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Global well-posedness for a multidimensional chemotaxis model in critical Besov spaces

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Abstract. We investigate the Cauchy problem of a multidimensional chemotaxis model with initial data in critical Besov spaces. The global existence and uniqueness of the strong solution is shown for initial data close to a constant equilibrium state.

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1. Introduction

In the present paper, we consider the Cauchy problem of the following multidimensional ($d \geqslant 2$) chemotaxis model

$$\begin{cases} \tilde{p}_t - \operatorname{div}(\tilde{p}\mathbf{q}) = \Delta\tilde{p}, \quad x \in \mathbb{R}^d,\ t > 0 \\ \mathbf{q}_t - \nabla\tilde{p} = 0 \\ (\tilde{p}(0,x), \mathbf{q}(0,x) = (\tilde{p}_0(x), \mathbf{q}_0(x)) \end{cases} \tag{1.1}$$

where $\tilde{p}(t,x)$ denotes the cell density, $\mathbf{q}(t,x) = -\nabla v/v$, and $v(t,x)$ is the chemical concentration.

Chemotaxis is a biological phenomenon in which somatic cells, bacteria, and other single-cell or multicellular organisms direct their movements according to certain chemicals in the environment where the cells reside. This is important for bacteria to find food by swimming toward the highest concentration of food molecules, or to flee from poisons. Many diverse disciplines involve chemotaxis models whose aspects include not only the mechanistic basis and biological foundations but also the modeling of specific systems and the mathematical analysis of the governing nonlinear equations. The Keller-Segel model of chemotaxis [11–13] has provided, as the authors of [19] wrote, a cornerstone for much of these works, its success being a consequence of its intuitive simplicity, analytical tractability, and capability to model the basic dynamics of chemotactic populations.

The following canonical formulation of the Keller-Segel type chemotaxis model has been extensively studied

$$\begin{cases} u_t = \operatorname{div}(D\nabla u - \chi u \nabla\Phi(v)) \\ \tau v_t = D_1\Delta v + g(u,v) \end{cases} \tag{1.2}$$

where $u(t,x)$ and $v(t,x)$ denote the cell density and the chemical concentration, respectively. $D > 0$ is the diffusion rate of the cells, and $D_1 \geqslant 0$ is the diffusion rate of the chemical substance. The constant χ, often referred as chemosensitivity, is a measure of the strength of chemical signals, and $\chi > 0$ (< 0) corresponds to attractive (repulsive) chemotaxis. The function $\Phi(v)$ is called the chemotactic potential function describing the mechanism of signal detection. $\tau \geqslant 0$ is a relaxation time scale such that $1/\tau$ is the rate toward the equilibrium. The function $g(u, v)$ describes the chemical kinetics.

For the Keller–Segel model (1.2), there are two limiting cases as follows. One is when the chemical substance relaxes so fast that it reaches its equilibrium instantaneously, that is, $\tau \to 0$ and (1.2) is reduced to a parabolic-elliptic system [10]. The other one is when the diffusion of the chemical substance is so small that it is negligible, that is, $D_1 \to 0$. In the current paper, we consider the latter case. If we take $D_1 = 0, D = 1, \tau = 1, \Phi(v) = \ln v, g(u, v) = -uv$, and $\chi > 0$ which corresponds to the attractive chemotaxis, then the model (1.2) becomes

$$\begin{cases} u_t = \operatorname{div}(D\nabla u - \chi u \nabla \ln v) \\ v_t = -uv \end{cases} \tag{1.3}$$

The system (1.1) is derived from (1.3) under transformations $u = \tilde{p}, \mathbf{q} = -\nabla v/v$, which was used in [22] for one-dimensional case and was extended to multidimensional cases in [19], and under scalings $\tilde{t} = \chi t, \tilde{x} = \sqrt{\chi}x$, and $\tilde{\mathbf{q}} = \sqrt{\chi}\mathbf{q}$ where tildes have been dropped from last three expressions. The model (1.3) has been studied in a lot of articles, for instance, in [18,20].

The one-dimensional version of the system (1.1) has been studied for both Cauchy problems and initial-boundary value problems, for instance, in [19,23] for bounded domains and in [15–17] for nonlinear stabilities of traveling waves. But there are few results for the multidimensional cases of the system (1.1). The initial-boundary value problems are studied in [5,6]. Recently, in [14], the local and global existence of the solution to the Cauchy problem is studied for the initial data $(p_0, \mathbf{q}_0) \in H^s$ with $s > d/2 + 1$.

In this paper, we are concerned the well-posedness of the solution to the Cauchy problem for initial data suitably close to a constant equilibrium state $(\bar{p}, \mathbf{0})$ in a functional space with minimal regularity order with the constant $\bar{p} > 0$. For this reason, we are going to use scaling considerations for (1.1) to guess which spaces may be critical. We observe that (1.1) is invariant by the transformation

$$(\tilde{p}_0(x), \mathbf{q}_0(x)) \to (\ell^2 \tilde{p}_0(\ell x), \ell \mathbf{q}_0(\ell x)),$$
$$(\tilde{p}(t, x), \mathbf{q}(t, x)) \to (\ell^2 \tilde{p}(\ell^2 t, \ell x), \ell \mathbf{q}(\ell^2 t, \ell x)).$$

Definition 1.1. *A functional space $E \subset \mathscr{S}'(\mathbb{R}^d) \times (\mathscr{S}'(\mathbb{R}^d))^d$ is called a critical space if the associated norm is invariant under the transformation $(\tilde{p}(\cdot), \mathbf{q}(\cdot)) \to (\ell^2 \tilde{p}(\ell \cdot), \ell \mathbf{q}(\ell \cdot))$ up to a constant independent of ℓ, where $\mathscr{S}'(\mathbb{R}^d)$ denotes the tempered distributions space on $\mathbb{R}^d$.*

Obviously, the homogeneous Sobolev spaces $\dot{H}^{d/2-2} \times (\dot{H}^{d/2-1})^d$ and the homogeneous Besov spaces $B^{d/2-2} \times (B^{d/2-1})^d$ (which will be defined in next section) are critical spaces for initial data. We shall decrease the regularity of the space H^s, provided $s > d/2+1$ in [14], and generalize the results to critical Besov spaces, that is, $B^{d/2-2} \times (B^{d/2-1})^d$ or its subspaces, for initial data.

Now, we state the main result of this paper as follows.

Theorem 1.1. *Let $d \geqslant 2$. Then there exist two positive constants α and M such that for all $(\tilde{p}_0 - \bar{p}, \mathbf{q}_0) \in B^{d/2-2} \times (B^{d/2-2,d/2-1})^d$ with some equilibrium state $\bar{p} > 0$ and $\|\tilde{p}_0 - \bar{p}\|_{B^{d/2-2}} + \|\mathbf{q}_0\|_{B^{d/2-2,d/2-1}} \leqslant \alpha$, there exists a unique solution $(p, \mathbf{q})$ to the system (1.1) such that*

$$(\tilde{p} - \bar{p}, \mathbf{q}) \in \mathcal{C}(\mathbb{R}^+; B^{d/2-2} \times (B^{d/2-2,d/2-1})^d) \cap L^1(\mathbb{R}^+; B^{d/2} \times (B^{d/2,d/2-1})^d)$$

and

$$\begin{aligned} &\|\tilde{p} - \bar{p}\|_{\tilde{L}^\infty(\mathbb{R}^+; B^{d/2-2})} + \|\mathbf{q}\|_{\tilde{L}^\infty(\mathbb{R}^+; B^{d/2-2,d/2-1})} \\ &\quad + \|\tilde{p} - \bar{p}\|_{L^1(\mathbb{R}^+; B^{d/2})} + \|\mathbf{q}\|_{L^1(\mathbb{R}^+; B^{d/2,d/2-1})} \\ &\quad \leqslant M(\|\tilde{p}_0 - \bar{p}\|_{B^{d/2-2}} + \|\mathbf{q}_0\|_{B^{d/2-2,d/2-1}}), \end{aligned}$$

where the mixed time-spatial space $\tilde{L}^\infty(\mathbb{R}^+; B^{s_1,s_2})$ will be defined in the next section.

The rest of the paper is organized as follows. In the second section, we recall some properties of the Littlewood–Paley decomposition and Besov spaces which we will use in this paper. In the third section, we reformulate the system and then derive *a priori* estimates for the linear system. The fourth section involves the proof of the existence of the solutions. And in the last section, we prove the uniqueness of the solution.

2. Littlewood–Paley theory and hybrid Besov spaces

This section is devoted to recall some properties of Littlewood–Paley theory and Besov spaces which will be used in this paper. For more details, one can see [8,9] and references therein.

Let $\psi : \mathbb{R}^d \to [0,1]$ be a radial smooth cutoff function valued in $[0,1]$ such that

$$\psi(\xi) = \begin{cases} 1, & |\xi| \leqslant 3/4 \\ \text{smooth}, & 3/4 < |\xi| < 4/3 \\ 0, & |\xi| \geqslant 4/3 \end{cases}$$

Let $\varphi(\xi)$ be the function

$$\varphi(\xi) := \psi(\xi/2) - \psi(\xi).$$

Thus, ψ is supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leqslant 4/3\}$, and φ is also a smooth cutoff function valued in $[0,1]$ and supported in the annulus $\{\xi \in \mathbb{R}^d : 3/4 \leqslant |\xi| \leqslant 8/3\}$. By construction, we have

$$\sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1, \quad \forall \xi \neq 0.$$

One can define the dyadic blocks as follows. For $k \in \mathbb{Z}$, let

$$\triangle_k f := \mathscr{F}^{-1}\varphi(2^{-k}\xi)\mathscr{F} f,$$

where $\mathscr{F}$ ($\mathscr{F}^{-1}$) denotes the Fourier (inverse) transformation.

The formal decomposition

$$f = \sum_{k\in\mathbb{Z}} \triangle_k f \tag{2.1}$$

is called homogeneous Littlewood–Paley decomposition. (2.1) is true modulo polynomials, in other words (cf. [21]), if $f \in \mathscr{S}'(\mathbb{R}^d)$, then $\sum_{k\in\mathbb{Z}} \triangle_k f$ converges modulo $\mathscr{P}[\mathbb{R}^d]$ and (2.1) holds in $\mathscr{S}'(\mathbb{R}^d)/\mathscr{P}[\mathbb{R}^d]$, where $\mathscr{P}[\mathbb{R}^d]$ denotes the space of all polynomials on $\mathbb{R}^d$.

Definition 2.1. *Let $s \in \mathbb{R}$. For $f \in \mathscr{S}'(\mathbb{R}^d)$, we write*

$$\|f\|_{B^s} = \sum_{k\in\mathbb{Z}} 2^{ks}\|\triangle_k f\|_{L^2}.$$

A difficulty comes from the choice of homogeneous spaces at this point. Indeed, $\|\cdot\|_{B^s}$ cannot be a norm on $\{f \in \mathscr{S}'(\mathbb{R}^d) : \|f\|_{B^s} < \infty\}$ because $\|f\|_{B^s} = 0$ means that f is a polynomial. This enforces us to adopt the following definition for homogeneous Besov spaces (cf. [8]).

Definition 2.2. *Let $s \in \mathbb{R}$ and $m = -[d/2 + 1 - s]$. If $m < 0$, then we define $B^s(\mathbb{R}^d)$ as*

$$B^s = \Big\{ f \in \mathscr{S}'(\mathbb{R}^d) : \|f\|_{B^s} < \infty \quad \textit{and} \quad f = \sum_{k\in\mathbb{Z}} \triangle_k f \ \textit{ in } \mathscr{S}'(\mathbb{R}^d) \Big\}.$$

If $m \geqslant 0$, we denote by $\mathscr{P}_m$ the set of d-variable polynomials of degree less than or equal to m and define

$$B^s = \Big\{ f \in \mathscr{S}'(\mathbb{R}^d)/\mathscr{P}_m : \|f\|_{B^s} < \infty \quad \textit{and} \quad f = \sum_{k\in\mathbb{Z}} \triangle_k f \ \textit{ in } \mathscr{S}'(\mathbb{R}^d)/\mathscr{P}_m \Big\}.$$

We also need hybrid Besov spaces for which regularity assumptions are different in low frequencies and high frequencies [8]. We are going to recall the definition of these new spaces and some of their main properties.

Definition 2.3. *Let* $s,\, t \in \mathbb{R}$. *We define*

$$\|f\|_{B^{s,t}} = \sum_{k \leqslant 0} 2^{ks} \|\triangle_k f\|_{L^2} + \sum_{k>0} 2^{kt} \|\triangle_k f\|_{L^2}.$$

Let $m = -[d/2+1-s]$, *we then define*

$$B^{s,t}(\mathbb{R}^d) = \{f \in \mathscr{S}'(\mathbb{R}^d) : \|f\|_{B^{s,t}} < \infty\}, \quad \text{if } m < 0,$$
$$B^{s,t}(\mathbb{R}^d) = \{f \in \mathscr{S}'(\mathbb{R}^d)/\mathscr{P}_m : \|f\|_{B^{s,t}} < \infty\}, \quad \text{if } m \geqslant 0.$$

Lemma 2.1. *We have the following inclusions for hybrid Besov spaces.*

(i) *We have* $B^{s,s} = B^s$.
(ii) *If* $s \leqslant t$ *then* $B^{s,t} = B^s \cap B^t$. *Otherwise,* $B^{s,t} = B^s + B^t$.
(iii) *The space* $B^{0,s}$ *coincides with the usual inhomogeneous Besov space* $B^s_{2,1}$.
(iv) *If* $s_1 \leqslant s_2$ *and* $t_1 \geqslant t_2$, *then* $B^{s_1,t_1} \hookrightarrow B^{s_2,t_2}$.

Let us now recall some useful estimates for the product in hybrid Besov spaces.

Lemma 2.2. *Let* $s_1, s_2, t_1, and t_2 \leqslant d/2$ *such that* $\min(s_1+s_2, t_1+t_2) > 0, f \in B^{s_1,t_1}$ *and* $g \in B^{s_2,t_2}$. *Then,* $fg \in B^{s_1+s_2-1,t_1+t_2-1}$ *and*

$$\|fg\|_{B^{s_1+s_2-d/2,t_1+t_2-d/2}} \lesssim \|f\|_{B^{s_1,t_1}} \|g\|_{B^{s_2,t_2}},$$

where (and throughout the paper) "$\lesssim$*" denotes "*$\leqslant C$*" for a universal constant* C *which may be different from each other in different arguments.*

In the context of this paper, we also need to use the interpolation spaces of hybrid Besov spaces together with a time space such as $L^\rho([0,T); B^{s,t})$. Thus, we have to introduce the mixed type time-spatial space (cf. [1,4]) which is a refinement of the space $L^\rho([0,T); B^{s,t})$.

Definition 2.4. *Let* $\rho \in [1,\infty], T \in (0,\infty]$ *and* $s,\, t \in \mathbb{R}$. *Then, we define*

$$\|f\|_{\tilde{L}^\rho([0,T);B^{s,t})} = \sum_{k \leqslant 0} 2^{ks} \|\triangle_k f\|_{L^\rho([0,T);L^2)} + \sum_{k>0} 2^{kt} \|\triangle_k f\|_{L^\rho([0,T);L^2)}.$$

Noting that Minkowski's inequality yields $\|f\|_{L^\rho([0,T);B^{s,t})} \leqslant \|f\|_{\tilde{L}^\rho([0,T);B^{s,t})}$, we define spaces $\tilde{L}^\rho([0,T); B^{s,t})$ as follows

$$\tilde{L}^\rho([0,T); B^{s,t}) = \{f \in L^\rho([0,T); B^{s,t}) : \|f\|_{\tilde{L}^\rho([0,T);B^{s,t})} < \infty\}.$$

If $T = \infty$, then we omit the subscript T from the notation $\tilde{L}^\rho([0,T); B^{s,t})$, that is, $\tilde{L}^\rho(B^{s,t})$ for simplicity. Let us observe that $L^1([0,T); B^{s,t}) = \tilde{L}^1([0,T); B^{s,t})$, but the embedding $\tilde{L}^\rho([0,T); B^{s,t}) \subset L^\rho([0,T); B^{s,t})$ is strict if $\rho > 1$.

We will use the following interpolation property which can be verified easily (cf. [1,2]).

Lemma 2.3. *Let* $s, t, s_1, t_1, s_2, and t_2 \in \mathbb{R}$ *and* $\rho, \rho_1, \rho_2 \in [1,\infty]$. *We have*

$$\|f\|_{\tilde{L}^\rho([0,T);B^{s,t})} \leqslant \|f\|^\theta_{\tilde{L}^{\rho_1}([0,T);B^{s_1,t_1})} \|f\|^{1-\theta}_{\tilde{L}^{\rho_2}([0,T);B^{s_2,t_2})},$$

where $\frac{1}{\rho} = \frac{\theta}{\rho_1} + \frac{1-\theta}{\rho_2}, s = \theta s_1 + (1-\theta)s_2,$ *and* $t = \theta t_1 + (1-\theta)t_2$.

3. Reformulation and a priori estimates

Let $p = \tilde{p} - \bar{p}$, then the system (1.1) can be rewritten as

$$\begin{cases} p_t - \bar{p}\mathrm{div}\mathbf{q} - \Delta p = \mathrm{div}(p\mathbf{q}) \\ \mathbf{q}_t - \nabla p = 0 \\ (p, \mathbf{q}|_{t=0}) = (p_0, \mathbf{q}_0) := (\tilde{p}_0 - \bar{p}, \mathbf{q}_0) \end{cases} \tag{3.1}$$

We investigate some *a priori* estimates for the linear system with a general function F

$$\begin{cases} p_t - \bar{p}\mathrm{div}\mathbf{q} - \Delta p = F \\ \mathbf{q}_t - \nabla p = 0 \\ (p, \mathbf{q})|_{t=0} = (p_0, \mathbf{q}_0) \end{cases} \tag{3.2}$$

First, we note that the solution of (3.2) behaves differently for low and high frequencies in view of the eigenvalues of the symbol for the evolution semigroup of (3.2). In fact, the solution of (3.2) can be written as the following integral form:

$$\begin{pmatrix} p(t) \\ \mathbf{q}(t) \end{pmatrix} = \mathrm{e}^{U(\partial)t} \begin{pmatrix} p_0 \\ \mathbf{q}_0 \end{pmatrix} + \int_0^t \mathrm{e}^{U(\partial)(t-\tau)} \begin{pmatrix} F(\tau) \\ \mathbf{0} \end{pmatrix} \mathrm{d}\tau, \tag{3.3}$$

where $U(\partial) = \mathscr{F}^{-1} U(\xi) \mathscr{F}$ with

$$U(\xi) = \begin{pmatrix} -|\xi|^2 & \bar{p} i\xi \\ i\xi & 0 \end{pmatrix}.$$

We compute the eigenvalue equation $\det(\lambda I - U(\xi)) = \lambda^{d-1}(\lambda^2 + |\xi|^2\lambda + \bar{p}|\xi|^2) = 0$ to obtain $d - 1$ zero eigenvalues corresponding to the irrotational parts of $\mathbf{q}$ and two non-zero eigenvalues $\lambda_\pm$ corresponding to p and the divergence part of $\mathbf{q}$.

For low frequencies, that is, $|\xi| < 2\sqrt{\bar{p}}$, we have

$$\lambda_\pm = -\frac{|\xi|^2}{2} \left(1 \pm i\sqrt{\frac{4\bar{p}}{|\xi|^2} - 1}\right),$$

so that we can expect a parabolic damping for low frequencies of p and $\mathbf{q}$.

For high frequencies, that is, $|\xi| \geqslant 2\sqrt{\bar{p}}$, the situation is quite different. We have now

$$\lambda_\pm = -\frac{|\xi|^2}{2} \left(1 \pm \sqrt{1 - \frac{4\bar{p}}{|\xi|^2}}\right),$$

and $\lambda_+ \sim -|\xi|^2$ and $\lambda_- \sim -\bar{p}$ as $|\xi| \to \infty$.

According to the above considerations, we may expect that the system (3.2) has a parabolic smoothing effect on p and on the low frequencies of $\mathbf{q}$, and a damping effect on the high frequencies of $\mathbf{q}$. In fact, we can prove the following proposition:

Proposition 3.1. *Let $s \in \mathbb{R}, T \in (0, \infty]$ and $(p, \mathbf{q})$ be a solution of (3.2) on $[0, T)$, then the following estimates hold*

$$\begin{aligned} &\|p\|_{\tilde{L}^\infty([0,T);B^{s-2})} + \|\mathbf{q}\|_{\tilde{L}^\infty([0,T);B^{s-2,s-1})} + \|p\|_{L^1([0,T);B^s)} + \|\mathbf{q}\|_{L^1([0,T);B^{s,s-1})} \\ &\quad \leqslant C(\|p_0\|_{B^{s-2}} + \|\mathbf{q}_0\|_{B^{s-2,s-1}} + \|F\|_{L^1([0,T);B^{s-2})}), \end{aligned}$$

where $C \geqslant 1$ depends only on d.

Proof. Applying the Littlewood–Paley operator $\triangle_k$ to the linear system (3.2) and denoting $p_k = \triangle_k p, \mathbf{q}_k = \triangle_k \mathbf{q}$ and $F_k = \triangle_k F$, we get

$$\begin{cases} \partial_t p_k - \bar{p}\mathrm{div}\mathbf{q}_k - \Delta p_k = F_k \\ \partial_t \mathbf{q}_k - \nabla p_k = 0 \\ (p_k(0,x), \mathbf{q}_k(0,x)) = (\triangle_k p_0(x), \triangle_k \mathbf{q}_0(x)) \end{cases} \tag{3.4}$$

Taking the inner products of the first equation of (3.4) with p_k, and the second one with $\mathbf{q}_k$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p_k\|_2^2 + \frac{\bar{p}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{q}_k\|_2^2 + \|\nabla p_k\|_2^2 = (F_k, p_k).$$

From the inner products of the first equation of (3.4) with $-\mathrm{div}\mathbf{q}_k$, and the second one with ∇p_k and $\Delta\mathbf{q}_k$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{q}_k, \nabla p_k) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{q}_k\|_2^2 + \bar{p}\|\mathrm{div}\mathbf{q}_k\|_2^2 - \|\nabla p_k\|_2^2 = -(F_k, \mathrm{div}\mathbf{q}_k).$$

Denote for $k \in \mathbb{Z}$ and $0 < A < 1$

$$\alpha_k^2 := \|p_k\|_2^2 + \bar{p}\|\mathbf{q}_k\|_2^2 + A\|\nabla\mathbf{q}_k\|_2^2 + 2A(\mathbf{q}_k, \nabla p_k).$$

Since $2|(q_k, \nabla p_k)| = 2|(\mathrm{div}\mathbf{q}_k, p_k)| \leqslant (A+1)\|\mathrm{div}\mathbf{q}_k\|_2^2/2 + 2\|p_k\|_2^2/(A+1)$, we have

$$\alpha_k \sim \|p_k\|_2 + \|\mathbf{q}_k\|_2 + \|\nabla\mathbf{q}_k\|_2. \tag{3.5}$$

Thus, there exists a constant c such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\alpha_k^2 + c\min(2^{2k}, 1)\alpha_k^2 \lesssim \|F_k\|_2\alpha_k,$$

which yields, by eliminating the term α_k from both sides, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_k + c\min(2^{2k}, 1)\alpha_k \lesssim \|F_k\|_2.$$

Integrating with respect to the time t, it follows that

$$\alpha_k(t) + c\min(2^{2k}, 1)\int_0^t \alpha_k(\tau)\mathrm{d}\tau \leqslant \alpha_k(0) + C\int_0^t \|F_k(\tau)\|_2\mathrm{d}\tau,$$

which implies, in view of (3.5), that

$$\begin{aligned} &\|p_k(t)\|_2 + \|\mathbf{q}_k(t)\|_2 + \|\nabla\mathbf{q}_k(t)\|_2 \\ &\quad + \min(2^{2k}, 1)\int_0^t (\|p_k(\tau)\|_2 + \|\mathbf{q}_k(\tau)\|_2 + \|\nabla\mathbf{q}_k(\tau)\|_2)\mathrm{d}\tau \\ &\lesssim \|p_k(0)\|_2 + \|\mathbf{q}_k(0)\|_2 + \|\nabla\mathbf{q}_k(0)\|_2 + \int_0^t \|F_k(\tau)\|_2\mathrm{d}\tau. \end{aligned}$$

Multiplying both sides by $2^{k(s-2)}$ and summing over $k \in \mathbb{Z}$, we get

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} 2^{k(s-2)}\|p_k\|_{L^\infty([0,T);L^2)} + \sum_{k\in\mathbb{Z}} 2^{k(s-2)}\|\mathbf{q}_k(t)\|_{L^\infty([0,T);L^2)} \\
&\quad + \sum_{k\in\mathbb{Z}} 2^{k(s-2)}\|\nabla\mathbf{q}_k(t)\|_{L^\infty([0,T);L^2)} \\
&\quad + \int_0^T \sum_{k\in\mathbb{Z}} 2^{k(s-2)} \min(2^{2k},1)\left(\|p_k(\tau)\|_2 + \|\mathbf{q}_k(\tau)\|_2 + \|\nabla\mathbf{q}_k(\tau)\|_2\right)\mathrm{d}\tau \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{k(s-2)}\|p_k(0)\|_2 + \sum_{k\in\mathbb{Z}} 2^{k(s-2)}\|\mathbf{q}_k(0)\|_2 + \sum_{k\in\mathbb{Z}} 2^{k(s-2)}\|\nabla\mathbf{q}_k(0)\|_2 \\
&\quad + \int_0^T \sum_{k\in\mathbb{Z}} 2^{k(s-2)}\|F_k(\tau)\|_2\mathrm{d}\tau. \qquad (3.6)
\end{aligned}$$

We also need the smoothing effect for high frequencies of p. From the inner product of the first equation of (3.4) with p_k, we also have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p_k\|_2^2 + \|\nabla p_k\|_2^2 \leqslant \bar{p}\|\mathrm{div}\mathbf{q}_k\|_2\|p_k\|_2 + \|F_k\|_2\|p_k\|_2,$$

which follows, as a similar argument as the above, that

$$\begin{aligned}
&\sum_{k\in\mathbb{N}} 2^{k(s-2)}\|p_k\|_{L^\infty([0,T);L^2)} + \int_0^T \sum_{k\in\mathbb{N}} 2^{ks}\|p_k(\tau)\|_2\mathrm{d}\tau \\
&\lesssim \sum_{k\in\mathbb{N}} 2^{k(s-2)}\|p_k(0)\|_2 + \bar{p}\int_0^T \sum_{k\in\mathbb{N}} 2^{ks}\|\mathbf{q}_k(\tau)\|_2 + \int_0^T \sum_{k\in\mathbb{N}} 2^{k(s-2)}\|F_k(\tau)\|_2\mathrm{d}\tau. \qquad (3.7)
\end{aligned}$$

Combining (3.6) with (3.7), we have the desired results. □

4. Existence of the solution

This section is devoted to the proof of the existence part in Theorem 1.1. The principle of the proof is very classical. We shall use the classical Friedrichs' regularization method (e.g. [3]) to construct the approximate solutions $(p^\ell, \mathbf{q}^\ell)_{\ell\in\mathbb{N}}$ to (3.1), and then we will use Proposition 3.1 to get some uniform bounds on $(p^\ell, \mathbf{q}^\ell)_{\ell\in\mathbb{N}}$ in order to get the strong convergence of the approximate solutions.

Step 1: The Friedrichs' approximation.

In order to construct the classical Friedrichs' approximation, we first define the frequency truncation operator $(\mathbb{F}_\ell)_{\ell\in\mathbb{N}}$ by

$$\mathbb{F}_\ell f := \mathscr{F}^{-1}\mathbf{1}_{B(1/\ell,\ell)}(\xi)\mathscr{F}f,$$

for any $f \in L^2(\mathbb{R}^d)$ where $B(1/\ell,\ell) := \{\xi \in \mathbb{R}^d : 1/\ell \leqslant |\xi| \leqslant \ell\}$. Then, we can define the following approximate system

$$\begin{cases} p_t^\ell - \bar{p}\mathrm{div}\mathbf{q}^\ell - \Delta p^\ell = F^\ell \\ \mathbf{q}_t^\ell - \nabla p^\ell = 0 \\ (p^\ell, \mathbf{q}^\ell)|_{t=0} = (p_\ell, \mathbf{q}_\ell) \end{cases} \qquad (4.1)$$

where

$$p_\ell = \mathbb{F}_\ell p_0, \quad \mathbf{q}_\ell = \mathbb{F}_\ell \mathbf{q}_0, \quad F^\ell = \mathbb{F}_\ell(\mathrm{div}(p^\ell\mathbf{q}^\ell)).$$

It is easy to check that it is an ordinary differential equation in $L^2_\ell \times (L^2_\ell)^d$ for every $\ell \in \mathbb{N}$, where $L^2_\ell = \{f \in L^2(\mathbb{R}^d) : \mathbb{F}_\ell f = f\}$. By the usual Cauchy-Lipschitz theorem, there is a strictly positive maximal time T^*_ℓ such that a unique solution $(p^\ell, \mathbf{q}^\ell)$ exists in $[0, T^*_\ell)$ which is continuous in time with value in $L^2_\ell \times (L^2_\ell)^d$.

Step 2: Uniform estimates.

Denote

$$E_0 = \|p_0\|_{B^{d/2-2}} + \|\mathbf{q}_0\|_{B^{d/2-2,d/2-1}},$$

and

$$\|(p,\mathbf{q})\|_{S^s_T} := \|p\|_{\tilde{L}^\infty([0,T);B^{s-2})} + \|\mathbf{q}\|_{\tilde{L}^\infty([0,T);B^{s-2,s-1})} + \|p\|_{L^1([0,T);B^s)} + \|\mathbf{q}\|_{L^1([0,T);B^{s,s-1})},$$

where we will omit the subscripts T if $T = \infty$, and let

$$T_\ell := \sup\{T \in [0, T^*_\ell) : \|(p^\ell, \mathbf{q}^\ell)\|_{S^{d/2}_T} \leqslant B\bar{C}E_0\},$$

where $\bar{C} \geqslant 1$ corresponds to the constant in Proposition 3.1, and the constant $B = 1/(2C\bar{C}^2E_0)$. Thus, by the continuity, we have $T_\ell > 0$.

From Lemma 2.2 and noticing that $d \geqslant 2$, we get

$$\|F^\ell\|_{L^1([0,T);B^{d/2-2})} \lesssim \|p^\ell \mathbf{q}^\ell\|_{L^1([0,T);B^{d/2-1})} \lesssim \|p^\ell\|_{L^1([0,T);B^{d/2})}\|\mathbf{q}^\ell\|_{\tilde{L}^\infty([0,T);B^{d/2-1})} \lesssim \|(p^\ell,\mathbf{q}^\ell)\|^2_{S^{d/2}_T}.$$

Thus,

$$\|(p^\ell,\mathbf{q}^\ell)\|_{S^{d/2}_T} \leqslant \bar{C}\left(E_0 + C\|(p^\ell,\mathbf{q}^\ell)\|^2_{S^{d/2}_T}\right) \leqslant \bar{C}(1 + C(B\bar{C})^2E_0)E_0,$$

where $C \geqslant 1$. Hence, we can choose $E_0 < 1/4C\bar{C}^2$, and then, we have $1 + CB^2\bar{C}^2E_0 < B$. Therefore, for any $T < T_\ell$, we have $\|(p^\ell,\mathbf{q}^\ell)\|_{S^{d/2}_T} \leqslant B\bar{C}E_0 \leqslant 1/2$. We claim that $T_\ell = T^*_\ell$. In fact, we have shown that $\|(p^\ell,\mathbf{q}^\ell)\|_{S^{d/2}_T} < B\bar{C}E_0$ for $T_\ell < T^*_\ell$. Thus, by continuity, for a sufficiently small constant $a > 0$, we can obtain $\|(p^\ell,\mathbf{q}^\ell)\|_{S^{d/2}_{T+a}} \leqslant B\bar{C}E_0$ which contradicts with the definition of T_ℓ.

Now, we show the approximate solution is a global one, that is, $T^*_\ell = \infty$. We assume $T^*_\ell < \infty$, then we have shown that $\|(p^\ell,\mathbf{q}^\ell)\|_{S^{d/2}_T} \leqslant B\bar{C}E_0$. In view of $p^\ell \in \tilde{L}^\infty([0,T^*_\ell);B^{d/2-2})$ and $\mathbf{q}^\ell \in \tilde{L}^\infty[0,(T^*_\ell);B^{d/2-2,d/2-1})$, it follows that $\|(p^\ell,\mathbf{q}^\ell)\|_{L^\infty([0,T^*_\ell);L^2_\ell)} < \infty$. Thus, we may continue the solution beyond T^*_ℓ by the Cauchy-Lipschitz theorem. This contradicts the definition of T^*_ℓ. Therefore, $(p^\ell,\mathbf{q}^\ell)_{\ell\in\mathbb{N}}$ is global in time.

Step 3: Compactness and convergence.

From (4.1), we can easily obtain that $\partial_t p^\ell \in (L^\infty + L^{4/3})(B^{d/2-5/2})$ and $\partial_t \mathbf{q}^\ell \in L^4(B^{d/2-5/2})$ since $d \geqslant 2$. Applying the Morrey embedding with respect to the time variable, we obtain that $(p^\ell,\mathbf{q}^\ell)$ is uniformly bounded in $\mathcal{C}^{1/4}(\mathbb{R}^+;B^{d/2-5/2}) \times (\mathcal{C}^{3/4}(\mathbb{R}^+;B^{d/2-5/2}))^d$.

Now we can use the Arzelà-Ascoli theorem to get the strong convergence of the approximate solutions. We need to localize the spatial space in order to utilize some compactness results of local Besov spaces. Let $(\chi_m)_{m\in\mathbb{N}}$ be a sequence of $\mathscr{D}(\mathbb{R}^d)$ cutoff functions supported in the ball $B(0,m+1)$ of $\mathbb{R}^d$ and equal to 1 in a neighborhood of $B(0,m)$. Then, by the uniform estimates we have obtained, we see that $(\chi_m p^\ell, \chi_m \mathbf{q}^\ell)_{\ell\in\mathbb{N}}$ is bounded in $S^{d/2}$ and uniformly equi-continuous in

$$(\mathcal{C}([0,T);B^{d/2-5/2}))^{1+d}$$

for any $m \in \mathbb{N}$ and $T > 0$. Moreover, the mapping $f \mapsto \chi_m f$ is compact from $B^{d/2-2}$ into $B^{d/2-5/2}$ and from $B^{d/2-2,d/2-1}$ into $B^{d/2-5/2}$.

Applying the Arzelà-Ascoli theorem to the family $(\chi_m p^\ell, \chi_m \mathbf{q}^\ell)_{\ell\in\mathbb{N}}$ on the time interval $[0,m]$, then we use the Cantor diagonal process. This finally provides us with a distribution $(p,\mathbf{q})$ continuous in time with values in $(B^{d/2-5/2})^{1+d}$ and a subsequence (which we still denote by the same notation) such that we have, for all $m \in \mathbb{N}$, that $(\chi_m p^\ell, \chi_m \mathbf{q}^\ell) \to (\chi_m p, \chi_m \mathbf{q})$ in $(\mathcal{C}([0,m];B^{d/2-5/2}))^{1+d}$ as $\ell \to \infty$. This obviously implies that $(p^\ell, \mathbf{q}^\ell)$ tends to $(p, \mathbf{q})$ in $\mathscr{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$. Coming back to the uniform estimates, we moreover obtain that $(p, \mathbf{q})$ belongs to $S^{d/2}$ and $\mathcal{C}^{1/4}(\mathbb{R}^+; B^{d/2-5/2}) \times (\mathcal{C}^{3/4}(\mathbb{R}^+; B^{d/2-5/2}))^d$. The convergence results stemming from this last result and interpolation argument enable us to pass to the limit in $\mathscr{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$ in the system (4.1) and to prove, in a standard way, that $(p, \mathbf{q})$ is indeed a solution of (3.1) with the initial data. In addition, the continuity in time of the solution is straightforward, and we omit the details.

5. Uniqueness

Finally, we prove the uniqueness of solutions. Let $(p_1, \mathbf{q}_1)$ and $(p_2, \mathbf{q}_2)$ be two solutions of (3.1) in S_T with the same initial data. Denote $(\delta p, \delta \mathbf{q}) = (p_2 - p_1, \mathbf{q}_2 - \mathbf{q}_1)$. Then they satisfy

$$\begin{cases} \partial_t \delta p - \bar{p}\mathrm{div}\delta\mathbf{q} - \Delta\delta p = \mathrm{div}(p_2\delta\mathbf{q}) + \mathrm{div}(\mathbf{q}_1\delta p) \\ \partial_t\delta\mathbf{q} - \nabla\delta p = \mathbf{0} \\ (\delta p, \delta\mathbf{q})|_{t=0} = (0, \mathbf{0}) \end{cases} \tag{5.1}$$

By Proposition 3.1, we have

$$\begin{aligned} \|(\delta p, \delta\mathbf{q})\|_{S_T^{d/2}} &\lesssim \|\mathrm{div}(p_2\delta\mathbf{q})\|_{L^1([0,T);B^{d/2-2})} + \|\mathrm{div}(\mathbf{q}_1\delta p)\|_{L^1([0,T);B^{d/2-2})} \\ &\lesssim \|p_2\|_{L^1([0,T);B^{d/2})}\|\delta\mathbf{q}\|_{\tilde{L}^\infty([0,T);B^{d/2-1})} + \|\mathbf{q}_1\|_{\tilde{L}^\infty([0,T);B^{d/2-1})}\|\delta p\|_{L^1([0,T);B^{d/2})} \\ &\leqslant C(\|p_2\|_{L^1([0,T);B^{d/2})} + \|\mathbf{q}_1\|_{\tilde{L}^\infty([0,T);B^{d/2-1})})\|(\delta p, \delta\mathbf{q})\|_{S_T^{d/2}}. \end{aligned}$$

Now, plugging this constant C to the definition of B, we have $C\|\mathbf{q}_1\|_{\tilde{L}^\infty([0,T);B^{d/2-1})} \leqslant 1/2$. On the other hand, taking T small enough, we have $C\|p_2\|_{L^1([0,T);B^{d/2})} \leqslant 1/3$. Then, it follows that $\|(\delta p, \delta\mathbf{q})\|_{S_T^{d/2}} \equiv 0$. Hence, $(p_1, \mathbf{q}_1)(t) = (p_2, \mathbf{q}_2)(t)$ on $[0, T)$. By a standard argument (e.g. [7]), we can conclude that $(p_1, \mathbf{q}_1)(t) = (p_2, \mathbf{q}_2)(t)$ on $\mathbb{R}^+$.

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