Global theory of one-frequency Schrödinger operators

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Regularity and chaos

In the study of classical dynamical systems, the main goal is the understanding of the long time behavior of observable (*positive measure*) parts of the phase space.

Two recurring themes arise:

- The persistence of quasiperiodic motion. Rigorously studied via perturbative techniques first introduced by Kolmogorov (KAM theory).
- The emergence of chaotic behavior. Few rigorous results.

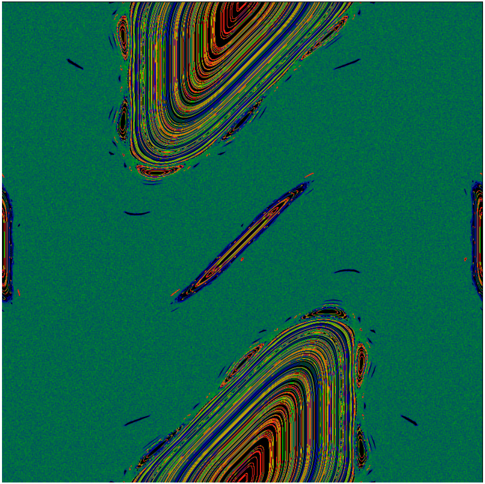
Uniform and nonuniform hyperbolicity

"Chaotic behavior" corresponds to the exponential growth of the derivative under iteration (positive *Lyapunov exponent*).

Can be produced in a controlled, but *robust*, way if the dynamics produces coherent stretching in certain directions: uniform hyperbolicity.

Serious difficulties arise outside the uniformly hyperbolic setting, when stretching and folding are combined.

Coexistence



Source: Wikipedia.

An accurate picture?

The standard map $(x, y) \mapsto (x + K \sin 2\pi y, y + x + K \sin 2\pi y)$ is the usual example of a dynamical system combining stretching and folding.

It is an area preserving map of the two-torus which is believed to be compatible with nonuniform hyperbolicity.

Whether this is actually so is unknown even when the stretching is very large.

The global point of view

The understanding of the entire phase/parameter space is even further away. For instance, it is possible that the typical behavior is always either quasiperiodic or chaotic?

The topic of this lecture involves, from the classical dynamics point of view, some of the simplest systems which are compatible with both KAM-like behavior and positive Lyapunov exponents.

Indeed the existence of both behaviors has been rigorously established 30 years ago. But a global picture has only emerged very recently.

Localization and transport

We will be interested on the quantum dynamics e^{itH} defined by a lattice Schrödinger operators $H = \Delta + V$ with ergodic potential. A central problem is to establish either localization or transport.

- Localization: the evolution of a unit vector $u \in \ell^2$ may remain in a compact set (thus $e^{itH}u$ is quasiperiodic). This is associated with ℓ^2 eigenfunctions, that is, point spectrum.
- Transport: mass escapes to infinity. In one-dimension, fastest average transport (ballistic motion) arising from absolutely continuous spectrum.

Coexistence?

Naturally, particularly interesting are situations that are compatible with both localization and transport, leading to finer questions regarding the phase-transition, e.g., mobility edges.

The Anderson Model (i.i.d. potential) is compatible with localization, irrespective of the dimension.

Transport is expected to be also possible in high dimensions, but is notoriously difficulty to analyze.

Going to one dimension in the Anderson model simplifies the analysis but destroys the possibility of transport: absolutely continuous spectrum is too fragile.

Quasiperiodicity

To get the possibility of transport in one dimension, we need to reduce the disorder. Quasiperiodic models provide the right amount of disorder.

We will be interested on the simplest case, of one-frequency potential:

$$(Hu)_n = u_{n+1} + u_{n-1} + V_n u_n, \quad V_n = v(n\alpha),$$

where v is an analytic function defined on $\mathbb{R}/\mathbb{Z}$ and α is irrational.

Contrary to i.i.d. models, the spectrum as a set tends to be quite complicated: it is often a Cantor set of positive measure.

The general case: local theories

Much of the general theory of one-frequency operators developed from 1970's to the 2000's can be understood as forming two local theories, corresponding to small and large potentials.

Keeping our focus on the localization/transport, we have now a very clean picture:

- *Any* one-frequency operator with *small potential* has purely absolutely continuous spectrum (A).
- A *typical* one-frequency operator with *large potential* has pure point spectrum (Bourgain-Goldstein).

The general case: local theories

As now currently understood, both local theories have precisely defined, robust, scopes (in terms of KAM and NUH behaviors).

They also encompass much more than just the description of spectral types.

The analysis of small potential dates back to the work of Dinaburg-Sinai, while the theory of large potentials was initially advanced by Sinai and Frohlich-Spencer.

Fundamental contributions (to one or both theories) were made by several authors: Herman, Eliasson, Bourgain, Goldstein, Schlag, Jitomirskaya, Fayad, Krikorian...

Global questions

Despite 30 years of developments, no approach existed to understand Schrödinger operators lying beyond the scope of the local theories (except for the Almost Mathieu Operator).
Basic questions:

- Local theories provide two robust regimes. Is there another (perhaps corresponding to robust singular continuous spectrum)? Or is there just some sort of "critical interface" separating the localization and transport regimes?
- What is the behavior of a typical one-frequency operator? What is the effect of critical energies?

A framework to address those questions was only developed in the last 4 years.

The Spectral Dichotomy Theorem

Theorem

A typical one-frequency operator splits as a direct sum of operators $H = H_+ \oplus H_-$ such that:

- H_+ *is within the scope of the local theory of large potentials,*
- H_- *is within the scope of the local theory of small potentials,*
- H_+ *and* H_- *have disjoint spectra.*

Thus, typically, there is no singular continuous spectrum. More surprisingly, there are no critical energies (and no mobility edges), and there are at most finitely many alternances between small and large-like behavior.

Control of the i.d.s. (Hölder, absolutely continuous) also follows.

Dynamics of cocycles

Ergodic Schrödinger operators in one-dimension are intimately connected to families of certain dynamical systems (cocycles), associated to each each eigenvalue equation $Hu = Eu$.

In the case of one-frequency operators, the dynamics takes place in the two-torus, and has the form

$$(\alpha, A) : (x, y) \mapsto (x + \alpha, A(x) \cdot y).$$

where A denotes an x-dependent $\mathrm{SL}(2, \mathbb{R})$ matrix acting projectively on y.

The fibered structure (over the x coordinate), and the projective behavior (on the y coordinate) make the dynamics particularly accessible to analysis.

Dynamics of cocycles

However, it is still complicated enough to allow both KAM-like behavior and positive Lyapunov exponents.

When Herman understood this he wrote about it in a paper with a really long title, probably to display his excitement:

Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnol'd et de Moser sur le tore de dimension 2.

The Lyapunov exponent

The *Lyapunov exponent* of a cocycle (α, A) is defined by

$$L = \lim \frac{1}{n} \int \ln \|A_n(x)\| dx.$$

where $A_n(x) = A(x + (n-1)\alpha) \cdots A(x)$.

Since $A \in \mathrm{SL}(2, \mathbb{R})$, $L \geq 0$. If $L > 0$, two situations arise, uniform or nonuniform hyperbolicity, according to whether the exponential growth of $\|A_n(x)\|$ is uniform w.r.t. x or not.

Uniform hyperbolicity is stable and simple to analyze, but it is its complement that is more relevant for spectral theory. Indeed:

The cocycle associated to $Hu = Eu$ is uniformly hyperbolic $\iff$ E is not in the spectrum.

Large potentials and NUH

The local theory of large potentials is actually the theory of nonuniform hyperbolicity.

Large potentials $\implies$ nonuniform hyperbolicity through the whole spectrum (Sorets-Spencer).

Nonuniform hyperbolicity in a region of the spectrum $\implies$ typical point spectrum in that region (Bourgain-Goldstein).

Positivity of the Lyapunov exponent is stable (Goldstein-Schlag, Bourgain-Jitomirskaya).

KAM-like behavior

A cocycle is called *reducible* if it can be turned into a constant one (A does not depend on x) by coordinate change.

It behaves as if it came from a constant potential.

Near constants, reducibility has been frequently studied through KAM techniques.

Reducibility is somewhat fragile due to small denominators issues pertaining to Diophantine approximation.

KAM-like behavior

A more robust concept is that of *almost reducibility*. A cocycle is called almost reducible if it can be made arbitrarily close to constant by a sequence of coordinate changes.

The coordinate changes themselves are allowed to diverge.

Almost reducibility provides a rigorous notion of “KAM-like” behavior in this setting.

KAM-like behavior

The local theory of small potentials is actually the theory of almost reducibility.

Any cocycle near a constant one is almost reducible (A).

Almost reducibility in a region of the spectrum $\Longrightarrow$ absolutely continuous spectrum in that region (A).

Almost reducibility is stable (A).

Spectral Dichotomy and dynamics

The Spectral Dichotomy Theorem is thus a consequence of a global result about the dynamics of cocycles.

Theorem

For any irrational α and for a typical v, and any E in the spectrum, the cocycle associated to $Hu = Eu$ is either nonuniformly hyperbolic or almost reducible.

This is the outcome of a series of works about the geometry, in the infinite dimensional parameter space, of the critical locus, and the relation between KAM-like behavior and Lyapunov exponents.

Focusing on the Lyapunov exponent

The starting point is a workable definition of three regimes, in the complement of uniform hyperbolicity, involving *complexification* of the dynamical system:

- *Supercritical* corresponding to positive Lyapunov exponent (NUH),
- *Subcritical* corresponding to zero Lyapunov exponent which is *stable under complexification*.
- *Critical* corresponding to zero Lyapunov which is *unstable under complexification*.

The focus on the Lyapunov exponent allows us to initially avoid the complications of understanding the domain of KAM-like behavior.

The geometry of the critical locus

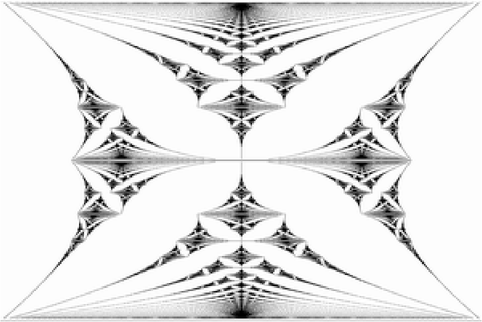
Theorem

Critical cocycles lie in a codimension-one stratified interface and have zero measure within it.

The later part of this result is quite surprising at first, since the spectrum has usually positive measure.

The spectrum becomes thinner at the critical locus, a result following from *renormalization* analysis.

The geometry of the critical locus



Source: Wikipedia.

Hofstadter butterfly, depicting thin critical spectrum in an exceptionally symmetric potential (the almost Mathieu operator).

This particular case can be largely (but not fully) analyzed by explicit computations.

The geometry of the critical locus

But the constraining of the critical locus to a codimension-one interface is also mysterious.

Indeed, a fundamental difficulty is the lack of standard regularity for the Lyapunov exponent as a function of parameters: it is not better than Hölder and sometimes worse, especially at the critical locus.

A fractal structure for the interface could be expected. Think of computing the boundary of $\{t \; : \; B(t) > 0\}$ for the Brownian motion on the line.

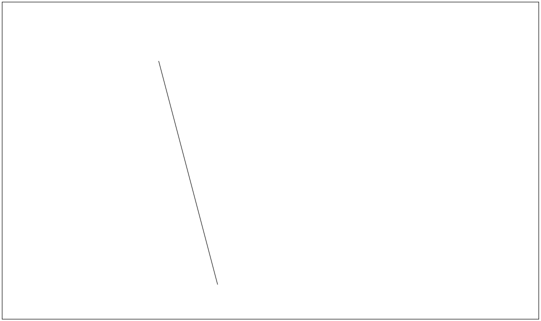
The geometry of the critical locus

The unexpected discovery of *stratified* regularity in 2008 brought many analytic tools into play.

Stratified regularity is compatible with bad conventional regularity since strata are allowed to be (and indeed are) complicated fractal sets.

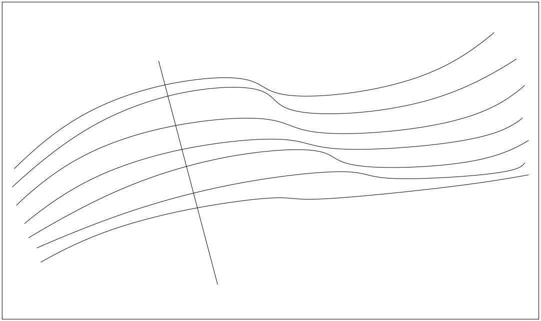
Stratification emerges from *quantization* of the acceleration of the Lyapunov exponent of complexifications.

Geometry of the critical locus



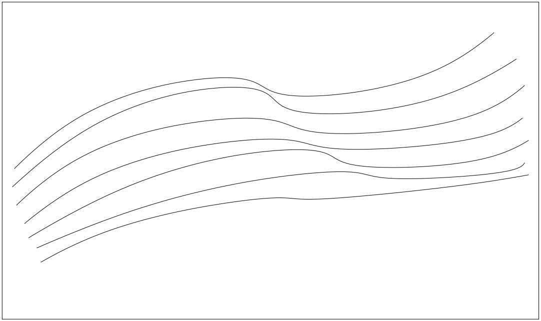
The family of cocycles associated to a given potential is a line in a Banach space of cocycles. From left to right, the nonlinearity increases.

Geometry of the critical locus



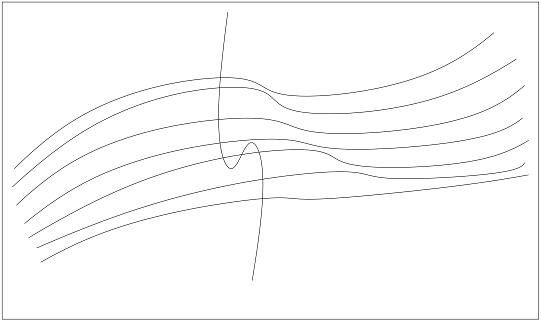
The spectrum corresponds to the intersection of this line with the bifurcation locus (complement of uniform hyperbolicity).

Geometry of the critical locus



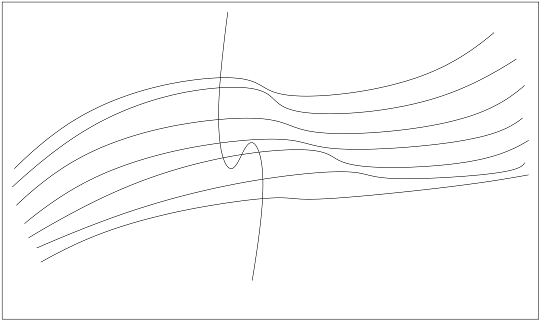
The bifurcation locus is roughly a codimension-one lamination. Leaves (corresponding to a fixed value of the i.d.s.) are not necessarily very regular and sometimes are branched. Transversely, it tends to be a Cantor set of positive measure.

Geometry of the critical locus



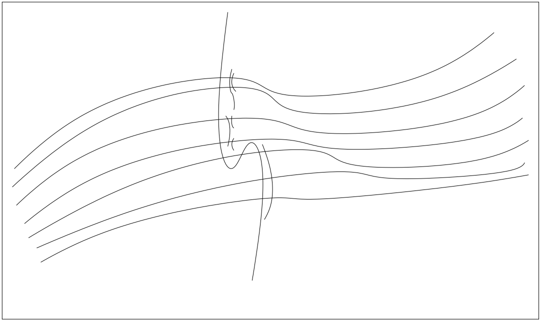
The critical locus lies in a codimension-one interface somewhere in the middle.

Geometry of the critical locus



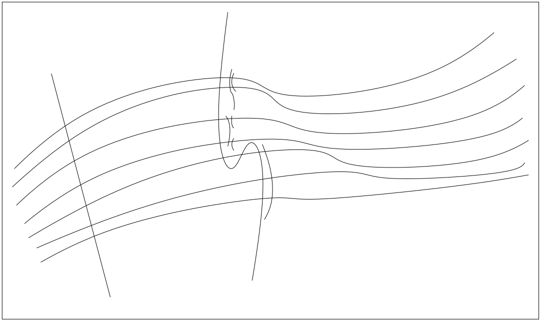
Subcritical and supercritical parts of each leaf of the bifurcation locus must be separated by the critical locus. But one may avoid the critical locus by going through the complement of the lamination.

Geometry of the critical locus



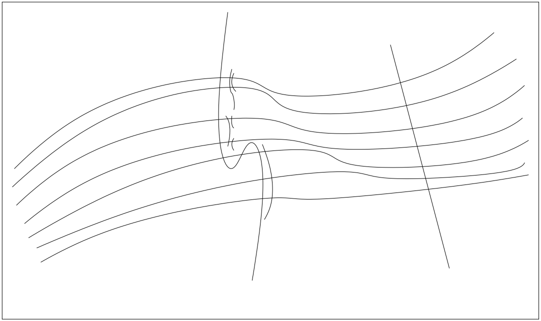
The interface is not really a single manifold. It is stratified, and higher order strata can accumulate on lower order ones.

Geometry of the critical locus



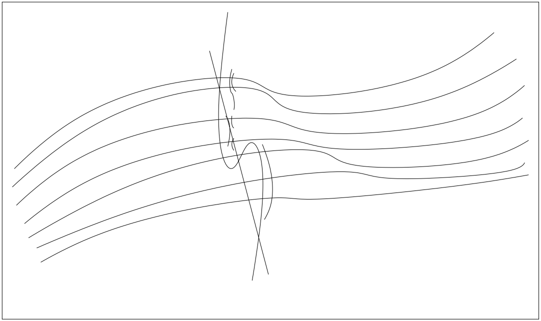
Small potentials avoid the interface and lie entirely on the subcritical regime.

Geometry of the critical locus



Large potentials avoid the interface and lie entirely on the supercritical regime.

Geometry of the critical locus



A general potential may hit the interface, but typically does so outside the actual critical locus.

Relation between KAM and the Lyapunov exponent

So far we have focused on the Lyapunov exponent, as a means to obtain a detailed picture of the parameter space. We must now connect the Lyapunov exponent with KAM behavior.

As discussed in the beginning, this is a general dynamical problem about which essentially nothing is known. Recently, we obtained a complete solution for the dynamics associated to one-frequency operators.

Theorem (Almost Reducibility Conjecture)

Subcritical cocycles are almost reducible.

This result contrasts with the bulk of reducibility theory since subcriticality is not in any way a perturbative assumption, so it lies beyond the scope of the KAM technology.

Relation between KAM and the Lyapunov exponent

Though renormalization methods could potentially apply, there are severe analytic difficulties which make it currently ineffective.

The analysis goes instead through detailed quantitative understanding of the complexified dynamics constrained by zero Lyapunov exponents, and their perturbations.

Changes of coordinates are constructed using the simpler structure of some approximating dynamics.

Their size must be compared with the quality of approximation. Bounds are obtained via a mix of soft and hard techniques.