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GLOBAL WEAK SOLUTIONS TO A GENERALIZED HYPERELASTIC-ROD WAVE EQUATION

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Abstract. We consider a generalized hyperelastic-rod wave equation (or generalized Camassa–Holm equation) describing nonlinear dispersive waves in compressible hyperelastic rods. We establish existence of a strongly continuous semigroup of global weak solutions for any initial data from $H^1(\mathbb{R})$. We also prove a "weak equals strong" uniqueness result.

1. Introduction and statement of main results

In recent years the so-called Camassa–Holm equation [3] has caught a great deal of attention. It is a nonlinear dispersive wave equation that takes the form

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial t \partial x^2} + 2\kappa \frac{\partial u}{\partial x} + 3u \frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + u \frac{\partial^3 u}{\partial x^3}, \quad t > 0, \ x \in \mathbb{R}. \tag{1.1}$$

When $\kappa > 0$ this equation models the propagation of unidirectional shallow water waves on a flat bottom, and $u(t, x)$ represents the fluid velocity at time t in the horizontal direction x [3, 21]. The Camassa–Holm equation possesses a bi-Hamiltonian structure (and thus an infinite number of conservation laws) [19, 3] and is completely integrable [3, 1, 11, 6]. Moreover, when $\kappa = 0$ it has an infinite number of solitary wave solutions, called *peakons* due to the discontinuity of their first derivatives at the wave peak, interacting like solitons:

$$u(t, x) = ce^{-|x-ct|}, \qquad c \in \mathbb{R}.$$

The solitary waves with $\kappa > 0$ are smooth, while they become peaked when $\kappa \to 0$. From a mathematical point of view the Camassa–Holm equation is well studied. Local well-posedness results are proved in [7, 20, 23, 29]. It is also known that there exist global solutions for a particular class of initial data and also solutions that blow up in finite time for a large class of initial data [5, 7, 10]. Here blow up means that the slope of the solution becomes unbounded while the solution itself stays bounded. More relevant for the present paper, we recall that existence and uniqueness results for global weak solutions of (1.1) with $\kappa = 0$ have been proved by Constantin and Escher [8], Constantin and Molinet [12], and Xin and Zhang [31, 32], see also Danchin [16, 17].

Here we are interested in the Cauchy problem for the nonlinear equation

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial t \partial x^2} + \frac{\partial}{\partial x}\left(\frac{g(u)}{2}\right) = \gamma \left(2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + u \frac{\partial^3 u}{\partial x^3}\right), \quad t > 0, \ x \in \mathbb{R}, \tag{1.2}$$

where the function $g : \mathbb{R} \to \mathbb{R}$ and the constant $\gamma \in \mathbb{R}$ are given. Observe that if $g(u) = 2\kappa u + 3u^2$ and $\gamma = 1$, then (1.2) is the classical Camassa–Holm equation. With $g(u) = 3u^2$, Dai [14, 13, 15] derived (1.2) as an equation describing finite length, small amplitude radial deformation waves in cylindrical compressible hyperelastic rods, and the equation is often referred to as the hyperelastic-rod wave equation. The constant γ is given in terms of the material constants and the prestress of the rod. We coin (1.2) the *generalized hyperelastic-rod wave equation*.

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In the derivation of the Camassa–Holm equation in the context of the shallow water waves [3, 21], the constant κ is proportional to the square root of water depth. Thus under normal circumstances it is not physical to set $\kappa = 0$. Although strictly speaking one does not have peakons in the shallow water model ($\kappa > 0$), one has them in Dai's model for compressible hyperelastic rods, since in this model $g(u) = 3u^2$ and $\gamma \in \mathbb{R}$. For $\gamma = 0$ and $g(u) = 3u^2$, the equation (1.2) becomes the regularized wave equation describing surface waves in channel [2]. The solutions are global, the equation has an Hamiltonian structure but is not integrable, and its solitary waves are not solitons.

A difference between the Camassa–Holm equation (1.1) (with $\kappa = 0$) and the generalized hyperelastic-rod wave equation (1.2) is that (the slope of) solitary wave solutions to (1.2) can blow up, while they cannot for (1.1). Solitary waves are bounded solutions of (1.2) of the form $u(t, x) = \varphi(x - ct)$, where c is the wave speed. It is not hard to check that $\varphi(\zeta)$, $\zeta = x - ct$, satisfies the ordinary differential equation $(\varphi')^2 = \frac{c\varphi^2 - G(\phi)}{c - \gamma\varphi}$, where $G(\xi) = \int_0^\xi g(\xi)d\xi$. From this expression it is clear that $|\varphi'|$ can become infinite. Notice however that for the Camassa–Holm equation (1.1) (with $\kappa = 0$), for which $G(u) = u^3$, it follows from the above equation that $(\varphi')^2 = \varphi$ (if $\varphi \neq c/\gamma$) and thus any solitary wave (peakon) φ belongs to $W^{1,\infty}$. Notice also that for (1.2) with $g(u) = 2\kappa u + 3u^2$, the above ordinary differential equation becomes $(\varphi')^2 = \phi^2 \frac{(c-\kappa) - \varphi}{c - \gamma\varphi}$, and choosing $\gamma = \frac{c}{c-\kappa}$, $c \neq \kappa$, we find the peakon solution

$$\varphi(\xi) = (c - \kappa)e^{-\sqrt{\frac{c-\kappa}{c}}|\xi|}. \tag{1.3}$$

From a mathematical point of view the generalized hyperelastic-rod wave equation (1.2) is much less studied than (1.1). Recently, Yin [33, 34, 35] (see also Constantin and Escher [9]) proved local well-posedness, global well-posedness for a particular class of initial data, and in particular that smooth solutions blow up in finite time (with a precise estimate of the blow-up time) for a large class of initial data. Lopes [27] proved stability of solitary waves for (1.2) with $\gamma = 1$, while Kalisch [22] studied the stability when $g(u) = 2\kappa u + 3u^3$ and $\gamma \in \mathbb{R}$. Qian and Tang [28] used the bifurcation method to study peakons and periodic cusp waves for (1.2) with $g(u) = 2\kappa u + au^2$, $\kappa, a \in \mathbb{R}$, $\gamma = 1$. When $a \neq 3$, $a > 0$, $\kappa \neq 0$, they found the following two peakon type solutions: $u(t, x) = \frac{6\kappa}{3-a} e^{-\sqrt{\frac{a}{3}}|x - \frac{6\kappa t}{3-a}|}$ and $u(t, x) = \frac{2\kappa}{a+1}(3ae^{-\sqrt{\frac{a}{3}}|x - \frac{2\kappa t}{a+1}|} - 2)$. When $a = 3$ and $\kappa \neq 0$ they also found a peakon type solution of the form $u(t, x) = \frac{3\kappa}{2} e^{-|x - \frac{\kappa t}{2}|} - \kappa$. For (1.2) with $g(u) = 3u^2$, Dai [15] has constructed explicitly a variety of traveling waves, including solitary shock (or peakon like) waves. To give an example, suppose $0 < \gamma < 3$ and pick any constant $c > 0$. Then the following peakon like function is a travelling wave solution: $u(t, x) = \frac{1}{2}(1 - \frac{1}{\gamma})c + \frac{c}{2}(\frac{3}{\gamma} - 1)e^{-\frac{1}{\sqrt{\gamma}}|x - ct - \zeta|}$, where ζ is a particular constant. Dai refers to this as a supersonic solitary shock wave. Although all the above displayed peakon type solutions belong to $W^{1,\infty}$ they do not all belong to $H^1(\mathbb{R})$ (some of them do not decay to zero at $\pm\infty$) and these cannot be encompassed by our theory.

Up to now there are no global existence results for weak solutions to the generalized hyperelastic-rod wave equation (1.2). Here we establish the existence of a global weak solution to (1.2) for any initial function u_0 belonging to $H^1(\mathbb{R})$. Furthermore, we prove the existence of a strongly continuous semigroup, which in particular implies stability of the solution with respect to perturbations of data in the equation as well as perturbation in the initial data. Our approach is based on a vanishing viscosity argument, showing stability of the solution when a regularizing term vanishes. This stability result is new even for the Camassa-Holm equation (1.1). Finally, we prove a "weak equals strong" uniqueness result. Here we follow closely the approach of Xin and Zhang [31] for the Camassa–Holm equation (1.1) with $\kappa = 0$.

Let us be more precise about our results. We shall assume

$$u|_{t=0} = u_0 \in H^1(\mathbb{R}), \tag{1.4}$$

and

$$g \in C^\infty(\mathbb{R}), \qquad g(0) = 0, \qquad \gamma > 0. \tag{1.5}$$

Observe that the case $\gamma = 0$ is much simpler than the one we are considering. Moreover, if $\gamma < 0$, peakons become antipeakons, so we can use a similar argument. The assumption of infinite differentiability of g is made just for convenience. In fact, locally Lipschitz continuity would be sufficient. Define

$$h(\xi) := \frac{1}{2}\big(g(\xi) - \gamma\xi^2\big)$$

for $\xi \in \mathbb{R}$. Formally, equation (1.2) is equivalent to the elliptic-hyperbolic system

$$\frac{\partial u}{\partial t} + \gamma u \frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = 0, \quad -\frac{\partial^2 P}{\partial x^2} + P = h(u) + \frac{\gamma}{2}\left(\frac{\partial u}{\partial x}\right)^2. \tag{1.6}$$

Moreover, since $e^{-|x|}/2$ is the Green's function of the operator $-\frac{\partial^2}{\partial x^2} + 1$, the equation (1.2) is equivalent to the integro-differential system

$$\frac{\partial u}{\partial t} + \gamma u \frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = 0, \quad P(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\left(h(u(t,y)) + \frac{\gamma}{2}\left(\frac{\partial u}{\partial x}(t,y)\right)^2\right)dy. \tag{1.7}$$

Motivated by this, we shall use the following definition of weak solution.

Definition 1.1. *We call $u\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ a weak solution of the Cauchy problem for* (1.2) *if*

(*i*) $u \in C([0,\infty)\times\mathbb{R}) \cap L^\infty\big((0,\infty); H^1(\mathbb{R})\big)$;
(*ii*) *u satisfies* (1.6) *in the sense of distributions;*
(*iii*) $u(0,x) = u_0(x)$, *for every* $x \in \mathbb{R}$;
(*iv*) $\|u(t,\cdot)\|_{H^1(\mathbb{R})} \le \|u_0\|_{H^1(\mathbb{R})}$, *for each* $t > 0$.

If, in addition, there exists a positive constant K_1 depending only on $\|u_0\|_{H^1(\mathbb{R})}$ such that

$$\frac{\partial u}{\partial x}(t,x) \le \frac{2}{\gamma t} + K_1, \qquad (t,x) \in (0,\infty)\times\mathbb{R}, \tag{1.8}$$

then we call u an admissible weak solution of the Cauchy problem for (1.2).

Our existence results are collected in the following theorem:

Theorem 1.2. *There exists a strongly continuous semigroup of solutions associated to the Cauchy problem* (1.2). *More precisely, let*

$$S\colon [0,\infty)\times(0,\infty)\times\mathcal{E}\times H^1(\mathbb{R}) \longrightarrow C([0,\infty)\times\mathbb{R}) \cap L^\infty\big([0,\infty); H^1(\mathbb{R})\big),$$

where

$$\mathcal{E} := \big\{g \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}) \mid g(0) = 0\big\}$$

be such that

(*j*) *for each $u_0 \in H^1(\mathbb{R})$, $\gamma > 0$, $g \in \mathcal{E}$ the map $u(t,x) = S_t(\gamma, g, u_0)(x)$ is an admissible weak solution of* (1.2);
(*jj*) *it is stable with respect to the initial condition in the following sense, if*

$$u_{0,n} \longrightarrow u_0 \ \text{in} \ H^1(\mathbb{R}),\ \gamma_n \longrightarrow \gamma,\ g_n' \longrightarrow g' \ \text{in} \ L^\infty(\mathcal{I}), \tag{1.9}$$

then

$$S(\gamma_n, g_n, u_{0,n}) \longrightarrow S(\gamma, g, u_0) \ \text{in} \ L^\infty([0,T]; H^1(\mathbb{R})), \tag{1.10}$$

for every $\{u_{0,n}\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R})$, $\{\gamma_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{E}$, $u_0 \in H^1(\mathbb{R})$, $\gamma > 0$, $g \in \mathcal{E}$, $T > 0$, where

$$\mathcal{I} := \frac{1}{\sqrt{2}}\left[-\sup_n \|u_{0,n}\|_{H^1(\mathbb{R})}, \sup_n \|u_{0,n}\|_{H^1(\mathbb{R})}\right].$$

Moreover, the following statements hold:

(*k*) *Estimate* (1.8) *is valid with* $K_1 := \sqrt{\frac{2}{\gamma}}\left(2\max_{|\xi|\le\sqrt{2}\|u_0\|_{H^1(\mathbb{R})}} \big|h(\xi)\big| + \frac{\gamma}{2}\|u_0\|^2_{H^1(\mathbb{R})}\right)^{1/2}$.

(*kk*) *There results*

$$\frac{\partial}{\partial x} S(\gamma, g, u_0) \in L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{R}), \tag{1.11}$$

for each $1 \le p < 3$.

(kkk) *The following identity holds in the sense of distributions on* $[0,\infty)\times\mathbb{R}$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\left[u^2+q^2\right]\right)+\frac{\partial}{\partial x}\left(u\left[\frac{\gamma}{2}q^2+P\right]+\frac{\gamma}{3}u^3-H(u)\right)=-\mu, \tag{1.12}$$

where $u=S(\gamma,g,u_0)$, $q=\frac{\partial}{\partial x}S(\gamma,g,u_0)$, $H'=h$, *the defect measure* μ *is a nonnegative Radon measure such that as* $R\to\infty$ *there holds* $Rq\,(q+R)\,\chi_{(-\infty,-R)}(q)\overset{\star}{\rightharpoonup}\mu$ *in the sense of measures and* $\mu([0,\infty)\times\mathbb{R})\leq\frac{1}{2}\|u_0\|_{H^1(\mathbb{R})}$.

We stress that the existence of a strongly continuous semigroup is new, even for the Camassa–Holm equation itself. In particular, this includes the stability of the solution with respect to perturbations in the initial data and the coefficients in the equation.

As in Xin and Zhang [31, 32] and their study of the Camassa–Holm equation (1.1) with $\kappa=0$, we prove existence of a global weak solution by establishing convergence as $\varepsilon\to0$ of a sequence of smooth viscous approximate solutions u_ε (see equation (2.1) below). Regarding the limiting process there is a an interesting mathematical issue: we need to prove that the derivative $q_\varepsilon=\partial u_\varepsilon/\partial x$, which a priori is only weakly compact, in fact converges strongly (along a subsequence). Strong convergence of q_ε is needed if we want to send ε to zero in the viscous problem and recover (1.2). To improve the weak convergence of q_ε to strong convergence we follow [31] closely when using renormalization theory for linear transport equations with non-smooth coefficients. The idea of renormalization goes back to DiPerna and Lions [18], and it has been developed further and applied by many authors (see Lions [25, 26], Xin and Zhang [31], and the references given therein for relevant information). In the process of improving weak convergence to strong convergence, the higher integrability estimate (1.11) for q_ε is crucial. It ensures that the weak limit of q_ε^2 does not contain singular measures (there are no concentration effects).

Regarding the optimality of (1.11), one should keep in mind that when a solution u blows up (necessarily in the sense that $|\partial u/\partial x|\to\infty$), say at $x=0$, then u must behave like $x^{2/3}$ and $\partial u/\partial x$ like $x^{-1/3}$, since $u(t,\cdot)\in H^1(\mathbb{R})$, in which case $\partial u/\partial x$ belongs to L^p_{loc} if and only if $1\leq p<3$.

Denote by u an (admissible) weak solution. If the associated defect measure μ defined in (1.12) vanishes, then we call u an *energy conservative (admissible) weak solution.* Xin and Zhang [32] proved a "weak equals strong" uniqueness result for energy conservative admissible weak solutions of the Camassa–Holm equation (1.1) when $\kappa=0$. Their result also contains the uniqueness result of Constantin and Molinet [12] as a special case. Herein we adapt the arguments of Xin and Zhang to prove a "weak equals strong" uniqueness result for the generalized hyperelastic-rod wave equation.

Theorem 1.3. *Suppose there exists a function* u *such that* (i), (ii), *and* (iii) *of Definition* 1.1 *hold and that there exists a function* $\beta\in L^2([0,T))$ *for all* $T>0$ *such that* $\left\|\frac{\partial u}{\partial x}(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\leq\beta(t)$ *for any* $t\geq0$. *Then energy conservative admissible weak solutions are unique.*

Whenever a sufficiently regular solution to (1.2) can be found (see [9, 15, 28, 33, 34, 35] for some situations where this happens), then Theorem 1.3 ensures that this solution is unique in the class energy conservative admissible weak solutions. Note that peakons are "sufficiently regular". For example, the peakon solution (1.3) is covered by our theory. One should compare Theorem 1.3 with the uniqueness/stability assertion in Theorem 1.2, which states that there is uniqueness in the class of vanishing viscosity solutions.

In passing, we mention that it is apparently not easy to prove existence and uniqueness results for (1.2) by adapting the methods in [8, 12] for the Camassa–Holm equation, which are based on studying the equation for the "vorticity" $m:=\left(1-\frac{\partial^2}{\partial x^2}\right)u$. In the present context the equation for m reads

$$\frac{\partial m}{\partial t}+\gamma u\frac{\partial m}{\partial x}+2\gamma\frac{\partial u}{\partial x}m=-\frac{1}{2}\frac{\partial}{\partial x}\left(g(u)-3\gamma u^2\right). \tag{1.13}$$

In the case of the Camassa–Holm equation (that is, $g(u)=3u^2$ and $\gamma=1$), the right-hand side of (1.13) vanishes, and assuming that $m|_{t=0}$ is a bounded nonnegative measure it is not difficult to see that $m(t,\cdot)\in L^1$ remains nonnegative at later times and consequently one can bound $\partial u/\partial x$

in L^∞ and $\partial^2 u/\partial x^2$ in L^1. Using these bounds one can in fact prove the existence and uniqueness of an energy conservative weak solution [8, 12]. In the general case ($g(u)$ is not equal to $3\gamma u^2$) it seems difficult to implement this strategy for proving existence and uniqueness results, and this fact has motivated us to use the "weak convergence" approach.

The remaining part of this paper is organized as follows: Section 2 is devoted to stating the viscous problem and a corresponding well-posedness result. In Sections 3 and 4 we establish respectively an Oleinik type estimate and a higher integrability estimate for the viscous approximants. Section 5 is devoted to proving basic compactness properties for the viscous approximants. Strong compactness of the derivative of the viscous approximants is obtained in Section 6, where also an existence result for (1.2) is stated. In Section 7 we prove the uniqueness of the vanishing viscosity limit, this defines a semigroup of solutions as stated in Theorem 1.2. In Section 8 we prove the continuity properties of the semigroup. Finally, in Section 9 we prove the uniqueness statement in Theorem 1.3.

2. Viscous approximants: Existence and energy estimate

We will prove existence of a weak solution to the Cauchy problem for (1.2) by proving compactness of a sequence of smooth functions $\{u_\varepsilon\}_{\varepsilon>0}$ solving the following viscous problems (see [4, Theorem 2.3]):

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + \gamma u_\varepsilon \dfrac{\partial u_\varepsilon}{\partial x} + \dfrac{\partial P_\varepsilon}{\partial x} = \varepsilon \dfrac{\partial^2 u_\varepsilon}{\partial x^2}, & t>0,\ x\in\mathbb{R},\\ -\dfrac{\partial^2 P_\varepsilon}{\partial x^2} + P_\varepsilon = h(u_\varepsilon) + \dfrac{\gamma}{2}\left(\dfrac{\partial u_\varepsilon}{\partial x}\right)^2, & t>0,\ x\in\mathbb{R},\\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x\in\mathbb{R}. \end{cases} \tag{2.1}$$

We shall assume that

$$\|u_{\varepsilon,0}\|_{H^1(\mathbb{R})} \le \|u_0\|_{H^1(\mathbb{R})},\ \varepsilon>0, \quad \text{and} \quad u_{\varepsilon,0}\to u_0 \text{ in } H^1(\mathbb{R}). \tag{2.2}$$

The starting point of our analysis is the following well-posedness result for (2.1).

Theorem 2.1. *Assume* (1.4) *and* (2.2). *Let $\varepsilon>0$, $u_{\varepsilon,0}\in H^\ell(\mathbb{R})$ and $\ell\ge 2$. Then there exists a unique solution $u_\varepsilon \in C(\mathbb{R}; H^\ell(\mathbb{R}))$ to the Cauchy problem* (2.1). *Moreover, for each $t\ge 0$,*

$$\begin{aligned} &\int_{\mathbb{R}} \left(u_\varepsilon^2 + \left(\frac{\partial u_\varepsilon}{\partial x}\right)^2\right)(t,x)dx \\ &\quad + 2\varepsilon \int_0^t \int_{\mathbb{R}} \left(\left(\frac{\partial u_\varepsilon}{\partial x}\right)^2 + \left(\frac{\partial^2 u_\varepsilon}{\partial x^2}\right)^2\right)(s,x)dxds = \|u_{\varepsilon,0}\|^2_{H^1(\mathbb{R})}, \end{aligned} \tag{2.3}$$

or

$$\|u_\varepsilon(t,\cdot)\|^2_{H^1(\mathbb{R})} + 2\varepsilon\int_0^t \|q_\varepsilon(s,\cdot)\|^2_{H^1(\mathbb{R})}\,ds = \|u_{\varepsilon,0}\|^2_{H^1(\mathbb{R})}.$$

Remark 2.2. *Due to* [24, Theorem 8.5], (2.2) *and* (2.3), *we have for each $t\ge 0$*

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2}}\,\|u_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R})} \le \frac{1}{\sqrt{2}}\|u_0\|_{H^1(\mathbb{R})}. \tag{2.4}$$

Proof of Theorem 2.1. From Theorem 2.3 in [4] we infer that (2.1) has a solution $u_\varepsilon \in C(\mathbb{R}; H^\ell(\mathbb{R}))$. Define

$$q_\varepsilon(t,x) := \frac{\partial u_\varepsilon}{\partial x}(t,x).$$

By (2.1), $q_\varepsilon = q_\varepsilon(t,x)$ is the solution of

$$\frac{\partial q_\varepsilon}{\partial t} + \gamma u_\varepsilon \frac{\partial q_\varepsilon}{\partial x} - \varepsilon\frac{\partial^2 q_\varepsilon}{\partial x^2} + \frac{\gamma}{2}q_\varepsilon^2 = h(u_\varepsilon) - P_\varepsilon, \quad q_\varepsilon(0,x) = \frac{\partial u_{\varepsilon,0}}{\partial x}(x), \tag{2.5}$$

for $t > 0$ and $x \in \mathbb{R}$. Multiply (2.1) by u_ε, (2.5) by q_ε, and add the resulting equations. After rearranging a bit, we derive the conservation law

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\left[u_\varepsilon^2+q_\varepsilon^2\right]\right)+\frac{\partial}{\partial x}\left(u_\varepsilon\left[\frac{\gamma}{2}q_\varepsilon^2+P_\varepsilon\right]+\frac{\gamma}{3}u^3-H(u)\right)=\frac{\varepsilon}{2}(u_\varepsilon^2+q_\varepsilon^2)_{xx}-\varepsilon q_\varepsilon^2-\varepsilon\left(\frac{\partial q_\varepsilon}{\partial x}\right)^2,$$

where $H' = h$. From this (2.3) follows easily. □

3. Viscous approximants: Oleinik type estimate

Lemma 3.1. *For each $t > 0$ and $x \in \mathbb{R}$,*

$$\frac{\partial u_\varepsilon}{\partial x}(t,x) \le \frac{2}{\gamma t} + C_2, \tag{3.1}$$

where $u_\varepsilon = u_\varepsilon(t,x)$ is the unique solution of (2.1), *and*

$$C_2 := \sqrt{\frac{2}{\gamma}}\left(2\max_{|\xi|\le\sqrt{2}\|u_0\|_{H^1(\mathbb{R})}}\left|h(\xi)\right|+\frac{\gamma}{2}\|u_0\|^2_{H^1(\mathbb{R})}\right)^{1/2}.$$

Proof. From (2.4),

$$\left\|h(u_\varepsilon)\right\|_{L^\infty([0,\infty)\times\mathbb{R})} \le \max_{|\xi|\le\sqrt{2}\|u_0\|_{H^1(\mathbb{R})}}\left|h(\xi)\right| := L_1 < \infty. \tag{3.2}$$

Moreover, since

$$\int_{\mathbb{R}} e^{-|x-y|}dy = 2, \qquad x \in \mathbb{R}, \tag{3.3}$$

again using (2.4), for each $t \ge 0$ and $x \in \mathbb{R}$,

$$|P_\varepsilon(t,x)| \le L_1 + \frac{\gamma}{4}\left\|\frac{\partial u_\varepsilon}{\partial x}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le L_1 + \frac{\gamma}{4}\|u_0\|^2_{H^1(\mathbb{R})} := L_2.$$

So, denoting $L := L_1 + L_2$, we have, from (2.5),

$$\frac{\partial q_\varepsilon}{\partial t} + \gamma u_\varepsilon\frac{\partial q_\varepsilon}{\partial x} - \varepsilon\frac{\partial^2 q_\varepsilon}{\partial x^2} + \frac{\gamma}{2}q_\varepsilon^2 \le L. \tag{3.4}$$

Let $f = f(t)$ be the solution of

$$\frac{df}{dt} + \frac{\gamma}{2}f^2 = L, \quad t > 0, \qquad f(0) = \left\|\frac{\partial u_{\varepsilon,0}}{\partial x}\right\|_{L^\infty(\mathbb{R})}. \tag{3.5}$$

Since, by (2.4) and (3.4), $f = f(t)$ is a super-solution of the parabolic initial value problem (2.5), due to the comparison principle for parabolic equations, we get

$$q_\varepsilon(t,x) \le f(t), \qquad t \ge 0,\ x \in \mathbb{R}. \tag{3.6}$$

Finally, consider the map $F(t) := \frac{2}{\gamma t}+\sqrt{\frac{2}{\gamma}L}$, $t > 0$. Observe that $\frac{dF}{dt}(t)+\frac{\gamma}{2}F^2(t)-L = \frac{2\sqrt{2L/\gamma}}{t} > 0$, for any $t > 0$, so that $F = F(t)$ is a super-solution of (3.5). Due to the comparison principle for ordinary differential equations, we get $f(t) \le F(t)$ for all $t > 0$. Therefore, by this and (3.6), the estimate (3.1) is proved. □

4. Viscous approximants: Higher integrability estimate

Lemma 4.1. *Let $0 < \alpha < 1$, $T > 0$, and $a, b \in \mathbb{R}$, $a < b$. Then there exists a positive constant C_3 depending only on $\|u_0\|_{H^1(\mathbb{R})}$, α, $T > 0$, a and b, but independent of ε, such that*

$$\int_0^T\int_a^b\left|\frac{\partial u_\varepsilon}{\partial x}(t,x)\right|^{2+\alpha}dtdx \le C_3, \tag{4.1}$$

where $u_\varepsilon = u_\varepsilon(t,x)$ is the unique solution of (2.1).

Proof. The proof is a variant of the proof found in Xin and Zhang [31]. Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function such that

$$0 \le \chi \le 1, \qquad \chi(x) = \begin{cases} 1, & \text{if } x \in [a,b], \\ 0, & \text{if } x \in (-\infty, a-1] \cup [b+1, \infty). \end{cases}$$

Consider also the map $\theta(\xi) := \xi\big(|\xi|+1\big)^{\alpha}$, $\xi \in \mathbb{R}$, and observe that, since $0 < \alpha < 1$,

$$\begin{aligned} \theta'(\xi) &= \big((\alpha+1)|\xi|+1\big)\big(|\xi|+1\big)^{\alpha-1}, \\ \theta''(\xi) &= \alpha \operatorname{sign}(\xi)\,\big(|\xi|+1\big)^{\alpha-2}\big((\alpha+1)|\xi|+2\big) \\ &= \alpha(\alpha+1)\operatorname{sign}(\xi)\,\big(|\xi|+1\big)^{\alpha-1} + (1-\alpha)\alpha\operatorname{sign}(\xi)\,\big(|\xi|+1\big)^{\alpha-2}, \end{aligned}$$

$$\big|\theta(\xi)\big| \le |\xi|^{\alpha+1} + |\xi|, \quad \big|\theta'(\xi)\big| \le (\alpha+1)|\xi|+1, \quad \big|\theta''(\xi)\big| \le 2\alpha, \tag{4.2}$$

$$\xi\theta(\xi) - \frac{1}{2}\xi^2\theta'(\xi) = \frac{1-\alpha}{2}\xi^2\big(|\xi|+1\big)^{\alpha} + \frac{\alpha}{2}\xi^2\big(|\xi|+1\big)^{\alpha-1} \ge \frac{1-\alpha}{2}\xi^2\big(|\xi|+1\big)^{\alpha}. \tag{4.3}$$

Multiplying (2.5) by $\chi\theta'(q_\varepsilon)$, using the chain rule, and integrating over $\Pi_T := [0,T]\times\mathbb{R}$, we get

$$\begin{aligned} &\int_{\Pi_T} \gamma\chi(x) q_\varepsilon \theta(q_\varepsilon)dtdx - \frac{\gamma}{2}\int_{\Pi_T} q_\varepsilon^2 \chi(x)\theta'(q_\varepsilon)dtdx \\ &\quad = \int_{\mathbb{R}} \chi(x)\,\big(\theta\big(q_\varepsilon(T,x)\big) - \theta\big(q_\varepsilon(0,x)\big)\big)\,dx - \int_{\Pi_T} \gamma u_\varepsilon \chi'(x)\theta(q_\varepsilon)dtdx \\ &\quad + \varepsilon\int_{\Pi_T} \frac{\partial q_\varepsilon}{\partial x}\chi'(x)\theta'(q_\varepsilon)dtdx + \varepsilon\int_{\Pi_T}\left(\frac{\partial q_\varepsilon}{\partial x}\right)^2 \chi(x)\theta''(q_\varepsilon)dtdx \\ &\quad - \int_{\Pi_T} \left(h(u_\varepsilon) - P_\varepsilon\right)\chi(x)\theta'(q_\varepsilon)dtdx. \end{aligned} \tag{4.4}$$

Observe that, by (4.3),

$$\begin{aligned} \int_{\Pi_T} \gamma\chi(x) q_\varepsilon\theta(q_\varepsilon)dtdx - \frac{\gamma}{2}\int_{\Pi_T} q_\varepsilon^2\chi(x)\theta'(q_\varepsilon)dtdx &= \int_{\Pi_T} \gamma\chi(x)\Big(q_\varepsilon\theta(q_\varepsilon) - \frac{1}{2}q_\varepsilon^2\theta'(q_\varepsilon)\Big)dtdx \\ &\ge \frac{\gamma(1-\alpha)}{2}\int_{\Pi_T} \chi(x) q_\varepsilon^2\big(|q_\varepsilon|+1\big)^{\alpha}dtdx. \end{aligned} \tag{4.5}$$

Let $t \ge 0$, since $0 < \alpha < 1$, using the Hölder inequality, (2.4) and the first part of (4.2),

$$\begin{aligned} \left|\int_{\mathbb{R}} \chi(x)\theta(q_\varepsilon)dx\right| &\le \int_{\mathbb{R}} \chi(x)\,\Big(|q_\varepsilon|^{\alpha+1} + |q_\varepsilon|\Big)\,dx \\ &\le \|\chi\|_{L^{2/(1-\alpha)}(\mathbb{R})}\|q_\varepsilon(t,\cdot)\|^{\alpha+1}_{L^2(\mathbb{R})} + \|\chi\|_{L^2(\mathbb{R})}\|q_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \\ &\le (b-a+2)^{(1-\alpha)/2}\|u_0\|^{\alpha+1}_{H^1(\mathbb{R})} + (b-a+2)^{1/2}\|u_0\|_{H^1(\mathbb{R})}, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \left|\int_{\Pi_T} \gamma u_\varepsilon\chi'(x)\theta(q_\varepsilon)dtdx\right| &\le \int_{\Pi_T} \gamma|u_\varepsilon||\chi'(x)|\,\big(|q_\varepsilon|^{\alpha+1} + |q_\varepsilon|\big)\,dtdx \\ &\le \int_{\Pi_T} \gamma\|u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}|\chi'(x)|\,\big(|q_\varepsilon|^{\alpha+1} + |q_\varepsilon|\big)\,dtdx \\ &\le \gamma\frac{\|u_0\|_{H^1(\mathbb{R})}}{\sqrt{2}}\int_0^T \big(\|\chi'\|_{L^{2/(1-\alpha)}(\mathbb{R})}\|q_\varepsilon(t,\cdot)\|^{\alpha+1}_{L^2(\mathbb{R})} \\ &\qquad + \|\chi'\|_{L^2(\mathbb{R})}\|q_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\big)dt \\ &\le \gamma T\frac{\|u_0\|_{H^1(\mathbb{R})}}{\sqrt{2}}\big(\|\chi'\|_{L^{2/(1-\alpha)}(\mathbb{R})}\|u_0\|^{\alpha+1}_{H^1(\mathbb{R})} + \|\chi'\|_{L^2(\mathbb{R})}\|u_0\|_{H^1(\mathbb{R})}\big). \end{aligned} \tag{4.7}$$

Moreover, observe that

$$\varepsilon\int_{\Pi_T} \frac{\partial q_\varepsilon}{\partial x}\chi'(x)\theta'(q_\varepsilon)dtdx = -\varepsilon\int_{\Pi_T} \theta(q_\varepsilon)\chi''(x)dtdx,$$

so, again by the Hölder inequality, (2.4) and the first part of (4.2),

$$
\begin{aligned}
\left|\varepsilon\int_{\Pi_T}\frac{\partial q_\varepsilon}{\partial x}\chi'(x)\theta(q_\varepsilon)dtdx\right| &\le \varepsilon\int_{\Pi_T}|\theta(q_\varepsilon)||\chi''(x)|dtdx\\
&\le \varepsilon\int_{\Pi_T}\left(|q_\varepsilon|^{\alpha+1}+|q_\varepsilon|\right)|\chi''(x)|dtdx \qquad (4.8)\\
&\le \varepsilon\int_0^T\left(\|\chi''\|_{L^{2/(1-\alpha)}(\mathbb{R})}\|q_\varepsilon(t,\cdot)\|^{\alpha+1}_{L^2(\mathbb{R})}+\|\chi''\|_{L^2(\mathbb{R})}\|q_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\right)dt\\
&\le \varepsilon T\left(\|\chi''\|_{L^{2/(1-\alpha)}(\mathbb{R})}\|u_0\|^{\alpha+1}_{H^1(\mathbb{R})}+\|\chi''\|_{L^2(\mathbb{R})}\|u_0\|_{H^1(\mathbb{R})}\right).
\end{aligned}
$$

Since $0<\alpha<1$, using (2.3) and the third part of (4.2),

$$
\varepsilon\left|\int_{\Pi_T}\left(\frac{\partial q_\varepsilon}{\partial x}\right)^2\chi(x)\theta''(q_\varepsilon)dtdx\right|\le 2\alpha\varepsilon\int_{\Pi_T}\left(\frac{\partial q_\varepsilon}{\partial x}\right)^2dtdx\le\alpha\|u_0\|^2_{H^1(\mathbb{R})}. \tag{4.9}
$$

As we showed in the proof of Lemma 3.1, there exists a constant $L>0$ depending only on $\|u_0\|_{H^1(\mathbb{R})}$ such that $\left\|h(u_\varepsilon)-P_\varepsilon\right\|_{L^\infty([0,\infty)\times\mathbb{R})}\le L$, so, since $0<\alpha<1$, using the second part of (4.2),

$$
\begin{aligned}
\left|\int_{\Pi_T}\left(h(u_\varepsilon)-P_\varepsilon\right)\chi(x)\theta'(q_\varepsilon)dtdx\right| &\le L\int_{\Pi_T}\chi(x)\left((\alpha+1)|q_\varepsilon|+1\right)dtdx \qquad (4.10)\\
&\le L\int_0^T\left((\alpha+1)\|\chi\|_{L^2(\mathbb{R})}\|q_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}+\|\chi\|_{L^1(\mathbb{R})}\right)dt\\
&\le LT\left((\alpha+1)(b-a+2)^{1/2}\|u_0\|_{H^1(\mathbb{R})}+(b-a+2)\right).
\end{aligned}
$$

From (4.4), (4.5), (4.6), (4.7), (4.8), (4.9) and (4.10), there exists a constant $c>0$ depending only on $\|u_0\|_{H^1(\mathbb{R})}$, α, $T>0$, a, and b, but independent of ε, such that

$$
\frac{\gamma(1-\alpha)}{2}\int_{\Pi_T}|q_\varepsilon|^2\chi(x)\left(|q_\varepsilon|+1\right)^\alpha dtdx\le c. \tag{4.11}
$$

Then

$$
\int_0^T\int_a^b\left|\frac{\partial u_\varepsilon}{\partial x}(t,x)\right|^{2+\alpha}dtdx\le\int_{\Pi_T}|q_\varepsilon|\chi(x)\left(|q_\varepsilon|+1\right)^{\alpha+1}dtdx\le\frac{2c}{\gamma(1-\alpha)},
$$

hence estimate (4.1) is proved. □

5. Viscous approximants: Basic compactness

Lemma 5.1. *There exists a positive constant C_4 depending only on $\|u_0\|_{H^1(\mathbb{R})}$ such that*

$$
\|P_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})},\ \|P_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})},\ \left\|\frac{\partial P_\varepsilon}{\partial x}(t,\cdot)\right\|_{L^\infty(\mathbb{R})},\ \left\|\frac{\partial P_\varepsilon}{\partial x}(t,\cdot)\right\|_{L^2(\mathbb{R})}\le C_4, \tag{5.1}
$$

where $u_\varepsilon=u_\varepsilon(t,x)$ is the unique solution of (2.1). In particular, $\{P_\varepsilon\}_\varepsilon$ is uniformly bounded in $L^\infty([0,\infty);W^{1,\infty}(\mathbb{R}))$ and $L^\infty([0,\infty);H^1(\mathbb{R}))$.

Proof. Define

$$
P_{1,\varepsilon}(t,x):=\frac{\gamma}{4}\int_{\mathbb{R}}e^{-|x-y|}q_\varepsilon^2dy,\qquad P_{2,\varepsilon}(t,x):=\frac{1}{2}\int_{\mathbb{R}}e^{-|x-y|}h\big(u_\varepsilon(t,y)\big)dy, \tag{5.2}
$$

and notice that $P_\varepsilon=P_{1,\varepsilon}+P_{2,\varepsilon}$. By (2.4) and (3.3),

$$
|P_{1,\varepsilon}(t,x)|\le\frac{\gamma}{4}\|u_\varepsilon(t,\cdot)\|^2_{H^1(\mathbb{R})}\le\frac{\gamma}{4}\|u_0\|^2_{H^1(\mathbb{R})}, \tag{5.3}
$$

$$
|P_{2,\varepsilon}(t,x)|\le\max_{|\xi|\le\|u_0\|_{H^1(\mathbb{R})}/\sqrt{2}}\big|h(\xi)\big|. \tag{5.4}
$$

Moreover, using (3.3) and the Tonelli theorem,

$$\int_{\mathbb{R}} |P_{1,\varepsilon}(t,x)|dx \le \frac{\gamma}{2}\|u_\varepsilon(t,\cdot)\|^2_{H^1(\mathbb{R})} \le \frac{\gamma}{2}\|u_0\|^2_{H^1(\mathbb{R})}. \tag{5.5}$$

From (3.3), (5.3), (5.5) and the Hölder inequality,

$$\int_{\mathbb{R}} |P_{1,\varepsilon}(t,x)|^2 dx \le \|P_{1,\varepsilon}\|_{L^\infty([0,\infty)\times\mathbb{R})}\|P_{1,\varepsilon}(t,\cdot)\|_{L^1(\mathbb{R})} \le \frac{\gamma^2}{8}\|u_0\|^4_{H^1(\mathbb{R})},$$

so that

$$\|P_{1,\varepsilon}(t,\cdot)\|_{L^2(\mathbb{R})} \le \frac{\gamma}{2\sqrt{2}}\|u_0\|^2_{H^1(\mathbb{R})}. \tag{5.6}$$

Using (1.4), (2.4), (3.3), the Tonelli theorem and the Hölder inequality,

$$\begin{aligned} \int_{\mathbb{R}} |P_{2,\varepsilon}(t,x)|^2 dx &\le \frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-|x-y|}dx\right)(h(u_\varepsilon(t,y)))^2\,dy \\ &\le \left(\max_{|\xi|\le\|u_0\|_{H^1(\mathbb{R})}/\sqrt{2}} (h'(\xi))^2\right)\int_{\mathbb{R}} u_\varepsilon^2(t,y)dy \\ &\le \left(\max_{|\xi|\le\|u_0\|_{H^1(\mathbb{R})}/\sqrt{2}} (h'(\xi))^2\right)\|u_0\|^2_{H^1(\mathbb{R})}. \end{aligned} \tag{5.7}$$

Finally, observing

$$\begin{aligned} \frac{\partial P_{1,\varepsilon}}{\partial x}(t,x) &= \frac{\gamma}{4}\int_{\mathbb{R}} \mathrm{sign}\,(y-x)\,e^{-|x-y|}\,(q_\varepsilon(t,y))^2\,dy, \\ \frac{\partial P_{2,\varepsilon}}{\partial x}(t,x) &= \frac{1}{2}\int_{\mathbb{R}} \mathrm{sign}\,(y-x)\,e^{-|x-y|}h\big(u_\varepsilon(t,y)\big)dy, \end{aligned}$$

and recalling $P_\varepsilon = P_{1,\varepsilon}+P_{2,\varepsilon}$, the claim is a direct consequence of (5.3), (5.4), (5.6), and (5.7). □

Lemma 5.2. *There exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ tending to zero and a function $u \in L^\infty([0,\infty);H^1(\mathbb{R}))\cap H^1([0,T]\times\mathbb{R})$, for each $T \ge 0$, such that*

$$u_{\varepsilon_j} \rightharpoonup u \quad \text{in } H^1([0,T]\times\mathbb{R}), \text{ for each } T\ge 0, \tag{5.8}$$

$$u_{\varepsilon_j} \to u \quad \text{in } L^\infty_{\mathrm{loc}}([0,\infty)\times\mathbb{R}), \tag{5.9}$$

where $u_\varepsilon = u_\varepsilon(t,x)$ is the unique solution of (2.1).

Proof. Fix $T > 0$. Observe that, from (2.1), $\frac{\partial u_\varepsilon}{\partial t} = \varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2} - \gamma u_\varepsilon\frac{\partial u_\varepsilon}{\partial x} - \frac{\partial P_\varepsilon}{\partial x}$, so, by (2.4), (2.3), (5.1), and the Hölder inequality,

$$\left\|\frac{\partial u_\varepsilon}{\partial t}\right\|_{L^2([0,T]\times\mathbb{R})} \le \sqrt{\frac{\varepsilon}{2}}\|u_0\|_{L^\infty(\mathbb{R})} + \frac{\gamma\sqrt{T}}{\sqrt{2}}\|u_0\|^2_{L^\infty(\mathbb{R})} + C_4\sqrt{T}. \tag{5.10}$$

Hence $\{u_\varepsilon\}$ is uniformly bounded in $H^1([0,T]\times\mathbb{R})\cap L^\infty([0,\infty);H^1(\mathbb{R}))$, and (5.8) follows.

Observe that, for each $0 \le s,\, t \le T$,

$$\|u_\varepsilon(t,\cdot)-u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}}\Big(\int_s^t \frac{\partial u_\varepsilon}{\partial t}(\tau,x)d\tau\Big)^2 dx \le \sqrt{|t-s|}\int_{\Pi(T)}\Big(\frac{\partial u_\varepsilon}{\partial t}(\tau,x)\Big)^2 d\tau dx.$$

Moreover, $\{u_\varepsilon\}$ is uniformly bounded in $L^\infty([0,T];H^1(\mathbb{R}))$ and $H^1(\mathbb{R}) \subset\subset L^\infty_{\mathrm{loc}}(\mathbb{R}) \subset L^2_{\mathrm{loc}}(\mathbb{R})$, then (5.9) is consequence of [30, Theorem 5]. □

Lemma 5.3. *The sequence $\{P_\varepsilon\}_\varepsilon$ is uniformly bounded in $W^{1,1}_{\mathrm{loc}}([0,\infty)\times\mathbb{R})$. In particular, there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ tending to zero and a function $P\in L^\infty([0,\infty);W^{1,\infty}(\mathbb{R}))$ such that for each $1 < p < \infty$*

$$P_{\varepsilon_j} \to P \quad \textit{strongly in } L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{R}). \tag{5.11}$$

Proof. We begin by proving that $\left\{\frac{\partial P_\varepsilon}{\partial t}\right\}_\varepsilon$ is uniformly bounded in $L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R})$. Fix $T>0$. We claim that

$$\left\{\frac{\partial P_{1,\varepsilon}}{\partial t}\right\}_\varepsilon \text{ is uniformly bounded in } L^1([0,T]\times\mathbb{R}), \tag{5.12}$$

$$\left\{\frac{\partial P_{2,\varepsilon}}{\partial t}\right\}_\varepsilon \text{ is uniformly bounded in } L^2([0,T]\times\mathbb{R}), \tag{5.13}$$

where $P_{1,\varepsilon}$ and $P_{2,\varepsilon}$ are defined in (5.2). We begin by proving (5.12). Observe that, from (2.5),

$$\begin{aligned}\frac{\partial P_{1,\varepsilon}}{\partial t}(t,x) &= \frac{\gamma}{2}\int_{\mathbb{R}} e^{-|x-y|} q_\varepsilon \frac{\partial q_\varepsilon}{\partial t}dy \\ &= \frac{\gamma}{2}\int_{\mathbb{R}} e^{-|x-y|}\Big(-\gamma q_\varepsilon u_\varepsilon \frac{\partial q_\varepsilon}{\partial x} + \varepsilon q_\varepsilon \frac{\partial^2 q_\varepsilon}{\partial x^2} - \frac{\gamma}{2}q_\varepsilon^3 + q_\varepsilon\left(h(u_\varepsilon)-P_\varepsilon\right)\Big)dy.\end{aligned} \tag{5.14}$$

Using $\frac{\gamma}{2}\frac{\partial}{\partial x}(u_\varepsilon q_\varepsilon^2) = \frac{\gamma}{2}q_\varepsilon^3 + \gamma q_\varepsilon u_\varepsilon \frac{\partial q_\varepsilon}{\partial x}$, $\frac{\partial}{\partial x}\Big(q_\varepsilon\frac{\partial q_\varepsilon}{\partial x}\Big) = q_\varepsilon\frac{\partial^2 q_\varepsilon}{\partial x^2} + \Big(\frac{\partial q_\varepsilon}{\partial x}\Big)^2$, (5.14), and integration by parts, we get

$$\begin{aligned}\frac{\partial P_{1,\varepsilon}}{\partial t}(t,x) &= \frac{\gamma}{4}\int_{\mathbb{R}} e^{-|x-y|}\left(-\frac{\gamma}{2}\frac{\partial}{\partial x}(u_\varepsilon q_\varepsilon^2) + \varepsilon\frac{\partial}{\partial x}\Big(q_\varepsilon\frac{\partial q_\varepsilon}{\partial x}\Big) - \varepsilon\left(\frac{\partial q_\varepsilon}{\partial x}\right)^2 + q_\varepsilon\left(h(u_\varepsilon)-P_\varepsilon\right)\right)dy \\ &= \frac{\gamma}{4}\int_{\mathbb{R}} e^{-|x-y|}\left(\operatorname{sign}(y-x)\left[\frac{\gamma}{2}u_\varepsilon q_\varepsilon^2 - \varepsilon q_\varepsilon\frac{\partial q_\varepsilon}{\partial x}\right] - \varepsilon\left(\frac{\partial q_\varepsilon}{\partial x}\right)^2 + q_\varepsilon\left(h(u_\varepsilon)-P_\varepsilon\right)\right)dy.\end{aligned}$$

Using (1.4), (2.3), (2.4), (5.1), the Tonelli theorem, and the Hölder inequality,

$$\int_{\mathbb{R}\times\mathbb{R}} e^{-|x-y|}|u_\varepsilon| q_\varepsilon^2 dxdy \le \sqrt{2}\|u_0\|_{H^1(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{H^1(\mathbb{R})} \le \sqrt{2}\|u_0\|^3_{H^1(\mathbb{R})},$$

$$\begin{aligned}\varepsilon\int_{\Pi_T\times\mathbb{R}} e^{-|x-y|}|q_\varepsilon|\left|\frac{\partial q_\varepsilon}{\partial x}\right| dtdxdy &\le \varepsilon\int_0^T \|u_\varepsilon(t,\cdot)\|^2_{H^1(\mathbb{R})}dt + \varepsilon\int_0^T\left\|\frac{\partial u_\varepsilon}{\partial x}(t,\cdot)\right\|^2_{H^1(\mathbb{R})}dt \\ &\le \left(\varepsilon T + \frac{1}{2}\right)\|u_0\|^2_{H^1(\mathbb{R})},\end{aligned}$$

$$\varepsilon\int_{\Pi_T\times\mathbb{R}} e^{-|x-y|}\left(\frac{\partial q_\varepsilon}{\partial x}\right)^2 dtdxdy \le 2\varepsilon\int_0^T\left\|\frac{\partial u_\varepsilon}{\partial x}(t,\cdot)\right\|^2_{H^1(\mathbb{R})}dt \le \|u_0\|^2_{H^1(\mathbb{R})},$$

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}} e^{-|x-y|}|q_\varepsilon||h(u_\varepsilon)|dxdy &\le \int_{\mathbb{R}} q_\varepsilon^2 dy + \max_{|\xi|\le\|u_0\|_{H^1(\mathbb{R})}/\sqrt{2}}\left(h'(\xi)\right)^2\int_{\mathbb{R}} u_\varepsilon^2 dy \\ &\le \left(1 + \max_{|\xi|\le\|u_0\|_{H^1(\mathbb{R})}/\sqrt{2}}\left(h'(\xi)\right)^2\right)\|u_0\|^2_{H^1(\mathbb{R})},\end{aligned}$$

$$\int_{\mathbb{R}\times\mathbb{R}} e^{-|x-y|}|q_\varepsilon||P_\varepsilon|dxdy \le \|u_\varepsilon(t,\cdot)\|^2_{H^1(\mathbb{R})} + \|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le \|u_0\|^2_{H^1(\mathbb{R})} + C_4^2.$$

It follows from these estimates that (5.12) holds.

We continue by proving (5.13). Observe that

$$\frac{\partial P_{2,\varepsilon}}{\partial t}(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}h'(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial t}dy, \tag{5.15}$$

so, using (1.4), (2.4), the Tonelli theorem and the Hölder inequality,

$$\left\|\frac{\partial P_{2,\varepsilon}}{\partial t}\right\|^2_{L^2(\Pi_T)} \le \max_{|\xi|\le\|u_0\|_{H^1(\mathbb{R})}/\sqrt{2}}\left(h'(\xi)\right)^2\left\|\frac{\partial u_\varepsilon}{\partial t}\right\|^2_{L^2(\Pi_T)}. \tag{5.16}$$

Then (5.13) is a direct consequence of (5.10).

Since the bound on $\left\{\frac{\partial P_\varepsilon}{\partial t}\right\}_\varepsilon$ is a consequence of (5.12) and (5.13), the family $\{P_\varepsilon\}_\varepsilon$ is bounded in $W^{1,1}_{\text{loc}}([0,\infty)\times\mathbb{R})$.

Finally, using also Lemma 5.1, we have the existence of a pointwise converging subsequence that is uniformly bounded in $L^\infty([0,\infty)\times\mathbb{R})$. Clearly, this implies (5.11). □

Lemma 5.4. *There exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ tending to zero and two functions $q\in L^p_{\text{loc}}([0,\infty)\times\mathbb{R})$, $\overline{q^2}\in L^r_{\text{loc}}([0,\infty)\times\mathbb{R})$ such that*

$$q_{\varepsilon_j}\rightharpoonup q \quad \text{in } L^p_{\text{loc}}([0,\infty)\times\mathbb{R}), \qquad q_{\varepsilon_j}\overset{\star}{\rightharpoonup} q \quad \text{in } L^\infty_{\text{loc}}([0,\infty);L^2(\mathbb{R})), \tag{5.17}$$

$$q^2_{\varepsilon_j}\rightharpoonup \overline{q^2} \quad \text{in } L^r_{\text{loc}}([0,\infty)\times\mathbb{R}), \tag{5.18}$$

for each $1<p<3$ and $1<r<\frac{3}{2}$. Moreover,

$$q^2(t,x)\le \overline{q^2}(t,x) \quad \text{for almost every } (t,x)\in[0,\infty)\times\mathbb{R} \tag{5.19}$$

and

$$\frac{\partial u}{\partial x}=q \quad \text{in the sense of distributions on } [0,\infty)\times\mathbb{R}. \tag{5.20}$$

Proof. Formulas (5.17) and (5.18) are direct consequences of Theorem 2.1 and Lemma 4.1. Inequality (5.19) is true thanks to the weak convergence in (5.18). Finally, (5.20) is a consequence of the definition of q_ε, Lemma 5.2, and (5.17). □

In the following, for notational convenience, we replace the sequences $\{u_{\varepsilon_j}\}_{j\in\mathbb{N}}$, $\{q_{\varepsilon_j}\}_{j\in\mathbb{N}}$, $\{P_{\varepsilon_j}\}_{j\in\mathbb{N}}$ by $\{u_\varepsilon\}_{\varepsilon>0}$, $\{q_\varepsilon\}_{\varepsilon>0}$, $\{P_\varepsilon\}_{\varepsilon>0}$, respectively.

In view of (5.17), we conclude that for any $\eta\in C^1(\mathbb{R})$ with η' bounded, Lipschitz continuous on $\mathbb{R}$ and any $1<p<3$ we have

$$\eta(q_\varepsilon)\rightharpoonup\overline{\eta(q)} \quad \text{in } L^p_{\text{loc}}([0,\infty)\times\mathbb{R}), \qquad \eta(q_\varepsilon)\overset{\star}{\rightharpoonup}\overline{\eta(q)} \quad \text{in } L^\infty_{\text{loc}}([0,\infty);L^2(\mathbb{R})). \tag{5.21}$$

Throughout this paper we use overbars to denote weak limits (the spaces in which these weak limits are taken should be clear from the context and thus they are not always explicitly stated).

Multiplying the equation in (2.5) by $\eta'(q_\varepsilon)$, we get

$$\begin{aligned}\frac{\partial}{\partial t}\eta(q_\varepsilon)+\frac{\partial}{\partial x}\left(\gamma u_\varepsilon\eta(q_\varepsilon)\right)-\varepsilon\frac{\partial^2}{\partial x^2}\eta(q_\varepsilon)-\varepsilon\eta''(q_\varepsilon)\left(\frac{\partial}{\partial x}\eta(q_\varepsilon)\right)^2&\\ =\gamma q_\varepsilon\eta(q_\varepsilon)-\frac{\gamma}{2}\eta'(q_\varepsilon)q_\varepsilon^2+\left(h(u_\varepsilon)-P_\varepsilon\right)\eta'(q_\varepsilon).&\end{aligned} \tag{5.22}$$

From (5.22), (2.3), and (2.4) it is not difficult to see that $t\mapsto\int_{\mathbb{R}}\phi(x)\eta(q_\varepsilon)(t,x)\,dx$ is uniformly bounded and continuous for any $\phi\in C^\infty(\mathbb{R})$ with compact support. In view of this and the second part of (5.21), it follows from, e.g., [25, App. C] that

$$\eta(q_\varepsilon)\rightharpoonup\overline{\eta(q)} \quad \text{in } C([0,T];L^2(\mathbb{R})_w), \text{ for any } T>0, \tag{5.23}$$

where $L^2(\mathbb{R})_w$ is the Lebesgue space $L^2(\mathbb{R})$ endowed with the weak topology.

Lemma 5.5. *For any convex $\eta\in C^1(\mathbb{R})$ with η' bounded, Lipschitz continuous on $\mathbb{R}$, we have*

$$\frac{\partial\overline{\eta(q)}}{\partial t}+\frac{\partial}{\partial x}\left(\gamma u\overline{\eta(q)}\right)\le\gamma\overline{q\eta(q)}-\frac{\gamma}{2}\overline{\eta'(q)q^2}+(h(u)-P)\overline{\eta'(q)}, \tag{5.24}$$

in the sense of distributions on $[0,\infty)\times\mathbb{R}$. Here $\overline{q\eta(q)}$ and $\overline{\eta'(q)q^2}$ denote the weak limits of $q_\varepsilon\eta(q_\varepsilon)$ and $\eta'(q_\varepsilon)q_\varepsilon^2$ in $L^r_{\text{loc}}([0,\infty)\times\mathbb{R})$, $1<r<\frac{3}{2}$, respectively. In addition, $t\mapsto\int_{\mathbb{R}}\phi(x)\overline{\eta(q)}(t,x)\,dx$ is continuous for any $\phi\in C^\infty(\mathbb{R})$ with compact support.

Proof. In (5.22), by convexity of η, (1.4), (5.9), (5.17), and (5.18), sending $\varepsilon\to0$ yields (5.24). □

Remark 5.6. *From* (5.17) *and* (5.18)*, it is clear that*

$$q = q_+ + q_- = \overline{q_+} + \overline{q_-}, \quad q^2 = (q_+)^2 + (q_-)^2, \quad \overline{q^2} = \overline{(q_+)^2} + \overline{(q_-)^2},$$

almost everywhere in $[0, \infty) \times \mathbb{R}$*, where* $\xi_+ := \xi\chi_{[0,+\infty)}(\xi)$*,* $\xi_- := \xi\chi_{(-\infty,0]}(\xi)$*,* $\xi \in \mathbb{R}$*. Moreover, by* (3.1) *and* (5.17)*,*

$$q_\varepsilon(t, x), q(t, x) \le \frac{2}{\gamma t} + C_2, \qquad t \ge 0, \ x \in \mathbb{R}. \tag{5.25}$$

Lemma 5.7. *There holds*

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}(\gamma u q) = \frac{\gamma}{2}\overline{q^2} + h(u) - P \quad \text{in the sense of distributions on } [0, \infty) \times \mathbb{R}. \tag{5.26}$$

Proof. Using (1.4), (5.9), (5.11), (5.17), and (5.18), the result (5.26) follows by $\varepsilon \to 0$ in (2.5). □

The next lemma contains a renormalized formulation of (5.26).

Lemma 5.8. *For any* $\eta \in C^1(\mathbb{R})$ *with* $\eta' \in L^\infty(\mathbb{R})$*,*

$$\frac{\partial \eta(q)}{\partial t} + \frac{\partial}{\partial x}\left(\gamma u \eta(q)\right) = \gamma q \eta(q) + \left(\frac{\gamma}{2}\overline{q^2} - \gamma q^2\right)\eta'(q) + \left(h(u) - P\right)\eta'(q), \tag{5.27}$$

in the sense of distributions on $[0, \infty) \times \mathbb{R}$*. In addition,* $t \mapsto \int_{\mathbb{R}} \phi(x)\eta(q)(t, x)\, dx$ *is continuous for any* $\phi \in C^\infty(\mathbb{R})$ *with compact support.*

Proof. Let $\{\omega_\delta\}_\delta$ be a family of mollifiers defined on $\mathbb{R}$. Denote $q_\delta(t, x) := (q(t, \cdot) \star \omega_\delta)(x)$. Here and in the following all convolutions are with respect to the x variable. According to Lemma II.1 of [18], it follows from (5.26) that q_δ solves

$$\frac{\partial q_\delta}{\partial t} + \gamma u \frac{\partial q_\delta}{\partial x} = \frac{\gamma}{2}\overline{q^2} \star \omega_\delta - \gamma q^2 \star \omega_\delta + h(u) \star \omega_\delta - P \star \omega_\delta + \rho_\delta, \tag{5.28}$$

where the error ρ_δ tends to zero in $L^1_{\mathrm{loc}}([0, \infty) \times \mathbb{R})$. Multiplying (5.28) by $\eta'(q_\delta)$, we get

$$\begin{aligned}\frac{\partial \eta(q_\delta)}{\partial t} + \frac{\partial}{\partial x}\left(\gamma u \eta(q_\delta)\right) = q\eta(q_\delta) &+ \frac{\gamma}{2}\left(\overline{q^2} \star \omega_\delta\right)\eta'(q_\delta) - \gamma\left(q^2 \star \omega_\delta\right)\eta'(q_\delta) \\ &+ \left(h(u) \star \omega_\delta\right)\eta'(q_\delta) - \left(P \star \omega_\delta\right)\eta'(q_\delta).\end{aligned} \tag{5.29}$$

Using the boundedness of η, η', we can send $\delta \to 0$ in (5.29) to obtain (5.27). The weak time continuity is standard. □

6. Strong convergence of q_ε and existence for (1.2)

Following [31], in this section we wish to improve the weak convergence of q_ε in (5.17) to strong convergence (and then we have an existence result for (1.2)). Roughly speaking, the idea is to derive a "transport equation" for the evolution of the defect measure $\left(\overline{q^2} - q^2\right)(t, \cdot) \ge 0$, so that if it is zero initially then it will continue to be zero at all later times $t > 0$. The proof is complicated by the fact that we do not have a uniform bound on q_ε from below but merely (5.25) and that in Lemma 4.1 we have only $\alpha < 1$.

Lemma 6.1. *There holds*

$$\lim_{t \to 0+} \int_{\mathbb{R}} q^2(t, x) dx = \lim_{t \to 0+} \int_{\mathbb{R}} \overline{q^2}(t, x) dx = \int_{\mathbb{R}} \left(\frac{\partial u_0}{\partial x}\right)^2 dx. \tag{6.1}$$

Proof. Since $u \in C([0, \infty) \times \mathbb{R})$ (see Lemma 5.2) and thanks to (5.20), it is not hard to see that $q(t, \cdot) \rightharpoonup \frac{\partial u_0}{\partial x}$ in $L^2(\mathbb{R})$ as $t \to 0+$, so that

$$\liminf_{t \to 0+} \int_{\mathbb{R}} q^2(t, x) dx \ge \int_{\mathbb{R}} \left(\frac{\partial u_0}{\partial x}(x)\right)^2 dx. \tag{6.2}$$

Moreover, from (2.2), (2.3), (5.9) and (5.18),

$$\int_{\mathbb{R}} u^2(t, x) dx + \int_{\mathbb{R}} \overline{q^2}(t, x) dx \le \int_{\mathbb{R}} u_0^2(x) dx + \int_{\mathbb{R}} \left(\frac{\partial u_0}{\partial x}\right)^2 dx,$$

and, again using the continuity of u (see Lemma 5.2), $\lim_{t\to 0+}\int_{\mathbb{R}} u^2(t,x)dx = \int_{\mathbb{R}} u_0^2 dx$. Hence

$$\limsup_{t\to 0+}\int_{\mathbb{R}}\overline{q^2}(t,x)dx \le \int_{\mathbb{R}}\left(\frac{\partial u_0}{\partial x}\right)^2 dx. \tag{6.3}$$

Clearly, (5.19), (6.2), and (6.3) imply (6.1). □

Lemma 6.2. *For each $R>0$,*

$$\lim_{t\to 0+}\int_{\mathbb{R}}\left(\overline{\eta_R^{\pm}(q)}(t,x)-\eta_R^{\pm}(q(t,x))\right)dx = 0, \tag{6.4}$$

where

$$\eta_R(\xi) := \begin{cases}\frac{1}{2}\xi^2, & \text{if } |\xi|\le R,\\ R|\xi|-\frac{1}{2}R^2, & \text{if } |\xi|>R,\end{cases} \tag{6.5}$$

and $\eta_R^+(\xi) := \eta_R(\xi)\chi_{[0,+\infty)}(\xi)$, $\eta_R^-(\xi) := \eta_R(\xi)\chi_{(-\infty,0]}(\xi)$, $\xi\in\mathbb{R}$.

Proof. Let $R>0$. Observe that

$$\overline{\eta_R(q)}-\eta_R(q) = \frac{1}{2}(\overline{q^2}-q^2)-\left(\overline{f_R(q)}-f_R(q)\right),$$

where $f_R(\xi) := \frac{1}{2}\xi^2-\eta_R(\xi)$, $\xi\in\mathbb{R}$. Since η_R and f_R are convex,

$$0\le\overline{\eta_R(q)}-\eta_R(q) = \frac{1}{2}\left(\overline{q^2}-q^2\right)-\left(\overline{f_R(q)}-f_R(q)\right)\le\frac{1}{2}\left(\overline{q^2}-q^2\right).$$

Then, from (6.1), $\lim_{t\to 0+}\int_{\mathbb{R}}\left(\overline{\eta_R(q)}(t,x)-\eta_R(q(t,x))\right)dx = 0$. Since, $\overline{\eta_R^{\pm}(q)}-\eta_R^{\pm}(q)\le\overline{\eta_R(q)}-\eta_R(q)$, the proof is done. □

Remark 6.3. *Let $R>0$. Then for each $\xi\in\mathbb{R}$*

$$\eta_R(\xi) = \frac{1}{2}\xi^2-\frac{1}{2}(R-|\xi|)^2\chi_{(-\infty,-R)\cup(R,\infty)}(\xi), \qquad \eta_R'(\xi) = \xi+(R-|\xi|)\operatorname{sign}(\xi)\,\chi_{(-\infty,-R)\cup(R,\infty)}(\xi),$$
$$\eta_R^+(\xi) = \frac{1}{2}(\xi_+)^2-\frac{1}{2}(R-\xi)^2\chi_{(R,\infty)}(\xi), \qquad (\eta_R^+)'(\xi) = \xi_+ +(R-\xi)\chi_{(R,\infty)}(\xi),$$
$$\eta_R^-(\xi) = \frac{1}{2}(\xi_-)^2-\frac{1}{2}(R+\xi)^2\chi_{(-\infty,-R)}(\xi), \qquad (\eta_R^-)'(\xi) = \xi_- -(R+\xi)\chi_{(-\infty,-R)}(\xi).$$

Lemma 6.4. *Assume* (1.4) *and* (2.2)*. Then for each $t\ge 0$*

$$\int_{\mathbb{R}}\left(\overline{(q_+)^2}-(q_+)^2\right)(t,x)dx\le 2\int_0^t\int_{\mathbb{R}}S(s,x)\left[\overline{q_+}(s,x)-q_+(s,x)\right]dsdx, \tag{6.6}$$

where $S(s,x) := h\big(u(s,x)\big)-P(s,x)$.

Proof. Let $R>C_2$ (see Lemma 3.1). Subtract (5.27) from (5.24) using the entropy η_R^+ (see Lemma 6.2). The result is

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\overline{\eta_R^+(q)}-\eta_R^+(q)\right)+\frac{\partial}{\partial x}\left(\gamma u\left[\overline{\eta_R^+(q)}-\eta_R^+(q)\right]\right)\\&\quad\le\gamma\left[\overline{q\eta_R^+(q)}-q\eta_R^+(q)\right]-\frac{\gamma}{2}\left[\overline{q^2(\eta_R^+)'(q)}-q^2(\eta_R^+)'(q)\right]\\&\qquad-\frac{\gamma}{2}\left(\overline{q^2}-q^2\right)(\eta_R^+)'(q)+S(t,x)\left[\overline{(\eta_R^+)'(q)}-(\eta_R^+)'(q)\right].\end{aligned} \tag{6.7}$$

Since η_R^+ is increasing and $\gamma\ge 0$, by (5.19),

$$-\frac{\gamma}{2}\left(\overline{q^2}-q^2\right)(\eta_R^+)'(q)\le 0. \tag{6.8}$$

Moreover, from Remark 6.3,

$$\gamma q\eta_R^+(q)-\frac{\gamma}{2}q^2(\eta_R^+)'(q) = -\frac{\gamma R}{2}q(R-q)\chi_{(R,\infty)}(q),$$

$$\gamma\overline{q\eta_R^+(q)} - \frac{\gamma}{2}\overline{q^2(\eta_R^+)'(q)} = -\frac{\gamma R}{2}\overline{q(R-q)\chi_{(R,\infty)}(q)}.$$

Therefore, due to (5.25),

$$\gamma q\eta_R^+(q) - \frac{\gamma}{2}q^2(\eta_R^+)'(q) = \overline{q\eta_R^+(q)} - \frac{1}{2}\overline{q^2(\eta_R^+)'(q)} = 0, \quad \text{in } \Omega_R := \left(\tfrac{1}{2}(R-C_2),\infty\right)\times\mathbb{R}. \tag{6.9}$$

Then from (6.7), (6.8), and (6.9) the following inequality holds in Ω_R:

$$\frac{\partial}{\partial t}\left(\overline{\eta_R^+(q)} - \eta_R^+(q)\right) + \frac{\partial}{\partial x}\left(\gamma u\left[\overline{\eta_R^+(q)} - \eta_R^+(q)\right]\right) \le S(t,x)\left[\overline{(\eta_R^+)'(q)} - (\eta_R^+)'(q)\right] \tag{6.10}$$

for each $t > 1/(2(R-C_2))$. In view of Remark 5.6 and due to (5.25),

$$\eta_R^+(q) = \frac{1}{2}(q_+)^2, \quad (\eta_R^+)'(q) = q_+, \quad \overline{\eta_R^+(q)} = \frac{1}{2}\overline{(q_+)^2}, \quad \overline{(\eta_R^+)'(q)} = \overline{q_+}, \quad \text{in } \Omega_R.$$

Inserting this into (6.10) and integrating the result over $(\frac{1}{2(R-C_2)},t)\times\mathbb{R}$ gives

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}\left[\overline{(q_+)^2}(t,x) - (q_+(t,x))^2\right]dx \le \int_{\mathbb{R}}\left[\overline{\eta_R^+(q)}(\tfrac{1}{2}(R-C_2),x) - \eta_R^+(q)(\tfrac{1}{2}(R-C_2),x)\right]dx\\ + \int_{\frac{1}{2}(R-C_2)}^t\int_{\mathbb{R}} S(s,x)\left[\overline{q_+}(s,x) - q_+(s,x)\right]dsdx,\end{aligned}$$

for each $t > \frac{1}{2}(R-C_2)$. Sending $R\to\infty$ and using Lemma 6.2, we get (6.6). □

Lemma 6.5. *For any $t \ge 0$ and any $R > 0$,*

$$\begin{aligned}&\int_{\mathbb{R}}\left[\overline{\eta_R^-(q)} - \eta_R^-(q)\right](t,x)dx\\ &\quad\le \frac{\gamma R^2}{2}\int_0^t\int_{\mathbb{R}}\overline{(R+q)\chi_{(-\infty,-R)}(q)}dsdx\\ &\quad- \frac{\gamma R^2}{2}\int_0^t\int_{\mathbb{R}}(R+q)\chi_{(-\infty,-R)}(q)dsdx + \gamma R\int_0^t\int_{\mathbb{R}}\left[\overline{\eta_R^-(q)} - \eta_R^-(q)\right]dsdx\\ &\quad+ \frac{\gamma R}{2}\int_0^t\int_{\mathbb{R}}\left[\overline{(q_+)^2} - q_+^2\right]dsdx + \int_0^t\int_{\mathbb{R}}S(s,x)\left[\overline{(\eta_R^-)'(q)} - (\eta_R^-)'(q)\right]dsdx.\end{aligned} \tag{6.11}$$

Proof. Let $R > 0$. By subtracting (5.27) from (5.24), using the entropy η_R^- (see Lemma 6.2), we deduce

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\overline{\eta_R^-(q)} - \eta_R^-(q)\right) + \frac{\partial}{\partial x}\left(\gamma u\left[\overline{\eta_R^-(q)} - \eta_R^-(q)\right]\right)\\ &\quad\le \gamma\left[\overline{q\eta_R^-(q)} - q\eta_R^-(q)\right] - \frac{\gamma}{2}\left[\overline{q^2(\eta_R^-)'(q)} - q^2(\eta_R^-)'(q)\right]\\ &\quad- \frac{\gamma}{2}(\overline{q^2} - q^2)(\eta_R^-)'(q) + S(t,x)\left[\overline{(\eta_R^-)'(q)} - (\eta_R^-)'(q)\right].\end{aligned} \tag{6.12}$$

Since $-R \le (\eta_R^-)' \le 0$ and $\gamma \ge 0$, by (5.19),

$$-\frac{\gamma}{2}\left(\overline{q^2} - q^2\right)(\eta_R^-)'(q) \le \frac{\gamma R}{2}\left(\overline{q^2} - q^2\right). \tag{6.13}$$

Using Remarks 5.6 and 6.3

$$\gamma q\eta_R^-(q) - \frac{\gamma}{2}q^2(\eta_R^-)'(q) = -\frac{\gamma R}{2}q(R+q)\chi_{(-\infty,-R)}(q), \tag{6.14}$$

$$\gamma\overline{q\eta_R^-(q)} - \frac{\gamma}{2}\overline{q^2(\eta_R^-)'(q)} = -\frac{\gamma R}{2}\overline{q(R+q)\chi_{(-\infty,-R)}(q)}. \tag{6.15}$$

Inserting (6.13), (6.14), and (6.15) into (6.12) gives

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\overline{\eta_R^-(q)} - \eta_R^-(q)\right) + \frac{\partial}{\partial x}\left(\gamma u\left[\overline{\eta_R^-(q)} - \eta_R^-(q)\right]\right)\\ &\quad\le -\frac{\gamma R}{2}\overline{q(R+q)\chi_{(-\infty,-R)}(q)} + \frac{\gamma R}{2}q(R+q)\chi_{(-\infty,-R)}(q)\end{aligned}$$

$$+\frac{\gamma R}{2}\left(\overline{q^2}-q^2\right)+S(t,x)\left[\overline{(\eta_R^-)'(q)}-(\eta_R^-)'(q)\right].$$

Integrating this inequality over $(0,t)\times\mathbb{R}$ yields

$$\begin{aligned}\int_{\mathbb{R}}\left[\overline{\eta_R^-(q)}-\eta_R^-(q)\right](t,x)dx\\ &\le -\frac{R}{2}\int_0^t\int_{\mathbb{R}}\overline{q(R+q)\chi_{(-\infty,-R)}(q)}dsdx\\ &+\frac{R}{2}\int_0^t\int_{\mathbb{R}}q(R+q)\chi_{(-\infty,-R)}(q)dsdx+\frac{R}{2}\int_0^t\int_{\mathbb{R}}\left[\overline{q^2}-q^2\right]dsdx\\ &+\int_0^t\int_{\mathbb{R}}S(s,x)\left[\overline{(\eta_R^-)'(q)}-(\eta_R^-)'(q)\right]dsdx.\end{aligned}\tag{6.16}$$

Using Remark 6.3,

$$\overline{\eta_R^-(q)}-\eta_R^-(q)=\frac{1}{2}\left(\overline{(q_-)^2}-(q_-)^2\right)+\frac{1}{2}(R+q)^2\chi_{(-\infty,-R)}(q)-\frac{1}{2}\overline{(R+q)^2\chi_{(-\infty,-R)}(q)}.$$

Hence, from Remark 5.6 and (6.16),

$$\begin{aligned}\int_{\mathbb{R}}\left[\overline{\eta_R^-(q)}-\eta_R^-(q)\right](t,x)dx\\ &\le -\frac{\gamma R}{2}\int_0^t\int_{\mathbb{R}}\overline{q(R+q)\chi_{(-\infty,-R)}(q)}dsdx\\ &+\frac{\gamma R}{2}\int_0^t\int_{\mathbb{R}}q(R+q)\chi_{(-\infty,-R)}(q)dsdx+\gamma R\int_0^t\int_{\mathbb{R}}\left[\overline{\eta_R^-(q)}-\eta_R^-(q)\right]dsdx\\ &-\frac{\gamma R}{2}\int_0^t\int_{\mathbb{R}}(R+q)^2\chi_{(-\infty,-R)}(q)dsdx+\frac{\gamma R}{2}\int_0^t\int_{\mathbb{R}}\overline{(R+q)^2\chi_{(-\infty,-R)}(q)}dsdx\\ &+\frac{\gamma R}{2}\int_0^t\int_{\mathbb{R}}\left[\overline{(q_+)^2}-q_+^2\right]dsdx+\int_0^t\int_{\mathbb{R}}S(s,x)\left[\overline{(\eta_R^-)'(q)}-(\eta_R^-)'(q)\right]dsdx,\end{aligned}$$

and applying twice the identity $\frac{R}{2}(R+q)^2-\frac{R}{2}q(R+q)=\frac{R^2}{2}(R+q)$ we deduce (6.11). □

Lemma 6.6. *There holds*

$$\overline{q^2}=q^2\quad\textit{almost everywhere in }[0,\infty)\times\mathbb{R}.\tag{6.17}$$

Proof. Adding (6.6) and (6.11) yields

$$\begin{aligned}\int_{\mathbb{R}}\Big(\frac{1}{2}\left[\overline{(q_+)^2}-(q_+)^2\right]+\left[\overline{\eta_R^-(q)}-\eta_R^-(q)\right]\Big)(t,x)dx\\ &\le\frac{\gamma R^2}{2}\int_0^t\int_{\mathbb{R}}\overline{(R+q)\chi_{(-\infty,-R)}(q)}dsdx-\frac{\gamma R^2}{2}\int_0^t\int_{\mathbb{R}}(R+q)\chi_{(-\infty,-R)}(q)dsdx\\ &+\gamma R\int_0^t\int_{\mathbb{R}}\left[\overline{\eta_R^-(q)}-\eta_R^-(q)\right]dsdx+\frac{\gamma R}{2}\int_0^t\int_{\mathbb{R}}\left[\overline{(q_+)^2}-q_+^2\right]dsdx\\ &+\int_0^t\int_{\mathbb{R}}S(s,x)\left(\left[\overline{q_+}-q_+\right]+\left[\overline{(\eta_R^-)'(q)}-(\eta_R^-)'(q)\right]\right)dsdx.\end{aligned}\tag{6.18}$$

Arguing as in the proof of Lemma 3.1, there exists a constant $L>0$, depending only on $\|u_0\|_{H^1(\mathbb{R})}$, such that

$$\|S\|_{L^\infty([0,\infty)\times\mathbb{R})}=\left\|h(u)-P\right\|_{L^\infty([0,\infty)\times\mathbb{R})}\le L.\tag{6.19}$$

By Remark 5.6 and 6.3,

$$q_++(\eta_R^-)'(q)=q-(R+q)\chi_{(-\infty,-R)}(q),\quad \overline{q_+}+\overline{(\eta_R^-)'(q)}=q-\overline{(R+q)\chi_{(-\infty,-R)}(q)},$$

so by the convexity of the map $\xi\mapsto\xi_++(\eta_R^-)'(\xi)$,

$$0\le\left[\overline{q_+}-q_+\right]+\left[\overline{(\eta_R^-)'(q)}-(\eta_R^-)'(q)\right]=(R+q)\chi_{(-\infty,-R)}(q)-\overline{(R+q)\chi_{(-\infty,-R)}(q)},$$

and, by (6.19),

$$S(s,x)\left([\overline{q_+}(s,x)-q_+(s,x)]+\left[\overline{(\eta_R^-)'(q)}-(\eta_R^-)'(q)\right]\right)$$
$$\le -L\left(\overline{(R+q)\chi_{(-\infty,-R)}(q)}-(R+q)\chi_{(-\infty,-R)}(q)\right).$$

Since $\xi\mapsto(R+\xi)\chi_{(-\infty,-R)}(\xi)$ is concave and choosing R large enough,

$$\begin{aligned}&\frac{\gamma R^2}{2}\overline{(R+q)\chi_{(-\infty,-R)}(q)}-\frac{\gamma R^2}{2}(R+q)\chi_{(-\infty,-R)}(q)\\ &\qquad+S(s,x)\left([\overline{q_+}(s,x)-q_+(s,x)]+\left[\overline{(\eta_R^-)'(q)}-(\eta_R^-)'(q)\right]\right)\\ &\qquad\le\left(\frac{\gamma R^2}{2}-L\right)\left(\overline{(R+q)\chi_{(-\infty,-R)}(q)}-(R+q)\chi_{(-\infty,-R)}(q)\right]\right)\le 0.\end{aligned}\tag{6.20}$$

Then, from (6.18) and (6.20),

$$\begin{aligned}0&\le\int_{\mathbb{R}}\left(\frac{1}{2}\left[\overline{(q_+)^2}-(q_+)^2\right]+\left[\overline{\eta_R^-(q)}-\eta_R^-(q)\right]\right)(t,x)dx\\ &\le\gamma R\int_0^t\int_{\mathbb{R}}\left(\frac{1}{2}\left[\overline{(q_+)^2}-q_+^2\right]+\left[\overline{\eta_R^-(q)}-\eta_R^-(q)\right]\right)dsdx,\end{aligned}$$

and using the Gronwall inequality and Lemmas 6.1 and 6.2 we conclude that

$$\int_{\mathbb{R}}\left(\frac{1}{2}\left[\overline{(q_+)^2}-(q_+)^2\right]+\left[\overline{\eta_R^-(q)}-\eta_R^-(q)\right]\right)(t,x)dx=0,\qquad\text{for each } t>0.$$

By the Fatou lemma, Remark 5.6, and (5.19), sending $R\to\infty$ yields

$$0\le\int_{\mathbb{R}}\left(\overline{q^2}-q^2\right)(t,x)dx\le 0,\qquad t>0,\tag{6.21}$$

and we see that (6.17) holds. □

Lemma 6.7. *Assume* (1.4) *and* (2.2)*. Then there exists an admissible weak solution of* (1.2)*, satisfying* (*k*)*,* (*kk*) *and* (*kkk*) *of Theorem* 1.2*.*

Proof. The conditions (*i*), (*iii*), (*iv*) of Definition 1.1 are satisfied, due to (2.2), (2.3) and Lemma 5.2. We have to verify (*ii*). Due to (6.17), we have

$$q_\varepsilon\ \to\ q\qquad\text{in } L^2_{\text{loc}}([0,\infty)\times\mathbb{R}).\tag{6.22}$$

Clearly (5.9), (5.11), and (6.22) imply that u is a distributional solution of (1.6). Finally, (*k*) and (*kk*) are consequence of Lemmas 3.1 and 4.1, respectively. For (*kkk*) we can argue as in [31], so let us just sketch the proof. Since $u\in L^\infty([0,T];H^1(\mathbb{R}))$, we can differentiate (1.2) to get an equation for q. Multiplying this equation by $\eta_R'(q)$ (for the definition of η_R, see Lemma 6.2) to get

$$\frac{\partial\eta_R(q)}{\partial t}+\frac{\partial}{\partial x}\left(\gamma u\eta_R(q)\right)=\gamma q\eta_R(q)-\frac{\gamma}{2}q^2\eta_R'(q)+\left(h(u)-P\right)\eta_R'(q).\tag{6.23}$$

From the definition of η_R,

$$\gamma\left(q\eta_R(q)-\frac{1}{2}q^2\eta_R'(q)\right)=\frac{\gamma R}{2}\left(q^2-Rq\right)\chi_{(R,\infty)}(q)-\frac{\gamma R}{2}\left(q^2+Rq\right)\chi_{(-\infty,-R)}(q)=:S_-^R+S_+^R.$$

By (1.8), it follows as in [31] that $\iint_{[0,\infty)\times\mathbb{R}}S_+^R dxdt\le C\|u_0\|_{H^1(\mathbb{R})}$ and thus, by integrating (6.23), $\iint_{[0,\infty)\times\mathbb{R}}S_-^R dxdt\le C$. The latter bound implies that along a subsequence $S_-^R\overset{\star}{\rightharpoonup}\mu$ in the sense of measures as $R\to\infty$, for some nonnegative Radon measure μ. By (1.8), $\iint_{[0,\infty)\times\mathbb{R}}S_+^R dxdt\to 0$ as $R\to\infty$. Hence sending $R\to\infty$ in (6.23) and adding the result to the equation obtained by multiplying (1.2) by u, we get (1.12). Finally, integrating (1.12) shows that the total mass of μ is bounded by $\|u_0\|_{H^1(\mathbb{R})}$. □

Remark 6.8. *It is possible to prove results similar to those obtained for* (1.6) *for slightly more general equations of the form*

$$\frac{\partial u}{\partial t}+\gamma u\frac{\partial u}{\partial x}+\frac{\partial P}{\partial x}=0,\quad -\alpha^2\frac{\partial^2 P}{\partial x^2}+P=h(u)+\frac{\gamma\alpha^2}{2}\left(\frac{\partial u}{\partial x}\right)^2, \tag{6.24}$$

where $\gamma\geq 0$, $\alpha>0$, and $h\colon\mathbb{R}\to\mathbb{R}$ is any locally Lipschitz continuous function with $h(0)=0$. The Green function of the operator $-\alpha^2\frac{\partial^2}{\partial x^2}+1$ is $e^{-|x|/\alpha}/2$. Formally, by letting $\alpha\to 0$, we recover the conservation law $\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}F(u)=0$, where the flux $F(u)$ is given by $F'(u)=\gamma u-h'(u)$. Hence (6.24) *may be viewed as a new type of regularization for one-dimensional conservation laws. We are currently investigating this singular limit problem.*

7. Uniqueness of the viscous limit: the semigroup

Here we prove the existence of the semigroup.

Lemma 7.1. *There exists a strongly continuous semigroup of solutions associated with the Cauchy problem* (1.2)

$$S\colon [0,\infty)\times(0,\infty)\times(\mathcal{E}\cap C^\infty(\mathbb{R}))\times H^1(\mathbb{R})\longrightarrow C([0,\infty)\times\mathbb{R})\cap L^\infty\big([0,\infty);H^1(\mathbb{R})\big),$$

namely, for each $u_0\in H^1(\mathbb{R})$, $\gamma>0$, $g\in\mathcal{E}$ the map $u(t,x)=S_t(\gamma,g,u_0)(x)$ is an admissible weak solution of (1.2)*. Moreover, (k), (kk), and (kkk) of Theorem* 1.2 *are satisfied.*

Clearly, this lemma is a direct consequence of the following lemma and of the lemmas in the previous sections.

Lemma 7.2. *Assume* (1.4), (1.5)*. Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$, $\{\mu_j\}_{j\in\mathbb{N}}\subset(0,\infty)$ and $u,v\in L^\infty([0,\infty);H^1(\mathbb{R}))\cap H^1([0,T]\times\mathbb{R})$, for each $T\geq 0$, be such that ε_j, $\mu_j\to 0$ and*

$$u_{\varepsilon_j}\to u,\quad u_{\mu_j}\to v,\quad \text{strongly in } L^\infty([0,\infty);\mathbb{R}), \tag{7.1}$$

then

$$u=v.$$

Proof. Let $t>0$, it is not restrictive to assume that

$$\|u_{0,\varepsilon}-u_{0,\mu}\|_{H^1(\mathbb{R})}\leq|\varepsilon-\mu|,\qquad \varepsilon,\ \mu>0. \tag{7.2}$$

From [4, Theorem 3.1] and (7.2), we have that

$$\begin{aligned}\|u_{\varepsilon_j}(t,\cdot)-u_{\mu_j}(t,\cdot)\|_{H^1(\mathbb{R})}&\\ \leq A(t,\varepsilon_j+\mu_j)\|u_{0,\varepsilon_j}-u_{0,\mu_j}\|_{H^1(\mathbb{R})}&+B(t,\varepsilon_j+\mu_j)|\varepsilon_j-\mu_j|\\ \leq\big(A(t,\varepsilon_j+\mu_j)+B(t,\varepsilon_j+\mu_j)\big)&|\varepsilon_j-\mu_j|,\end{aligned} \tag{7.3}$$

with

$$A(t,\varepsilon_j+\mu_j)=\mathcal{O}\big(e^{t/(\varepsilon_j+\mu_j)}\big),\qquad B(t,\varepsilon_j+\mu_j)=\mathcal{O}\big(e^{t/(\varepsilon_j+\mu_j)}\big).$$

Choose now a subsequence $\{\varepsilon_{j_n}\}_{n\in\mathbb{N}}$ of $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that

$$\mu_n\leq\varepsilon_{j_n}\leq\mu_n+e^{-1/\mu_n^2},\qquad n\in\mathbb{N}.$$

Clearly,

$$\|u_{\varepsilon_{j_n}}(t,\cdot)-u_{\mu_n}(t,\cdot)\|_{H^1(\mathbb{R})}\to 0, \tag{7.4}$$

which concludes the proof. □

8. Stability of the semigroup and proof of Theorem 1.2

Here we prove the stability of the semigroup and then conclude the proof of Theorem 1.2.

Lemma 8.1. *The semigroup S defined on $[0,\infty)\times(0,\infty)\times(\mathcal{E}\cap C^\infty(\mathbb{R}))\times H^1(\mathbb{R})$ satisfies the stability property (jj) of Theorem 1.2.*

Proof. Fix $\varepsilon>0$. Denote S^ε the semigroup associated to the viscous problem (2.1). Choose $\{u_{0,n}\}_{n\in\mathbb{N}}\subset H^1(\mathbb{R})$, $\{\gamma_n\}_{n\in\mathbb{N}}\subset(0,\infty)$, $\{g_n\}_{n\in\mathbb{N}}\subset\mathcal{E}\cap C^\infty(\mathbb{R})$, $u_0\in H^1(\mathbb{R})$, $\gamma>0$, $g\in\mathcal{E}\cap C^\infty(\mathbb{R})$ satisfying (1.9). The initial data satisfy $u_{0,\varepsilon,n},\ u_{0,\varepsilon}\in H^\ell(\mathbb{R})$, $\ell\geq 2$, the condition (2.2), and

$$\|u_{0,\varepsilon,n}-u_{0,\varepsilon}\|_{H^1(\mathbb{R})}\leq\|u_{0,n}-u_0\|_{H^1(\mathbb{R})}. \tag{8.1}$$

Finally, write

$$u_{\varepsilon,n}:=S^\varepsilon(\gamma_n,g_n,u_{0,n}),\qquad u_n:=S(\gamma_n,g_n,u_{0,n}),\qquad u:=S(\gamma,g,u_0).$$

Let $t>0$, then

$$\begin{aligned}\|u_n(t,\cdot)-u(t,\cdot)\|_{H^1(\mathbb{R})}\leq{}&\|u_n(t,\cdot)-u_{\varepsilon,n}(t,\cdot)\|_{H^1(\mathbb{R})}\\&+\|u_{\varepsilon,n}(t,\cdot)-u_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R})}+\|u_\varepsilon(t,\cdot)-u(t,\cdot)\|_{H^1(\mathbb{R})},\end{aligned} \tag{8.2}$$

so

$$\begin{aligned}0\leq\liminf_n\|u_n(t,\cdot)-u(t,\cdot)\|_{H^1(\mathbb{R})}\leq{}&\liminf_{\varepsilon,n}\|u_n(t,\cdot)-u_{\varepsilon,n}(t,\cdot)\|_{H^1(\mathbb{R})}\\&+\liminf_{\varepsilon,n}\|u_{\varepsilon,n}(t,\cdot)-u_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R})}\\&+\liminf_{\varepsilon}\|u_\varepsilon(t,\cdot)-u(t,\cdot)\|_{H^1(\mathbb{R})}.\end{aligned} \tag{8.3}$$

From Lemma 5.2 we know that

$$\liminf_{\varepsilon,n}\|u_n(t,\cdot)-u_{\varepsilon,n}(t,\cdot)\|_{H^1(\mathbb{R})}=0, \tag{8.4}$$

$$\liminf_{\varepsilon}\|u_\varepsilon(t,\cdot)-u(t,\cdot)\|_{H^1(\mathbb{R})}=0. \tag{8.5}$$

We claim that

$$\liminf_{\varepsilon,n}\|u_{\varepsilon,n}(t,\cdot)-u_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R})}=0. \tag{8.6}$$

Using [4, Theorem 3.1] and (8.1), we have that

$$\|u_{\varepsilon,n}(t,\cdot)-u_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R})}\leq A(t,\varepsilon)\|u_{0,n}-u_0\|_{H^1(\mathbb{R})}+B(t,\varepsilon)\big(\|g_n-g\|_{L^\infty(\mathcal{I})}+|\gamma_n-\gamma|\big), \tag{8.7}$$

with

$$A(t,\varepsilon)=\mathcal{O}(e^{T/\varepsilon}),\qquad B(t,\varepsilon)=\mathcal{O}(e^{T/\varepsilon}),\quad t\in[0,T].$$

Define

$$\varepsilon_n:=\frac{T}{|\log(k_n)|},\qquad k_n:=\max\left\{\|u_{0.n}-u_0\|_{H^1(\mathbb{R})}^{1/2},\|g_n-g\|_{L^\infty(\mathcal{I})}^{1/2},|\gamma_n-\gamma|^{1/2}\right\},$$

clearly

$$\liminf_{\varepsilon,n}\|u_{\varepsilon,n}(t,\cdot)-u_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R})}\leq\liminf_n\|u_{\varepsilon_n,n}(t,\cdot)-u_{\varepsilon_n}(t,\cdot)\|_{H^1(\mathbb{R})}, \tag{8.8}$$

and

$$\lim_n A(\varepsilon_n,t)\|u_{0,n}-u_0\|_{H^1(\mathbb{R})}=\lim_n B(\varepsilon_n,t)\|g_n-g\|_{L^\infty(\mathcal{I})}=0. \tag{8.9}$$

Then (8.6) is consequence of (8.7), (8.8), and (8.9). From (8.3), (8.4), (8.5), and (8.6), we get

$$\lim_n\|u_n(t,\cdot)-u(t,\cdot)\|_{H^1(\mathbb{R})}=0.$$

□

Proof of Theorem 1.2. It is direct consequence of Lemmas 7.1 and 8.1. □

9. Proof of Theorem 1.3

By inspecting the proof of Lemma 1 in [32], we see that the following lemma holds:

Lemma 9.1. *Let* $u \in L^\infty([0,\infty); H^1(\mathbb{R})) \cap C([0,\infty)\times\mathbb{R})$ *satisfy* $\frac{\partial u}{\partial x}(t,x) \le \frac{p}{\gamma t} + C$ *for all* $t>0$, $x \in \mathbb{R}$. *If* $p \in [0,2)$, *then there exists a unique solution* $\Phi_t(x) \in L^\infty_{\text{loc}}([0,\infty); C^{1-p/2}(\mathbb{R}))$ *of*

$$\frac{d}{dt}\Phi_t(x) = \gamma u(t,\Phi_t(x)), \qquad \Phi_t(x)|_{t=0} = x. \tag{9.1}$$

If $p=2$ *and in addition there holds*

$$\lim_{t\to 0+} \|u(t,\,\cdot\,) - u_0\|_{H^1(\mathbb{R})} = 0, \tag{9.2}$$

then (9.1) *possesses a unique solution* $\Phi_t(x) \in L^\infty_{\text{loc}}([0,\infty); C(\mathbb{R}))$.

We remark that for an admissible weak solution u there automatically holds $u(t,\,\cdot\,) \to u_0$ in $H^1(\mathbb{R})$. To see this, notice first that $u(t,\,\cdot\,) \rightharpoonup u_0$ in $H^1(\mathbb{R})$. This is an easy consequence of (1.2) and Appendix C in [25]. Hence $\liminf_{t\to 0+}\|u(t,\,\cdot\,)\|_{H^1(\mathbb{R})} \ge \|u_0\|_{H^1(\mathbb{R})}$. In view of this and *(iv)* of Definition 1.1, we conclude that $\lim_{t\to 0+}\|u(t,\,\cdot\,)\|_{H^1(\mathbb{R})} = \|u_0\|_{H^1(\mathbb{R})}$. Our claim now follows thanks to the well known fact that "weak plus norm convergence" implies strong convergence.

By inspecting the proof of Lemma 2 in [32], we see that also the following lemma holds:

Lemma 9.2. *Let* u *satisfy the assumptions of Lemma* 9.1, *and consider the problem*

$$\frac{\partial f}{\partial t} + \gamma u \frac{\partial f}{\partial x} = g, \qquad f|_{t=0} = f_0. \tag{9.3}$$

Suppose either

$$f \in L^\infty([0,\infty); H^1_{\text{loc}}(\mathbb{R})), \quad g \in L^1_{\text{loc}}([0,\infty); L^\infty(\mathbb{R}))$$

or

$$f \in L^\infty([0,\infty); W^{1,1}(\mathbb{R})), \quad \lim_{t\to 0+}\left\| \frac{\partial f}{\partial x}(t,\,\cdot\,) - \frac{\partial f_0}{\partial x}\right\|_{L^1(\mathbb{R})} = 0,$$

$$g \in L^1_{\text{loc}}([0,\infty); L^p(\mathbb{R})) \quad \textit{for all } p \ge p_0 \textit{ with } p_0 \textit{ sufficiently large.}$$

Then, for any $t>0$, $\|f(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})} \le \|f_0\|_{L^\infty(\mathbb{R})} + \int_0^t \|g(s,\,\cdot\,)\|_{L^\infty(\mathbb{R})} ds$.

Proof of Theorem 1.3. Since the proof is very similar to that in Xin and Zhang [32], we will only sketch it and refer to [32] for further details.

Let u_1 be an energy conservative admissible weak solution of the Cauchy problem for (1.2). Let u_2 be a weak solution of the Cauchy problem for (1.2) and suppose there exists a function $\beta_2(t)$ in $L^2([0,T))$ for all $T>0$ such that $\left\|\frac{\partial u_2}{\partial x}(t,\,\cdot\,)\right\|_{L^\infty(\mathbb{R})} \le \beta_2(t)$. Let us use the notation $q_i = \frac{\partial u_i}{\partial x}$, $i=1,2$. It is an easy calculation to see that u_2 is in fact an energy conservative weak solution. Indeed, we have (see [32, p. 1834])

$$\lim_{R\to\infty}\int_0^T\int_{\mathbb{R}} R\, q_2^2\, \chi_{\{\xi||\xi|\ge R\}}(q_2)\,dx\,dt \le \lim_{R\to\infty}\frac{\int_0^T (\beta_2(t))^2\,dt}{R}\|q_2\|_{L^\infty([0,\infty);L^2(\mathbb{R}))} = 0$$

for all $T>0$. By (1.2), $w := u_1 - u_2$ satisfies

$$\frac{\partial w}{\partial t} + \gamma u_1 \frac{\partial w}{\partial x} = -\gamma w \frac{\partial u_2}{\partial x} - \frac{\partial}{\partial x}(P_1 - P_2), \qquad w|_{t=0} \equiv 0. \tag{9.4}$$

As $w \in L^\infty([0,\infty); H^1(\mathbb{R}))$, we can apply Lemma 9.2 to obtain

$$\|w(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})} \le \int_0^t \left(\beta_2(s)\|w(s,\,\cdot\,)\|_{L^\infty(\mathbb{R})} + \left\|\frac{\partial}{\partial x}(P_1-P_2)(s,\,\cdot\,)\right\|_{L^\infty(\mathbb{R})}\right) ds. \tag{9.5}$$

Let

$$e_i := \frac{1}{2}\left(u_i^2 + q_i^2\right), \quad e := e_1 - e_2,$$

and observe that

$$h(u_i) + \frac{\gamma}{2}q_i^2 = e_i + \frac{1}{2}g(u_i) - \gamma u_i^2, \quad i = 1,2.$$

In what follows, let Λ^{-1} denote the pseudo-differential operator $\frac{\partial}{\partial x}\left(1-\frac{\partial^2}{\partial x^2}\right)^{-1}$. Since

$$P_i = \left(1-\frac{\partial^2}{\partial x^2}\right)^{-1}\left(h(u_i)+\frac{\gamma}{2}q_i^2\right),$$

we find

$$\frac{\partial}{\partial x}(P_1-P_2) = \gamma\Lambda^{-1}e + \Lambda^{-1}\left\{\frac{1}{2}\left(g(u_1)-g(u_2)\right) - \gamma\left(u_1^2-u_2^2\right)\right\},$$

using the identity

$$\Lambda^{-1}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\Lambda^{-1} = \left(1-\frac{\partial^2}{\partial x^2}\right)^{-1} - I. \tag{9.6}$$

Clearly,

$$\left\|\Lambda^{-1}\left\{\frac{1}{2}\left(g(u_1)-g(u_2)\right)(s,\cdot) - \gamma\left(u_1^2-u_2^2\right)(s,\cdot)\right\}\right\|_{L^\infty(\mathbb{R})} \le C\|w(s,\cdot)\|_{L^\infty(\mathbb{R})}.$$

Therefore

$$\|w(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \int_0^t \left(\left\|\Lambda^{-1}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})} + (C+\beta_2(s))\,\|w(s,\cdot)\|_{L^\infty(\mathbb{R})}\right)ds. \tag{9.7}$$

We now set out to estimate $\left\|\Lambda^{-1}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})}$. By definition, the energy density e_i satisfies

$$\frac{\partial e_i}{\partial t} + \frac{\partial}{\partial x}\left(\gamma u_i e_i\right) = -\frac{\partial}{\partial x}\left(u_i\left[P_i - \frac{\gamma}{2}u_i^2\right] + \frac{\gamma}{3}u_i^3 - H(u_i)\right). \qquad i=1,2, \tag{9.8}$$

and thus e satisfies

$$\begin{aligned}\frac{\partial e}{\partial t} + \frac{\partial}{\partial x}\left(\gamma u_1 e + \gamma w e_2\right) = -\frac{\partial}{\partial x}\Big(&u_1\left[P_1-\frac{\gamma}{2}u_1^2\right] - u_2\left[P_2-\frac{\gamma}{2}u_2^2\right]\\ &+\frac{\gamma}{3}u_1^3 - \frac{\gamma}{3}u_2^3 - H(u_1) + H(u_2)\Big).\end{aligned} \tag{9.9}$$

Furthermore, we find that $f := \Lambda^{-1}e$ satisfies $\frac{\partial f}{\partial t} + \gamma u_1\frac{\partial f}{\partial x} = g$, where

$$\begin{aligned} g := {}&-\gamma w e_2 + \left(u_1\left[P_1-\frac{\gamma}{2}u_1^2\right] - u_2\left[P_2-\frac{\gamma}{2}u_2^2\right] + \frac{\gamma}{3}u_1^3 - \frac{\gamma}{3}u_2^3 - H(u_1)+H(u_2)\right)\\ &-\left(1-\frac{\partial^2}{\partial x^2}\right)^{-1}\Big\{\gamma u_1 e + \gamma w e_2 + \Big(u_1\left[P_1-\frac{\gamma}{2}u_1^2\right] - u_2\left[P_2-\frac{\gamma}{2}u_2^2\right]\\ &\qquad\qquad + \frac{\gamma}{3}u_1^3 - \frac{\gamma}{3}u_2^3 - H(u_1)+H(u_2)\Big)\Big\}\\ &+\gamma u_1\left(1-\frac{\partial^2}{\partial x^2}\right)^{-1} e.\end{aligned}$$

Since $\Lambda^{-1}e \in L^\infty([0,\infty); W^{1,1}(\mathbb{R}))$ and, in view of (9.6) and (9.2), $\lim_{t\to 0+}\left\|\frac{\partial}{\partial x}\Lambda^{-1}e(t,\cdot)\right\|_{L^1(\mathbb{R})} = 0$. Hence we can apply Lemma 9.2 to deduce

$$\|\Lambda^{-1}e(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \int_0^t \|g(s,\cdot)\|_{L^\infty(\mathbb{R})}ds. \tag{9.10}$$

To estimate $\|g(s,\cdot)\|_{L^\infty(\mathbb{R})}$, we can proceed as in [32]. The final result is

$$\begin{aligned}\|\Lambda^{-1}e(t,\cdot)\|_{L^\infty(\mathbb{R})} \le C\int_0^t \big(&\|\Lambda^{-1}e(s,\cdot)\|_{L^\infty(\mathbb{R})} + (1+\beta_2(s))\,\|w(s,\cdot)\|_{L^\infty(\mathbb{R})}\\ &+\|(P_1-P_2)(s,\cdot)\|_{L^\infty(\mathbb{R})}\big)\,ds.\end{aligned} \tag{9.11}$$

We have

$$\|(P_1-P_2)(s,\cdot)\|_{L^\infty(\mathbb{R})} \le C\left(\left\|\left(1-\frac{\partial^2}{\partial x^2}\right)^{-1}e(s,\cdot)\right\|_{L^\infty(\mathbb{R})} + \|w(s,\cdot)\|_{L^\infty(\mathbb{R})}\right). \tag{9.12}$$

Repeating the calculations in [32] we find

$$\left\|\left(1-\frac{\partial^2}{\partial x^2}\right)^{-1} e(s,\,\cdot\,)\right\|_{L^\infty(\mathbb{R})} \le C\left(\|\Lambda^{-1}e(s,\,\cdot\,)\|_{L^\infty(\mathbb{R})} + \left\|\left(1-\frac{\partial^2}{\partial x^2}\right)^{-2} e(s,\,\cdot\,)\right\|_{L^2(\mathbb{R})}\right). \tag{9.13}$$

To estimate the last term on the right-hand side, we apply $\left(1-\frac{\partial^2}{\partial x^2}\right)^{-2}$ to equation (9.9). We then obtain $\frac{\partial}{\partial t}\left(1-\frac{\partial^2}{\partial x^2}\right)^{-2} e = \tilde{g}$, where

$$\tilde{g} := -\frac{\partial}{\partial x}\left(1-\frac{\partial^2}{\partial x^2}\right)^{-2}\Big\{\gamma u_1 e + \gamma w e_2 + \Big(u_1[P_1 - \frac{\gamma}{2}u_1^2] - u_2[P_2 - \frac{\gamma}{2}u_2^2] + \frac{\gamma}{3}u_1^3 - \frac{\gamma}{3}u_2^3 - H(u_1) + H(u_2)\Big)\Big\}.$$

From this it follows that $\left\|\left(1-\frac{\partial^2}{\partial x^2}\right)^{-2} e(s,\,\cdot\,)\right\|_{L^2(\mathbb{R})} \le \int_0^t \|\tilde{g}(s,\,\cdot\,)\|_{L^2(\mathbb{R})}$. To estimate $\|\tilde{g}(s,\,\cdot\,)\|_{L^2(\mathbb{R})}$, we proceed once more as in [32]. The final result is

$$\begin{aligned}&\left\|\left(1-\frac{\partial^2}{\partial x^2}\right)^{-2} e(t,\,\cdot\,)\right\|_{L^2(\mathbb{R})}\\ &\quad\le C\int_0^t \left(\|w(s,\,\cdot\,)\|_{L^\infty(\mathbb{R})} + \|\Lambda^{-1}e(s,\,\cdot\,)\|_{L^\infty(\mathbb{R})}\left\|\left(1-\frac{\partial^2}{\partial x^2}\right)^{-2} e(s,\,\cdot\,)\right\|_{L^2(\mathbb{R})}\right) ds.\end{aligned} \tag{9.14}$$

Introduce the quantity

$$y(t) = \|w(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})} + \|\Lambda^{-1}e(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})} + \left\|\left(1-\frac{\partial^2}{\partial x^2}\right)^{-2} e(t,\,\cdot\,)\right\|_{L^2(\mathbb{R})}.$$

Then, from (9.7), (9.10), (9.11), (9.12), (9.13), and (9.14),

$$y(t) \le C\int_0^t \left(1 + (\beta_2(s))^2\right) y(s)ds,$$

and thus by Gronwall's inequality we conclude that $y(t) = 0$.

□

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