays are reported in [18–22] and the references therein. For instance, the synchronization problems of discrete-time stochastic complex networks with delays were discussed in [18, 19]. As to the continuous case, Cao et al. [22] designed an adaptive feedback controller to solve the synchronization problem for an array of linearly stochastically coupled networks with time delays. Wang et al. [21] investigated the delay-dependent synchronization criterion for continuous stochastic complex networks with time-delays. And as a special class of stochastic delayed complex networks, stochastic neural networks with delays have been widely studied in [23–25].

On the other hand, for stochastic complex (or neural) networks with time delays, a very active research topic is to obtain the delay-dependent conditions. The reason is that the delay-dependent condition makes use of the information on the size of time delays, and the delay-dependent condition is generally less conservative than the delay-independent one [29–31]. However, when we used the existing effective methods, such as the model transformation method [29, 30] and the free-weighting-matrix method [31], to give the delay-dependent condition for stochastic delay systems including stochastic complex (or neural) networks with time delays, one important question will appear: How do we deal with the expectations of stochastic cross terms containing the Itô integral? Let us give an example to illustrate it. Considering the following stochastic functional differential equation:

$$dx(t) = f(t, x_t)\,dt + g(t, x_t)\,dw(t) \qquad (1)$$

on $t \geq 0$ with the initial data $x_0 = \xi \in L^2_{\mathcal{F}_0}([-h, 0]; \mathcal{R}^n)$ and h is the time delay of (1). Since the Newton–Leibniz is not tenable in the stochastic case [32] and (1) is just a symbolic expression, we must use the corresponding stochastic integral equation

$$x(t) = \xi(0) + \int_0^t f(s, x_s)\,ds + \int_0^t g(s, x_s)\,dw(s) \qquad (2)$$

to obtain the delay-dependent condition. Then there will appear the following stochastic cross-terms:

$$x(t)^T \mathfrak{M} \int_{t-h}^t g(s, x_s)\,dw(s),$$

$$x(t-h)^T \mathfrak{N} \int_{t-h}^t g(s, x_s)\,dw(s),$$

$$\left(\int_{t-h}^t f(s, x_s)\,ds\right)^T \mathfrak{L} \int_{t-h}^t g(s, x_s)\,dw(s).$$

It is still very difficult to calculate the expectations of these stochastic cross terms up until now. The results in [23–25] resorted to bounding techniques, which obviously can bring the conservatism. Some papers such as [21, 26–28] considered that the expectations of these stochastic terms are all equal to zero. However, these results are not still given by strict mathematical proofs, and we can find examples to illustrate that expectations of some stochastic cross terms are not equal to zero in Remark 1. Therefore, in order to obtain the delay-dependent synchronization criterion with less conservatism for stochastic delayed complex networks, there is a strong need to investigate the expectations of stochastic cross terms containing the Itô integral to avoid the mistake in [21] first.

Motivated by the above discussions, this paper aims to investigate the delay-dependent synchronization criterion for stochastic complex networks with time delays. The main contributions of this paper are summarized as follows: (1) Expectations of stochastic cross terms containing the Itô integral are investigated by stochastic analysis techniques in Lemma 2. We prove that the expectation of $x(t-h)^T \mathfrak{N} \int_{t-h}^t g(s, x_s)\,dw(s)$ is equal to zero, and expectations of other stochastic cross terms are not. (2) Based on this lemma, this paper establishes a delay-dependent synchronization criterion that guarantees the globally asymptotically mean-square synchronization of stochastic delayed complex networks. In the derivation process, the mathematical development avoids bounding stochastic cross terms. Thus, the method leads to a simple criterion with less conservatism, and a numerical example is provided to demonstrate the effectiveness of the proposed approach.

Notation: Throughout the paper, unless otherwise specified, we will employ the following notation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\mathcal{E}(\cdot)$ be the expectation operator with respect to the probability measure. If A is a vector or matrix, its transpose is denoted by A^T. If P is a square matrix, $P > 0$ $(P < 0)$ means that is a symmetric positive (negative) definite matrix of appropriate dimensions while $P \geq 0$ $(P \leq 0)$ is a symmetric positive (negative) semidefinite matrix. I stands for the identity matrix of appropriate dimensions. Denote by $\lambda_{\min}(\cdot)$ the minimum eigenvalue of a given matrix. Let $|\cdot|$ denote the Euclidean norm of a vector and its induced norm of a matrix. Unless explicitly specified, matrices are assumed to have real entries and compatible dimensions. $L^2(\Omega)$ denotes the space of all random variables X with $\mathcal{E}|X|^2 < \infty$; it is a Banach space with norm $\|X\|_2 = (\mathcal{E}|X|^2)^{1/2}$. Let $h > 0$ and $C([-h,0];\mathcal{R}^n)$ denote the family of all continuous $\mathcal{R}^n$-valued functions φ on $[-h,0]$ with the norm $\|\varphi\| = \sup\{|\varphi(\theta)| : -h \leq \theta \leq 0\}$. Let $L^2_{\mathcal{F}_0}([-h,0];\mathcal{R}^n)$ be the family of all $\mathcal{F}_0$-measurable $C([-h,0];\mathcal{R}^n)$-valued random variables ϕ such that $\mathcal{E}(\|\phi\|^2) < \infty$, and $\mathcal{L}^2([a,b];\mathcal{R}^n)$ be the family of all $\mathcal{R}^n$-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\leq t\leq b}$ such that $\int_a^b |f(t)|^2\,dt < \infty$ a.s. Let $\mathcal{M}^2([a,b];\mathcal{R}^n)$ be the family of processes $\{f(t)\}_{a\leq t\leq b}$ in $\mathcal{L}^2([a,b];\mathcal{R}^n)$ such that $\mathcal{E}(\int_a^b |f(t)|^2\,dt) < \infty$, and $\mathcal{M}^2([a,b])$ is the 1-dimension case of $\mathcal{M}^2([a,b];\mathcal{R}^n)$.

2 Problem formulation and preliminaries

Consider the following complex dynamical networks consisting of N identical nodes with stochastic perturbations:

$$
\begin{aligned}
dx_i(t) = {} & \Big[Ax_i(t) + Bf\big(x_i(t)\big) + Cf\big(x_i(t-h)\big) \\
& + \sum_{j=1}^{N} g_{ij}\Gamma x_j(t) + \sum_{j=1}^{N} h_{ij}\Upsilon x_j(t-h) \Big] dt \\
& + \sigma_i\big(t, x_i(t), x_i(t-h)\big)\,dw(t), \\
& i = 1,2,\ldots,N
\end{aligned} \tag{3}
$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{\text{in}}(t)]^T \in \mathcal{R}^n$ is the state vector of the ith network at time t; the scalar $h > 0$ denotes the time delay; A denotes a known connection matrix, B and C denote, respectively, the connection weight matrix and the delayed connection weight matrix; $\Gamma, \Upsilon \in \mathcal{R}^{n\times n}$ are matrices describing the inner-coupling between the subsystems at time t and $t-h$, respectively; $G = (g_{ij})_{N\times N}$ and $H = (h_{ij})_{N\times N}$ are the out-coupling configuration matrices representing the coupling strength and the topological structure of the complex networks. $\sigma_i(\cdot,\cdot,\cdot) : \mathcal{R}\times\mathcal{R}^n\times\mathcal{R}^n \to \mathcal{R}^n$ is the noise intensity function vector and $w(t)$ is a scalar standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. And $f(x_i(t)) = (f_1(x_{i1}(t)), \ldots, f_n(x_{\text{in}}(t)))^T$ is an unknown but sector-bounded nonlinear function.

The initial conditions associated with system (3) are given by

$$
x_i(s) = \varphi_i(s), -h \leq s \leq 0, \quad i = 1,2,\ldots,N, \tag{4}
$$

where $\varphi_i(s) \in L^2_{\mathcal{F}_0}([-h,0];\mathcal{R}^n)$.

Let

$$
\begin{aligned}
x(t) &= \big(x_1(t)^T, \ldots, x_N(t)^T\big)^T, \\
F\big(x(t)\big) &= \big(f\big(x_1(t)\big)^T, \ldots, f\big(x_N(t)\big)^T\big)^T, \\
F\big(x(t-h)\big) &= \big(f\big(x_1(t-h)\big)^T, \ldots, f\big(x_N(t-h)\big)^T\big)^T, \\
\sigma(t) &= \big(\sigma_1\big(t, x_1(t), x_1(t-h)\big)^T, \\
&\qquad \ldots, \sigma_N\big(t, x_N(t), x_N(t-h)\big)^T\big)^T.
\end{aligned}
$$

With the Kronecker product "⊗" for matrices, system (3) can be rearranged as

$$
\begin{aligned}
dx(t) = {} & \big[(I_N \otimes A + G \otimes \Gamma)x(t) \\
& + (H \otimes \Upsilon)x(t-h) + (I_N \otimes B)F\big(x(t)\big) \\
& + (I_N \otimes C)F\big(x(t-h)\big)\big]\,dt + \sigma(t)\,dw(t).
\end{aligned} \tag{5}
$$

Throughout this paper, the following assumptions, definitions, and propositions are needed to prove our main results.

Definition 1 [33] Let $\{\eta(t)\}_{a\leq t\leq b}$ is a stochastic process and belongs to $\mathcal{M}^2([a,b])$, then its Itô integral (from a to b) is defined by

$$\int_a^b \eta(t)\,dw(t) = \lim_{n\to\infty} \int_a^b \eta_n(t)\,dw(t) \quad \big(\lim \text{ in } L^2(\Omega)\big),$$

where $\{\eta_n(t)\}_{a\le t\le b}$ $(n = 1, 2, \ldots)$ are the step stochastic processes and belong to $\mathcal{M}^2([a, b])$ such that

$$\lim_{n\to\infty} \mathcal{E}\left(\int_a^b \big|\eta(t) - \eta_n(t)\big|^2\, dt\right) = 0.$$

Definition 2 [21] The stochastic delayed complex network (3) is globally asymptotically synchronized in the mean square if, for all $\varphi_i, \varphi_j \in L^2_{\mathcal{F}_0}([-h, 0]; \mathcal{R}^n)$. The following holds:

$$\lim_{t\to\infty} \mathcal{E}\big\{|x_i(t,\varphi_i) - x_j(t,\varphi_j)|^2\big\} = 0, \quad 1 \le i < j \le N.$$

Definition 3 [32] Let $\{\mathcal{F}_t\}_{t\in T}$ be an increasing family of σ-algebras of subset of Ω. A stochastic process $\{X_t\}_{t\in T}$ is said to be adapted to $\{\mathcal{F}_t\}_{t\in T}$ if for each t, the random variable X_t is $\mathcal{F}_t$-measurable.

Assumption 1 *For $\forall x, y \in \mathcal{R}^n$, the nonlinear function $f(\cdot)$ is assumed to satisfy the following condition*:

$$\big(f(x) - f(y) - U(x - y)\big)^T \times \big(f(x) - f(y) - V(x - y)\big) \le 0, \tag{6}$$

where U and V are real constant matrices with $V - U$ being symmetric and positive definite.

Assumption 2 *The outer-coupling configuration matrices of the complex networks* (3) *satisfy*

$$g_{ij} = g_{ji} \ge 0, \qquad h_{ij} = h_{ji} \ge 0 \quad (i \ne j),$$
$$g_{ii} = -\sum_{j=1, j\ne i}^{N} g_{ij}, \qquad h_{ii} = -\sum_{j=1, j\ne i}^{N} h_{ij}, \quad i, j = 1, 2, \ldots, N.$$

Assumption 3 *The noise intensity function vector $\sigma_i : \mathcal{R} \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$ satisfies the Lipschitz condition, i.e., there exist constant matrices W_1 and W_2 of appropriate dimensions such that*

$$\big|\sigma_i(t, x_1, y_1) - \sigma_j(t, x_2, y_2)\big|^2 \le \big|W_1(x_1 - x_2)\big|^2 + \big|W_2(y_1 - y_2)\big|^2 \tag{7}$$

for all $i, j = 1, 2, \ldots, N$ and $x_1, y_1, x_2, y_2 \in \mathcal{R}^n$.

Proposition 1 [33] *Let $\{\mathfrak{h}(t)\}_{a\le t\le b}$ is a stochastic process and belong to $\mathcal{M}^2([a, b])$, then*

$$\mathcal{E}\left(\int_a^b \mathfrak{h}(t)\,dw(t)\right) = 0.$$

Proposition 2 [13] *The Kronecker product has the following properties*:

$$(\alpha A) \otimes B = A \otimes (\alpha B),$$
$$(A + B) \otimes C = A \otimes C + B \otimes C,$$
$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD),$$
$$(A \otimes B)^T = A^T \otimes B^T.$$

Proposition 3 [17] *Let $\mathcal{U} = (\alpha_{ij})_{n\times n}$, $P \in \mathcal{R}^{m\times m}$, $x = (x_1^T, x_2^T, \ldots, x_n^T)^T$, $y = (y_1^T, y_2^T, \ldots, y_n^T)^T$, where $x_i = (x_{i1}, x_{i2}, \ldots, x_{\text{im}})^T \in \mathcal{R}^m$, $y_i = (y_{i1}, y_{i2}, \ldots, y_{\text{im}})^T \in \mathcal{R}^m$ $(i = 1, 2, \ldots, n)$. If $\mathcal{U} = \mathcal{U}^T$ and each row sum of $\mathcal{U}$ is equal to zero, then*

$$x^T(\mathcal{U} \otimes P)y = -\sum_{1\le i<j\le n} \alpha_{ij}(x_i - x_j)^T P(y_i - y_j). \tag{8}$$

3 Main results

Then we give the following lemmas which will be used in the proof of our main results.

Lemma 1 *If ζ is a bounded and $\mathcal{F}_a$-measurable random variable and $\{\eta(t)\}_{a\le t\le b}$ is a stochastic process which belongs to $\in \mathcal{M}^2([a, b])$, then*

$$\int_a^b \zeta\eta(t)\,dw(t) = \zeta \int_a^b \eta(t)\,dw(t). \tag{9}$$

Proof Since ζ is a bounded and $\mathcal{F}_a$-measurable random variable, it is easily to verify $\{\zeta\eta(t)\}_{a\le t\le b} \in \mathcal{M}^2([a, b])$.

Step 1: If $\{\eta(t)\}_{a\le t\le b}$ is a step stochastic process, then we let without loss of generality,

$$\eta(t) = \sum_{i=1}^{n} \chi_{i-1} 1_{[t_{i-1}, t_i)}(t),$$

where $t_0 = a, t_n = b, \chi_{i-1}$ is $\mathcal{F}_{t_{i-1}}$-measurable and $\mathcal{E}(\chi_{i-1}^2) < \infty$. In this case,

$$\int_a^b \zeta \eta(t)\,dw(t) = \sum_{i=1}^n \zeta \chi_{i-1}\big(w(t_i) - w(t_{i-1})\big) = \zeta \sum_{i=1}^n \chi_{i-1}\big(w(t_i) - w(t_{i-1})\big) = \zeta \int_a^b \eta(t)\,dw(t). \tag{10}$$

Step 2: If $\{\eta(t)\}_{a\le t\le b} \in \mathcal{M}^2([a,b])$ is not a step stochastic process, then by Definition 1, we can know that there exists a sequence of step stochastic processes in $\mathcal{M}^2([a,b])$: $\{\eta_1(t)\}_{a\le t\le b}, \{\eta_2(t)\}_{a\le t\le b}, \dots, \{\eta_n(t)\}_{a\le t\le b}, \dots$ such that

$$\int_a^b \eta(t)\,dw(t) = \lim_{n\to\infty} \int_a^b \eta_n(t)\,dw(t) \quad \big(\lim \text{ in } L^2(\Omega)\big), \tag{11}$$

where $\{\eta(t)\}_{a\le t\le b}, \{\eta_n(t)\}_{a\le t\le b}$ satisfies

$$\lim_{n\to\infty} \mathcal{E}\left(\int_a^b |\eta(t) - \eta_n(t)|^2\,dt\right) = 0. \tag{12}$$

Since ζ is bounded, and by Definition 1 and (11)–(12), it is easy to prove that

$$\int_a^b \zeta\eta(t)\,dw(t) = \lim_{n\to\infty} \int_a^b \zeta\eta_n(t)\,dw(t) \quad \big(\lim \text{ in } L^2(\Omega)\big), \tag{13}$$

$$\zeta \int_a^b \eta(t)\,dw(t) = \lim_{n\to\infty} \zeta \int_a^b \eta_n(t)\,dw(t) \quad \big(\lim \text{ in } L^2(\Omega)\big). \tag{14}$$

From Step 1, we know that for each step stochastic process $\{\eta_n(t)\}_{a\le t\le b}$, we have

$$\int_a^b \zeta\eta_n(t)\,dw(t) = \zeta \int_a^b \eta_n(t)\,dw(t). \tag{15}$$

Therefore, it is easy to know

$$\lim_{n\to\infty} \int_a^b \zeta\eta_n(t)\,dw(t) = \lim_{n\to\infty} \zeta \int_a^b \eta_n(t)\,dw(t) \quad \big(\lim \text{ in } L^2(\Omega)\big). \tag{16}$$

Then, we can see that by (13), (14), and (16)

$$\int_a^b \zeta\eta(t)\,dw(t) = \zeta \int_a^b \eta(t)\,dw(t). \tag{17}$$

This completes the proof. □

Lemma 2 *Considering the following stochastic delay differential equation*:

$$dx(t) = f(t, x_t)\,dt + g(t, x_t)\,dw(t), \tag{18}$$

on $t \ge 0$ *with the initial data* $x_0 = \xi \in L^2_{\mathcal{F}_0}([-h, 0]; \mathcal{R}^n)$, *where* $h > 0$ *is the time delay in* (18). $f(\cdot,\cdot)$ *and* $g(\cdot,\cdot)$ *satisfy the local Lipschitz condition and the linear growth condition. Let* $x(t)$ *to be the solution of* (18), $\mathfrak{N}$ *is a any compatible dimension matrix*, *we have*

$$\mathcal{E}\left(x(t-h)^T \mathfrak{N}\left[\int_{t-h}^t g(s, x_s)\,dw(s)\right]\right) = 0, \quad t \ge h. \tag{19}$$

Proof Since $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ satisfy the local Lipschitz condition and the linear growth condition, it is easy to verify that, for $\forall T > 0$, the stochastic delay differential equation has a unique continuous solution on $[-h, T]$ denoted by $\{x(t)\}_{-h\le t\le T}$, that is adapted to $\{\mathcal{F}_t\}_{-h\le t\le T}$ and $\{x(t)\}_{-h\le t\le T} \in \mathcal{M}^2([-h, T])$. Therefore, we can easily know that for $t \ge h$, $x(t-h)$ is a bounded random variable and $x(t-h)$ is $\mathcal{F}_{t-h}$-measurable from Definition 3. Therefore, by Lemma 1, it is easy to obtain

$$x(t-h)^T \mathfrak{N}\left[\int_{t-h}^t g(s, x_s)\,dw(s)\right] = \int_{t-h}^t x(t-h)^T \mathfrak{N} g(s, x_s)\,dw(s), \quad t \ge h. \tag{20}$$

From Proposition 1, we can prove (19). □

Remark 1 Lemma 1 has proved

$$\mathcal{E}\left(x(t-h)^T \mathfrak{N}\left[\int_{t-h}^t g(s, x_s)\,dw(s)\right]\right) = 0, \quad t \ge h.$$

However, for any compatible dimension matrix $\mathfrak{N}$, it cannot prove

$$\mathcal{E}\left(x(t)^T \mathfrak{N}\left[\int_{t-h}^t g(s, x_s)\,dw(s)\right]\right) = 0,$$

$$\mathcal{E}\left(\left(\int_{t-h}^{t} f(s,x_s)\,ds\right)^T \mathfrak{N}\left[\int_{t-h}^{t} g(s,x_s)\,dw(s)\right]\right)=0,$$
$$t\geq h.$$

For example, considering the following stochastic equation:

$$dx(t)=dw(t), \tag{21}$$

which has a one solution $x(t)=w(t)$. However, we can easily verify that

$$\begin{aligned}&\mathcal{E}\left(x(t)^T\mathfrak{N}\int_{t-h}^{t} g(s,x_s)\,dw(s)\right)\\&\quad=\mathcal{E}\left(w(t)\mathfrak{N}\int_{t-h}^{t} dw(s)\right)=\mathfrak{N}h\neq 0,\quad \forall \mathfrak{N}\neq 0.\end{aligned} \tag{22}$$

Then we can also consider the following one-dimension Langevin equation [32]:

$$dx(t)=f(t,x_t)\,dt+g(t,x_t)\,dw(t),\quad x(0)=\xi,$$

where $f(t,x_t)=-x(t)$, $g(t,x_t)=1$. Obviously, it has a solution

$$x(t)=e^{-(t-u)}x(u)+\int_{u}^{t} e^{-(t-s)}\,dw(s),\quad u\leq t, \tag{23}$$

or

$$x(t)=e^{-t}x(0)+\int_{0}^{t} e^{-(t-s)}\,dw(s). \tag{24}$$

Then by (23), we can see that

$$\begin{aligned}&\mathcal{E}\left(x(t)\mathfrak{N}\int_{t-h}^{t} g(s,x_s)\,dw(s)\right)\\&\quad=\mathcal{E}\left(\left(e^{-h}x(t-h)+\int_{t-h}^{t} e^{-(t-s)}\,dw(s)\right)\right.\\&\qquad\left.\times\mathfrak{N}\left[\int_{t-h}^{t} dw(s)\right]\right)\\&\quad=e^{-h}\mathcal{E}\left(x(t-h)\mathfrak{N}\left[\int_{t-h}^{t} dw(s)\right]\right)\\&\qquad+\mathcal{E}\left(\int_{t-h}^{t} e^{-(t-s)}\,dw(s)\mathfrak{N}\int_{t-h}^{t} dw(s)\right)\\&\quad=0+\mathfrak{N}\int_{t-h}^{t}\mathcal{E}\left(e^{-(t-s)}\right)ds\\&\quad=\mathfrak{N}-\mathfrak{N}e^{-h}\neq 0,\quad \forall N\neq 0.\end{aligned} \tag{25}$$

and

$$\begin{aligned}&\mathcal{E}\left(\int_{t-h}^{t} f(s,x_s)\,ds\mathfrak{N}\left[\int_{t-h}^{t} g(s,x_s)\,dw(s)\right]\right)\\&\quad=\mathcal{E}\left(\left(x(t)-x(t-h)-\int_{t-h}^{t} g(s,x_s)\,dw(s)\right)\right.\\&\qquad\left.\times\mathfrak{N}\left[\int_{t-h}^{t} g(s,x_s)\,dw(s)\right]\right)\\&\quad=\mathcal{E}\left(x(t)\mathfrak{N}\int_{t-h}^{t} g(s,x_s)\,dw(s)\right)\\&\qquad-\mathcal{E}\left(x(t-h)\mathfrak{N}\int_{t-h}^{t} g(s,x_s)\,dw(s)\right)\\&\qquad-\mathcal{E}\left(\int_{t-h}^{t} g(s,x_s)\,dw(s)\mathfrak{N}\int_{t-h}^{t} g(s,x_s)\,dw(s)\right)\\&\quad=\mathfrak{N}\left(1-e^{-h}\right)-0-\mathfrak{N}\int_{t-h}^{t} ds\\&\quad=\mathfrak{N}\left(1-e^{-h}-h\right)\neq 0,\quad \forall\mathfrak{N}\neq 0.\end{aligned} \tag{26}$$

Recently, some papers such as [21, 26–28] considered that the expectations of these stochastic terms are all equal to zero. However, this is not the case. From above examples and Lemma 2, we can see that $x(t-h)^T\mathfrak{N}\int_{t-h}^{t} g(s,x_s)\,dw(s)$ is the only one, whose expectation is equal to zero.

Now, we are in the position to present our main results for the synchronization criterion of the delayed complex networks with stochastic perturbations.

Theorem 1 *Under the Assumptions 1–3, the dynamic system (3) is globally asymptotically synchronized in the mean square if there exist matrices $P>0$, $Q>0$, $R>0$, $Z>0$, S and scalars $\epsilon>0$, $\lambda>0$ such that the following LMIs hold for all $1\leq i<j\leq N$*

$$P<\lambda I, \tag{27}$$

$$\Xi=\begin{pmatrix}\Xi_{11} & 0 & \Xi_{13} & PC & \Xi_{15} & 0\\ * & \Xi_{22} & 0 & 0 & \Xi_{25} & \Xi_{26}\\ * & * & \Xi_{33} & 0 & B^TS^T & 0\\ * & * & * & -R & C^TS^T & 0\\ * & * & * & * & hZ-S-S^T & 0\\ * & * & * & * & * & -hZ\end{pmatrix}<0, \tag{28}$$

where

$$\Xi_{11} = PA + A^T P - Ng_{ij} P\Gamma - Ng_{ij}\Gamma^T P + \lambda W_1^T W_1 + Q - \epsilon U^T V - \epsilon V^T U,$$
$$\Xi_{13} = PB + \epsilon U^T + \epsilon V^T,$$
$$\Xi_{15} = A^T S^T - Ng_{ij}\Gamma^T S^T,$$
$$\Xi_{22} = -Nh_{ij} P\Upsilon - Nh_{ij}\Upsilon^T P + \lambda W_2^T W_2 - Q,$$
$$\Xi_{25} = -Nh_{ij}\Upsilon^T S^T,$$
$$\Xi_{26} = -hNh_{ij}\Upsilon^T P,$$
$$\Xi_{33} = R - 2\epsilon I.$$

Proof First, set

$$y(t) = (I_N \otimes A + G \otimes \Gamma)x(t) + (H \otimes \Upsilon)x(t-h) + (I_N \otimes B)F\big(x(t)\big) + (I_N \otimes C)F\big(x(t-h)\big), \tag{29}$$

then (3) can be rewritten as

$$dx(t) = y(t)\,dt + \sigma(t)\,dw(t). \tag{30}$$

Integrating the system equation of (30) on both sides from $t-h$ to t, we can have

$$x(t) - x(t-h) = \int_{t-h}^{t} y(s)\,ds + \int_{t-h}^{t} \sigma(s)\,dw(s). \tag{31}$$

Consider the following Lyapunov functional for the systems (30):

$$V(x_t,t) = x(t)^T(U\otimes P)x(t) + \int_{t-h}^{t} x(s)^T(U\otimes Q)x(s)\,ds + \int_{-h}^{0}\int_{t+\theta}^{t} y(s)^T(U\otimes Z)y(s)ds d\theta + \int_{t-h}^{t} F\big(x(s)\big)^T(U\otimes R)F\big(x(s)\big)\,ds, \quad t \ge h, \tag{32}$$

where

$$U = \begin{pmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & N-1 \end{pmatrix}.$$

Then, by Itô's formula, the stochastic differential $dV(x_t,t)$ can be obtained

$$dV(x_t,t) = \mathcal{L}V(x_t,t)\,dt + 2x(t)^T(U\otimes P)\sigma(t)\,dw(t),$$

where

$$\begin{aligned}\mathcal{L}V(x_t,t) &= 2x(t)^T(U\otimes P)y(t) + \sigma(t)^T(U\otimes P)\sigma(t) \\ &\quad + x(t)^T(U\otimes Q)x(t) \\ &\quad - x(t-h)^T(U\otimes Q)x(t-h) \\ &\quad + F\big(x(t)\big)^T(U\otimes R)F\big(x(t)\big) \\ &\quad - F\big(x(t-h)\big)^T(U\otimes R)F\big(x(t-h)\big) \\ &\quad + hy(t)^T(U\otimes Z)y(t) \\ &\quad - \int_{t-h}^{t}\big[y(s)^T(U\otimes Z)y(s)\big]\,ds. \end{aligned}\tag{33}$$

By (31), we have

$$\begin{aligned} &2x(t)^T(U\otimes P)y(t) \\ &\quad = 2x(t)^T(U\otimes P)\big[(I_N\otimes A + G\otimes\Gamma)x(t) \\ &\qquad + (I_N\otimes B)F\big(x(t)\big) + (I_N\otimes C)F\big(x(t-h)\big)\big] \\ &\qquad + 2x(t)^T(U\otimes P)(H\otimes\Upsilon)x(t-h) \\ &\quad = 2x(t)^T(U\otimes P)\big[(I_N\otimes A + G\otimes\Gamma)x(t) \\ &\qquad + (I_N\otimes B)F\big(x(t)\big) + (I_N\otimes C)F\big(x(t-h)\big)\big] \\ &\qquad + 2\bigg[x(t-h) + \int_{t-h}^{t} y(s)\,ds \\ &\qquad + \int_{t-h}^{t}\sigma(s)\,dw(s)\bigg]^T(U\otimes P)(H\otimes\Upsilon)x(t-h). \end{aligned}\tag{34}$$

Using Lemma 2, we can obtain

$$\begin{aligned} &\mathcal{E}\big[2x(t)^T(U\otimes P)y(t)\big] \\ &\quad = \mathcal{E}\Big[2x(t)^T(U\otimes P)\big((I_N\otimes A + G\otimes\Gamma)x(t) \\ &\qquad + (I_N\otimes B)F\big(x(t)\big) + (I_N\otimes C)F\big(x(t-h)\big)\big) \end{aligned}$$

$$+2\Big(x(t-h)+\int_{t-h}^{t}y(s)\,ds\Big)^T \times (U\otimes P)(H\otimes\Upsilon)x(t-h)\Big]. \tag{35}$$

By (29), we can have that for any matrices S,

$$2y(t)^T(U\otimes S)\big[(I_N\otimes A+G\otimes\Gamma)x(t) + (H\otimes\Upsilon)x(t-h)+(I_N\otimes B)F\big(x(t)\big) + (I_N\otimes C)F\big(x(t-h)\big)-y(t)\big]=0. \tag{36}$$

From (33)–(36) and by Propositions 2 and 3, it is easy to obtain

$$\begin{aligned}
&\mathcal{E}\big(\mathcal{L}V(x_t,t)\big)\\
&=\mathcal{E}\Big(\frac{1}{h}\int_{t-h}^{t}\big[2x(t)^T(U\otimes P)\big((I_N\otimes A\\
&\quad+G\otimes\Gamma)x(t)\\
&\quad+(I_N\otimes B)F\big(x(t)\big)+(I_N\otimes C)F\big(x(t-h)\big)\big)\\
&\quad+2\big(x(t-h)+hy(s)\big)^T(U\otimes P)\\
&\quad\times(H\otimes\Upsilon)x(t-h)\\
&\quad+\sigma(t)^T(U\otimes P)\sigma(t)+x(t)^T(U\otimes Q)x(t)\\
&\quad-x(t-h)^T(U\otimes Q)x(t-h)\\
&\quad+F\big(x(t)\big)^T(U\otimes R)F\big(x(t)\big)\\
&\quad-F\big(x(t-h)\big)^T(U\otimes R)F\big(x(t-h)\big)\\
&\quad+hy(t)^T(U\otimes Z)y(t)-hy(s)^T(U\otimes Z)y(s)\\
&\quad+2y(t)^T(U\otimes S)((I_N\otimes A+G\otimes\Gamma)x(t)\\
&\quad+(H\otimes\Upsilon)x(t-h)+(I_N\otimes B)F\big(x(t)\big)\\
&\quad+(I_N\otimes C)F\big(x(t-h)\big)-y(t))\big]\,ds\Big)\\
&=\mathcal{E}\Big(\frac{1}{h}\int_{t-h}^{t}\Big[\sum_{1\le i<j\le N}\big(2\big(x_i(t)-x_j(t)\big)^T\\
&\quad\times(PA-Ng_{ij}P\Gamma)\big(x_i(t)-x_j(t)\big)\\
&\quad+2\big(x_i(t)-x_j(t)\big)^T PB\big(f\big(x_i(t)\big)-f\big(x_j(t)\big)\big)\\
&\quad+2\big(x_i(t)-x_j(t)\big)^T PC\big(f\big(x_i(t-h)\big)\\
&\quad-f\big(x_j(t-h)\big)\big)\\
&\quad-2\big(x_i(t-h)-x_j(t-h)\big)^T(Nh_{ij}P\Upsilon)\\
&\quad\times\big(x_i(t-h)-x_j(t-h)\big)\\
&\quad-2h\big(y_i(s)-y_j(s)\big)^T Nh_{ij}P\\
&\quad\times\Upsilon\big(x_i(t-h)-x_j(t-h)\big)\\
&\quad+\big(\sigma_i\big(t,x_i(t),x_i(t-h)\big)\\
&\quad-\sigma_j\big(t,x_j(t),x_j(t-h)\big)\big)^T\\
&\quad\times P\big(\sigma_i\big(t,x_i(t),x_i(t-h)\big)\\
&\quad-\sigma_j\big(t,x_j(t),x_j(t-h)\big)\big)\\
&\quad+\big(x_i(t)-x_j(t)\big)^T Q\big(x_i(t)-x_j(t)\big)\\
&\quad-\big(x_i(t-h)-x_j(t-h)\big)^T\\
&\quad\times Q\big(x_i(t-h)-x_j(t-h)\big)\\
&\quad+\big(f\big(x_i(t)\big)-f\big(x_j(t)\big)\big)^T\\
&\quad\times R\big(f\big(x_i(t)\big)-f\big(x_j(t)\big)\big)\\
&\quad-\big(f\big(x_i(t-h)\big)-f\big(x_j(t-h)\big)\big)^T\\
&\quad\times R\big(f\big(x_i(t-h)\big)\\
&\quad-f\big(x_j(t-h)\big)\big)+h\big(y_i(t)-y_j(t)\big)^T\\
&\quad\times Z\big(y_i(t)-y_j(t)\big)\\
&\quad-h\big(y_i(s)-y_j(s)\big)^T Z\big(y_i(s)\\
&\quad-y_j(s)\big)+2\big(y_i(t)-y_j(t)\big)^T\\
&\quad\times(SA-Ng_{ij}S\Gamma)\big(x_i(t)-x_j(t)\big)\\
&\quad-2\big(y_i(t)-y_j(t)\big)^T(Nh_{ij}S\Upsilon)\\
&\quad\times\big(x_i(t-h)-x_j(t-h)\big)\\
&\quad+2\big(y_i(t)-y_j(t)\big)^T\\
&\quad\times SB\big(f\big(x_i(t)\big)-f\big(x_j(t)\big)\big)\\
&\quad+2\big(y_i(t)-y_j(t)\big)^T SC\big(f\big(x_i(t-h)\big)\\
&\quad-f\big(x_j(t-h)\big)\big)-2\big(y_i(t)-y_j(t)\big)^T\\
&\quad\times S\big(y_i(t)-y_j(t)\big)\big)\Big]\,ds\Big).
\end{aligned}\tag{37}$$

According to Assumption 3 and (27), it is clear that

$$\begin{aligned}
&\big(\sigma_i\big(t,x_i(t),x_i(t-h)\big)-\sigma_j\big(t,x_j(t),x_j(t-h)\big)\big)^T\\
&\quad\times P\big(\sigma_i\big(t,x_i(t),x_i(t-h)\big)\\
&\quad-\sigma_j\big(t,x_j(t),x_j(t-h)\big)\big)
\end{aligned}$$

$$\leq \lambda\big(x_i(t)-x_j(t)\big)^T W_1^T W_1\big(x_i(t)-x_j(t)\big) + \lambda\big(x_i(t-h)-x_j(t-h)\big)^T \times W_2^T W_2\big(x_i(t-h)-x_j(t-h)\big). \quad (38)$$

From Assumption 1, it can be derived that

$$\begin{aligned} 0 \leq\ & 2\epsilon\big(x_i(t)-x_j(t)\big)^T U^T\big(f(x_i(t))-f(x_j(t))\big) \\ & + 2\epsilon\big(f(x_i(t))-f(x_j(t))\big)^T V\big(x_i(t)-x_j(t)\big) \\ & - 2\epsilon\big(x_i(t)-x_j(t)\big)^T U^T V\big(x_i(t)-x_j(t)\big) \\ & - 2\epsilon\big(f(x_i(t))-f(x_j(t))\big)^T \\ & \times \big(f(x_i(t))-f(x_j(t))\big). \end{aligned} \quad (39)$$

Combining (37)–(39), we have

$$\mathcal{E}\big(\mathcal{L}V(x_t,t)\big) \leq \mathcal{E}\left[\frac{1}{h}\int_{t-h}^{t} \sum_{1\leq i<j\leq N} \xi_{ij}^T \Xi \xi_{ij}\, ds\right]. \quad (40)$$

where

$$\xi_{ij} = \begin{pmatrix} x_i(t)-x_j(t) \\ x_i(t-h)-x_j(t-h) \\ f(x_i(t))-f(x_j(t)) \\ f(x_i(t-h))-f(x_j(t-h)) \\ y_i(t)-y_j(t) \\ y_i(s)-y_j(s) \end{pmatrix}.$$

From (27)–(28), it is guaranteed that all the subsystems in (3) are globally asymptotically synchronized in the mean square by Definition 2. The proof is completed. □

Remark 2 For the delay-dependent synchronization criterion of stochastic complex networks with time delays, Wang et al. [21] has investigated this problem. However, we should point out here that Wang et al. [21] made a mistake when dealing with the expectations of stochastic cross terms. In fact, (25) in [21] was obtained by taking the expectation of (21) in [21]. In this derivation process, Wang et al. [21] considered that

$$\mathcal{E}\left[\sum_{1\leq i<j\leq N}\big(-2\big(\gamma_i(t)-\gamma_j(t)\big)^T M\big(\Omega_i(t)-\Omega_j(t)\big)\big)\right] = 0, \quad (41)$$

where $\gamma_i(t), \gamma_j(t), \Omega_i(t), \Omega_j(t)$ are given in [21]. But (41) is not tenable. For instance, if we let $r=1$ in Theorem 1 of [21], it is the simplest case of the delay fractioning approach used in [21]. Under this situation, we can see that

$$\mathcal{E}\left[\sum_{1\leq i<j\leq N}\big(-2\big(\gamma_i(t)-\gamma_j(t)\big)^T M\big(\Omega_i(t)-\Omega_j(t)\big)\big)\right] = \mathcal{E}\left[-2x(t)^T(U\otimes M)\int_{t-\tau}^{t}\sigma(s)\,dw(s)\right]$$

is not always equal to 0 (see the Remark 1 of this paper), where $x(t), \sigma(t), U$ are also same as the ones in [21]. Therefore, the result in [21] is not correct.

Remark 3 In order to deal with the stochastic cross terms containing the Itô integral, cross-terms bounding techniques was adopted in [23–25]. It may increase the conservatism. In the derivation process of Theorem 1, we do not use any bounding technique to deal with stochastic cross terms. Therefore, our method can show less conservatism.

4 Numerical example

In this section, we present a simulation example to illustrate the effectiveness of our approach. Consider the following complex network consisting of three identical nodes:

$$dx_i(t) = \left[Ax_i(t) + Bf\big(x_i(t)\big) + Cf\big(x_i(t-h)\big) + \sum_{j=1}^{3} g_{ij}\Gamma x_j(t) + \sum_{j=1}^{3} h_{ij}\Upsilon x_j(t-h)\right]dt + \sigma_i\big(t, x_i(t), x_i(t-h)\big)\,dw(t)$$

for all $i = 1, 2, 3$, where $x_i(t) = [x_{i1}(t), x_{i2}(t)]^T \in \mathcal{R}^2$ is the state vector of the ith subsystem, and let

$$A = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0.1 \\ 0.2 & -0.1 \end{pmatrix},$$

$$C = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

The out-coupling configuration matrices G, H and inner-coupling matrices Γ, Υ are chosen as

$$G = \begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix},$$

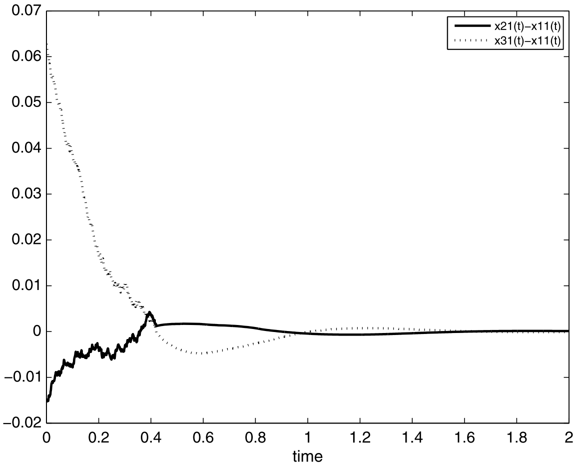

Fig. 1 State error of $x_{i1}(t) - x_{11}(t)$, $i = 2, 3$

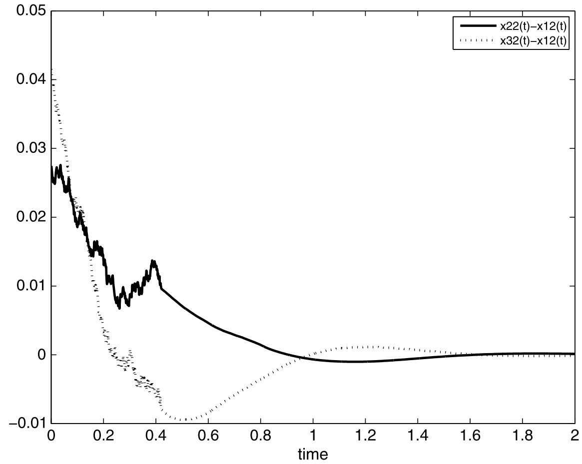

Fig. 2 State error of $x_{i2}(t) - x_{12}(t)$, $i = 2, 3$

$$H = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 2 & -3 \end{pmatrix},$$

$$\Gamma = \begin{pmatrix} 0.2 & 0 \\ 0.1 & 0.2 \end{pmatrix}, \qquad \Upsilon = \begin{pmatrix} 0.2 & 0.1 \\ 0.2 & 0.2 \end{pmatrix}.$$

The noise intensity function vector $\sigma(\cdot,\cdot,\cdot)$ is of the following form:

$$\sigma\big(t, x(t), x(t-h)\big) = \begin{pmatrix} \sqrt{0.1} & 0 & \sqrt{0.2} & 0 \\ 0 & \sqrt{0.1} & 0 & \sqrt{0.2} \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix},$$

and the nonlinear function $f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)))^T = (\tanh(x_{i1}(t)), \tanh(x_{i2}(t)))^T$. Thus, the matrices U, V, W_1, W_2, in Assumptions 1 and 3 are

$$U = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$W_1 = \begin{pmatrix} \sqrt{0.2} & 0 \\ 0 & \sqrt{0.2} \end{pmatrix}, \qquad W_2 = \begin{pmatrix} \sqrt{0.4} & 0 \\ 0 & \sqrt{0.4} \end{pmatrix}.$$

According to Theorem 1, the allowable maximum delay h that guarantees the globally asymptotically mean-square synchronization of the delayed stochastic complex networks, is 0.42. When we randomly choose the initial states in $[0, 1] \times [0, 1]$, the synchronization errors are plotted in Figs. 1 and 2, which can confirm that the stochastic complex dynamical system (3) is globally synchronized in the mean square.

5 Conclusions

In this paper, the problem of the delay-dependent synchronization criterion for stochastic complex networks with time delays is investigated. First, this paper is concerned with expectations of stochastic cross terms containing the Itô integral. We prove that among stochastic cross terms, $x(t-h)^T \mathfrak{N} \int_{t-h}^{t} g(s, x_s)\, dw(s)$ is the only one, whose expectation is equal to zero. Then, based on this, the paper establishes a delay-dependent synchronization criterion for stochastic delayed complex networks. In the derivation process, the mathematical development avoids bounding stochastic cross terms. Thus, the method leads to a simple criterion with less conservatism, and a numerical example is provided to demonstrate the effectiveness of the proposed approach.

On the other hand, it is worth mentioning that there are still some important problems to solve for stochastic complex networks with time delays. (1) Most of the stochastic complex networks considered in existing results are perturbed by the Brown noises. However, there are many other stochastic noises such as Poisson noises and Lévy noises in the real world. Thus, it is very important to investigate the dynamic behaviors, such as the synchronization phenomena, for complex networks perturbed by the Poisson noise and Lévy noises in future researches. (2) As a very important kind of stochastic delayed complex networks, the neutral-type stochastic delayed complex networks are still not investigated up to now. Since the neutral-type stochastic systems with time delays have many applications in practice, it is necessary to discuss the dynamical behaviors of neutral-type stochastic delayed complex networks in future researches.

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