Commun. Math. Phys. 154, 245–263 (1993)

Communications in Mathematical Physics

Global Weak Solutions of the Vlasov–Maxwell System with Boundary Conditions*

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Received September 30, 1991; in revised form December 26, 1991

Abstract. Boundaries occur naturally in physical systems which satisfy the Vlasov–Maxwell system. Assume perfect conductor boundary conditions for Maxwell, and either specular reflection or partial absorption for Vlasov. Then weak solutions with finite energy exist for all time.

§0. Introduction

We study the initial and boundary value problem of both the non-relativistic and relativistic Vlasov–Maxwell system. We shall prove the global existence of weak solutions under various boundary conditions.

Let Ω be an open set in R^3 with $C^{1,\mu}$ boundary, for some $\mu > 0$. Consider the non-relativistic Vlasov–Maxwell system:

$$\begin{cases} \partial_t f_\beta + \dfrac{v}{m_\beta}\cdot\nabla_x f_\beta + \dfrac{e_\beta}{m_\beta}\left(E + \dfrac{1}{c}v\times B\right)\cdot\nabla_v f_\beta = 0, \quad 1 \leqq \beta \leqq N \\ \partial_t E - c\,\mathrm{curl}\,B = -j = -4\pi\sum\limits_\beta e_\beta \int\limits_{\mathbf{R}^3} f_\beta\,dv\,, \\ \partial_t B + c\,\mathrm{curl}\,E = 0\,, \end{cases} \tag{VM}$$

where $0 < t < \infty$, $x \in \Omega$ and $v \in R^3$, with the constraints

$$\begin{cases} \mathrm{div}\,E = \rho = 4\pi\sum\limits_\beta e_\beta \int\limits_{\mathbf{R}^3} f_\beta\,dv\,, \\ \mathrm{div}\,B = 0\,. \end{cases} \tag{0.1}$$

* This research is supported in part by NSF Grant DMS 90-23864

The initial conditions are

$$\begin{cases} f_\beta(0,x,v) = f_{0\beta}(x,v) \quad \text{for } 1 \leqq \beta \leqq N, \quad E(0,x) = E_0(x), B(0,x) = B_0(x)\,, \\ \operatorname{div} E_0 = \rho_0 \quad \text{and} \quad \operatorname{div} B_0 = 0\,. \end{cases} \tag{0.2}$$

The boundary conditions are

$$\begin{cases} E \times \vec{n} = 0\,, \\ f_\beta(t,x,v) = a_\beta(t,x,v)(Kf_\beta(t,x,v)) + g_\beta(t,x,v), \quad 1 \leqq \beta \leqq N, \end{cases} \tag{0.3}$$

for $x \in \partial\Omega$ and $n \cdot v < 0$, where n is the outward normal vector of $\partial\Omega$ at x. Here the reflection operator is defined as

$$Kf(t,x,v) = f(t,x,v - 2(v \cdot \vec{n})\vec{n})\,, \tag{0.4}$$

where $\vec{v} - 2(v \cdot \vec{n})\vec{n}$ is the reflected vector of $\vec{v}$ respect to $\vec{n}$. Also N is the number of different types of particles with charges e_β and masses m_β, c is the speed of light. The absorption coefficient $a_\beta(t,x,v)$ and the boundary source $g_\beta(t,x,v)$ are two given functions on $n \cdot v < 0$ satisfying either one of the following conditions:

1. Purely specular reflection condition:

$$a_\beta(t,x,v) \equiv 1, \quad g_\beta(t,x,v) \equiv 0\,. \tag{0.5}$$

2. Partially absorbing condition:

$$0 \leqq a_\beta(t,x,v) \leqq a_0 < 1, \quad g_\beta(t,x,v) \geqq 0\,, \tag{0.6}$$

where a_0 is a constant. The purely absorbing condition is $a_\beta \equiv 0$ and $g_\beta \equiv 0$.

These are two typical kinds of the boundary conditions for transport equations. The assumed condition $E \times \vec{n} \equiv 0$ comes naturally from physics when Ω is surrounded by a perfect conductor. The integrated energy for the non-relativistic case is

$$\mathscr{E}_T = 4\pi \sum_\beta \int_{(0,T)\times\Omega\times R^3} (1 + |v|^2) m_\beta f_\beta \, dt\, dx\, dv + \int_{(0,T)\times\Omega} (E^2 + B^2)\, dt\, dx\,. \tag{0.7}$$

Let $\chi_T(\cdot)$ be the characteristic function of [0, T]. Our main results are as follows.

Theorem 0.1 (*Non-relativistic case*). *Let $\partial\Omega \in C^{1,\mu}$, for some $\mu > 0$. Let $f_{0\beta} \geqq 0$ a.e., for $1 \leqq \beta \leqq N$, and let E_0 and $B_0 \in L^2(\Omega)$ satisfy* $\operatorname{div} E_0 = \rho_0$ *and* $\operatorname{div} B_0 = 0$ *in the sense of distributions. Assume $f_{0\beta}(1 + |v|^2) \in L^1$. In the purely specular case* (0.5), *assume $f_\beta \in L^\infty \cap L^1$. In the partially absorbing case* (0.6), *assume $f_{0\beta} \in L^p$, $\chi_T g_\beta \in L^p$, $\chi_T g_\beta (1 + |v|)^2 \in L^1$, for some $2 \leqq p \leqq \infty$ and all $T < \infty$. Then there exist a weak solution of* (VM) *in $0 < t < \infty$, $x \in \Omega$, $v \in R^3$ with finite energy $\mathscr{E}_T$, for all $T < \infty$. Moreover, if $f_{0\beta} \in L^q$, $\chi_T g_\beta \in L^q$, for all $T < \infty$, then $\chi_T f_\beta \in L^q$, for all $T < \infty$, where $2 \leqq q \leqq \infty$.*

The relativistic Vlasov–Maxwell system (RVM) is the same as (VM) except that $\frac{v}{m_\beta}$ is replaced by $\hat{v} = \frac{v}{\sqrt{m_\beta^2 + \frac{|v|^2}{c^2}}}$. (See [GS1].) The energy is the same as for (VM) except that $(1 + |v|^2)m_\beta$ is replaced by $2c^2\left(m_\beta^2 + \frac{|v|^2}{c^2}\right)^{\frac{1}{2}}$. We have a parallel theorem (Theorem 5.1).

This paper is a first attempt to describe the plasma-wall interaction. An important potential application is to a tokamak. However, there are several sources of particle fluxes to the wall, such as ions and electrons that diffuse across the confining field, runaway electrons, and neutral particles that are injected into the plasma from the wall. According to [St], "the physics of the transport processes within the plasma core and boundary regions and the atomic physics of the plasma-wall interaction are sufficiently complex and the experimental evidence is sufficiently limited, that it is very difficult to confidently predict the magnitude and energy distribution of the particle fluxes to the wall." Because of this uncertainty, it is useful to remark that our proof works if we replace the second condition in (0.3) by $f_\beta = \mathscr{K} f_\beta + g_\beta$, and eliminate (0.4), (0.5), (0.6), where $\mathscr{K}$ is any linear operator: $L^p(\{n \cdot v > 0\}) \to L^p(\{n \cdot v < 0\})$, with $\|\mathscr{K}\| < 1$, assuming that $2 \leqq p < \infty$.

Arsenev [A] first proved the global existence of weak solutions of the Vlasov–Poisson system. Using a velocity averaging argument, DiPerna and Lions [DL] proved the global existence of the weak solutions of the Cauchy problem of the Vlasov–Maxwell system. Regularity of the global weak solutions with regular initial data for the Vlasov–Maxwell systems (VM) and (RVM) were proved earlier by Glassey, Strauss and Schaeffer in [GS1, GS2 and GSc], but they require some restrictions on the data. In the Vlasov–Poisson case, regularity without extra restrictions on the data have recently been proved by [Pf, H, Sc and PL]. Greengard and Raviart [GR] proved the uniqueness and existence of weak solutions for the one-dimensional stationary Vlasov–Poisson system with boundary conditions. The case of linear transport equations have been studied by many mathematicians. In particular, Beals and Protopopescu [BP] gave a unified formulation in a general setting. Cooper and Strauss [CS] treated the general initial-boundary value problem for the Maxwell system in time-dependent domains.

Even in the case of the full (VM) or (RVM) system without the boundary, as in [DL], the questions of uniqueness, regularity and conservation of energy are open, unless the data is restricted as in [GS1, GS2 and GSc]. We have some positive and negative results on these questions, which will appear in a later paper.

To prove the existence of the weak solution, we first approximate the phase space $\Omega \times \mathbf{R}^3$ by a sequence of bounded domains. In each bounded domain, we approximate a cut-off problem by a sequence of linear Vlasov equations and linear Maxwell systems with suitable new initial and boundary conditions. Using the results of [BP], we get a sequence of weak solutions (Sect. 2). We take the weak limits of the solutions of the linear problems and obtain the energy estimate by the compactness results of [DL] (Sect. 3). Then we get the weak solution of the partial absorption problem as the limit of the solutions of the cut-off problems. We approximate the purely specular problem by partial absorption problems (Sect. 4). Finally we treat the relativistic case (Sect. 5).

1. Notation and Weak Formulation

Definition 1.1. *Let $\Pi = (0, \infty) \times \Omega \times \mathbf{R}^3$, where Ω is an open set in $\mathbf{R}^3$ with $C^{1,\mu}$ bioundary, $\mu > 0$. Let n be the outward normal vector of $\partial\Omega$ at x. Let*

$$\gamma^{\pm} = \{(t, x, v) \in (0, \infty) \times \partial\Omega \times \mathbf{R}^3 \mid \pm \vec{n} \cdot v > 0\} , \tag{1.1}$$

$$\gamma^{0} = \{(t, x, v) \in (0, \infty) \times \partial\Omega \times \mathbf{R}^3 \mid \vec{n} \cdot v = 0\} . \tag{1.2}$$

For any $T > 0$, let $\Pi_1 = (0, T) \times \Omega_1 \times V_1 \in \Pi$. Let

$$\gamma_1^{\pm} = \gamma^{\pm} \cap \overline{\Pi_1} . \tag{1.3}$$

Let $|\cdot|_{p;\Pi_1}$ be the L^p norm on Π_1, and let $|\cdot|_{p;\gamma_1^{\pm}(\beta)}$ *be the L^p norm on $\gamma_1^{\pm}$ with respect to the measure $|d\gamma_\beta|$, where*

$$d\gamma_\beta = \left(\vec{n} \cdot \frac{v}{m_\beta}\right) d\sigma_x \, dv \, dt , \tag{1.4}$$

where $d\sigma_x$ is the standard surface measure of $\partial\Omega_1$, and $1 \leqq p \leqq +\infty$.

Most of the estimates in this paper depend on any fixed T, but the solutions are defined for $0 \leqq t < \infty$.

Definition 1.2. *The integrated energy in a region is*

$$\mathscr{E}(f_\beta, E, B, \Omega_1, V_1, T) = 4\pi \sum_\beta m_\beta \int_{\Pi_1} (1 + |v|^2) f_\beta \, dt \, dx \, dv + \int_{(0,T)\times\Omega_1} (E^2 + B^2) \, dt \, dx . \tag{1.5}$$

We also define the initial-boundary energy as

$$\mathscr{E}_0(f_{0\beta}, E_0, B_0, g_\beta, \Omega, T) = 4\pi \sum_\beta m_\beta \int_{\Omega\times\mathbf{R}^3} (1 + |v|^2) f_{0\beta} \, dx \, dv + \int_\Omega (E_0^2 + B_0^2) \, dx$$
$$- \sum_\beta m_\beta \int_{\gamma^-} \chi_T (1 + |v|^2) g_\beta \, d\gamma_\beta, \quad \text{for a fixed } T > 0 . \tag{1.6}$$

Definition 1.3. *The test function spaces are*

$$\mathscr{V} = \{\alpha(t, x, v) \in C_c^\infty([0, \infty) \times \mathbf{R}^3 \times \mathbf{R}^3; \mathbf{R}^1) \mid$$
$$\operatorname{supp} \alpha \subset\subset \{[0, \infty) \times \bar{\Omega} \times \mathbf{R}^3\} \setminus \{(0 \times \partial\Omega) \cup \gamma^0\}\} , \tag{1.7}$$

$$\mathscr{M} = \{(\vec{\psi}, \vec{\varphi}) \mid \vec{\psi} \in C_c^\infty([0, \infty) \times \Omega; \mathbf{R}^3), \vec{\varphi} \in C_c^\infty([0, \infty) \times \mathbf{R}^3; \mathbf{R}^3)\} . \tag{1.8}$$

Definition 1.4. (*Test functionals*). *Let Π_1 be as in Definition* 1.1. *Let $f_\beta \in L^1_{\mathrm{loc}}(\Pi_1)$, $f_{0\beta} \in L^1_{\mathrm{loc}}(\Omega \times \mathbf{R}^3)$, $f_\beta^+ \in L^1_{\mathrm{loc}}(\gamma_1^+)$, and $g_\beta \in L^1_{\mathrm{loc}}(\gamma_1^-)$, with respect to $d\gamma_\beta$, for $1 \leqq \beta \leqq N$. Let E and $B \in L^1_{\mathrm{loc}}((0, T) \times \Omega)$, and E_0 and $B_0 \in L^1_{\mathrm{loc}}(\Omega)$. Let $\alpha_\beta \in \mathscr{V}$,* *and* $(\vec{\psi}, \vec{\varphi}) \in \mathcal{M}$. *Define*

$$A_\beta(f_\beta, f_\beta^+, E, B, \alpha_\beta, \Omega_1, V_1) = -\int_{\Omega_1 \times V_1} f_{0\beta}\alpha_\beta(0, x, v)\,dx\,dv - \int_{\Pi_1} (\partial_t \alpha_\beta + \nabla_x \alpha_\beta \cdot \vec{v}$$
$$+ (E + v \times B) \cdot \nabla_v \alpha_\beta) f_\beta \,dt\,dx\,dv + \int_{\gamma_1^+} \alpha_\beta f_\beta^+ \,d\gamma_\beta$$
$$+ \int_{\gamma_1^-} \alpha_\beta \{a_\beta K f_\beta^+ + g_\beta\}\,d\gamma_\beta\,, \tag{1.9}$$

$$C(E, B, \vec{\psi}, j, \Omega_1, V_1) = -\int_0^T \int_{\Omega_1} E \cdot \partial_t \vec{\psi}\,dt\,dx - \int_{\Omega_1} \vec{\psi}(0, x) \cdot E_0\,dx$$
$$- \int_0^T \int_{\Omega_1} \operatorname{curl} \vec{\psi} \cdot B\,dt\,dx + \int_0^T \int_{\Omega_1} \vec{\psi} \cdot j\,dt\,dx\,, \tag{1.10}$$

where $j = 4\pi \int_{V_1} \sum_\beta v e_\beta f_\beta \,dv$, and

$$D(E, B, \vec{\varphi}, \Omega_1, V_1) = -\int_0^T \int_{\Omega_1} B \cdot \partial_t \vec{\varphi}\,dt\,dx - \int_{\Omega_1} \vec{\varphi}(0, x) \cdot B_0\,dx$$
$$+ \int_0^T \int_{\Omega_1} \operatorname{curl} \vec{\varphi} \cdot E\,dt\,dx\,. \tag{1.11}$$

Definition 1.5. (*Weak solutions*). Let $f_\beta \geqq 0$ a.e., $f_\beta \in L^1_{\mathrm{loc}}(\Pi)$, $f_\beta^+ \geqq 0$ a.e., $f_\beta^+ \in L^1_{\mathrm{loc}}(\gamma^+)$ *and* $E, B \in L^1_{\mathrm{loc}}((0, \infty) \times \Omega)$. *They are a weak solution* of (VM) *with conditions* (0.1) *through* (0.3), *if* $\forall \alpha_\beta \in \mathcal{V}$, $\forall (\vec{\psi}, \vec{\varphi}) \in \mathcal{M}$, $1 \leqq \beta \leqq N$,

$$\begin{cases} A_\beta(f_\beta, f_\beta^+, E, B, \alpha_\beta, \Omega, \mathbf{R}^3) = 0\,, \\ C(E, B, \vec{\psi}, j, \Omega, R^3) = 0, \quad D(E, B, \vec{\varphi}, \Omega, R^3) = 0\,, \\ \operatorname{div} E = \rho \text{ and } \operatorname{div} B = 0 \quad \text{in the sense of distributions}\,. \end{cases} \tag{1.12}$$

Since γ^0 has zero surface measure (see [GMP]), it is omitted.

Lemma 1.1. $\operatorname{div} B = 0$ *is implied by the other conditions in* (1.12).

Proof. For any $\zeta \in C_c^\infty([0, \infty) \times \Omega; \mathbf{R})$. Assume $\zeta(t, x) = 0$ when $t > T$. Plug $\vec{\varphi} = \int_0^t \nabla \zeta\,d\tau - \int_0^T \nabla \zeta\,dt$ into $D(E, B, \vec{\varphi}, \Omega, R^3) = 0$ and the lemma follows.

Lemma 1.2. *Suppose that* $f_{0\beta}$, E_0, g *and* a *are smooth, that* $E_0 \times n = 0$, *on* $\partial\Omega$, *that* f_β, f_β^+, E *and* B *are a weak solution of* (VM), *that* $f_\beta \in C^1(\Pi)$, *that* $f_\beta^+ \in C^1(\gamma^+)$, *that* $E, B \in C^1((0, \infty) \times \Omega)$, *that* f_β *have continuous extensions to* $\gamma^- \cup \gamma^+ \cup \{t = 0\}$, *and that* E *and* B *have continuous extensions to* $[0, T] \times \bar{\Omega}$, *for all* $T < \infty$. *Then* f_β, f_β^+, E *and* B *is a classical solution of* (VM) *with classical initial and boundary conditions on* $\gamma^+ \cup \gamma^- \cup \{t = 0\}$.

The proof is standard.

Remark. If $\vec{n} \cdot B_0 = 0$ in the weak sense on $\partial\Omega$, then $\vec{n} \cdot B = 0$ follows for all t. Since $\operatorname{div} B_0 = 0$, the weak form of $\vec{n} \cdot B_0 = 0$ is $\int_\Omega \nabla \zeta \cdot B_0\,dx = 0$, $\forall \zeta \in C_c^\infty(R^3)$. Choose a test function ζ such that $\zeta(t, x) = 0$ when $t > T$. Plug $\vec{\psi} = \int_0^t \nabla \zeta\,dz - \int_0^T \nabla \zeta\,dt$ into $D = 0$. We get $-\int_0^T \int_\Omega \nabla \zeta \cdot B\,dt\,dx = 0$, which is the weak form of $\vec{n} \cdot B = 0$.

2. An Approximate Solution for (VM)

For notational simplicity in Sects. 2–4 we take only one species of particle, drop the subscript β and set the basic constants equal to unity. Under these assumptions, we treat the partially absorbing problem of (VM) in the next two sections. Let's assume $f_0(1+|v|^2) \in L^1(\Omega \times R^3)$, $f_0 \in L^\infty(\Omega \times R^3)$, $f_0 \geqq 0$ a.e., $\chi_T(1+|v|^2)g \in L^1(\gamma^-)$, $\chi_T g \in L^\infty(\gamma^-)$, $g \geqq 0$ a.e., for all $T < \infty$, $0 \leqq a(t,x,v) \leqq a_0 < 1$, and $E_0, B_0 \in L^2(\Omega)$. In order to get good estimates, we cut the physical space Ω to Ω_N and the velocity space R^3 to V_N, where

$$\Omega_N = \{x \in \Omega \,|\, |x| < N\}, \quad V_N = \{v \in R^3 \,|\, |v| < N\} .$$

Let $\Pi_N = (0,N) \times \Omega_N \times V_N$, and $\bar{\Pi}_N$ be its closure. For fixed N, we will define a sequence of functions by solving a sequence of linear problems, (2.11) and (2.17) below.

The first approximations are $E^0 = E_0$ and $B^0 = B_0$. Suppose we already know B^k and $E^k \in L^2((0,N) \times \Omega_N)$, *for some* $k \geqq 0$. Let B^k_*, $B^k_* \in C_c^\infty((0,N) \times \Omega_N)$ such that $|E^k_* - E^k|_{2;(0,N)\times\Omega_N} \leqq \frac{1}{2k}$, $|B^k_k - B^k|_{2;(0,N)\times\Omega_N} \leqq \frac{1}{2k}$. The linear equation satisfied by f^{k+1} will be defined following the procedure of [BP], as follows.

Definition 2.1. *For fixed k, let $(t,x,v) \in \bar{\Pi}_N$. The path $\Gamma^k(s;t,x,v)$ is the solution $(t(s), x(s), v(s))$ of the system*

$$\frac{dx}{ds} = v, \quad \frac{dv}{ds} = E^k_* + v \times B^k_*, \quad \frac{dt}{ds} = 1 \tag{2.1}$$

which passes through the point (t,x,v) when $s=t$, extended over the maximal s-interval for which the path lies in $\bar{\Pi}_N$. By the length of this path we mean the length of the maximal s-interval over which the path remains in $\bar{\Pi}_N$.

Definition 2.2. (*Incoming and outgoing sets*). *Let $D^-(D^+)$ be the subset of $\partial\Pi_N$ consisting of the left* (resp. *right*) *limits* (*in the parameter s*) *of all maximal paths with initial values in $\bar{\Pi}_N$. Keep in mind that $D^\pm$ depend on k and N.*

We also define $\gamma_N^\pm = \gamma^\pm \cap \bar{\Pi}_N$. Clearly from (2.1) we have

$$\gamma_N^\pm \subset D^\pm, \quad \{t=0\} \subset D^-, \quad \{t=T\} \subset D^+ . \tag{2.2}$$

We also define the following sets, which are also dependent on k and N.

$$\begin{cases} F^- = D^- \backslash (\gamma_N^- \cup \{t=0\}), & R^- = D^- \backslash \{t=0\} = F^- \cup \gamma_N^- , \\ F^+ = D^+ \backslash (\gamma_N^+ \cup \{t=T\}), & R^+ = D^+ \backslash \{t=T\} = F^+ \cup \gamma_N^+ , \end{cases} \tag{2.3}$$

Definition 2.3. *By $\Phi = \Phi^k = \Phi^k_N$ we denote the test function space of the linear Vlasov equation. It consists of all the Borel functions ϕ on $\bar{\Pi}_N$ with the following three properties*:

1. *ϕ is continuously differentiable in the variable s along the path $\Gamma^k(s;t,x,v)$.*
2. *ϕ and $Y\phi$ are bounded, where $Y = \partial_t + v \cdot \nabla_x + (E^k_* + v \times B^k_*) \cdot \nabla_v$.*
3. *Among all the paths which meet the support of ϕ, there is a positive lower bound to their lengths inside $\bar{\Pi}_N$. This lower bound may depend on k and N.*

If ϕ is smooth, then properties 1 and 2 are obviously satisfied. But we want to allow ϕ to be discontinuous in some directions. Notice that the test function space depends on k and N. The following lemma shows that $\mathscr{V}$ belong to every Φ^k after being cut off.

Lemma 2.1. *Given $E_*^k, B_*^k \in C_c^\infty((0, N)\times\Omega_N)$, and let $X_N = \{|x| < N\}$, let*

$$\mathscr{V}_N = \{\alpha \in \mathscr{V} \mid \operatorname{supp}\alpha \subset\subset [0, N)\times(\bar{\Omega}\cap X_N)\times V_N\} . \tag{2.4}$$

Then $\mathscr{V}_N \subset \Phi^k$, for all k.

Proof. Choose $\alpha \in \mathscr{V}_N$. Let $K_2 =$ the v-projection of $\operatorname{supp}\alpha$. Then $K_2 \subset\subset V_N$. Let $K_1 =$ the x-projection of $(\operatorname{supp}\alpha \cap \{t = 0\})$, then $K_1 \subset\subset \Omega_N$. From the definition of $\mathscr{V}$, and from (2.4), we know that α vanishes on a neighborhood of F^-. Now we let

$$d_0 = \min\{d(K_1, \Omega_N^c), d(K_2, V_N^c), d(\operatorname{supp}\alpha, F^-), d(\operatorname{supp}\alpha, \gamma^0), d(\operatorname{supp}\alpha, X_N^c)\} , \tag{2.5}$$

where d is the Euclidean distance in $R\times R^3\times R^3$. Certainly $d_0 > 0$ and α satisfies properties 1 and 2. Let $\Gamma^k(s; t, x, v)$ be a path in $\bar{\Pi}_N$ which meets $\operatorname{supp}\alpha$. Because $\frac{dt}{ds} = 1$, and $t \in [0, N]$, the path must emanate from some point $(t', x', v') \in D^-$. So we can write the path as $\Gamma^k(s; t, x, v)$, where $(t, x, v) \in D^-$. We shall find a lower bound for $|s - t|$.

Case 1. $d((t, x, v), \operatorname{supp}\alpha) \geqq \frac{d_0}{4}$. Since the velocity is bounded, clearly there is a lower bound of time to cover the distance $\frac{d_0}{4}$.

Case 2. $d((t, x, v), \operatorname{supp}\alpha) < \frac{d_0}{4}$. From (2.5), we know

$$|x| \leqq N - \frac{3d_0}{4} \quad \text{and} \quad |v| \leqq N - \frac{3d_0}{4} . \tag{2.6}$$

By (2.1), we have $\frac{d}{ds}|v|^2 = v\cdot E_*^k \leqq |v(s)|^2 + |E_*^k|^2$. By the boundedness of E_*^k and by Gronwall's inequality, $|v(s)|^2 \leqq (|v| + C_4(s - t)^{1/2})^2$, where C_3 and C_4 are constants depending only on $\|E_*^k\|_\infty$. Let $s_0 = t + \min\left(\frac{d_0^2}{4C_4^2}, \frac{d_0}{2N}\right)$. If $s \geqq s_0$, then s_0 clearly is a lower bound for $|s - t|$. If $t \leqq s \leqq s_0$, we know from (2.6) that $|v(s)| \leqq |v| + \frac{d_0}{2} < N$, and $|x(s)| \leqq |x| + \frac{d_0}{2} < N$. By (2.5) and (2.3), we know $(t, x, v) \in \{t = 0\} \cup \gamma_N^-$.

Now we treat two different situations. In case $(t, x, v) \in (t = 0\}$, then from (2.5),

$$d(x, \Omega_N^c) \geqq d(\operatorname{supp}\alpha, \Omega_N^c) - d(\operatorname{supp}\alpha, x) \geqq d_0 - \frac{d_0}{4} = \frac{3d_0}{4} .$$

Since $|x(s) - x| \leqq \frac{d_0}{2}$, $d(x(s), \Omega_N^c) \geqq d(x, \Omega_N^c) - d(x(s), x) \geqq \frac{3d_0}{4} - \frac{d_0}{2} = \frac{d_0}{4}$. Hence, $(s, x(s), v(s)) \in \overline{\Pi_N}$, when $0 = t \leqq s \leqq s_0$. Thus s_0 is a lower bound for the length of the trajectory. Finally in case $(t, x, v) \in \gamma_N^-$, then we know that $n\cdot v < 0$. From (2.5), we have

$$d((t, x, v), \gamma^0) \geqq d(\operatorname{supp}\alpha, \gamma^0) - d(\operatorname{supp}\alpha, (t, x, v)) \geqq d_0 - \frac{d_0}{4} = \frac{3d_0}{4} .$$

Hence there is a $\delta > 0$, such that

$$n\cdot v \leqq -\delta, \quad \text{for all such } (t, x, v)\,. \tag{2.7}$$

Since $\partial\Omega \in C^{1,\mu}$, $\mu > 0$, we know that $|y-x|^{-1}|(y-x)\cdot n|$ is uniformly small, provided that $|y-x|$ is small, and $x, y \in \partial\Omega \cap \bar{X}_N$. Now let $x \in \partial\Omega$, and $|x| \leqq N - \frac{3d_0}{4}$. It is easy to show that there is a $\frac{d_0}{2} > \eta > 0$, such that

$$\text{if } |y-x| < \eta \quad \text{and} \quad (y-x)\cdot n \leqq -\frac{\delta|y-x|}{2N}, \text{ then we have } y \in \bar{\Omega}_N\,. \tag{2.8}$$

Now let $s_1 = t + \min\left\{s_0, \frac{\eta}{2N}, \frac{\delta}{2(\|E_*^k\|_\infty + N\|B_*^k\|_\infty)}\right\}$. If $t \leqq s \leqq s_1$, $|x(s) - x| \leqq \int_t^s |v(\tau)|\,d\tau \leqq (s-t)N < \frac{\eta}{2}$. From (2.7),

$$(x(s)-x)\cdot n = \int_t^s v(\tau)\cdot n\,d\tau = \int_t^s \left[v(t) + \int_t^\tau v'(\xi)\,d\xi\right]\cdot n d\tau$$

$$\leqq -(s-t)\delta + (s-t)^2(\|E_*^k\|_\infty + N\|B_*^k\|_\infty) \leqq -(s-t)\frac{\delta}{2}\,.$$

Hence $(x(s)-x)\cdot n \leqq -\frac{\delta|x(s)-x|}{2N}$. By (2.8), $x(s) \in \bar{\Omega}_N$. Since $v(s) \in V_N$, $(s, x(s), v(s)) \in \bar{\Omega}_N$ and $s_1 - t$ is a lower bound for the trajectory. Therefore we conclude there is a lower bound for the trajectory in every case. Q.E.D.

Lemma 2.2. There are two unique positive Borel measures $\mu^\pm$ on $D^\pm$, such that

$$\int_{\Pi_N} Y\phi\,dt\,dx\,dv = \int_{D^+} \phi\,d\mu^+ - \int_{D^-} \phi\,d\mu^-, \forall\phi \in \Phi\,. \tag{2.9}$$

Moreover, $d\mu^-$ restricted to $\{t = 0\}$ is $dx\,dv$ and $d\mu^\pm$ restricted to $\gamma_N^\pm$ is $|d\gamma|$. These measures depend on k and N.

Proof. Equation (2.9) is proved in Lemma 3.1 in Chapter XI of [GMP]. Choose $\alpha = \phi \in \mathscr{V}_N \subset \Phi$. By (2.3) we know $\alpha = 0$ on $F^\pm$. Therefore we can replace $D^\pm$ by $\gamma_N^\pm$ in (2.9). Applying the divergence theorem to the left side of (2.9), we deduce the rest of the lemma.

Definition 2.4 (*Trace*). *If u and Yu belong to $L^p(\Pi_N)$, the trace of u is a pair of functions $u^\pm$ in $L^p_{\mathrm{loc}}(D^\pm, d\mu^\pm)$, such that $\forall\phi \in \Phi$,*

$$\langle Yu, \phi\rangle + \langle u, Y\phi\rangle = \int_{D^+} u\phi\,d\mu^+ - \int_{D^-} u\phi\,d\mu^-\,. \tag{2.10}$$

From Prop. 1 of [BP], we know that trace of u exist and is unique. Now we are ready to define our f^{k+1} as a unique solution of the linear Vlasov equation

$$\partial_t f^{k+1} + v\cdot\nabla_x f^{k+1} + (E_*^k + v \times B_*^k)\cdot\nabla_v f^{k+1} = 0 \tag{2.11}$$

with the initial and boundary conditions

$$\begin{cases} f^{k+1}(0,x,v) = f_0(x,v), \quad f^{k+1}|_{F^-} = 0\,, \\ f^{k+1}|_{\gamma_N^-} = a(t,x,v)(Kf^{k+1})|_{\gamma_N^+} + g(t,x,v)\,, \end{cases} \tag{2.12}$$

satisfied in the sense of trace. More precisely, we have the following

Lemma 2.3. *Given E_*^k and B_*^k, there exist two unique nonnegative functions $f^{k+1} \in L^p(\Pi_N)$ and $f_+^{k+1} \in L^p(D^+)$, such that $\forall \phi \in \Phi$,*

$$\int_{\Pi_N} (Y\phi) f^{k+1}\, dt\, dx\, dv = \int_{D^+} \phi f_+^{k+1}\, d\mu^+ - \int_{\gamma_N^-} \phi(aKf_+^{k+1} + g)\, d\mu^- - \int_{t=0} \phi f_0\, dx\, dv\,. \tag{2.13}$$

Proof. Recall from (2.3) that $R^- = F^- \cup \gamma_N^-$. Define

$$\bar{g} = \begin{cases} g & \text{in } \gamma_N^- \\ 0 & \text{in } F^- \end{cases} \quad \text{and} \quad \bar{K}q = \begin{cases} aKq & \text{in } \gamma_N^- \\ 0 & \text{in } F_N^- \end{cases}$$

for any function $q(t,x,v)$ on R^-. Now we can write part of (2.12) as $f^{k+1}|_{R^-} = \bar{K}(f^{k+1}|_{R^+}) + \bar{g}$. By changing v-variables, we have

$$\begin{aligned} \|\bar{K}q\|_{L^p(R^-,d\mu^-)} &= \left\{ \int_{\gamma_N^-} |aKq|^p |d\gamma| \right\}^{\frac{1}{p}} = \left\{ -\int_{\gamma_N^-} a(t,x,v)^p |q(t,x,v-2(n\cdot v)n)|^p d\gamma \right\}^{\frac{1}{p}} \\ &= \left\{ \int_{\gamma_N^+} a(t,x,v-2(n\cdot v)v)^p |q(t,x,v)|^p d\gamma \right\}^{\frac{1}{p}} = \left\{ \int_{\gamma_N^+} (Ka)^p |q|^p d\gamma \right\}^{\frac{1}{p}} \\ &\leqq a_0 \|q\|_{L^p(R^+,d\mu^+)} < \|q\|_{L^p(R^+,d\mu^+)}\,. \end{aligned}$$

Also $\bar{K}q \geqq 0$, if $q \geqq 0$. Now our lemma is an immediate consequence of the following special case of the theorems in [BP].

Theorem 1 and 2 of [BP]. *Suppose that $\mathscr{K}: L^p(R^+, d\mu^+) \to L^p(R^-, d\mu^-)$, $1 \leqq p < \infty$, has operator norm less than 1. For any $f_0 \in L^p(\Omega_N \times V_N)$, $g \in L^p(R^-)$, the linear transport problem*

$$Yu = 0 \quad \text{in } \Pi_N, \quad u|_{t=0} = f_0 \quad u^- = \mathscr{K}u^+ + g \quad \text{on } R^-$$

has a unique solution $u \in L^p(\Pi_N)$ with unique trace $u^\pm \in L^p(\partial \Pi_N)$. Moreover, if $\forall q \in L^p(R^-)$, $q \geqq 0$ implies $\mathscr{K}q \geqq 0$. Then the solution $u \geqq 0$ if $f_0 \geqq 0$ and $g \geqq 0$.

Lemma 2.4. *The solution from the previous lemma has the following properties:*

$$\forall \alpha \in \mathscr{V}_N, \quad A(f^{k+1}, f_+^{k+1}, E_*^k, B_*^k, \alpha, \Omega_N, V_N) = 0\,, \tag{2.14}$$

$$(1-a_0)^{\frac{1}{p}} |\chi_T f_+^{k+1}|_{p;\gamma_N^+} + |\chi_T f^{k+1}|_{p;\Pi_N} \leqq 2e^T (|f_0|_{p;\Pi_0} + (1-a_0)^{-1} |\chi_T g|_{p;\gamma^-}) \tag{2.15}$$

for $1 \leqq p \leqq \infty$, where $\Pi_0 = \Omega_N \times V_N$, and $\chi_T(\cdot)$ is the characteristic function of $[0,T]$, $0 < T \leqq N$, Here we set $(1-a_0)^{\frac{1}{p}} = 1$ if $p = \infty$. Furthermore,

$$\int_{\Pi_N} e^{-t}(1+|v|^2)\chi_T f^{k+1}\, dt\, dx\, dv \leqq \int_{\Pi_0} (1+|v|^2) f_0\, dx\, dv - \int_{\gamma^-} \chi_T (1+|v|^2) g\, d\gamma + 2\int_{\Pi_N} e^{-t}\chi_T E_*^k v f^{k+1}\, dt\, dx\, dv\,. \tag{2.16}$$

Proof. By Lemma 2.1 and 2.3, if $\alpha \in \mathscr{V}_N \subset \Phi$, we recall that $\alpha = 0$ on $F^{\pm}$ and on $t = T$. Plugging α into (2.13), we have

$$\begin{aligned} 0 &= -\int_{\Pi_N} f^{k+1} Y\alpha \, dt\, dv\, dx + \int_{D^+} f_+^{k+1} \alpha \, d\mu^+ - \int_{\gamma_N^-} \alpha(aKf_+^{k+1} + g)\, d\mu^- - \int_{t=0} \alpha f_0 \, dx\, dv \\ &= -\int_{\Pi_N} Y\alpha f^{k+1} \, dt\, dx\, dv + \int_{\gamma_N^+} \alpha f_+^{k+1} d\gamma + \int_{\gamma_N^-} (aKf_+^{k+1} + g)\alpha \, d\gamma - \int_{t=0} \alpha f_0 \, dx\, dv \\ &= A(f^{k+1}, f_+^{k+1}, E_*^k, B_*^k, \alpha, \Omega_N, V_N) \end{aligned}$$

which proves (2.14). Next, let $\Pi_N^T = \Pi_N \cap ((0, T) \times \Omega_N \times V_N)$, and let $D_T^{\pm}$ be its incoming (outgoing) sets. Multiplying (2.14) by e^{-t}, we have $Y(e^{-t} f^{k+1}) + e^{-t} f^{k+1} = 0$ in $\mathscr{D}(\Pi_N^T)$. Since $e^{-t} f^{k+1}$ has $e^{-t} f_{\pm}^{k+1}$ as its trace, by Prop. 1 of [BP],

$$\int_{D_T^+} (e^{-t} f_+^{k+1})^p \, d\mu^+ + p \int_{\Pi_N^T} (e^{-t} f^{k+1})^p \, dt\, dx\, dv = \int_{D_T^-} (e^{-t} f_-^{k+1})^p \, d\mu^- ,$$

for $1 \leqq p < \infty$. Let's write out every term above explicitly. Notice that

$$\begin{aligned} D_T^+ &= \{t = T\} \cup \{F^+ \cap \Pi_N^T\} \cup \{\gamma_N^+ \cap \Pi_N^T\} , \\ D_T^- &= \{t = 0\} \cup \{F^- \cap \Pi_N^T\} \cup \{\gamma_N^- \cap \Pi_N^T\} . \end{aligned}$$

Since $aKf_+^{k+1} + g$ is the trace for f^{k+1} on γ_N^-, and since $f^{k+1} \geqq 0$ and $f_+^{k+1} \geqq 0$, we get

$$\begin{aligned} &\int_{t=T} (e^{-t} f_+^{k+1})^p \, d\mu^+ + \int_{F^+} \chi_T (e^{-t} f_+^{k+1})^p \, d\mu^+ + \int_{\gamma_N^+} \chi_T (e^{-t} f_+^{k+1})^p \, d\gamma + p \int_{\Pi_N^T} (e^{-t} f^{k+1})^p \, dz \\ &\quad = -\int_{\gamma_N^-} \chi_T [e^{-t}(aKf_+^{k+1} + g)]^p \, d\gamma + \int_{\Pi_0} f_0^p \, dx\, dv + \int_{F^-} \chi_T (e^{-t} f_-^{k+1})^p \, d\mu^- , \end{aligned}$$

where $dz = dt\, dx\, dv$. Notice that the first and second terms are nonnegative, and the last term vanishes. We estimate the first term on the right as

$$\begin{aligned} 0 &\leqq -\int_{\gamma_N^-} \chi_T [e^{-t}(aKf_+^{k+1} + g)]^p \, d\gamma = \int_{\gamma_N^+} \chi_T [e^{-t}((Ka) f_+^{k+1} + Kg)]^p \, d\gamma \\ &\leqq \int_{\gamma_N^+} \chi_T [e^{-t}(a_0 f_+^{k+1} + (1 - a_0)(1 - a_0)^{-1} Kg)]^p \, d\gamma \\ &\leqq a_0 \int_{\gamma_N^+} \chi_T (e^{-t} f_+^{k+1})^p \, d\gamma + (1 - a_0) \int_{\gamma_N^+} \chi_T [(1 - a_0)^{-1} e^{-t} Kg]^p \, d\gamma . \end{aligned}$$

Therefore

$$\begin{aligned} &(1 - a_0) \int_{\gamma_N^+} \chi_T (e^{-t} f_+^{k+1})^p \, d\gamma + \int_{\Pi_N} \chi_T (e^{-t} f^{k+1})^p \, dt\, dx\, dv \\ &\quad \leqq \int_{\Pi_0} f_0^p \, dx\, dv - \int_{\gamma_N^-} [e^{-t}(1 - a_0)^{-1} \chi_T g]^p \, d\gamma . \end{aligned}$$

This proves (2.15) for $1 \leqq p < \infty$. Since all the measures here are finite, when $p \to \infty$, (2.15) is valid. Finally we multiply (2.11) by $e^{-t}(1 + |v|^2)$ on Π_N^T to get

$$Y\{e^{-t}(1 + |v|^2) f^{k+1}\} + e^{-t}(1 + |v|^2) f^{k+1} = 2e^{-t} f^{k+1} v E_*^k ,$$

where $e^{-t}f^{k+1}vE_*^k \in L^1(\Pi_N)$. Noticing $e^{-t}(1+|v|^2)f^{k+1}$ has trace $e^{-t}(1+|v|^2)f_\pm^{k+1}$, repeating the argument as above, we get (2.16). Q.E.D.

We now define E^{k+1} and B^{k+1} as the weak solution of the linear Maxwell system

$$\begin{cases} \partial_t E^{k+1} - \operatorname{curl} B^{k+1} = -j^{k+1} = -\int\limits_{V_N} v f^{k+1}\,dv\,, \\ \partial_t B^{k+1} + \operatorname{curl} E^{k+1} = 0 \end{cases} \tag{2.17}$$

with initial data E_0, B_0 and boundary data $E^{k+1} \times n = 0$. More precisely, we have

Lemma 2.5. *For fixed k, suppose* $j^{k+1} = \int_{V_N} v f^{k+1}\,dv \in L^\infty((0,N)\times\Omega_N)$. *There exist* E^{k+1} *and* $B^{k+1} \in L^2((0,N)\times\Omega_N)$, *such that* $\forall(\vec{\psi},\vec{\varphi}) \in \mathscr{M}$, *and* $\operatorname{supp}\vec{\psi} \subset\subset [0,N)\times\Omega_N$,

$$C(E^{k+1}, B^{k+1}, j^{k+1}, \vec{\psi}, \Omega_N, V_N) = 0, \quad D(E^{k+1}, B^{k+1}, \vec{\varphi}, \Omega_N, V_N) = 0\,, \tag{2.18}$$

$$\int\limits_{(0,N)\times\Omega_N} \chi_T e^{-t}[|E^{k+1}|^2 + |B^{k+1}|^2]\,dt\,dx \leqq \int\limits_{\Omega}(E_0^2 + B_0^2)\,dx - 2\int\limits_{\Pi_N} \chi_T e^{-t} E^{k+1} v f^{k+1}\,dt\,dx\,dv\,, \tag{2.19}$$

where $0 < T \leqq N$.

Sketch of the proof. Notice that Ω_N has a Lipschitz boundary which is not necessarily C^2. In case $\partial\Omega_N$ happens to be C^2, the proof is standard. See [CS]. In the case $\partial\Omega$ is not C^2, by [N], we deduce our lemma by approximating Ω_N with smooth domains.

We summarize our constructions as follows.

Lemma 2.6. *There is a well-defined sequence* f^k, f_+^k, E^k, B^k *satisfying* (2.14), (2.15), (2.16), (2.18) *and* (2.19).

3. The Cut Off Nonlinear Problem

In this section we let $k \to \infty$. This process will result in the following lemma.

Lemma 3.1. *There exist* f, f^+, E, B *such that* $\forall \alpha \in \mathscr{V}_N$, $\forall(\vec{\psi},\vec{\varphi}) \in \mathscr{M}$ *with* $\operatorname{supp}\vec{\psi} \subset\subset [0,N)\times\Omega_N$, *we have*

$$A(f, f^+, E, B, \alpha, \Omega_N, V_N) = 0\,, \tag{3.1}$$

$$C(E, B, j, \vec{\psi}, \Omega_N, V_N) = 0, \quad D(E, B, \vec{\varphi}, \Omega_N, V_N) = 0\,, \tag{3.2}$$

$$(1-a_0)^{\frac{1}{p}}|\chi_T f^+|_{p;\gamma_N^+} + |\chi_T f|_{p;\Pi_N} \leqq 2e^T(|f_0|_{p;\Pi_0} + (1-a_0)^{-1}|\chi_T g|_{p;\gamma^-}) \tag{3.3}$$

$$\mathscr{E}(f, E, B, \Omega_N, V_N, T) \leqq e^T \mathscr{E}_0(T), \quad \text{where } 1 \leqq p \leqq \infty, \quad 0 \leqq T \leqq N\,. \tag{3.4}$$

Proof. By (2.15) and (2.19), there exist weak limits f, f^+, E, B and subsequences such that $f^k \rightharpoonup f$ weakly in $L^p(\Pi_N)$, $f_+^k \overset{*}{\rightharpoonup} f^+$ weakly * in $L^p(\gamma_N^+)$, $E^k \rightharpoonup E$ weakly in $L^2((0,N)\times\Omega_N)$, and $B^k \rightharpoonup B$ weakly in $L^2((0,N)\times\Omega_N)$ for $1 \leqq p \leqq \infty$. Since $E_*^k \rightharpoonup E$ and $B_*^k \rightharpoonup B$ weakly in $L^2((0,N)\times\Omega_N)$, we get (3.3) by weak lower semicontinuity. In order to prove the lemma, we have to consider the limit of (2.14), (2.16), (2.18) and (2.19), when $k \to \infty$. Since in (2.18) every term is linear, (3.2) is valid. Our main task is to prove (3.1), for which we take the limit in (2.14). It is sufficient to consider only these two delicate terms as follows. We claim that for each $\alpha \in \mathscr{V}_N$,

$$\lim_{k\to\infty} \int_{\gamma_N^-} \alpha(a(Kf_+^{k+1}) + g)d\gamma = \int_{\gamma_N^-} \alpha(aKf^+ + g)d\gamma \,, \tag{3.5}$$

$$\lim_{k\to\infty} \int_{\Pi_N} (E_*^k + v\times B_*^k)\nabla_v \alpha f^{k+1}\,dt\,dx\,dv = \int_{\Pi_N} (E + v\times B)\nabla_v \alpha f\,dt\,dx\,dv \,, \tag{3.6}$$

Proof of the claim. For (3.5), we change v-variables, take the weak limit, and then change v-variables back again. Thus,

$$\lim_{k\to\infty} \int_{\gamma_N^-} \alpha(a(Kf_+^{k+1}) + g)d\gamma = \lim_{k\to\infty} \int_{\gamma_N^+} (-1)K\alpha((Ka)f_+^{k+1} + Kg)d\gamma$$

$$= -\int_{\gamma_N^+} (K\alpha)[(Ka)f^+ + Kg]\,d\gamma$$

$$= \int_{\gamma_N^-} \alpha(aKf^+ + g)d\gamma \,.$$

This proves (3.5). For (3.6), fix any $\eta \in C_c^\infty(\Pi_N)$, $0 \leqq \eta \leqq 1$. From our construction,

$$\begin{aligned}\partial_t(\eta f^{k+1}) + v\cdot\nabla_x(\eta f^{k+1}) &= -\operatorname{div}_v(\eta(E_*^k + v\times B_*^k)f^{k+1}) \\ &\quad + \nabla_v\eta\cdot(E_*^k + v\times B_*^k)f^{k+1} \\ &\quad + \eta_t f^{k+1} + v\cdot\nabla_x\eta f^{k+1} = h^k\end{aligned} \tag{3.7}$$

in $\mathscr{D}'$. Noticing that h^k is a bounded sequence in $L^2(R\times R_x^3,\, H^{-1}(R_v^3))$, by the averaging lemma of DiPerna and Lions, ([DL]), we deduce that $\forall\phi(v) \in C_c^\infty(R^3)$, $\int \eta f^{k+1}\phi(v)\,dv$ is bounded in $H^{\frac14}((0,\infty)\times R^3)$. Hence $\int \eta f^{k+1}\phi(v)\,dv$ is compact in L^2. So there is a subsequence (still denoted by f^{k+1}), such that $\int \eta f^{k+1}\phi(v)\,dv \to \int \eta f\phi(v)\,dv$ strongly in L^2 as $k\to\infty$. By a density argument, we can assume α of the form $\alpha_1(v)\alpha_2(t,x)$, where $\alpha_1(v) = 0$, if $|v| \geqq N$, and $\alpha_2(t,x)$ may not vanish on the boundary. We wish to show that $\int \nabla\alpha f^{k+1}\,dv$ converges strongly in $L^2((0,N)\times\Omega_N)$. We break up $f^{k+1} - f = \eta(f^{k+1} - f) + (1-\eta)(f^{k+1} - f)$, and estimate these two terms separately. We have

$$\left[\int_{(0,N)\times\Omega_N} \alpha_2^2 \left\{\int_{V_N} (1-\eta)(f^{k+1} - f)\nabla_v\alpha_1\,dv\right\}^2 dt\,dx\right]^{\frac12} \leqq C\left[\int_{\Pi_N} (1-\eta)^2\,dt\,dv\,dx\right]^{\frac12}, \tag{3.8}$$

by (2.15), we can choose C depends only on $\alpha_1, \alpha_2, f_0, a_0$ and g. Now for any $\varepsilon > 0$, we choose η such that $C[\int_{\Pi_N}(1-\eta)^2\,dt\,dv\,dx]^{\frac12} < \varepsilon/2$. Then for this fixed η, we choose k so large that $[\int_{(0,N)\times\Omega_N} \alpha_2^2(\int_{V_n} \eta(f^{k+1} - f)\nabla\alpha_1\,dv)^2\,dt\,dx]^{\frac12} < \varepsilon/2$. Thus we have shown

$$\lim_{k\to\infty}\left[\int_{(0,N)\times\Omega_N} \alpha_2^2\left(\int_{V_N} (f^{k+1} - f)\nabla\alpha_1\,dv\right)^2 dt\,dx\right]^{\frac12} = 0 \,.$$

Now in (3.6), $\int_{(0,N)\times\Omega_N} E_*^k(\int_{V_N}\nabla\alpha f^{k+1}\,dv)\,dt\,dx$ converges because one factor converges weakly in L^2 and the other converges strongly in L^2. The second term in (3.6) is

$$\int_{\Pi_N}(v\times B_*^k)\cdot\nabla_v\alpha f^{k+1}\,dt\,dx\,dv = \int_{(0,N)\times\Omega} B_*^k\left(\int_{V_N}(v\times\nabla_v\alpha)f^{k+1}\,dv\right)dt\,dx\,.$$

Regarding $v\times\nabla_v\alpha$ as another test function, we deduce our claim.

Finally let's consider the limit of (2.16) and (2.19). By exactly the same method as in the proof of (3.6), both $\int_{\Pi_N}\chi_T E_*^k e^{-t}vf^{k+1}\,dt\,dx\,dv$ and $\int_{\Pi_N}\chi_T E^k e^{-t}vf^k\,dt\,dx\,dv$ converge to $\int_{\Pi_N}\chi_T E e^{-t}vf\,dt\,dx\,dv$. The reason is that $\chi_T e^{-t}v$ now behaves like a test function, since our Π_N is bounded. Letting $k\to\infty$ in (2.16) and (2.19), using the weak lower semicontinuity, then adding (2.16) and (2.19) together, we finally prove (3.4). Q.E.D.

4. Solution of (VM)

In this section, we begin with the partial absorption problem and conclude with the specular case. We let $N\to\infty$ in the cut off problems. The method is similar to the previous section.

Theorem 4.1. *Suppose* $\partial\Omega\in C^{1,\mu}$, *for some* $\mu>0$. *Let* $f_0(1+|v|^2)\in L^1(\Omega\times R^3)$, $f_0\in L^\infty(\Omega\times R^3)$, *and* $f_0\geqq 0$ a.e.. *Let* $0\leqq a(t,x,v)\leqq a_0<1$, $\chi_T g(t,x,v)\in L^\infty(\gamma^-)$, $(1+|v|^2)\chi_T g\in L^1(\gamma^-)$, *for all* $0<T<\infty$ *and* $g\geqq 0$ a.e. *Let* E_0 *and* $B_0\in L^2(\Omega)$ *with the constraint conditions* $\operatorname{div}B_0=0$, $\operatorname{div}E_0=\rho_0$ *in* $\mathscr{D}'(\Omega)$. *Then there is a weak solution* f, f^+, B, E *of the partial absorption problem, with*

$$\begin{cases}(1-a_0)^{1/p}|\chi_T f^+|_{p;\gamma^+}+|\chi_T f|_{p;\Pi}\leqq 2e^T(|f_0|_{p;\Pi_0}+(1-a_0)^{-1}|\chi_T g|_{p,\gamma^-})\\ \mathscr{E}(f,E,B,\Omega,T)\leqq e^T\mathscr{E}_0(T),\quad \text{for } 1\leqq p\leqq\infty,\quad \forall T<\infty\,.\end{cases}\tag{4.1}$$

Proof. We now have sequences f_N, f_N^+, B_N, E_N satisfying (3.1), (3.2), (3.3) and (3.4). We extend the functions f_N, f_N^+, E_N, B_N by 0 outside $\Pi\backslash\Pi_N$. The extended functions still satisfy (3.1) to (3.4) in the cut off domains. Abusing notation, we call them f_N, f_N^+, E_N and B_N again. It is easy to show that there exist measurable functions f, f^+, E and B defined for $0\leqq t<\infty$, and subsequences (still denoted by N), such that

$$\begin{aligned}&\chi_T f_N\rightharpoonup\chi_T f \quad\text{weakly in } L^2(\Pi),\ \chi_T f_N^+\overset{*}{\rightharpoonup}\chi_T f^+,\quad\text{weakly * in } L^\infty(\gamma^+),\\ &\chi_T E_N\rightharpoonup\chi_T E\quad\text{weakly in } L^2((0,\infty)\times\Omega),\\ &\chi_T B_N\rightharpoonup\chi_T B\quad\text{weakly in } L^2((0,\infty)\times\Omega),\ \text{for all } T.\end{aligned}\tag{4.2}$$

By weak lower semicontinuity for f, f^+, E and B, (4.1) is valid. We also have $\forall\alpha\in\mathscr{V}_N$, $\forall(\vec\psi,\vec\varphi)\in\mathscr{M}$ with $\operatorname{supp}\vec\psi\subset\subset[0,T)\times\Omega_N$,

$$\begin{cases}A(f_N,f_N^+,E_N,B_N,\alpha,\Omega,R^3)=A(f_N,f_N^+,E_N,B_N,\alpha,\Omega_N,V_N)=0\,,\\ C(E_N,B_N,j_N,\vec\psi,\Omega,R^3)=C(E_N,B_N,j_N,\vec\psi,\Omega_N,V_N)=0\,,\\ D(E_N,B_N,\vec\varphi,\Omega,R^3)=D(E_N,B_N,\vec\varphi,\Omega_N,V_N)=0\,.\end{cases}\tag{4.3}$$

Now fix any $\alpha\in\mathscr{V}$, and $(\vec\psi,\vec\varphi)\in\mathscr{M}$. There exists a $r>0$, such that $\operatorname{supp}\alpha\subset\subset[0,r)\times(\bar\Omega_N\cap X_N)\times V_N$ when $N>r$, and $\operatorname{supp}\vec\psi\subset\subset[0,r)\times\Omega_N$ when $N>r$.

In other words, (4.3) holds for any α and $(\vec{\psi}, \vec{\varphi})$ when $N > r$, r depending on α and $\vec{\psi}$. When letting $N \to \infty$ in (4.3), we only need to consider

$$\int_{(0,r)\times\Omega} \vec{\psi} j_N \, dt\, dx \quad \text{and} \quad \int_{\Pi} (E_N + v \times B_N)(\nabla_v \alpha) f_N \, dt\, dx\, dv \,. \tag{4.4}$$

In the former integral, we break up $j_N = \int_{R^3} v f_N \, dv$ into its parts over $|v| \leqq R$ and $|v| > R$. Then we get $|\int_{(0,r)\times\Omega\times(|v|>R)} v f_N \, dt\, dx\, dv| \leqq \frac{C}{R} |\int (1 + |v|^2) f_N \, dt\, dx\, dv| \leqq \frac{C'}{R}$, thanks to the energy inequality (3.4). $\forall \varepsilon > 0$, choose R big enough that $\frac{C'}{R} < \frac{\varepsilon}{2}$. For this fixed R, by (4.2) we have $\int_{(0,r)\times\Omega\times\{|v|\leqq R\}} \vec{\psi} v f_N \, dt\, dx\, dv \to \int_{(0,r)\times\Omega\times\{|v|\leqq R\}} \vec{\psi} v f \, dt\, dx\, dv$. Hence the former one goes to the correct limit.

For the latter integral in (4.4), we follow the proof of (3.6). We first choose $\eta \in C_c^\infty(\Pi)$, $0 \leqq \eta \leqq 1$, such that (3.8) holds for $(f_N - f)$. There exists $N_0(\eta)$ such that when $N \geqq N_0(\eta)$, $\operatorname{supp} \eta \subset\subset \Pi_N$. Therefore the rest of the argument for (3.6) holds with $N \geqq N_0(\eta)$ and $T = N_0(\eta)$. Hence A, C and D are zero with f, f^+, E and B.

Finally, in order to complete the proof of Theorem 4.1, we only need to check $\operatorname{div} E = \rho$ in the sense of distributions. To accomplish this, we use the following observation, which is a fundamental motivation of this paper.

Lemma 4.2. *If* $A = C = D = 0$ *holds with* $f \in L^\infty((0,T)\times\Omega\times R^3)$, $E \in L^2((0,T)\times\Omega)$, *and* $\chi_T(1+|v|^2)^{\frac{1}{2}} f \in L^1(\Pi)$, $\forall 0 \leqq T < \infty$. *Then* $\operatorname{div} E = \rho$.

Proof of Lemma 4.2. Under above assumptions, we claim that $A = 0$ holds for any $\alpha(t,x) \in C_c^\infty([0,\infty)\times\Omega)$ which is independent of v and vanishes near $\partial\Omega$. In fact, let $b_N(|v|) \in C_c^\infty(R^3)$, with $b_N = 1$, for $|v| \leqq N$, $b_N = 0$, for $|v| \geqq 2N$, and $|\nabla b_N| \leqq \frac{2}{N}$. Then $b_N\alpha \in \mathscr{V}$. Assume $\alpha(t,x) = 0$ if $t \geqq T$. Plugging it into $A = 0$, we know that $-\int b_N \alpha f_0 \, dx\, dv - \int b_N \partial_t \alpha f \, dt\, dx\, dv - \int f b_N \vec{v} \cdot \nabla_x \alpha \, dt\, dx\, dv - \int (E + v \times B) f \nabla_v b_N \alpha \, dt\, dx\, dv$ equal to zero, since there is no boundary term. Notice that $\nabla_v b_N$ is parallel to $\vec{v}$, so that $v \times B \cdot \nabla_v b_N = 0$. When $N \to \infty$, only the last integral will present any difficulty. Noticing that the volume of $\{N \leqq |v| \leqq 2N\}$ is $O(N^3)$, we have

$$\begin{aligned}
|\int E f \nabla_v b_N \alpha \, dt\, dv\, dx| &\leqq \left(\int_{(0,T)\times\Omega} |E|^2 \, dt\, dx \right)^{\frac{1}{2}} \left(\int_{(0,T)\times\Omega} \left(\int f \nabla_v b_N \alpha \, dv\right)^2 dx\, dt \right)^{\frac{1}{2}} \\
&\leqq C \left(\int_{(0,T)\times\Omega} \left(\int_{N \leqq |v| \leqq 2N} f \frac{2}{N} dv \right)^2 dx\, dt \right)^{\frac{1}{2}} \\
&\leqq C \left(\int_{(0,T)\times\Omega\times\{N\leqq|v|\leqq 2N\}} N f^2 \, dv\, dt\, dx \right)^{\frac{1}{2}} \\
&\leqq C \left(\int_{(0,T)\times\Omega\times\{N\leqq|v|\leqq 2N\}} (1+|v|^2)^{\frac{1}{2}} f \, dt\, dv\, dx \right)^{\frac{1}{2}} .
\end{aligned}$$

The last integral goes to 0 as $N \to \infty$. This proves $A = 0$ for such α.

Now for any $\alpha(t, x) \in C_c^\infty([0, \infty) \times \Omega)$. Notice that $\tilde{\alpha}(t, x) = \int_0^t \alpha d\tau - \int_0^T \alpha\, dt$ is again a test function of the form given above. Replacing α by $\tilde{\alpha}$, and $(\psi, \vec{\varphi})$ by $(\int_0^t \nabla\alpha\, d\tau - \int_0^T \nabla\alpha\, dt, 0)$ in A, C, and D, we deduce $\operatorname{div} E = \rho$ from the $\operatorname{div} E_0 = \rho_0$. Q.E.D.

Now we can extend our results about the partial absorption problem.

Theorem 4.3. *Suppose* $0 \leqq T < \infty$, $f_0 \in L^p(\Omega \times R^3)$, $\chi_T g \in L^p(\gamma^-)$ *for some* $2 \leqq p < \infty$, *where* f_0 *and* $\chi_T g$ *are not bounded. Let the other assumptions of Theorem* 4.1 *remain the same. Then there is a weak solution* f, f^+, E, *and* B *with given data* f_0, a, g, E_0, B_0, *and satisfy* (4.1) *for above* p.

Proof. We first treat the case of $2 < p < \infty$. *Let* $f_0^{(n)} \in L^\infty(\Omega \times R^3)$, $\chi_T g^{(n)} \in L^\infty(\gamma^-)$, such that $\| f_0^{(n)} - f_0 \|_{L^1 \cap L^p(\Omega \times R^3)} \to 0$, $\|\chi_T(g^{(n)} - g)\|_{L^1 \cap L^p(\gamma^-)} \to 0$ when $n \to \infty$. By Theorem 4.1, there exist weak solutions $f^{(n)}$, $f^{(n)+}$, $E^{(n)}$, $B^{(n)}$ with given data $f_0^{(n)}$, E_0, B_0, a, $g^{(n)}$. Also $f^{(n)}$, $f^{(n)+}$, $E^{(n)}$, and $B^{(n)}$ satisfy the estimate (4.1) with the fixed p. Hence there exist weak limits f, f^+, B and E of the corresponding sequences such that (4.2) holds for these functions. By weak lower semicontinuity, f, f^+, B and E satisfy (4.1) with the fixed p. To prove they are weak solutions, the only difficult term is again the nonlinear one $\int (E^{(n)} + v \times B^{(n)}) f^{(n)} \nabla_v \alpha\, dt\, dx\, dv$. By the L^p boundedness of $f^{(n)}$, we can follow the proof as in (3.6). We only need to replace (3.8) by

$$I = \left[\int_{(0,\infty)\times\Omega} \alpha_2^2 \left(\int (1-\eta)(f^{(n)} - f)\nabla\alpha_1\, dv\right)^2 dt\, dx \right]^{\frac{1}{2}}$$

$$\leqq C\left[\int_{\operatorname{supp}\alpha_2} \left(\int_{\operatorname{supp}\alpha_1} (1-\eta)^2\, dv\right)\left(\int_{\operatorname{supp}\alpha_1} (f^{(n)} - f)^2\, dv\right) dt\, dx \right]^{\frac{1}{2}}. \tag{4.5}$$

Next we use Hölder inequality with $\dfrac{1}{q} + \dfrac{2}{p} = 1$, $p > 2$. Since $f^{(n)}$, $f \in L^p$, we get

$$I \leqq C\left[\int_{\operatorname{supp}\alpha_2 \times \operatorname{supp}\alpha_1} (1-\eta)^{2q}\, dt\, dx\, dv \right]^{\frac{1}{2q}} \left[\int_{\operatorname{supp}\alpha_2 \times \operatorname{supp}\alpha_1} (|f^{(n)}|^p + |f|^p)\, dt\, dx\, dv \right]^{\frac{1}{p}}$$

$$\leqq C\left[\int_{\operatorname{supp}\alpha_2 \times \operatorname{supp}\alpha_1} (1-\eta)^{2q}\, dt\, dx\, dv \right]^{\frac{1}{2q}},$$

where C depends on α_1, α_2, f_0, a_0 and g. We thus conclude the proof for $p > 2$ by the same arguments in (3.6).

Now let $p = 2$, and we repeat the process of case 1. It suffices to show that I in (4.5) can be arbitrarily small. To this end, we follow the idea of [DL] and [S2]. It is easy to show if f_0 and $\chi_T g$ in L^2, then there exist a C^2 function $\theta(u)$: $[0, \infty] \to [0, \infty)$, such that $\theta(0) = 0$, $\theta(u_1 + u_2) \leqq \theta(u_1) + \theta(u_2)$, $\lim_{u \to \infty} \theta(u) = \infty$, and

$$\int_{\Omega \times R^3} \theta(f_0) f_0^2\, dx\, dv < \infty, \quad \int_{\gamma^-} \chi_T \theta(g) g^2 |d\gamma| < C_T < \infty .$$

With this θ, we claim that both $\int_\Pi \theta(f^{(n)})(f^{(n)})^2 \chi_T\, dt\, dx\, dv$ and $\int_\Pi \theta(f) f^2 \chi_T\, dt\, dx\, dv$ are uniformly bounded on n, where $f^{(n)}$ and f are the same as in case 1.

Proof of the claim. It suffices to show that (2.15) holds for $\Theta(f_N^{(n),k+1})$, where $\Theta(u)=\theta(u)u^2$. Since $Y(e^{-t}\Theta(f^{k+1}))+e^{-t}\Theta(f^{k+1})=0$, by the same proof in pp. 380–382 of [GMP], it is easy to show that $e^{-t}\Theta(f_N^{(n),k+1})$ has trace $e^{-t}\Theta(f_{N,\pm}^{(n),k+1})$. By Prop. 1 of [BP],

$$\int_{D_T^+} e^{-t}\Theta(f_{N,+}^{(n),k+1})\,d\mu^+ + \int_{\Pi_N^T} e^{-t}\Theta(f_N^{(n),k+1})\,dt\,dx\,dv = \int_{D_T^-} e^{-t}\Theta(f_{N,-}^{(n),k+1})\,d\mu^- . \tag{4.6}$$

From the same argument as in (2.15), by the boundary condition, we get $\int_{D_T^-}=\int_{\gamma^-}+\int_{t=0}$. By the properties of $\theta(u)$, for any integer $M>1$, we write the integral over γ^- as

$$\int_{\gamma^-}\chi_T e^{-t}\Theta(aKf_{N,+}^{(n),k+1}+g)|d\gamma| = \int_{Mg\leqq Kf_{N,+}^{(n),k+1}} + \int_{Mg\geqq Kf_{N,+}^{(n),k+1}} = I_1+I_2 .$$

Since $a\leqq a_0<1$, $a_0+\dfrac{1}{M}<1$ when M is large. Since $\theta(u)$ is increasing,

$$I_1\leqq \int_{\gamma^-}\chi_T e^{-t}\Theta\left(\left[a_0+\frac{1}{M}\right]Kf_{N,+}^{(n),k+1}\right)|d\gamma| \leqq \int_{\gamma^-}\chi_T e^{-t}\Theta(Kf_{N,+}^{(n),k+1})|d\gamma| ,$$

which can be killed by the same integral on γ^+. Since $\theta((M+1)u)\leqq(M+1)\theta(u)$,

$$I_2\leqq \int_{Mg\geqq Kf_{N,+}^{(n),k+1}} \leqq \int_{\gamma^-}\chi_T e^{-t}\Theta((M+1)g)2|d\gamma| \leqq C(M)\int_{\gamma^-}\chi_T e^{-t}\Theta(g)|d\gamma| .$$

The remaining terms in left side of (4.6) are nonnegative, hence $\int_{\Pi_N} e^{-t}\chi_T\Theta(f_N^{n,k+1})\,dt\,dx\,dv$ is uniformly bounded on n and k. Thus our claim follows through the limiting procedure as $k, N\to\infty$ from Sects. 2 to 4.

Now we can estimate (4.5) by the standard method. Since

$$I\leqq C\left[\int_{\operatorname{supp}\alpha_2}\left(\int_{\operatorname{supp}\alpha_1}(1-\eta)^2\,dv\right)\left(\int_{\operatorname{supp}\alpha_1}(f^{(n)})^2+f^2\,dv\right)dt\,dx\right]^{\frac{1}{2}},$$

it suffices to estimate $f^{(n)}$ and f separately. The integral with $f^{(n)}$ is split to two parts, $\int_{f^{(n)}\leqq M}$ and $\int_{f^{(n)}\geqq M}$. The term with $f^{(n)}\leqq M$ is bounded by $CM[\int(1-\eta)^2\,dt\,dx\,dv]^{\frac{1}{2}}$, and the term with $f^{(n)}\geqq M$ is bounded by $\left[\dfrac{C}{\theta(M)}\int\theta(f^n)(f^n)^2\,dt\,dx\,dv\right]^{\frac{1}{2}}$. For any $\varepsilon>0$, we first choose M large, such that $\int_{f^{(n)}\geqq M}<\dfrac{\varepsilon}{4}$, then for this fixed M, choosing η such that $\int_{f^{(n)}\leqq M}<\dfrac{\varepsilon}{4}$. It is the same for f. Hence I can be arbitrarily small. Q.E.D.

Next we study the purely specular problem. Now the boundary condition for the Vlasov equation is $f_{\gamma^-}=Kf_{\gamma^+}$. In other words, $a(t,x,v)\equiv 1$, $g\equiv 0$.

Theorem 4.4. *Suppose* $\partial\Omega\in C^{1,\mu}$, *for some* $\mu>0$. *Let* $f_0\in L^\infty\cap L^1(\Omega\times R^3)$, $f_0(1+|v|^2)\in L^\infty(\Omega\times R^3)$, *and* $f_0\geqq 0$, a.e.. *Let* $E_0\in L^2(\Omega)$, $B_0\in L^2(\Omega)$ *with constraint conditions* $\operatorname{div}E_0=\rho_0$, $\operatorname{div}B_0=0$ *in* $\mathscr{D}'(\Omega)$. *Then there is a weak solution*

f, f^+, E, B of the purely specular problem. Moreover

$$\begin{cases} |\chi_T f|_{p;\Pi} \leqq 2e^T |f_0|_{p;\Pi_0}, & \text{for } 1 \leqq p \leqq \infty \ , \\ |\chi_T f^+|_{\infty;\gamma^+} \leqq 2e^T |f_0|_{\infty;\Pi_0}, & \text{for } p = \infty \quad \forall 0 \leqq T < \infty \ , \\ \mathscr{E}(f, E, B, \Omega, R^3, T) \leqq e^T \mathscr{E}_0(T) \ . \end{cases} \tag{4.7}$$

Proof. Choose $0 < a_m < 1$, $\lim_{m\to\infty} a_m = 1$, where a_m is a constant. For fixed m, consider the partial absorption problem (VM) with the boundary conditions $E^{(m)} \times \vec{n} = 0$ and $f^{(m)}_{\gamma^-} = a_m f^{(m)}_{\gamma^+}$, and with initial values f_0, B_0 and E_0. Since $g \equiv 0$, by Theorem 4.1, there is a solution $f^{(m)}$, $f^{(m)^+}$, $E^{(m)}$ and $B^{(m)}$ satisfying (4.7) for all m. Now as $m \to \infty$, there are global weak limits f, f^+, E, B of corresponding sequences such that (4.2) holds. So we get (4.7) by weak lower semicontinuity. Since in (4.7) the constants are independent of m, we get the correct limit by the same method in Lemma 3.1. Q.E.D.

5. Relativistic Case

Let Π, $\gamma^\pm$ and γ^0 be the same as before. When the particles with which we are concerned move very fast, we have to consider the following (RVM) system [GS 1]:

$$\begin{cases} \partial_t f_\beta + \hat{v}_\beta \cdot \nabla_x f_\beta + e_\beta \left(E + \dfrac{1}{c} \hat{v}_\beta \times B \right) \cdot \nabla_v f_\beta = 0, \quad 1 \leqq \beta \leqq N \\ \partial_t E - c \operatorname{curl} B = -j = -4\pi \sum\limits_\beta e_\beta \int \hat{v}_\beta \cdot f_\beta \, dv \\ \partial_t B + c \operatorname{curl} E = 0 \end{cases} \tag{RVM}$$

with the constraint conditions

$$\operatorname{div} E = \rho = 4\pi \sum_\beta e_\beta \int f_\beta \, dv, \quad \operatorname{div} B = 0 \tag{5.1}$$

and with the same initial and boundary conditions as (VM), (0.4) through (0.8). Here

$$\hat{v}_\beta = \frac{v}{\sqrt{\left(m_\beta^2 + \dfrac{|v|^2}{c^2} \right)}}, \quad 1 \leqq \beta \leqq N \ .$$

We shall make the following definitions for (RVM).

The *surface measure* $d\gamma_\beta$ is the same as (1.4) except that $\dfrac{v}{m_\beta}$ is replaced by $\hat{v}_\beta$, $1 \leqq \beta \leqq N$. The *test function spaces for* (RVM) remain the same as (1.9), (1.10) and (1.11). The *energies $\mathscr{E}$ and $\mathscr{E}_0$ for* (RVM) are the same as (1.5) and (1.6) except that the factor $(1 + |v|^2) m_\beta$ is replaced by $2c^2 \sqrt{m_\beta^2 + \dfrac{|v|^2}{c^2}}$, $1 \leqq \beta \leqq N$. For the new $d\gamma_\beta$, the *test functionals* and the definition of a *weak solution for* (RVM) are the same as in (1.12), except that $\dfrac{v}{m_\beta}$ is replaced by $\hat{v}_\beta$. With the new definitions, Lemma 1.1 and

Lemma 1.2 are still valid for (RVM). Our major results about (RVM) are parallel to those for (VM). We summarize these results in the following Theorem.

Theorem 5.1. (*Relativistic case*). *Let* $\partial\Omega \in C^{1,\mu}$, *for some* $\mu > 0$. *Let* $f_{0,\beta} \geqq 0$ a.e., $1 \leqq \beta \leqq N$. *Let* E_0 *and* B_0 *in* $L^2(\Omega)$ *satisfy* $\operatorname{div} E_0 = \rho_0$, $\operatorname{div} B_0 = 0$ *in the sense of distributions. There are two kinds of conditions for* $f_{0\beta}$.

(1) *If* $a_\beta \equiv 1$, *and* $g_\beta \equiv 0$, *let* $f_{0\beta} \in L^\infty \cap L^1(\Omega \times R^3)$, $f_{0\beta}\sqrt{(1 + |v|^2)} \in L^1(\Omega \times R^3)$.
(2) *If* $0 \leqq a_\beta(t, x, v) \leqq a_0 < 1$, *and* $g_\beta(t, x, v) \geqq 0$ a.e., *let* $f_{0\beta}\sqrt{(1 + |v|^2)} \in L^1(\Omega \times R^3)$, $f_{0\beta} \in L^p(\Omega \times R^3)$, $\chi_T g_\beta \in L^p(\gamma^-)$ *and* $\chi_T g_\beta \sqrt{(1 + |v^2)} \in L^1(\gamma^-)$, *for* $2 \leqq p \leqq \infty$, $0 < T < \infty$.

Then there is a weak solution of (VM), *denoted* f_β, f_β^+, E *and* B *with finite energy. Moreover, if* $f_{0\beta} \in L^q(\Omega \times R^3)$, $\chi_T g_\beta \in L^p(\gamma^-)$, *then* $\chi_T f_\beta \in L^q(\Pi)$, *where* $2 \leqq q \leqq \infty$.

Sketch of the proof of Theorem 5.1. We follow the arguments step by step from Sects. 2 to 4 with some suitable changes. We first modify the definitions in Sect. 2. Definition 2.2 remains the same for (RVM). Definition 2.1, 2.2 and 2.3 still make sense if we define $\hat{Y} = \partial_t + \hat{v}\cdot\nabla_x + (E_*^k + \hat{v}\times B_*^k)\cdot\nabla_v$ instead of $Y = \partial_t + v\cdot\nabla_x + (E_*^k + v\times B_*^k)\cdot\nabla_v$, where $\hat{v} = \dfrac{v}{\sqrt{(1 + |v|^2)}}$. Under these new definitions, Lemma 2.1 and Lemma 2.2 now are valid. Hence Definition 2.4 makes sense for (RVM) with $\hat{Y}$ and $d\hat{\mu}^\pm$, where $d\hat{\mu}^\pm$ is the new measure in Lemma 2.2. In order to prove Lemma 2.3 for (RVM), it suffices to modify the results of [BP] as follows.

Theorem 1′ and Theorem 2′. Suppose that $\mathscr{K} : L^p(R^+, d\hat{\mu}^+) \mapsto L^p(R^-, d\hat{\mu}^-)$, $1 \leqq p < \infty$, *has operator norm less than* 1. *For any* $f_0 \in L^p(\Omega_N \times V_N)$, $g \in L^p(R^-)$, *the linear transport problem*

$$\hat{Y}u = 0 \quad in\ \Pi_N, \quad u|_{t=0} = f_0, \quad u^+ = \mathscr{K}u^- + g \quad on\ R^-$$

has a unique solution $u \in L^p$ *with trace* $u^\pm \in L^p$. *Moreover, if* $\forall q \in L^p(R^-)$, *that* $q \geqq 0$ *implies* $\mathscr{K}q \geqq 0$. *Then the solution* $u \geqq 0$ *if* $f_0 \geqq 0$ *and* $q \geqq 0$.

Proof of these two Theorems. Theorem 1 and 2 of [BP] are exactly the same as Theorem 4.3 and Theorem 4.4 in Chapter XI of [GMP]. We deduce these Theorems by the same proofs as in Theorem 4.3 and 4.4 of [GMP].

So our Lemma 2.3 for (RVM) follows easily. Therefore our Lemma 2.4 holds by multiplying (2.11) with $2(1 + |v|^2)^{\frac{1}{2}}$. Now (2.16) takes the form

$$2\int_{\Pi_N} e^{-t}(1 + |v|^2)^{\frac{1}{2}}\chi_T f^{k+1}\,dt\,dx\,dv \leqq 2\int_{\Pi_0}(1 + |v|^2)^{\frac{1}{2}} f_0\,dx\,dv - 2\int_{\gamma^-}\chi_T(1 + |v|^2)^{\frac{1}{2}} g\,d\gamma + 2\int_{\Pi_N} e^{-t}\chi_T E_*^k \hat{v} f^{k+1}\,dt\,dx\,dv\,.$$

Lemma 2.5 is also valid with $\hat{v}$ in (2.19). In Sect. 3, we need the relativistic version of DiPerna—Lions's Lemma, see [S2]. Then Lemma 3.1 is true for (RVM) by using the same argument. In Sect. 4, since we get $\sqrt{(1 + |v|^2)} f \in L^1(\Pi)$ for (RVM), the first term in (4.4) goes to the correct limit with $j = \int \hat{v} f\,dv$. So is Lemma 4.1. Using the same method as in Theorem 4.1, Corollary 4.3 and Theorem 4.4, we establish Theorem 5.1. Q.E.D.

Acknowledgements. The author is truly indebted to Prof. W.A. Strauss, his advisor, for his guidance, encouragement and patience. The author also would like to thank Prof. J. Pipher for pointing out the reference [N] for him.

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Communicated by S.-T. Yau