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GLOBAL WELL-POSEDNESS FOR THE MAXWELL-KLEIN GORDON EQUATION IN $4+1$ DIMENSIONS. SMALL ENERGY.

JOACHIM KRIEGER, JACOB STERBENZ, AND DANIEL TATARU

Abstract. We prove that the critical Maxwell-Klein Gordon equation on $\mathbb{R}^{4+1}$ is globally well-posed for smooth initial data which are small in the energy. This reduces the problem of global regularity for large, smooth initial data to precluding concentration of energy.

1. Introduction

Let $\mathbb{R}^{4+1}$ be the five dimensional Minkowski space equipped with the standard Lorentzian metric $g = \mathrm{diag}(1,-1,-1,-1,-1)$. Denote by $\phi : \mathbb{R}^{4+1} \to \mathbb{C}$ a scalar function, and by $A_\alpha : \mathbb{R}^{4+1} \to \mathbb{R}$, $\alpha = 1\ldots,4$, a real valued connection form, interpreted as taking values in $isu(1)$. Introducing the curvature tensor

$$F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

as well as the covariant derivative

$$D_\alpha \phi := (\partial_\alpha + iA_\alpha)\phi$$

the *Maxwell-Klein Gordon system* are the Euler-Lagrange equations associated with the formal Lagrangian action functional[1]

$$\mathcal{L}(A_\alpha, \phi) := \frac{1}{2}\int_{\mathbb{R}^{4+1}} \big(\frac{1}{2}D_\alpha\phi\overline{D^\alpha\phi} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\big)\,dxdt;$$

here we are using the standard convention for raising indices. Introducing the covariant wave operator

$$\Box_A := D^\alpha D_\alpha$$

we can write the Maxwell-Klein Gordon system in the following form

$$\begin{aligned} \partial^\beta F_{\alpha\beta} &= -J_\alpha := \Im(\phi\overline{D_\alpha\phi}), \\ \Box_A \phi &= 0 \end{aligned} \tag{1}$$

One key feature of this system is the underlying *Gauge invariance*, which is manifested by the fact that if (A_α, ϕ) is a solution, then so is $(A_\alpha - \partial_\alpha\chi, e^{i\chi}\phi)$. This allows us to impose an

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[1] The Lagrangian below should also contain a mass term, which we choose to neglect here; thus the system under consideration might be more aptly named "the Maxwell-scalar field system". For historical reasons we have chosen to retain the "Maxwell-Klein Gordon" terminology.

additional Gauge condition, and we shall henceforth impose the *Coulomb Gauge condition* which requires

$$\sum_{j=1}^{4} \partial_j A_j = 0 \tag{2}$$

The MKG-CG system can be written explicitly in the following form

$$\Box A_i = \mathcal{P}_i J_x \tag{3a}$$

$$\Box_A \phi = 0 \tag{3b}$$

$$\Delta A_0 = J_0 \tag{3c}$$

$$\Delta \partial_t A_0 = \nabla^i J_i, \tag{3d}$$

where the operator $\mathcal{P}$ denotes the Leray projection onto divergence free vector fields,

$$\mathcal{P} = I - \nabla \Delta^{-1} \nabla$$

The last equation (3d) is a consequence of (3c) due to the divergence free condition on the moments $\partial^\alpha J_\alpha = 0$, which in turn follows from (3b). However, it plays a role in the sequel so it is added here for convenience.

Here it is assumed that the data $A_j(0, \cdot)$ satisfy the vanishing divergence relation (2). We remark that given an arbitrary finite energy data set for the MKG problem, one can find a gauge equivalent data set of comparable size which satisfies the Coulomb gauge condition. This argument only involves solving linear elliptic pde's, and is omitted. The key question now is to decide whether a family of data

$$(A, \phi)[0] := (A_\alpha(0, \cdot), \partial_t A_\alpha(0, \cdot), \phi(0, \cdot), \partial_t \phi(0, \cdot))$$

which satisfy the compatibility conditions required by the two equations for A_0 above can be extended to a global-in-time solution for the Maxwell-Klein-Gordon system. The key for deciding this question is the criticality character of the system. Note that the *energy*

$$E(A, \phi) := \int_{\mathbb{R}^4} \big(\frac{1}{4} \sum_{\alpha,\beta} F_{\alpha\beta}^2 + \frac{1}{2} \sum_{\alpha} |D_\alpha \phi|^2 \big)\, dx$$

is preserved under the flow (1), and in our $4+1$-dimensional setting it is also invariant under the *natural scaling*

$$\phi(t, x) \to \lambda \phi(\lambda t, \lambda x),\ A_\alpha(t, x) \to \lambda A(\lambda t, \lambda x)$$

This means the 4+1-MKG system is *energy critical*, and in recent years a general approach to the large data Cauchy problem associated with energy critical wave equations has emerged. The first key step in this approach consists in establishing an essentially optimal global well-posedness result for data which are small in the energy norm, which is usually the optimal small-data global well-posedness result achievable. In this paper, we set out to prove this for the $4+1$-dimensional Maxwell-Klein-Gordon system.

Theorem 1. *a) Let $(A, \phi)[0]$ be a C^∞ Coulomb data set satisfying*

$$E(A, \phi) < \epsilon_* \tag{4}$$

for a sufficiently small universal constant $\epsilon_* > 0$. Then the system (1) *admits a unique global smooth solution (A, ϕ) on $\mathbb{R}^{4+1}$ with these data.*

b) In addition, the data to solution operator extends continuously[2] *on the set* (4) *to a map*

$$\dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4) \ni (A, \phi)[0] \to (A, \phi) \in C(\mathbb{R}; \dot{H}^1(\mathbb{R}^4)) \cap \dot{C}^1(\mathbb{R}, L^2(\mathbb{R}^4))$$

We remark that the same result holds in all higher dimensions for small data in the scale invariant space $\dot{H}^{\frac{n}{2}-1} \times \dot{H}^{\frac{n}{2}-2}$. This has already been known in dimensions $n \geq 6$, see [21]. We have chosen to restrict our exposition to the more difficult case $n = 4$ in order to keep the notations simple, but our analysis easily carries over to dimension $n = 5$. On the other hand, we do not know whether a similar result holds in dimension $n = 3$.

Before explaining some more details of our approach, we recall here earlier developments on this problem, and how our approach relates to these. Considering the case of general spatial dimension n and denoting the *critical Sobolev exponent* by $s_c = \frac{n}{2} - 1$ (thus $s_c = 1$ for $n = 4$, corresponding to the energy), a global regularity result for data which are smooth and small in $\dot{H}^{s_c}$ was established in dimensions $n \geq 6$ in the work [21] which served as inspiration to the present work. We note in passing that a result analogous to [21] was established in [18] for the Yang-Mills system in dimensions $n \geq 6$, and the present work most likely also admits a corresponding analogue for the $4 + 1$-dimensional Yang-Mills problem. The global regularity question for the physically relevant $n = 3$ case of the Yang-Mills problem had been established earlier in the groundbreaking work [6]. Observe that this problem is *energy sub-critical*.

In the context of MKG, the result [21] had been preceded by a number of works which aimed at improving the *local wellposedness* of MKG in the $n = 3$ case, beginning with [9], followed by [5], and more recently [19]; the latter in particular established an essentially optimal local well-posedness result by exploiting a subtle cancellation feature of MKG, which also plays a role in the present work. We also mention the recent result [8] which establishes global regularity for energy sub-critical data in the $n = 3$ case, in the spirit of earlier work by Bourgain [3].

In the higher dimensional case $n \geq 4$, an essentially optimal local well-posedness result for a model problem closely related to MKG was obtained in [14]. This model problem does not display the crucial cancellation feature of the precise MKG-system which enable us here to go all the way to the critical exponent and global regularity. We mention also that essentially optimal local well-posedness for the exact MKG-system was obtained in [17]. Finally, the recent work [24] established global regularity of equations of MKG-type with data small in a weighted but scaling invariant Besov type space, in the case $n = 4$.

The present paper follows a similar strategy as [21] : one observes that the spatial Gauge connection components A_j, $j = 1, 2, 3$, which are governed by the first equation (3a), may in fact be decomposed into a free wave part and an inhomogeneous term (the second term in the Duhamel formula for A) which in fact obeys a better l^1-Besov type bound (while energy corresponds to l^2)

$$A_j = A_j^{\text{free}} + A_j^{\text{nonlin}}$$

This is important for handling the key difficulty of the MKG-system , which is the equation for ϕ, i. e. the second equation (3b). In fact, we shall verify that the contribution of the term A^{nonlin} to the difficult magnetic interaction term $2iA_j\partial_j\phi$ *in the low-high frequency interaction case* can be suitably bounded when combined with the term $2iA_0\partial_t\phi$, an observation coming from [19]. However, the contribution of the free term A^{free} to the magnetic interaction term

[2]here the continuity is locally in time

is nonperturbative and cannot be handled in this manner. Thus, following the example set in [21], we retain this term into the covariant wave operator[3] $\Box_{A^{\text{free}}}$. More precisely, we shall define a suitable paradifferential wave operator $\Box_A^p$ which incorporates the 'leading part' of $\Box_{A^{\text{free}}}$ while relegating the rest to the source terms on the right.

The key novelty of this paper then is the development of a functional calculus, involving in particular $X^{s,b}$-type as well as atomic null-frame spaces developed in other contexts, for solutions of the general inhomogeneous 'covariant' wave equation

$$\Box_A^p = f$$

This refined functional calculus is necessary to control the nonlinear interaction terms, which become significantly more delicate in the critical dimension than in the setting studied in [21]. In particular, the Strichartz norms themselves appear far from sufficient to handle the present situation. The above covariant wave equation will be solved by means of a suitable approximate parametrix, and we show that this parametrix satisfies many of the same bounds as the usual free wave parametrix, in particular encompassing refined square sum type microlocalized Strichartz norms as well as null-frame spaces. We expect the calculus developed here to be of fundamental importance in other contexts, such as the regularity question of the critical Yang-Mills system and related problems from mathematical physics.

2. Technical preliminaries

Throughout the sequel we shall rely on Littlewood-Paley calculus, both in space as well as space-time. In particular, we constantly invoke the standard Littlewood-Paley localizers P_k, $k \in \mathbf{Z}$, which are defined by

$$\widehat{P_k f} = \chi(\frac{|\xi|}{2^k})\hat{f}(\xi)$$

for functions f defined on $\mathbb{R}^4$. Here χ is a smooth bump function, supported on $[\frac{1}{4}, 4]$, which satisfies the key condition $\sum_{k\in\mathbf{Z}} \chi(\frac{\xi}{2^k}) = 1$ if $\xi > 0$. To measure proximity of the space-time Fourier support to the light cone, we use the concept of *modulation*. Thus we introduce the multipliers Q_j, $j \in \mathbf{Z}$, via

$$\widehat{Q_j f}(\tau, \xi) = \chi(\frac{\big||\tau| - |\xi|\big|}{2^j})\hat{f}(\tau, \xi)$$

with the same χ as before, where $\hat{}$ in this context denotes the space-time Fourier transform. We then refer to 2^j as the modulation of the function. On occasion we shall also use multipliers S_l, which restrict the *space-time frequency* to size $\sim 2^l$. These multipliers allow us to introduce a variety of norms. In particular, for any $p \in [1, \infty)$, we set for any norm $\|\cdot\|_S$

$$\big\|F\big\|_{l^p S} := \big(\sum_{k\in\mathbf{Z}} \big\|P_k F\big\|_S^p\big)^{\frac{1}{p}}$$

We also have the following $X^{s,b}$-type norms, applied to functions localized to spatial frequency $\sim 2^k$:

$$\big\|F\|_{X_p^{s,r}} := 2^{sk}\big(\sum_{j\in\mathbf{Z}} \big[2^{rj}\big\|Q_j F\big\|_{L^2_{t,x}}\big]^p\big)^{\frac{1}{p}},\ p \in [1, \infty),$$

[3]Here we set $A_0^{\text{free}} = 0$.

with the obvious analogue

$$\big\|F\big\|_{X^{s,r}_\infty} := 2^{sk}\sup_{k\in\mathbf{Z}} 2^{rj}\big\|Q_j F\big\|_{L^2_{t,x}}$$

For more refined norms, we shall also have to use multipliers P^ω_l, which localize the homogeneous variable $\frac{\xi}{|\xi|}$ to caps $\omega \subset S^3$ of diameter 2^l, by means of smooth cutoffs. In these situations, we shall assume that for each l a uniformly (in l) finitely overlapping covering of S^3 by caps ω has been chosen with appropriate cutoffs subordinate to these caps. Similar comments apply to the multipliers $P_{\mathcal{C}^{l'}_{k'}}$ which localize to rectangular boxes and will be defined below.

Given a norm $\|\cdot\|_S$ with corresponding space S, we denote by S_k the space of functions in S which are localized to frequency $\sim 2^k$. Furthermore, we denote by

$$S_{k,\pm}$$

the subspace of functions in S_k with Fourier support in the half-space $\tau >< 0$, with τ the Fourier variable dual to t.

3. Function Spaces

There are three function spaces we work with: N, N^*, and S. These are set up to that their dyadic subspaces N_k, N^*_k, and S_k satisfy the following relations:

$$N_k \ = \ L^1(L^2) + X_1^{0,-\frac{1}{2}} \ , \qquad X_1^{0,\frac{1}{2}} \subseteq S_k \subseteq N^*_k \ , \tag{5}$$

Then define:

$$\| F \|^2_N \ = \ \sum_k \| P_k F \|^2_{N_k} \ .$$

We also define $S^\sharp_k$ by

$$\|u\|_{S^\sharp_k} = \|\Box u\|_{N_k} + \|\nabla u\|_{L^\infty L^2}$$

On occasion we need to separate the two characteristic cones $\{\tau = \pm|\xi|\}$. Thus we define

$$\begin{aligned} &N_{k,\pm}, && N_k = N_{k,+}\cap N_{k,-}\\ &S^\sharp_{k,\pm}, && S^\sharp_k = S^\sharp_{k,+} + S^\sharp_{k,-}\\ &N^*_{k,\pm}, && N^*_k = N^*_{k,+} + N^*_{k,-} \end{aligned}$$

Our space S_k scales like L^2 free waves, and is defined by:

$$\| \phi \|^2_{S_k} \ = \ \| \phi \|^2_{S^{str}_k} + \| \phi \|^2_{S^{ang}_k} + \| \phi \|^2_{X^{0,\frac{1}{2}}_\infty} \ ,$$

where:

$$S^{str}_k \ = \ \cap_{\frac{1}{q}+\frac{3/2}{r}\leqslant\frac{3}{4}} 2^{(\frac{1}{q}+\frac{4}{r}-2)k} L^q(L^r) \ , \qquad \| \phi \|^2_{S^{ang}_k} \ = \ \sup_{l<0}\sum_\omega \| P^\omega_l Q_{<k+2l}\phi \|^2_{S^\omega_k(l)} \ , \tag{6}$$

The angular sector norms $S^\omega_k(l)$ are essentially the same as in the wave maps context, see e.g. [27], and defined shortly. Our space of solutions scales like free waves with $\dot{H}^1$ data, with a high modulational gain as in [25]. Thus we set:

$$\| \phi \|^2_{S^1} \ = \ \sum_k \| \nabla_{t,x} P_k \phi \|^2_{S_k} + \| \Box\phi \|^2_{\ell^1 L^2(\dot{H}^{-\frac{1}{2}})} \ . \tag{7}$$

For later reference, we shall also use the norms

$$\|\phi\|_{S^N} := \|\nabla_{t,x}^{N-1}\phi\|_{S^1},\ N \geq 2$$

Returning to $S_k^\omega(l)$, we define these as usual except that we need to add additional square information over smaller radially directed blocks $\mathcal{C}_{k'}(l')$ dimensions $2^{k'} \times (2^{k'+l'})^3$ with appropriate dyadic gains. First define:

$$\begin{aligned}
\| \phi \|_{PW_\omega^\pm(l)} &= \inf_{\phi=\int\phi^{\omega'}} \int_{|\omega-\omega'|\leqslant 2^l} \| \phi^{\omega'} \|_{L^2_{\pm\omega'}(L^\infty_{(\pm\omega')^\perp})} d\omega' \ , \\
\| \phi \|_{NE} &= \sup_\omega \| \not\nabla_\omega \phi \|_{L^\infty_\omega(L^2_{\omega^\perp})} \ ,
\end{aligned}$$

where the norms are with respect to $\ell_\omega^\pm = t \pm \omega\cdot x$ and the transverse variable, while $\not\nabla_\omega$ denotes spatial differentiation in the $(\ell_\omega^+)^\perp$ plane. Now set:

$$\begin{aligned}
(8)\quad \| \phi \|^2_{S_k^\omega(l)} &= \| \phi \|^2_{S_k^{str}} + 2^{-2k}\| \phi \|^2_{NE} + 2^{-3k}\sum_\pm \| Q^\pm\phi \|^2_{PW_\omega^\mp(l)} \\
&+ \sup_{\substack{k'\leqslant k, l'\leqslant 0 \\ k+2l\leqslant k'+l'\leqslant k+l}} \sum_{\mathcal{C}_{k'}(l')} \Big(\| P_{\mathcal{C}_{k'}(l')}\phi \|^2_{S_k^{str}} + 2^{-2k}\| P_{\mathcal{C}_{k'}(l')}\phi \|^2_{NE} \\
&\qquad + 2^{-2k'-k}\| P_{\mathcal{C}_{k'}(l')}\phi \|^2_{L^2(L^\infty)} + 2^{-3(k'+l')}\sum_\pm \| Q^\pm P_{\mathcal{C}_{k'}(l')}\phi \|^2_{PW_\omega^\mp(l)} \Big) \ .
\end{aligned}$$

We remark that an important feature of these norms is that the time-like oriented $L^2(L^\infty)$ block norm gains dyadically from the length of $\mathcal{C}_{k'}(l')$ in the radial direction, while the null oriented $L^2_\omega(L^\infty_{\omega^\perp})$ norms gain from the size of $\mathcal{C}_{k'}(l')$ in the angular direction. We also remark that a useful feature of this setup is:

$$(9)\qquad \Big(\sum_{\mathcal{C}_{k'}} \| P_{\mathcal{C}_{k'}} P_k\phi \|^2_{L^2(L^\infty)}\Big)^{\frac12} \lesssim 2^{k'} 2^{\frac12 k} \| P_k\phi \|_{S_k^{ang}\cap X_\infty^{0,\frac12}} \ ,$$

where $\mathcal{C}_{k'}$ are a finitely overlapping set of cubes of side length $2^{k'}$. This follows by splitting $P_k\phi = Q_{<k'}P_k\phi + Q_{\geqslant k'}P_k\phi$ and using the $S_k^\omega\big(\frac12(k'-k)\big)$ norm for the first term and the $X_\infty^{0,\frac12}$ norm and Bernstein's inequality for the second.

Next we describe some auxiliary spaces of $L^1(L^\infty)$ which will be useful for decomposing the non-linearity. The first is for the hyperbolic part of the solution:

$$\| \phi \|_{Z^{hyp}} = \sum_k \| P_k\phi \|_{Z_k^{hyp}} \ , \quad \| \phi \|^2_{Z_k^{hyp}} = \sup_{l<C} \sum_\omega 2^l \| P_l^\omega Q_{k+2l}\phi \|^2_{L^1(L^\infty)} \ .$$

Note that as defined this space already scales like $\dot{H}^1$ free waves. In addition, note the following useful embedding which is a direct consequence of Bernstein's inequality:

$$(10)\qquad \Box^{-1}\ell^1 L^1(L^2) \subseteq Z^{hyp} \ .$$

The second is for the elliptic part of the solution:

$$\| A \|_{Z_k^{ell}} = \sum_{j<k+C} \| Q_j A \|_{L^1(L^\infty)} \ , \qquad \| A \|_{Z^{ell}} = \sum_k \| P_k A \|_{Z_k^{ell}} \ .$$

We remark that the only purpose of this last norm is to handle sums over Q_j in product estimates involving A in $L^1(L^\infty)$.

Finally, the function spaces for A_0 are much easier to describe as the A_0 equation is elliptic:

$$\|A_0\|_{Y^1}^2 = \|\nabla_{x,t} A_0\|_{L^\infty L^2}^2 + \|A_0\|^2_{L^2_t \dot{H}^{\frac{3}{2}}_x} + \|\partial_t A_0\|^2_{L^2_t \dot{H}^{\frac{1}{2}}_x} .$$

We also have the derivative norms

$$\|A_0\|_{Y^N} := \|\nabla^{N-1}_{t,x} A_0\|_{Y^1},\ N \geq 2$$

4. Decomposing the non-linearity; Statement and use of the core multilinear estimates

Recalling the definition of the currents $J_\alpha = -\Im(\phi \overline{D_\alpha \phi})$ we write the MKG-CG system again here as:

$$\Box A_i = \mathcal{P}_i J_x , \tag{11a}$$

$$\Box_A \phi = 0 , \tag{11b}$$

$$\Delta A_0 = J_0 , \tag{11c}$$

This system will be solved iteratively using the following scheme. We initialize

$$A_i^{(1)} = A_i^{\text{free}}, \qquad A_0^{(1)} = 0, \qquad \phi^{(1)} = \phi^{\text{free}},$$

where A_i^{free} and ϕ^{free} solve the flat wave equation with initial data $A_i[0]$, respectively $\phi[0]$. Given $A_\alpha^{(m)}$, $\phi_\alpha^{(m)}$ and their associated currents $J_\alpha^{(m)}$, we define the next iteration via the equations

$$\Box A_i^{(m+1)} = \mathcal{P}_i J_x^{(m)} , \tag{12a}$$

$$\Box_{A^{(m)}} \phi^{(m+1)} = 0 , \tag{12b}$$

$$\Delta A_0^{(m+1)} = J_0^{(m)} , \tag{12c}$$

with the same initial data $A_i[0]$, respectively $\phi[0]$. Assuming small energy for the initial data $A_i[0]$ and $\phi[0]$ as in (4), we will prove that this Picard type iteration converges in the space S^1. Our starting point is the linear bound

$$\|A_x^{(1)}\|_{S^1} + \|\phi^{(1)}\|_{S^1} \leq C_0 \epsilon_* \tag{13}$$

Then we will inductively establish the bound

$$\|A_x^{(m+1)} - A_x^{(m)}\|_{l^1 S^1} + \|\phi^{(m+1)} - \phi^{(m)}\|_{S^1} + \|A_0^{(m+1)} - A_0^{(m)}\|_{Y^1} \leq (C\epsilon_*)^m \tag{14}$$

for a universal constant $C \geq 2C_0$.

Assuming this holds, passing to the limit as $n \to \infty$ we obtain a Coulomb solution (A, ϕ) to the MKG equation which satisfies the bound

$$\|A_x^{\text{nonlin}}\|_{l^1 S^1} + \|\phi\|_{S^1} + \|A_0\|_{Y^1} \lesssim \epsilon_* \tag{15}$$

The same argument proves uniqueness. If two solutions $(A^{(0)}, \phi^{(0)})$ and $(A^{(1)}, \phi^{(1)})$ have the same Cauchy data for A_x, then the same A_x^{free} is used for both. Thus applying the same series of estimates in this section to the difference of the two exact solutions rather than two approximate solutions we obtain the bound

$$\|A_x^{(0)} - A_x^{(1)}\|_{l^1 S^1} + \|\phi^{(0)} - \phi^{(1)}\|_{S^1} + \|A_0^{(0)} - A_0^{(1)}\|_{Y^1} \lesssim \|\phi^{(0)}[0] - \phi^{(1)}[0]\|_{\dot{H}^1 \times L^2} \tag{16}$$

This bound proves both uniqueness and Lipschitz dependence of the solution with respect to $\phi[0]$. The continuous dependence with respect to $A_x[0]$ is a more delicate issue and will be explained in section 5.

The estimate (14)(m) will follow from (14)($< m$). Summing up (14)($< m$) and (13), we can easily add to our induction hypothesis the bound

$$\|A_x^{(n)} - A_x^{\text{free}}\|_{l^1 S^1} + \|\phi^{(n)}\|_{S^1} + \|A_0^{(n)}\|_{Y^1} \leq 2C_0\epsilon_*, \qquad n \leq m \tag{17}$$

provided that ϵ_* is small enough.

A simple but very useful observation is that, since all iterated $\phi^{(m)}$ solve covariant wave equations, it follows that the associated moments are divergence free, $D^\alpha J_\alpha^{(m)} = 0$. Then, differentiating the equation (12c), we obtain

$$\Delta \partial_t A_0^{(m+1)} \ = \ \partial^i J_i^{(m)} \tag{18}$$

which will be used to estimate the high modulations of A_0. This is the reason why we have completely avoided using $\phi^{(m)}$ in (12b), even though some of the terms in there will be treated perturbatively.

A second fact to keep in mind is that while in terms of formulas this is a one step iteration, in terms of estimates this is really a two step iteration. Precisely, in order to obtain good bounds for $\phi^{(m+1)}$ we will need to reiterate "bad" portions of A_α in terms of $\phi^{(m-1)}$ (but not $A^{(m-1)}$). All this is explained in detail below.

To decompose the non-linearity, the following device will be useful. If $\mathcal{M}(D_{t,x}, D_{t,y})$ is any bilinear operator we set:

$$\begin{aligned}
\mathcal{H}_k \mathcal{M}(\phi, \psi) \ &= \ \sum_{j<k+C} Q_j P_k \mathcal{M}(Q_{<j-C}\phi, Q_{<j-C}\psi) \ , \\
\mathcal{H}_k^* \mathcal{M}(\phi, \psi) \ &= \ \sum_{j<k+C} Q_{<j-C} \mathcal{M}(Q_j P_k \phi, Q_{<j-C}\psi) \ ,
\end{aligned}$$

Now we describe the function spaces and multilinear estimates for each iterative piece of the non-linearity. In this Section we only state the main estimates, and reduce them to appropriate dyadic bounds. These will be proved later in the paper.

4.1. **Estimates for the A_i.** We split the spatial potentials into homogeneous and inhomogeneous parts with respect to $t = 0$ as follows, $A_i^{(m)} = A_i^{\text{free}} + A_i^{\text{nonlin},(m)}$ where the second part solves the linear equation

$$\Box A_i^{\text{nonlin},(m+1)} = -\mathcal{P}_i\big(\phi^{(m)} \nabla_x \phi^{(m)} + |\phi^{(m)}|^2 A_x^{(m)}\big), \qquad A_i^{nonlin}[0] \ = \ 0 \tag{19}$$

In order to establish the $l^1 S^1$ bound in the first term of (14) it suffices to work directly with this equation. However, $A^{(m)}$ also appears in the $\phi^{(m+1)}$ equation (12b), and in order to be able to treat its contribution perturbatively there we also need to control its Z^{hyp} norm (see the subsection below devoted to ϕ). This works for the most part, but there is a part of $A_i^{\text{nonlin},(m)}$ which requires a more delicate treatment, which is:

$$\mathcal{H} A_i^{\text{nonlin},(m)} \ = \ - \sum_{\substack{k,k_i: \\ k<\min\{k_1,k_2\}-C}} \Box^{-1} \mathcal{H}_k \mathcal{P}_i(\phi_{k_1}^{(m-1)} \nabla_x \phi_{k_2}^{(m-1)}) \ . \tag{20}$$

For the good part of $A_i^{\mathrm{nonlin},(m)}$ we will establish the difference bound

(21)
$$\|(A_i^{\mathrm{nonlin},(n)} - \mathcal{H}A_i^{\mathrm{nonlin},(n)}) - (A_i^{\mathrm{nonlin},(n-1)} - \mathcal{H}A_i^{\mathrm{nonlin},(n-1)})\|_{Z^{hyp}} \leq (C\epsilon_*)^{n-1}, \qquad 3 \leq n \leq m+1$$

as well as the summed estimate

$$\|A_i^{\mathrm{nonlin},(n)} - \mathcal{H}A_i^{\mathrm{nonlin},(n)}\|_{Z^{hyp}} \lesssim C_0^2\epsilon_*^2, \qquad 2 \leq n \leq m+1 \tag{22}$$

On the other hand the bad part $\mathcal{H}A_i^{\mathrm{nonlin},(m)}$ will be considered later in combination with the similar part of A_0.

As seen above, we need estimates not only for the equation (19), but also for differences of two consecutive iterations. Thus we are led to consider a more general equation of the form

$$\Box B_i \ = \ \mathcal{P}_i\big(\phi^{(1)}\nabla_x\phi^{(2)} + \phi^{(3)}\phi^{(4)}A_i^{(1)}\big) \ , \quad B_i^{nonlin}[0] \ = \ 0 \ . \tag{23}$$

and its corresponding bad part,

$$\mathcal{H}B_i \ = \ - \sum_{\substack{k,k_i: \\ k<\min\{k_1,k_2\}-C}} \Box^{-1}\mathcal{H}_k\mathcal{P}_i(\phi_{k_1}^{(1)}\nabla_x\phi_{k_2}^{(2)}) \ . \tag{24}$$

Then we will show:

Proposition 4.1. *One has the following iterative estimates for the functions B_i and $\mathcal{H}B_i$ defined above:*

$$\| B_i \|_{\ell^1 S^1} \ \lesssim \ \| \phi^{(1)} \|_{S^1} \| \phi^{(2)} \|_{S^1} + \| A_i^{(1)} \|_{S^1} \| \phi^{(3)} \|_{S^1} \| \phi^{(4)} \|_{S^1} \ , \tag{25}$$

$$\| B_i - \mathcal{H}B_i \|_{Z^{hyp}} \ \lesssim \ \| \phi^{(1)} \|_{S^1} \| \phi^{(2)} \|_{S^1} \ . \tag{26}$$

In combination with our induction hypothesis, the estimate (25) leads to the l^1S^1 bound for the first term in (14). Similarly, the estimate (26) yields the Z^{hyp} bounds for the good part of $A_i^{\mathrm{nonlin},(m)}$ in (21) and (22).

Decomposition of the proof of estimates (25) *and* (26). For estimate (25) we begin with the high modulational bounds:

$$\| \phi^{(1)}\nabla_{t,x}\phi^{(2)} \|_{\ell^1 L^2(\dot{H}^{-\frac{1}{2}})} \ \lesssim \ \| \phi^{(1)} \|_{S^1} \| \phi^{(2)} \|_{S^1} \ , \tag{27}$$

$$\| \phi^{(3)}\phi^{(4)}\phi^{(5)} \|_{\ell^1 L^2(\dot{H}^{-\frac{1}{2}})} \ \lesssim \ \| \phi^{(3)} \|_{S^1} \| \phi^{(4)} \|_{S^1} \| \phi^{(5)} \|_{S^1} \ . \tag{28}$$

Next, for the cubic terms in A_i^{nonlin} both (25) and (26) follow once we can show:

$$\| \phi^{(3)}\phi^{(4)}\phi^{(5)} \|_{\ell^1 L^1(L^2)} \ \lesssim \ \| \phi^{(3)} \|_{S^1} \| \phi^{(4)} \|_{S^1} \| \phi^{(5)} \|_{S^1} \ . \tag{29}$$

Returning to the quadratic part, we employ the notation:

$$\mathcal{N}_{ij}(\phi^{(1)}, \phi^{(2)}) \ = \ \partial_i\phi^{(1)}\partial_j\phi^{(2)} - \partial_j\phi^{(1)}\partial_i\phi^{(2)} \ ,$$

which allows us to write:

$$\mathcal{P}_j(\phi^{(1)}\nabla_x\phi^{(2)}) \ = \ \Delta^{-1}\nabla^i\mathcal{N}_{ij}(\phi^{(1)}, \phi^{(2)}) \ . \tag{30}$$

Then for the quadratic part of (25) it suffices to show:

$$\| \nabla_x\Delta^{-1}\mathcal{N}(\phi^{(1)}, \phi^{(2)}) \|_{\ell^1 N} \ \lesssim \ \| \phi^{(1)} \|_{S^1} \| \phi^{(2)} \|_{S^1} \ , \tag{31}$$

where $\mathcal{N}$ denotes any instance of the $\mathcal{N}_{ij}$. Finally, making an analogous definition to (24) for each term in (30) we need to show:

$$\| (I-\mathcal{H})\nabla_x \Delta^{-1}\mathcal{N}(\phi^{(1)},\phi^{(2)}) \|_{\Box Z^{hyp}} \lesssim \| \phi^{(1)} \|_{S^1} \| \phi^{(2)} \|_{S^1} . \tag{32}$$

□

4.2. **Estimates for A_0 and $\partial_t A_0$.** The estimates for temporal potentials are analogous to those above, but using instead the equations (12c) and (18). In expanded form these are written as

$$\begin{aligned} \Delta A_0^{(m+1)} &= -\Im(\phi^{(m)}\partial_t\bar{\phi}^{(m)}) - A_0^{(m)}|\phi^{(m)}|^2 \\ \Delta\partial_t A_0^{(m+1)} &= -\partial^i\Im(\phi^{(m)}\partial_i\bar{\phi}^{(m)}) - \partial^i(A_i^{(m)}|\phi^{(m)}|^2) \end{aligned}$$

In a first approximation, from $A_0^{(m)}$ we isolate to output of bilinear *high* $\times$ *high* $\to$ *low* interactions, namely

$$A_0^{(m+1),hh} = \sum_{\substack{k,k_i:\\ k<\min\{k_1,k_2\}-C}} \Delta^{-1}P_k(\phi_{k_1}^{(m)}\partial_t\phi_{k_2}^{(m)}) .$$

To bring the analysis to the same level as in the case of the A_j's, in analogy with (20), we also define the component $\mathcal{H}A_0^{(m+1)}$ of $A_0^{(m+1),hh}$,

$$\mathcal{H}A_0^{(m+1)} = -\sum_{\substack{k,k_i:\\ k<\min\{k_1,k_2\}-C}} \Delta^{-1}\mathcal{H}_k(\phi_{k_1}^{(m)}\partial_t\phi_{k_2}^{(m)}) . \tag{33}$$

In addition to the estimates included in (14), we also need the some similar bounds for $A^{(m+1),hh)}$ as well as bounds for the good differences for $3\le n\le m+1$:

$$\begin{aligned} \|A_0^{(n),hh} - A_0^{(n-1),hh}\|_{\ell^1 L^2(\dot{H}^{\frac{3}{2}})} &\le (C\epsilon)^{n-1} \\ \|(A_0^{(n)} - A_0^{(n),hh}) - (A_0^{(n-1)} - A_0^{(n-1),hh})\|_{\ell^1 L^1(L^\infty)} &\le (C\epsilon)^{n-1} \\ \|(\mathcal{H}A_0^{(n)} - A_0^{(n),hh}) - (\mathcal{H}A_0^{(n-1)} - A_0^{(n-1),hh})\|_{Z^{ell}} &\le (C\epsilon)^{n-1} \end{aligned} \tag{34}$$

Passing to differences we arrive at a system of equations with multilinear inhomogeneities of the form:

$$\Delta B_0 = \phi^{(1)}\partial_t\phi^{(2)} + \phi^{(3)}\phi^{(4)}A_0^{(1)} , \tag{35}$$

$$\Delta\partial_t B_0 = \nabla_x\big(\phi^{(1)}\nabla_x\phi^{(2)} + \phi^{(3)}\phi^{(4)}A_i^{(1)}\big) . \tag{36}$$

As above we write:

$$B_0^{hh} = \sum_{\substack{k,k_i:\\ k<\min\{k_1,k_2\}-C}} P_k(\phi_{k_1}^{(1)}\partial_t\phi_{k_2}^{(2)}) .$$

respectively

$$\mathcal{H}B_0 = -\sum_{\substack{k,k_i:\\ k<\min\{k_1,k_2\}-C}} \Delta^{-1}\mathcal{H}_k(\phi_{k_1}^{(1)}\partial_t\phi_{k_2}^{(2)}) . \tag{37}$$

Then we will show:

Proposition 4.2. *One has the following estimates for B_0 and $\partial_t B_0$ defined above:*

$$\| \nabla_{t,x} B_0 \|_{L^\infty(L^2)} \lesssim \| \phi^{(1)} \|_{S^1} \| \phi^{(2)} \|_{S^1} + \| A_\alpha^{(1)} \|_{L^\infty(\dot H^1)} \| \phi^{(3)} \|_{S^1} \| \phi^{(4)} \|_{S^1} , \tag{38}$$

$$\| (B_0, B_0^{hh}) \|_{\ell^1 L^2(\dot H^{\frac{3}{2}})} \lesssim \| \phi^{(1)} \|_{S^1} \| \phi^{(2)} \|_{S^1} + \| A_0^{(1)} \|_{L^2(\dot H^{\frac{3}{2}})} \| \phi^{(3)} \|_{S^1} \| \phi^{(4)} \|_{S^1} , \tag{39}$$

$$\| \partial_t B_0 \|_{\ell^1 L^2(\dot H^{\frac{1}{2}})} \lesssim \| \phi^{(1)} \|_{S^1} \| \phi^{(2)} \|_{S^1} + \| A_i^{(1)} \|_{S^1} \| \phi^{(3)} \|_{S^1} \| \phi^{(4)} \|_{S^1} , \tag{40}$$

$$\| B_0 - B_0^{hh} \|_{\ell^1 L^1(L^\infty)} \lesssim \| \phi^{(1)} \|_{S^1} \| \phi^{(2)} \|_{S^1} + \| A_0^{(1)} \|_{L^2(\dot H^{\frac{3}{2}})} \| \phi^{(3)} \|_{S^1} \| \phi^{(4)} \|_{S^1} , \tag{41}$$

$$\| B_0^{hh} - \mathcal{H} B_0 \|_{Z^{ell}} \lesssim \| \phi^{(1)} \|_{S^1} \| \phi^{(2)} \|_{S^1} . \tag{42}$$

The bound for the last term in (14) follows from (38), (39) and (40). The three bounds in (34) are consequences of (39), (41) and (42).

Outline of proof of estimates (39)–(41). Estimate (38) follows from Sobolev's embedding, while the proofs of (39) and (40) either follow immediately from (27) and (28) above, or in the case of A_0^{hh} from the estimates used to produce those bounds. Estimate (41) follows from similar considerations. In detail, the estimates (38) - (41) will be proved in subsection 4.4. On the other hand (42) is more microlocal in nature and follows from calculations similar to those in the proof of (32) listed above. It will be proved as a consequence of Theorem 12.2 below. □

4.3. **Estimates for ϕ.** We now turn to the heart of the matter, which is the covariant wave equation (18) for $\phi^{(m+1)}$. This cannot be viewed as an equation of type $\Box\phi = \textit{perturbative}$, and therein lies the difficulty. To address this issue we identify the nonperturbative part, and add it to the main operator $\Box$ to obtain a paradifferential type magnetic d'Alembertian. Precisely, we define the leading order paradifferential approximation to the covariant $\Box_A$ equation using only the free part A_i^{free} of $A_i^{(m)}$, which is independent of n:

$$\Box_A^p = \Box - 2i \sum_k P_{<k-C} A^{\text{free},j} P_k \partial_j , \tag{43}$$

Fortunately, this operator does not depend on m. Then the equation for $\phi^{(m+1)}$ takes the form

$$\Box_A^p \phi^{(m+1)} = \mathcal{M}(A^{(m)}, \phi^{(m+1)}) \tag{44}$$

where $\mathcal{M}(A^{(m)}, \phi^{(m+1)})$ is given by

$$\begin{aligned} \mathcal{M}(A^{(m)}, \phi^{(m+1)}) &= 2i\big(A^{(m),\alpha}\partial_\alpha \phi^{(m+1)} - \sum_k P_{<k-C} A^{\text{free},j} \partial_j P_k \phi^{(m+1)}\big) \\ &\quad - \big(i\partial_t A_0^{(m)} \phi^{(m+1)} + A^{(m),\alpha} A_\alpha^{(m)} \phi^{(m+1)}\big) \\ &:= \mathcal{M}^1(A^{(m)}, \phi^{(m+1)}) + \mathcal{M}^2(A^{(m)}, \phi^{(m+1)}) \end{aligned} \tag{45}$$

We further write

$$\mathcal{M}^1 = \mathcal{N} + \mathcal{N}_0$$

where $\mathcal{N}$ contains the terms of the form $A^j \partial_j \phi$ and $\mathcal{N}_0$ contains the terms of the form $A^0 \partial_0 \phi$. We remark that $\mathcal{N}$ exhibits a null structure. Indeed, using the divergence free character of A_i we write:

$$A_j = \nabla^i \Delta^{-1} F_{ij} , \qquad F_{ij} = \nabla_i A_j - \nabla_j A_i ,$$

so that:

$$\mathcal{N}(A_x,\phi) = A^i\partial_i\phi \ = \ \frac{1}{2}\sum_{i<j}\mathcal{N}_{ij}(\nabla_i\Delta^{-1}A_j,\phi) \ .$$

From here on we take the last expression as the definition of $\mathcal{N}$. This will allow us to retain the null form while discarding the divergence free condition on finer decompositions of A_x.

Now the idea is to first produce a parametrix for $\Box_A^p$, and then to estimate the right hand side of (44) perturbatively. The linear estimate for the magnetic wave operator $\Box_A^p$ is one of the key points of the paper, and has the following form:

Theorem 4.3 (Linear estimates for ϕ)**.** *Let $\Box_A^p$ be the paradifferential gauge-covariant wave operator defined on line* (43)*, and suppose that $\Box A^{free}=0$ with $\|\, A^{free}[0]\,\|_{\dot H^1\times L^2}\leqslant \epsilon$. If ϵ is sufficiently small then we have:*

$$\|\,\phi\,\|_{S^1} \ \lesssim \ \|\,\phi[0]\,\|_{\dot H^1\times L^2} + \|\,\Box_A^p\phi\,\|_{N\cap l^1L^2(\dot H^{-\frac12})} \ . \tag{46}$$

Section 6 is devoted to the proof of this result.

In order to solve the equation (44) it remains to estimate the the right hand side of (44),

$$\|\mathcal{M}(A^{(m)},\phi^{(m+1)})\|_{N\cap L^2(\dot H^{-\frac12})}\lesssim \epsilon_*\|\phi^{(m+1)}\|_{S^1} \tag{47}$$

which is applied with $\phi=\phi^{(m)}$. In order to estimate the difference $\phi^{(m+1)}-\phi^{(m)}$ we need in addition to show that

$$\|\mathcal{M}(A^{(m)},\phi)-\mathcal{M}(A^{(m-1)},\phi)\|_{N\cap L^2(\dot H^{-\frac12})}\lesssim (C\epsilon)^{m-1}\|\phi\|_S \tag{48}$$

which is then applied to $\phi=\phi^{(m)}$. To prove (47) and (48) we peel off some easier cases before we arrive at the heart of the matter.

Step 1:(*The $\mathcal{M}^2$ term.*) For this it suffices to have the following estimates:

$$\|\,\partial_t A_0\phi\,\|_{\ell^1L^2(\dot H^{-\frac12})\cap L^1(L^2)} \ \lesssim \ \|\,\partial_t A_0\,\|_{L^2(\dot H^{\frac12})\cap L^\infty(L^2)}\|\,\phi\,\|_{S^1} \ , \tag{49}$$

$$\|\,A_0^{(1)}A_0^{(2)}\phi\,\|_{\ell^1L^2(\dot H^{-\frac12})\cap L^1(L^2)} \ \lesssim \ \prod_{i=1,2}\|\,A_0^{(i)}\,\|_{L^2(\dot H^{\frac32})\cap L^\infty(\dot H^1)}\|\,\phi\,\|_{S^1} \ . \tag{50}$$

which are proved using only global Strichartz type bounds and Sobolev embeddings.

Step 2:(*High modulation bounds for $\mathcal{M}^1$*) To establish the $L^2(\dot H^{-\frac12})$ bound for $\mathcal{M}^1$ we use (27) and the following two estimates:

$$\sum_k\|\,P_{<k-C}A_i^{free}\,\nabla_xP_k\phi\,\|_{L^2(\dot H^{-\frac12})} \ \lesssim \ \|\,A_i^{free}\,\|_{S^1}\|\,\phi\,\|_{S^1} \ , \tag{51}$$

$$\|\,A_0\partial_t\phi\,\|_{\ell^1L^2(\dot H^{-\frac12})} \ \lesssim \ \|\,A_0\,\|_{L^2(\dot H^{\frac32})}\|\,\phi\,\|_{S^1} \ . \tag{52}$$

It remains to prove the N bounds for $\mathcal{M}^1$. At first we separately consider $\mathcal{N}$ and $\mathcal{N}_0$.

Step 3:(*Peeling off the good parts of $\mathcal{N}$*) The first step will be to show that we can restrict our attention to *low* $\times$ *high* interactions in $\mathcal{N}$. For this we define the component $\mathcal{N}^{lowhi}$ by

$$\mathcal{N}^{lowhi}(A_x,\phi) = \ \sum_k\mathcal{N}(P_{<k-C}A_i,P_k\phi) \ ,$$

To estimate the difference $\mathcal{N}-\mathcal{N}^{lowhi}$ we will prove the null form estimate

$$\|\,\mathcal{N}(A_x,\phi)-\mathcal{N}^{lowhi}(A_x,\phi)\,\|_N \ \lesssim \ \|\,A_x\,\|_{S^1}\|\,\phi\,\|_{S^1} \ . \tag{53}$$

We remark that once this is done, the free part A_x^{free} has been taken care of and will no longer appear.

For the low-high interactions in $\mathcal{N}$ we still have the null structure to use, but the balance of frequencies is unfavourable. For the most part we can still bound $\mathcal{N}^{lowhi}$ via a null form bilinear $S^1 \times S^1$ estimate; the exception to this is the case of *high* $\times$ *low* $\to$ *low* modulation interactions. We group these in an expression denoted by $\mathcal{H}^*\mathcal{N}^{lowhi}$, which is given by

$$\mathcal{H}^*\mathcal{N}^{lowhi}(A_j, \phi) \;=\; \sum_{\substack{k,k': \\ k'<k-C}} \mathcal{H}^*_{k'}\mathcal{N}(A_j, P_k\phi) \ ,$$

For the difference we will prove the bilinear null form estimate

$$\| \mathcal{N}^{lowhi}(A_x, \phi) - \mathcal{H}^*\mathcal{N}^{lowhi}(A_x, \phi) \|_N \;\lesssim\; \| A_i^{(1)} \|_{\ell^1 S^1} \| \phi^{(1)} \|_{S^1} \ , \tag{54}$$

in the last section of the paper.

The *high* $\times$ *low* $\to$ *low* modulation interactions contained in $\mathcal{H}^*\mathcal{N}^{lowhi}$ can no longer be estimated using the S^1 norm of A_x, instead we need the stronger Z^{hyp} norm. This leads us to introduce the expression

$$\mathcal{H}^*\mathcal{N}^{lowhi}(\mathcal{H}A_x, \phi)$$

In view of the estimates (26), (22) and (21), the N bound for the expression

$$\mathcal{H}^*\mathcal{N}^{lowhi}(A_x^{(m)}, \phi) - \mathcal{H}^*\mathcal{N}^{lowhi}(\mathcal{H}A_x^{(m-1)}, \phi),$$

as well as the corresponding differences, is a consequence of the following

$$\| \mathcal{H}^*\mathcal{N}^{lowhi}(A_x, \phi) \|_N \;\lesssim\; \| A_x \|_{Z^{hyp}} \| \phi \|_{S^1} \ . \tag{55}$$

After peeling all the good contributions, the only part of $\mathcal{N}$ which has not been estimated so far is

$$\mathcal{H}^*\mathcal{N}^{lowhi}(\mathcal{H}A_x^{(m)}, \phi^{(m+1)})$$

Step 4:(*Peeling off the good parts of* $\mathcal{N}_0$) The arguments here follow the same lines as before. However, since A_0 solves an elliptic equation, a larger portion of all cases can be treated in a direct fashion via Strichartz estimates and Sobolev embeddings, which leaves less in terms of bilinear estimates to be proved later. Recall that $\mathcal{N}_0(A_0, \phi) = A_0 \partial_t \phi$. Modifying slightly the setup before we set

$$\mathcal{N}_0^{lowhi}(A_0, \phi) = \sum_k \mathcal{N}_0(P_{<k-C}Q_{<k-C}A_0, P_k\phi) \ ,$$

Then the difference is estimated via

$$\|\mathcal{N}_0(A_0, \phi) - \mathcal{N}_0^{lowhi}(A_0, \phi)\|_{L^1L^2} \lesssim \|A_0\|_{Y^1} \|\phi\|_S \tag{56}$$

The extra step we can take in the case of $\mathcal{N}_0$ is to replace $A_0^{(n)}$ by $A_0^{(n),hh}$. By (39) and the second part of (34) the difference is estimated using

$$\|\mathcal{N}_0^{lowhi}(A_0, \phi) - \mathcal{N}_0^{lowhi}(A_0^{hh}, \phi)\|_{L^1L^2} \lesssim \|A_0 - A_0^{hh}\|_{\ell^1 L^1 L^\infty} \|\phi\|_S \tag{57}$$

The last two steps are similar to the case of A_x. First we introduce

$$\mathcal{H}^*\mathcal{N}_0^{lowhi}(A_0^{hh}, \phi) \;=\; \sum_{\substack{k,k': \\ k'<k-C}} \mathcal{H}^*_{k'}\mathcal{N}(A_0^{hh}, P_k\phi) \ ,$$

In view of (39) and the first part of (34), the corresponding differences are estimated using

$$\| \mathcal{N}_0^{lowhi}(A_0^{hh},\phi) - \mathcal{H}^*\mathcal{N}_0^{lowhi}(A_0^{hh},\phi) \|_N \lesssim \| A_0^{hh} \|_{\ell^1 L^2(\dot H^{\frac{3}{2}})} \| \phi \|_{S^1} \ , \tag{58}$$

Finally, replace this by $\mathcal{H}^*\mathcal{N}_0^{lowhi}(\mathcal{H}A_0,\phi)$. By (42) and the third part of (34) for the difference it suffices to have the estimate

$$\| \mathcal{H}^*\mathcal{N}_0^{lowhi}(A_0^{hh},\phi) - \mathcal{H}^*\mathcal{N}_0^{lowhi}(\mathcal{H}A_0,\phi) \|_N \lesssim \| A_0^{(1),hh} - \mathcal{H}A_0^{(1)} \|_{Z^{ell}} \| \phi^{(1)} \|_{S^1} \ . \tag{59}$$

After peeling all the good contributions, the only part of $\mathcal{N}_0$ which has not been estimated so far is

$$\mathcal{H}^*\mathcal{N}_0^{lowhi}(\mathcal{H}A_0^{(m)},\phi^{(m+1)})$$

Step 5:(*Reduction of the remaining terms to quadrilinear null form bounds*) So far we have one portion of $\mathcal{N}$ and one portion of $\mathcal{N}_0$ left to estimate. We collect the two together in the expression

$$R^{(n+1)} = -\mathcal{H}^*\mathcal{N}_0^{lowhi}(\mathcal{H}A_x^{(m)},\phi^{(m+1)}) + \mathcal{H}^*\mathcal{N}^{lowhi}(\mathcal{H}A_0^{(m)},\phi^{(m+1)})$$

Here we recall that $\mathcal{H}A_x^{(m)}$ and $\mathcal{H}A_0^{(m)}$ are bilinear expressions in $\phi^{(m-1)}$.

The key idea is that there is a cancellation that occurs between the two terms above, which is why they need to be treated together rather than separately. The trilinear estimate that needs to be proved in this case for the trilinear expression

$$R(\phi^{(1)},\phi^{(2)},\phi^{(3)}) = -\mathcal{H}^*\mathcal{N}_0^{lowhi}(\mathcal{H}A_x(\phi^{(1)},\phi^{(2)}),\phi^{(3)}) + \mathcal{H}^*\mathcal{N}^{lowhi}(\mathcal{H}A_0(\phi^{(1)},\phi^{(2)}),\phi^{3})$$

is

$$\| R(\phi^{(1)},\phi^{(2)},\phi^{(3)}) \|_N \ \lesssim \ \| \phi^{(1)} \|_{S^1} \| \phi^{(2)} \|_{S^1} \| \phi^{(3)} \|_{S^1} \ , \tag{60}$$

Expanding everything into $\phi^{(i)}$, and performing some algebraic manipulations[4] we have:

$$R(\phi^{(1)},\phi^{(2)},\phi^{(3)}) \ = \ \mathcal{Q}_1 - \mathcal{Q}_2 - \mathcal{Q}_3 \ ,$$

where

$$\mathcal{Q}_1 \ = \ \mathcal{H}^*\big(\Box^{-1}\mathcal{H}(\phi^{(1)}\partial_\alpha\phi^{(2)})\cdot\partial^\alpha\phi^{(3)}\big) \ , \tag{61}$$

$$\mathcal{Q}_2 \ = \ \mathcal{H}^*\big(\Delta^{-1}\Box^{-1}\partial_t\partial_\alpha\mathcal{H}(\phi^{(1)}\partial^\alpha\phi^{(2)})\cdot\partial_t\phi^{(3)}\big) \ , \tag{62}$$

$$\mathcal{Q}_3 \ = \ \mathcal{H}^*\big(\Delta^{-1}\Box^{-1}\partial_\alpha\partial^i\mathcal{H}(\phi^{(1)}\partial_i\phi^{(2)})\cdot\partial^\alpha\phi^{(3)}\big) \ . \tag{63}$$

For each of these we prove (60) separately, which concludes our estimates for $\mathcal{M}^{quad}$.

4.4. **Proof of the L^p product estimates.** Before continuing with the main thrust of the paper, we pause here to dispense with the easiest cases of the multilinear estimates listed above. These are (27), (28), (29), (38), (39), (40), (41), (51), (52), (57), (49), and (50). For the most part these can be broken down into the following estimates, which are immediate consequences of Hölder's and Bernstein's inequalitites:

Lemma 4.4 (Core generic product estimates)**.** *One has the following dyadic bounds:*

$$\| P_k(A_{k_1}\phi_{k_2}) \|_{L^1(L^2)} \ \lesssim \ 2^{\delta(k-\max\{k_i\})}2^{-\delta|k_1-k_2|} \| A_{k_1} \|_{L^2(\dot H^{\frac{1}{2}})} \| \phi_{k_2} \|_{L^2(\dot W^{6,\frac{1}{6}})} \ , \tag{64}$$

$$\| P_k(\phi_{k_1}^{(1)}\phi_{k_2}^{(2)}) \|_{L^2(\dot H^{-\frac{1}{2}})} \ \lesssim \ 2^{\delta(k-\max\{k_i\})}2^{-\delta|k_1-k_2|} \| \phi_{k_1}^{(1)} \|_{L^\infty(L^2)} \| \phi_{k_2}^{(2)} \|_{L^2(\dot W^{6,\frac{1}{6}})} \ , \tag{65}$$

[4]See the Appendix for a calculation done without the clutter of abstract index notation.

Proof of estimate (27). This follows at once from summing over (65) and the inclusion:

$$S^1 \ \subseteq \ L^2(\dot{W}^{6,\frac{1}{6}}) \ . \tag{66}$$

□

Proof of estimate (28). This follows from the inclusion $S^1 \cdot S^1 \subseteq L^\infty(L^2)$, a simple consequence of energy estimates and $\dot{H}^1 \subseteq L^4$, and then summing over (65). □

Proof of estimate (29). This follows from summing over (64) after using (66) and the bilinear embedding:

$$B \cdot B \subseteq L^2(\dot{H}^{\frac{1}{2}}) \ , \qquad B \ = \ L^2(\dot{W}^{6,\frac{1}{6}}) \cap L^\infty(\dot{H}^1) \ . \tag{67}$$

This last estimate is a straightforward application of trichotomy, putting the high frequency term in the energy norm. The standard details are left to the reader. □

Proof of estimate (38). This is immediate from Sobolev embeddings involving $L^4, L^{\frac{4}{3}}, L^2$. □

Proof of estimates (39)*,* (40)*,* (51)*, and* (52). These follow from (65) as in estimates (27) and (28) above. We use:

$$L^2(\dot{H}^{\frac{3}{2}}) \ \subseteq \ L^2(\dot{W}^{6,\frac{1}{6}}) \ , \tag{68}$$

as a replacement for (66) when necessary. □

Proof of estimate (41). First notice that the desired bound for the cubic terms follows from (64) and estimates similar to those above, and $\Delta^{-1}\ell^1 L^1(L^2) \subseteq \ell^1 L^1(L^\infty)$. On the other hand, for the quadratic term one has:

$$\| P_k(\phi^{(1)}_{k_1}\partial_t\phi^{(2)}_{k_2}) \|_{L^1(L^\infty)} \ \lesssim \ 2^{2k}2^{-\frac{1}{2}|k_1-k_2|}\| \phi^{(1)}_{k_1} \|_{L^2(\dot{W}^{6,\frac{1}{6}})}\| \partial_t\phi^{(1)}_{k_1} \|_{L^2(\dot{W}^{6,\frac{1}{6}})} \ .$$

when $k = \max\{k_i\} + O(1)$. □

Proof of estimate (49). This is again an immediate consequence of (64), (65), and (66) □

Proof of estimate (50). For the $L^2(\dot{H}^{-\frac{1}{2}})$ we can use (65) once we place the product of A_0 in $L^\infty(L^2)$ via $\dot{H}^1 \subseteq L^4$. On the other hand the $L^1(L^2)$ estimate comes from (64) and using (67) for the produce of A_0 and (66) for ϕ. □

Proof of estimate (56). First decompose:

$$A_0\partial_t\phi - \mathcal{N}_0^{lowhi}(A_0,\phi) \ = \ T_1 + T_2 \ ,$$

where:

$$T_1 \ = \ \sum_k P_{\geqslant k-C}A_0\partial_t\phi_k \ ,$$

$$T_2 \ = \ P_{<k-C}Q_{>k-C}A_0\partial_t\phi_k \tag{69}$$

We'll show each of $T_i \in L^1(L^2)$. For T_1 this follows from (64) and using:

$$\| P_{\geqslant k-C}A_0 \|_{L^2(\dot{H}^{\frac{1}{2}})} \ \lesssim \ \sum_{k'} 2^{-k'}\| P_{k'}A_0 \|_{L^2(\dot{H}^{\frac{3}{2}})} \ , \quad \| \partial_t\phi_k \|_{L^2(\dot{W}^{6,\frac{1}{6}})} \ \lesssim \ 2^k\| \phi_k \|_{S^1} \ .$$

The bound for T_2 is similar except we use:

$$\| P_{<k-C} Q_{\geqslant k-C} A_0 \|_{L^2(\dot H^{\frac12})} \lesssim 2^{-k} \| \partial_t A_0 \|_{L^2(\dot H^{\frac12})} .$$

The bound for $\|T_2\|_{L^1 L^2}$ follows from (41), placing the second factor in $L^\infty(L^2)$. □

Proof of estimate (57). This follows from (41) by placing again the second factor in $L^\infty(L^2)$. □

We remark that at this point we have reduced the proof of our main Theorem 1 to the demonstration of the linear estimates in Theorem 4.3 in Section 6, and then proving the multilinear estimates (31), (32), (42), (53), (54), (55), (58), (59), and finally (60) for each of the expressions (61)–(63). These bounds are dealt with in the last section of the paper.

5. Higher regularity and continuous dependence

In the previous section we have constructed solutions which are small in S^1 for all small energy data. Here we complete the proof of Theorem 1 by discusing the remaining issues, namely higher regularity and continuous dependence. The ideas here are fairly standard, and in particular closely resemble the similar proof for wave maps, see [30]. For that reason we merely outline the arguments that follow.

5.1. **Higher regularity.** The goal here is to establish the bound

$$\|A_x^{\mathrm{nonlin}}\|_{l^1 S^N} + \|\phi\|_{S^N} + \|A_0\|_{Y^N} \lesssim \|(A_x[0], \phi[0])\|_{\dot H^N \times \dot H^{N-1}} \tag{70}$$

for all small energy data MKG solutions. This is done by a standard iterative procedure, and we explain here how to obtain bounds on $\nabla_x A_x$, $\nabla_x A_0$, $\nabla_x \phi$. In fact, this is accomplished by using the same estimates as before. Commencing with A_x, A_0, recall that we have

$$\Box A_i^{\mathrm{nonlin},(m+1)} = -\mathcal{P}_i\big(\phi^{(m)} \nabla_x \phi^{(m)} + |\phi^{(m)}|^2 A_x^{(m)}\big)$$

whence we see that $\nabla A_i^{\mathrm{nonlin},(m+1)}$ satisfies a wave equation whose source term is a multilinear expression like the one for $\Box A_i^{\mathrm{nonlin},(m+1)}$ but with one factor $\phi^{(m)}$ or $A_x^{(m)}$ replaced by $\nabla_x \phi^{(m)}$, respectively $\nabla_x A_x^{(m)}$. Using the same multilinear estimates as before, one then concludes that

$$\big\|\nabla_x A_i^{\mathrm{nonlin},(m+1)} - \nabla_x A_i^{\mathrm{nonlin},(m)}\big\|_{l^1 S^1} \lesssim (C\epsilon_*)^m,$$

$$\big\|(1-\mathcal{H})\nabla_x A_i^{\mathrm{nonlin},(m+1)} - (1-\mathcal{H})\nabla_x A_i^{\mathrm{nonlin},(m)}\big\|_{Z^{hyp}} \lesssim (C\epsilon_*)^m$$

with implicit constant depending on $\|\nabla_x \phi[0,\cdot]\|_{\dot H^1 \times L^2} + \|\nabla_x A_i[0]\|_{\dot H^1 \times L^2}$. A similar argument applies to $\nabla_x A_0$, $\nabla_x \partial_t A_0$. It remains to bound $\nabla_x \phi^{(m+1)}$. Here we recall (44)

$$\Box_A^p \phi^{(m+1)} = \mathcal{M}(A^{(m)}, \phi^{(m+1)}),$$

which leads to

$$\Box_A^p(\nabla_x \phi^{(m+1)}) = \nabla_x \mathcal{M}(A^{(m)}, \phi^{(m+1)}) + 2i \sum_k \nabla_x P_{<k-C} A^{free,j} P_k \partial_j \phi^{(m+1)}$$

From (131), we find with $k_1 < k_2 - C$

$$\big\|\nabla_x P_{k_1} A^{free,j} \partial_j P_{k_2} \phi^{(m+1)}\big\|_N \lesssim 2^{-\delta|k_1-k_2|} \|P_{k_1} A^{free,j}\|_{S^1} \|\nabla_x P_{k_2} \phi^{(m+1)}\|_{S^1},$$

from which we easily infer

$$\Big\|2i\sum_k \nabla_x P_{<k-C}A^{free,j}P_k\partial_j\phi^{(m+1)}\Big\|_N \lesssim \|A_x^{free}\|_{S^1}\|\nabla_x\phi\|_{S^1}$$

On the other hand, the term

$$\nabla_x\mathcal{M}(A^{(m)},\phi^{(m+1)})$$

is again a multilinear expression in $\phi^{(k)}, A^{(k)}, \nabla_x\phi^{(k)}, \nabla_x A^{(k)}$, $k\in\{m,m-1\}$, with at most one derivative factor in each monomial. Thus the multilinear estimates from above apply.

5.2. **Frequency envelope bounds.** The S^1 bounds for the small data solutions can be supplemented with corresponding frequency envelope bounds in a standard manner. Precisely, suppose that $\{c_k\}$ is a $\dot H^1\times L^2$ frequency envelope for the data, in the sense that

$$\|P_k(A_x[0],\phi[0])\|_{\dot H^1\times L^2}\leq c_k, \qquad \|(A_x[0],\phi[0])\|_{\dot H^1\times L^2}\approx \|c_k\|_{l^2}\leq\epsilon\ll 1 \tag{71}$$

Assume also that $\{c_k\}$ is slowly varying,

$$|c_k/c_j|\leq 2^{\delta|j-k|}, \qquad \delta\ll 1$$

Then we have a similar bound for the solutions,

$$\|P_kA_x^{\text{nonlin}}\|_{S^1}\lesssim c_k^2, \qquad \|P_k\phi\|_{S^1}+\|P_kA_0\|_{Y^1}\lesssim c_k \tag{72}$$

The proof of these bounds is a straightforward consequence of the estimates in the previous section, since we have off-diagonal decay in all of our multilinear estimates.

5.3. **Weak Lipschitz dependence on data.** While establishing Lipschitz dependence on data in the energy norm would be desirable, this does not seem to hold because of the free wave component of the magnetic wave equation. Instead, we have a weaker bound for the difference of two solutions $(A_x^{(1)},\phi^{(1)})$ and $(A_x^{(2)},\phi^{(2)})$, of the form

$$\|A_x^{(1)}-A_x^{(2)}\|_{S^{1-\delta}}+\|\phi^{(1)}-\phi^{(2)})\|_{S^{1-\delta}}+\|A_0^{(1)}-A_0^{(2)}\|_{Y^{1-\delta}}\lesssim \|(A_x^{(1)}-A_x^{(2)},\phi^{(1)}-\phi^{(2)})[0]\|_{\dot H^{1-\delta}\times\dot H^{-\delta}} \tag{73}$$

for small positive δ.

This is equivalent to a similar bound for the linearized equation. All the components of the nonlinearity for which we have off-diagonal decay in the multilinear estimates cause no difficulty, and impose no sign restriction on δ. The only difficulty arises in the paradifferential magnetic wave equation

$$\Box_A^p\phi=0$$

Denoting by (B,ψ) the corresponding linearized variables, we are led to consider the equation

$$\Box_A^p\psi=-\sum_k B_{j,<k}\partial_j\phi_k$$

where B is an $H^{1-\delta}$ free wave. By the bilinear estimate (54), for the right hand side we have the bound

$$\|\sum_k B_{j,<k}\partial_j\phi_k\|_{N^{-\delta}}\lesssim \|B[0]\|_{\dot H^{1-\delta}\times\dot H^{-\delta}}\|\phi\|_{S^1}, \qquad \delta>0$$

which leads to the desired linearized bound

$$\|\psi\|_{S^{1-\delta}}\lesssim \|\psi[0]\|_{\dot H^{1-\delta}\times\dot H^{-\delta}}+\|B[0]\|_{\dot H^{1-\delta}\times\dot H^{-\delta}}\|\phi\|_{S^1}, \qquad \delta>0$$

5.4. **Approximation by smooth solutions.** Consider a small energy Coulomb initial data $(A_x, \psi)[0]$ and its regularizations $(A_x^{(m)}, \psi^{(m)})[0] = P_{<m}(A_x, \psi)[0]$. Denote by (A_x, ϕ), respectively $(A_x^{(m)}, \psi^{(m)})$ the corresponding solutions. Our aim here is to prove the following:

Lemma 2. *Let $\{c_k\}$ be a slowly varying frequency envelope for $(A_x, \psi)[0]$ in the energy norm $\dot{H}^1 \times L^2$. Then*

$$\|(A_x^{(m)} - A_x, \psi^{(m)} - \psi)\|_{S^1}^2 \lesssim \sum_{k>m} c_k^2 \tag{74}$$

The proof of the lemma is straightforward, by using (73) to bound the fequencies less tham m for the difference, and by using (72) for the higher frequencies of each of the terms.

5.5. **Continuous dependence.** Given a convergent sequence of small data $(A_x^k, \psi^k)[0] \to (A_x, \psi)[0]$ in $\dot{H}^1 \times L^2$, we consider their frequency envelopes $\{c^{k,m}\}$ respectively c^k in l^2. Due to the above convergence, it follows that

$$\lim_{m_0 \to \infty} \sum_{m>m_0} |c^{k,m}|^2 = 0, \qquad \text{uniformly in } k \tag{75}$$

We consider the corresponding regularized data $(A_x^{k,(m)}, \psi^{k,(m)})[0]$ and the corresponding solutions $(A^{k,(m)}, \psi^{k,(m)})$. By the relation (75) and (74) it follows that

$$(A^{k,(m)}, \psi^{k,(m)}) \to (A^k, \psi^k) \ \text{ in } S^1, \qquad \text{uniformly in k}$$

On the other hand by the well-posedness theory for smooth data we have the convergence

$$(A^{k,(m)}, \psi^{k,(m)}) \to (A^{(m)}, \psi^{(m)}) \ \text{ in } H^N_{loc}$$

Combining the two we obtain the desired local in time convergence

$$(A^k, \psi^k) \to (A, \psi) \ \text{ in } S^1_{loc}.$$

6. Renormalization

In what follows we work with the selfadjoint paradifferential covariant wave operator

$$\Box^p_A = \Box - 2i \sum_k A^j_{<k-C} \partial_j P_k \tag{76}$$

Here $\nabla_x \cdot A = 0$ are Coulomb gauge potentials which solve the free wave equation $\Box A = 0$. Our first goal is to prove estimates and construct a parametrix for the frequency localized evolution

$$\Box^p_A \phi = f, \qquad (\phi(0), \phi_t(0)) = (g, h) \tag{77}$$

with all functions ϕ, f, g, h localized at frequency 1.

In general $A \neq 0$, so one cannot in addition have $dA = 0$; in other words the derivative interaction cannot be removed via a physical space gauge transformation. The situation changes drastically if one views the gauge potentials as pseudo-differential operators. This stems from the fact that when viewed microlocally all connections have zero curvature, because they contain only one component for each fixed direction in phase space. To write the gauge interaction in terms of a potential we need to solve:

$$A \cdot \xi \ = \ -\tau \psi_t + \xi \psi_x.$$

This is impossible, but if we further restrict ourselves to the region where $\tau \approx \pm|\xi|$, we are left with the exact solution:

$$A \cdot \xi \ = \ d\psi_{\pm} \cdot (\mp|\xi|, \xi) \ , \qquad \psi_{\pm}(t,x,\xi) \ = \ -L^{\omega}_{\pm}\Delta^{-1}_{\omega^{\perp}}(A\cdot\omega) \ .$$

Here $\psi_{\pm}$ are real valued and

$$\omega = \frac{\xi}{|\xi|}, \qquad L^{\omega}_{\pm} = \partial_t \pm \omega\cdot\nabla_x, \qquad \Delta_{\omega^{\perp}} = \Delta - (\omega\cdot\nabla_x)^2,$$

Also we have used the relation $L^{\omega}_{+}L^{\omega}_{-} = \Box + \Delta_{\omega^{\perp}}$. As defined ψ_+ is associated to the upper cone $\{\tau = |\xi|\}$ and ψ_- is associated to the lower cone $\{\tau = -|\xi|\}$.

Quantizing this, one can write the reduced covariant wave equation approximately as:

$$\Box^p_A \ \approx \ \Box - 2i(\partial^{\alpha}\psi_{\pm})(t,x,D)\partial_{\alpha} \ , \qquad \pm\tau > 0 \ .$$

which suggests that one should be be able to remove the gauge interaction through the pseudodifferential conjugation:

$$\Box^p_A \ \approx e^{-i\psi_{\pm}}(t,x,D)\Box e^{-i\psi_{\pm}}(D,y,s) \qquad \pm\,\tau > 0 \ .$$

Applying this algorithm directly does not work well for two reasons. First, the symbol we obtain is not localized at frequency $\ll 1$. Secondly, the symbol Ψ defined above is too singular due to degeneracy of the operator $\Delta^{-1}_{\omega^{\perp}}$; this corresponds exactly to parallel frequencies in A and ϕ.

To remedy the latter issue we take advantage of the fact that in the bilinear null form estimates there is a small angle gain. This allows us to tightly cut off the small angle interactions between A and ϕ in the construction of Ψ, though not uniformly with respect to the A frequencies. Precisely, we define the dyadic portions of ψ by

$$\psi_{k,\pm}(t,x,\xi) \ = \ -L^{\omega}_{\pm}\Delta^{-1}_{\omega^{\perp}}(\Pi^{\omega}_{>\delta k}A_k\cdot\omega) \tag{78}$$

Then the full ψ's are defined by

$$\psi_{\pm} := \psi_{<0,\pm}$$

For the renormalization we will use frequency localized versions of $e^{\pm i\psi_{\pm}}$, namely the operators

$$e^{-i\psi_{\pm}}_{<0}(t,x,D), \qquad e^{i\psi_{\pm}}_{<0}(D,y,s)$$

Here we use $P(x,D)$ for the left quantization, and $P(D,y)$ for the right quantization. Also the subscript <0 stands for the space-time frequency localization at frequencies $\ll 1$.

Our main goal now is to show that the renormalizations are compatible with the S and N spaces. Below we use the notation

$$\Box^p_{A_{<0}} := \Box - 2iA^j_{<-C}\partial_j P_0$$

and analogously for $\Box^p_{A_{<k}}$.

Theorem 3. *The frequency localized renormalization operators have the following mapping properties with* $Z \in \{N_0, L^2, N_0^*\}$*:*

$$e^{\pm i\psi_\pm}_{<0}(t,x,D): \quad Z \to Z\ , \tag{79}$$

$$\partial_t e^{\pm i\psi_\pm}_{<0}(t,x,D): \quad Z \to Z\ , \tag{80}$$

$$e^{-i\psi_\pm}_{<0}(t,x,D)e^{i\psi_\pm}_{<0}(D,y,s) - I: \quad Z \to \epsilon Z, \tag{81}$$

$$e^{-i\psi_\pm}_{<0}(t,x,D)\Box - \Box^p_{A_{<0}} e^{-i\psi_\pm}_{<0}(t,x,D): \quad N^*_{0,\pm} \to \epsilon N_{0,\pm}\ . \tag{82}$$

$$e^{-i\psi_\pm}_{<0}(t,x,D): \quad S_0^\sharp \to S_0\ , \tag{83}$$

The above theorem allows us to construct an approximate solution for (76) as follows:

$$\begin{aligned} \phi_{app} = \frac{1}{2}\sum_{\pm} e^{-i\psi_\pm}_{<0}(t,x,D)\frac{1}{|D|}e^{\pm it|D|}e^{i\psi_\pm}_{<0}(D,y,0)(|D|g \pm h) \\ + e^{-i\psi_\pm}_{<0}(t,x,D)\frac{1}{|D|}K^\pm e^{\pm i\psi_\pm}_{<0}(D,y,s)f \end{aligned} \tag{84}$$

where

$$K^\pm f(t) = \int_0^t e^{\pm i(t-s)|D|} f(s)ds$$

represents the solution to

$$(\partial_t \mp i|D|)u = f, \qquad u(0) = 0$$

Here if we drop the $e^{\pm i\phi_\pm}$ operators we simply have the expression of the exact solution for the constant coefficient wave equation. Thus the idea is that these operators approximately conjugate the covariant wave flow to the constant coefficient wave flow.

Theorem 4. *Assume that* f, g, h *are localized at frequency* 1*, and also that* f *is localized at modulation* $\lesssim 1$*. Then* ϕ_{app} *is an approximate solution for* $\Box^p_{A_{<0}}\phi = f$, $\phi(0) = (f,g)$*, in the sense that*

$$\|\phi_{app}\|_{S_0} \lesssim \|f\|_{N_0} + \|g\|_{L^2} + \|h\|_{L^2} \tag{85}$$

and

$$\|\phi_{app}[0] - (g,h)\|_{L^2} + \|\Box^p_{A_{<0}}\phi_{app} - f\|_{N_0} \ll \|f\|_{N_0} + \|g\|_{L^2} + \|h\|_{L^2} \tag{86}$$

We remark that ϕ_{app} constructed above has the same localization at frequency 1 and modulation $\lesssim 1$. We also remark that the actual parametrix construction only requires the estimates (79)-(83), but yields a weaker form of (85) with S_0 replaced by N_0^*. Then the estimate (83) serves to provide the additional S_0 regularity. An easy consequence of this is the following

Theorem 5. *For all* $f \in N$ *and* $(g,h) \in \dot H^1 \times L^2$ *the solution to the paradifferential covariant wave equation* (77) *is defined globally and satisfies*

$$\|\phi\|_{S^1} \lesssim \|f\|_{N \cap l^1 L^2 \dot H^{-\frac{1}{2}}} + \|g\|_{\dot H^1} + \|h\|_{L^2} \tag{87}$$

Again, using only (79)-(83) suffices but yields a weaker form of (85) with S replaced by N^*.

For the remainder of the section we use Theorem 3 to prove Theorems 4, 5. The proof of Theorem 3 is completed in Sections 8, 9, after some preliminaries in Section 7.

Proof of Theorem 4. The estimate (85) follows directly by concatenating (79) and (83).

It remains to consider (86). We begin with the initial data. For the position we have

$$\phi_{app}(0) - g = \frac{1}{2}\sum_{\pm} e^{-i\psi_\pm}_{<0}(0,x,D)\frac{1}{|D|}e^{i\psi_\pm}_{<0}(D,y,0)(|D|g \pm h) - g$$
$$= \frac{1}{2}\sum_{\pm}\left[e^{-i\psi_\pm}_{<0}(0,x,D)\frac{1}{|D|}e^{i\psi_\pm}_{<0}(D,y,0) - \frac{1}{|D|}\right](|D|g \pm h)$$

and we can apply the L^2 version of (81).

For the velocity we have

$$\partial_t\phi_{app}(0) - h = \frac{1}{2}\sum_{\pm} \pm e^{-i\psi_\pm}_{<0}(0,x,D)e^{i\psi_\pm}_{<0}(D,y,0)(|D|g \pm h) - h$$
$$+ [\partial_t e^{-i\psi_\pm}_{<0}](0,x,D)\frac{1}{|D|}e^{i\psi_\pm}_{<0}(D,y,0)(|D|g \pm h) +$$
$$+ e^{-i\psi_\pm}_{<0}(0,x,D)\frac{1}{|D|}[\partial_t e^{i\psi_\pm}_{<0}](D,y,0)(|D|g \pm h)$$
$$\pm e^{-i\psi_\pm}_{<0}(0,x,D)e^{i\psi_\pm}_{<0}(D,y,0)f(0)$$

The first line is rewritten as

$$\frac{1}{2}\sum_{\pm}\left[e^{-i\psi_\pm}_{<0}(0,x,D)e^{i\psi_\pm}_{<0}(D,y,0) - 1\right](\pm|D|g + h)$$

and then we can use (81). For the second and third lines we use (80). Finally, for the last line we use (81) twice, along with the bound

$$\|f(0)\|_{L^2} \lesssim \|f\|_{N_0}$$

derived by Bernstein's inequality due to the unit modulation localization of f.

Lastly, we consider the error estimate. We have

$$\Box^p_{A_{<0}}\phi_{app} - f = \sum_{\pm}[\Box^p_{A_{<0}} e^{-i\psi_\pm}_{<0}(t,x,D) - e^{-i\psi_\pm}_{<0}(t,x,D)\Box]\phi_\pm$$
$$\pm \frac{1}{2}e^{-i\psi_\pm}_{<0}(t,x,D)\frac{D_t \pm |D|}{|D|}e^{i\psi_\pm}_{<0}(D,y,s)f - f$$
$$= \sum_{\pm}[\Box^p_{A_{<0}} e^{-i\psi_\pm}_{<0}(t,x,D) - e^{-i\psi_\pm}_{<0}(t,x,D)\Box]\phi_\pm$$
$$+ \frac{1}{2}[e^{-i\psi_\pm}_{<0}(t,x,D)e^{i\psi_\pm}_{<0}(D,y,s) - 1]f$$
$$\pm \Big[\frac{1}{2}[e^{-i\psi_\pm}_{<0}(t,x,D)\frac{1}{|D|}e^{i\psi_\pm}_{<0}(D,y,s) - \frac{1}{|D|}]\partial_t f$$
$$+ e^{-i\psi_\pm}_{<0}(t,x,D)\frac{1}{|D|}[\partial_t e^{i\psi_\pm}_{<0}](D,y,s))]f\Big]$$

where

$$\phi_\pm = \frac{1}{|D|}[e^{\pm it|D|}e^{i\psi_\pm}_{<0}(D,y,0)(|D|g \pm h) + K^\pm e^{\pm i\psi_\pm}_{<0}(D,y,s)]$$

For the first line we bound $\phi_\pm$ in $N_\pm^*$ via (79) and $\Box^{-1}$ estimates, and then use the conjugation bound (82). For the second and third we use (81) in N. Finally for the fourth line we use (80) in N. $\Box$

Proof of Theorem 5. Consider first the frequency localized problems (with k referring to spatial frequency)

$$\Box^p_{A_{<k}} \phi_k = f_k,\ \phi_k(0) = (g_k, h_k),\ k \in \mathbf{Z}$$

We solve each of these approximately and re-assemble the solutions $\phi_{app,k}$ to the full approximate solution $\phi_{app} = \sum_k \phi_{app,k}$. We cannot immediately apply the preceding theorem, since we make no further assumption on the space-time Fourier support of the source f. To remedy this, we split

$$f_k = f_k^{hyp} + f_k^{ell},$$

where f_k^{hyp} is supported in the region $||\tau| - |\xi|| \lesssim 2^k$.Then we solve the two problems

$$\Box^p_{A_{<k}} \phi_k^1 = f_k^{hyp}, \qquad \Box^p_{A_{<k}} \phi_k^2 = f_k^{ell}$$

approximately, the first by using the previous theorem, the second by neglecting the magnetic term $2iA^j_{<k}\partial_j \phi_k^2$. We then solve the second equation by 'division by the symbol', i. e. we set (from now on $k = 0$ by scaling invariance and we omit the subscripts)

$$\phi^2 := \Box^{-1} f^{ell}$$

where the operator $\Box^{-1}$ is defined by multiplication with $\frac{1}{\tau^2 - |\xi|^2}$ on the Fourier side. Then the bound

$$\|\phi^2\|_{S_0} \lesssim \|f^{ell}\|_{N_0}$$

is immediate, and we reduce to solving the problem

$$\Box^p_{A_{<0}} \phi^1 = f^{hyp}, \qquad \phi^1(0) = (g, h) - \Box^{-1} f^{ell}(0)$$

which we do by invoking Theorem 4.

It remains to control the additional error generated by our approximate solution ϕ^2, which is

$$2iA^j_{<0}\partial_j \phi^2$$

This is estimated by

$$\|2iA^j_{<0}\partial_j \phi^2\|_{L^1_t L^2_x \cap L^2 \dot{H}^{-\frac{1}{2}}} \lesssim \|A^j_{<0}\|_{L^2_t L^\infty_x} \|\phi^2\|_{L^2_{t,x} \cap L^\infty_t L^2_x} \ll \|f^{ell}\|_{N_0}$$

It follows that our approximate solution $\phi_{app} = \sum_k \phi_{app,k}$ satisfies the conditions

$$\|\Box^p_A \phi_{app} - f\|_{N \cap l^1 L^2 \dot{H}^{-\frac{1}{2}}} + \|\phi_{app}(0) - (g,h)\|_{\dot{H}^1 \times L^2} \ll \|f\|_{N \cap l^1 L^2 \dot{H}^{-\frac{1}{2}}} + \|(g,h)\|_{\dot{H}^1 \times L^2}$$

Also, observe that the preceding Theorem 4 implies

$$\|\phi_{app}\|_{S^1} \lesssim \|f\|_{N \cap l^1 L^2 \dot{H}^{-\frac{1}{2}}} + \|(g,h)\|_{\dot{H}^1 \times L^2}$$

The proof is now completed by simple iterative application of the preceding to the successive errors. $\Box$

7. Decomposable Spaces and Some Symbol Bounds

7.1. **Review of the Basic Decomposable Calculus.** First we discuss the notion of decomposable function spaces and estimates. Recall that a zero homogeneous symbol $c(t,x;\xi)$ is said to be in "decomposable $L^q(L^r)$" if $c=\sum_\theta c^{(\theta)}$, $\theta\in 2^{-\mathbb{N}}$, and:

$$\sum_\theta \| \, c^{(\theta)} \, \|_{D_\theta\left(L^q_t(L^r_x)\right)} \ < \ \infty \ , \tag{88}$$

where:

$$\| \, c^{(\theta)} \, \|_{D_\theta\left(L^q_t(L^r_x)\right)} \ = \ \Big\| \Big(\sum_{k=0}^{10n} \sum_\phi \ \sup_\omega \ \| \, b^\phi_\theta \ (\theta\nabla_\xi)^k \ c^{(\theta)} \, \|^2_{L^r_x} \Big)^{\frac{1}{2}} \Big\|_{L^q_t} \ . \tag{89}$$

Here $b^\phi_\theta(\xi)$ denotes a cutoff on a solid angular sector $\big|\xi|\xi|^{-1}-\phi\big|\leqslant\theta$ for a fixed $\phi\in\mathbb{S}^{n-1}$, and the sum is taken over a uniformly finitely overlapping collection. We define $\| \, b \, \|_{DL^q(L^r)}$ as the infimum over all sums (88). In [18] it is shown that the following Hölder type inequality holds:

$$\| \prod_{i=1}^m b_i \, \|_{DL^q(L^r)} \ \lesssim \ \prod_{i=1}^m \| \, b_i \, \|_{DL^{q_i}(L^{r_i})} \ , \quad (q^{-1},r^{-1}) = \sum_i (q_i^{-1}, r_i^{-1}) \ . \tag{90}$$

In the sequel we only need a special case of decompositions provided in terms of these norms:

Lemma 7.1 (Decomposability Lemma)**.** *Let $A(t,x;D)$ be any pseudodifferential operator with symbol $a(t,x;\xi)$. Suppose A satisfies the fixed time bound:*

$$\sup_t \| \, A(t,x;D) \, \|_{L^2\to L^2} \ \lesssim \ 1 \ . \tag{91}$$

Then for any symbol $c(t,x;\xi)\in DL^q(L^r)$ one has the space-time bounds:

$$\begin{gathered} \| \, (ac)(t,x;D) \, \|_{L^{q_1}L^2\to L^{q_2}(L^{r_2})} \lesssim \| \, c \, \|_{DL^q(L^r)} \ , \\ \frac{1}{q_1}+\frac{1}{q}=\frac{1}{q_2}, \qquad \frac{1}{2}+\frac{1}{r}=\frac{1}{r_2}, \qquad 1\le q_1,q_2,q,r,r_2\le\infty \end{gathered} \tag{92}$$

Proof. Due to the l^1 summation over θ in (88) it suffices to consider the case $c=c_\theta$ for a fixed θ. We further decompose

$$c_\theta(t,x,\xi)=\sum_\phi c^\phi_\theta(t,x,\xi), \qquad c^\phi_\theta(t,x,\xi):=b^\phi_\theta(\xi)c^{(\theta)}(t,x,\xi)$$

By (89) each c^ϕ_θ is supported in an angle θ sector, and it is smooth on the scale of its support. Thus by a Fourier series decomposition we can separate variables and represent

$$c^\phi_\theta(t,x,\xi)=\sum_{j>0} d^{\phi,j}_\theta(t,x)e^{\phi,j}_\theta(\xi),$$

where

$$\|d^{\phi,j}_\theta(t,x)\|_{L^r_x}\lesssim j^{-N}\sum_{k=0}^{10n} \| \, b^\phi_\theta \ (\theta\nabla_\xi)^k \ c^{(\theta)} \, \|_{L^r_x}, \qquad |e^{\phi,j}_\theta|\le 1$$

Due to the rapid decay with respect to j it suffices to consider the contribution to c_θ^ϕ coming from a single j, say $j = 1$. Then c_θ has the form

$$c_\theta = \sum_\phi d_\theta^\phi(t,x) e_\theta^\phi(\xi),$$

where

$$\|d_\theta^\phi\|_{L_t^q l_\phi^2 L_x^r} \lesssim 1, \qquad |e_\theta^\phi| \leq 1 \tag{93}$$

Then we can represent

$$(ac)(t,x,D)u = \sum_\phi d_\theta^\phi(t,x) \cdot A(t,x,D) e_\theta^\phi(D)u$$

The second factor above inherits the L^2 norm from u due to (91) and the square summation in ω due to the sector decomposition. Thus

$$\|A(t,x,D)e_\theta^\phi(D)u\|_{L_t^{q_1} l_\omega^2 L_x^2} \lesssim \|u\|_{L^2}$$

The estimate for $(ac)(t,x,D)u$ follows by combining the last bound with (93). □

7.2. **A Decomposable Calculus for Pseudodifferential Products.** In the sequel it will also be useful for us treat estimates for products of operators in a modular way. Recall that if $a(x,\xi)$ and $b(x,\xi)$ are symbols, then $a^r b^r - (ab)^r \approx i(\partial_x a \partial_\xi b)^r$. This formula is not exact, but it leads to an estimate:

Lemma 7.2 (Decomposable product calculus)**.** *Let $a(x,\xi)$ and $b(x,\xi)$ be smooth symbols. Then:*

$$\| \, a^r b^r - (ab)^r \, \|_{L^r(L^2) \to L^q(L^2)} \lesssim \| \, (\nabla_x a)^r \, \|_{L^r(L^2) \to L^{p_1}(L^2)} \| \, \nabla_\xi b \, \|_{D_1 L^{p_2}(L^\infty)} \tag{94}$$

where $q^{-1} = \sum p_i^{-1}$. Furthermore, if $b = b(\xi)$ is a smooth compactly supported multiplier, then for any two translation invariant spaces X, Y one has:

$$\| \, a^r b^r - (ab)^r \, \|_{X \to Y} \lesssim \| \, (\nabla_x a)^r \, \|_{X \to Y} \ . \tag{95}$$

Proof. To prove the first estimate (94) we write the kernel $K(x,y)$ of the difference $a^r b^r - (ab)^r$ as follows:

$$\begin{aligned} K(x,y) &= c_n \int_{\mathbb{R}^{3n}} e^{i(x-z)\cdot\xi} e^{i(z-y)\cdot\eta} a(z,\xi)\big(b(y,\eta) - b(y,\xi)\big) dz d\xi d\eta \\ &= c_n \int_0^1 \int_{\mathbb{R}^{3n}} e^{i(x-z)\cdot\xi} e^{i(z-y)\cdot\eta} a(z,\xi) \nabla_\xi b\big(y, s\eta + (1-s)\xi\big) \cdot (\eta - \xi) dz d\xi d\eta ds \ , \\ &= c_n i \int_0^1 \int_{\mathbb{R}^{3n}} e^{i(x-z)\cdot\xi} e^{i(z-y)\cdot\eta} \nabla_x a(z,\xi) \cdot \nabla_\xi b\big(y, s\eta + (1-s)\xi\big) dz d\xi d\eta ds \ , \\ &= i \frac{1}{(2\pi)^n} \int_0^1 \int_{\mathbb{R}^n} T_{(s-1)\Xi} \nabla_x a^r(x,D) T_{-s\Xi} \widehat{\nabla_\xi b}(\cdot, \Xi) ds d\Xi \ , \end{aligned} \tag{96}$$

where we have used Fourier inversion:

$$\nabla_\xi b\big(y, s\eta + (1-s)\xi\big) \ = \ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(s\eta + (1-s)\xi)\cdot\Xi} \ \widehat{\nabla_\xi b}(y, \Xi) \, d\Xi \ .$$

From the smoothness of b on the unit scale in ξ we have that the weight function $\widehat{\nabla_\xi b}$ obeys the estimate:

$$\int_{\mathbb{R}^n} \| \widehat{\nabla_\xi b}(\cdot, \Xi) \|_{L^{p_2}(L^\infty)} d\Xi \ \lesssim \ \| b \|_{D_1 L^{p_2}(L^\infty)} \ .$$

Then (94) is a direct consequence of Hölder's inequality and the the translation invariance of L^p spaces.

Note that the proof of (95) also follows from the identity (96) directly. because in this case the weight function $\widehat{\nabla_\xi b}$ is a constant in y. □

7.3. **Some Symbols Bounds for Phases.** For its use in the sequel, we list out a number of decomposable estimates for the phase $\psi(t,x;\xi)$ used to define our microlocal gauge transformations:

Lemma 7.3 (Decomposable estimates for ψ)**.** *Let the phase $\psi(t,x;\xi)$ be defined as in* (78)*, and its angular components $\psi^{(\theta)} = \Pi_\theta^\omega \psi(t,x;\xi)$, where $\omega = |\xi|^{-1}\xi$. Then for $q \geq 2$ and $2/q + 3/r \leq 1$ one has:*

$$\| (\psi_k^{(\theta)}, 2^{-k}\nabla_{t,x}\psi_k^{(\theta)}) \|_{DL^q(L^r)} \ \lesssim \ 2^{-(\frac{1}{q}+\frac{4}{r})k}\theta^{\frac{1}{2}-\frac{2}{q}-\frac{3}{r}}\epsilon, \tag{97}$$

In particular

$$\| (\psi_k, 2^{-k}\nabla_{t,x}\psi_k) \|_{DL^q(L^\infty)} \ \lesssim \ 2^{-\frac{1}{q}k}\epsilon \ , \qquad\qquad q > 4 \ , \tag{98}$$

$$\| \nabla_{t,x}\psi_k \|_{DL^2(L^r)} \ \lesssim \ 2^{(\frac{1}{2}-\frac{4}{r}-\delta(\frac{1}{2}+\frac{3}{r}))k}\epsilon \ , \qquad\qquad r \geq 6 \ , \tag{99}$$

Proof. Notice that the last two estimates follow from the first by summing over dyadic $2^{-\delta k} \leq \theta \lesssim 1$. For the first bound we interchange the t integration and the ω summation to obtain:

$$\begin{aligned}\| (\psi_k^{(\theta)}, 2^{-k}\nabla_{t,x}\psi_k^{(\theta)}) \|_{DL^q(L^r)} \ &\lesssim \theta^{-2}2^{-k}\big(\sum_\omega \| \Pi_\theta^\omega(D)A\cdot\omega \|^2_{L^q(L^r)}\big)^{\frac{1}{2}} \\ &\lesssim \theta^{-1}2^{-k}\big(\sum_\omega \| \Pi_\theta^\omega(D)A \|^2_{L^q(L^r)}\big)^{\frac{1}{2}},\end{aligned}$$

where at the second step we have used the Coulomb gauge to gain another factor of θ.

Now we conclude using the Strichartz estimates. In four space dimensions the Strichartz sharp range is given by $\frac{2}{q}+\frac{3}{r_0}=\frac{3}{2}$. Moreover, on an angular sector of size θ Bernstein's inequality gives the embedding $\Pi_\theta^\omega(D)P_k L^{r_0} \subseteq \theta^{3(\frac{1}{r_0}-\frac{1}{r})}2^{4(\frac{1}{r_0}-\frac{1}{r})k}L^r$. Thus:

$$\big(\sum_\omega \| \Pi_\theta^\omega(D)A_k \|^2_{L^q(L^r)}\big)^{\frac{1}{2}} \ \lesssim \ \theta^{\frac{3}{2}-\frac{2}{q}-\frac{3}{r}}2^{(1-\frac{1}{q}-\frac{4}{r})k}\| A_k \|_{S_k} \ ,$$

and the second estimate follows. □

We wrap this section up by proving some additional symbol type bounds for the phases ψ. These involve the variation over the physical space variables:

Lemma 7.4 (Additional symbol bounds for ψ)**.** *Let ψ be as above. Then one has:*

$$|\psi_{<k}(t,x;\xi) - \psi_{<k}(s,y;\xi)| \lesssim \epsilon\log(1+2^k(|t-s|+|x-y|)), \tag{100}$$

$$|\psi(t,x;\xi) - \psi(s,y;\xi)| \lesssim \epsilon\log(1+|t-s|+|x-y|) \tag{101}$$

$$|\partial_\xi^\alpha(\psi(t,x;\xi) - \psi(s,y;\xi))| \lesssim \ \epsilon\langle(t-s,x-y)\rangle^{|\alpha-\frac{1}{2}|\sigma}, 1 \leqslant \alpha \leqslant \sigma^{-1}. \tag{102}$$

Proof. We decompose as before

$$\psi_{<k}(t,x;\xi) = \sum_{j<k} \sum_{\theta > 2^{\sigma j}} \psi_j^{(\theta)}(t,x,\xi)$$

For each fixed θ and j we have by the definition of ψ and the Coulomb gauge condition

$$|\psi_j^{(\theta)}(t,x,\xi)| \lesssim \theta^{-1} 2^{-j} \sup_{\omega} \|\Pi_\theta^\omega A_j\|_{L^\infty}$$

Then by energy estimates for A and Bernstein's inequality we obtain

$$|\psi_j^{(\theta)}(t,x,\xi)| \lesssim \theta^{\frac12} \|A_j[0]\|_{H^1\times L^2}, \qquad |\psi_j(t,x,\xi)| \lesssim \|A_j[0]\|_{H^1\times L^2} \tag{103}$$

A similar argument leads to

$$|\partial_{t,x}\psi_j^{(\theta)}(t,x,\xi)| \lesssim 2^j \theta^{\frac12} \|A_j[0]\|_{H^1\times L^2}, \qquad |\partial_{t,x}\psi_j(t,x,\xi)| \lesssim 2^j \|A_j[0]\|_{H^1\times L^2} \tag{104}$$

Differentiating with respect to ξ yields θ^{-1} factors,

$$|\partial_\xi^\alpha \psi_j^{(\theta)}(t,x,\xi)| \lesssim \theta^{\frac12 - |\alpha|} \|A_j[0]\|_{H^1\times L^2}, \qquad |\partial_{x,t}\partial_\xi^\alpha \psi_j^{(\theta)}(t,x,\xi)| \lesssim 2^j \theta^{\frac12-|\alpha|} \|A_j[0]\|_{H^1\times L^2}.$$

For the bound (100) we use both (103) and (7.3) to write for $j \leq k$

$$|\psi_{<k}(t,x;\xi) - \psi_{<k}(s,y;\xi)| \lesssim 2^j(|t-s| + |x-y|) + |k-j|$$

and then optimize the choice of j.

The proof of (102) is similar.

□

8. L^2 estimates for the gauge transformations

In this section we prove three core L^2 based estimates for the gauge transformations $e^{\pm i\psi_\pm}$. These will serve as building blocks in later sections.

8.1. **Oscillatory integral estimates.** In order to prove various estimates involving the operators $e_{<0}^{-i\psi_\pm}(t,x,D)$ and $e_{<0}^{i\psi_\pm}(D,y,s)$ we need to obtain pointwise kernel bounds for operators of the form

$$T^a = e^{-i\psi_\pm}(t,x,D) a(D) e^{\pm i(t-s)|D|} e^{i\psi_\pm}(D,y,s)$$

where a is localized at frequency 1. The kernel of the operator T_a is given by the oscillatory integral

$$K^a(t,x;s,y) = \int e^{-i\psi_\pm}(t,x,\xi) a(\xi) e^{\pm i(t-s)|\xi|} e^{i\xi(x-y)} e^{i\psi_\pm}(\xi,y,s) d\xi$$

Our main estimates for such kernels are as follows:

Proposition 6. *a) Assume that a is a smooth bump on the unit scale. Then the kernel K_a satisfies*

$$|K_a(t,x;s,y)| \lesssim \langle t-s\rangle^{-\frac32} \langle |t-s| - |x-y|\rangle^{-N} \tag{105}$$

b) Let $a = a_C$ be a bump function on a rectangular region C of size $2^k \times (2^{k+l})^3$ with $k \leq l \leq 0$. Then

$$|K_a(t,x;s,y)| \lesssim 2^{4k+3l} \langle 2^{2(k+l)}(t-s)\rangle^{-\frac32} \langle 2^k(|t-s| - |x-y|)\rangle^{-N} \tag{106}$$

If in addition $x-y$ and C have a 2^{k+l} angular separation then

$$|K_a(t,x;s,y)| \lesssim 2^{4k+3l}\langle 2^{2(k+l)}|t-s|\rangle^{-N}\langle 2^k(|t-s|-|x-y|)\rangle^{-N} \tag{107}$$

Proof. a) Away from a conic neighbourhood of the cone $\{|t-s| = \pm|x-y|\}$ the phase

$$\Psi = \pm(t-s)|\xi| + \xi(x-y) - (\psi_\pm(t,x,\xi) - \psi_\pm(s,y,\xi))$$

is nondegenerate due to (102) with $|\alpha| = 1$. Hence repeated integration by parts yields

$$|K^a(t,x,s,y)| \lesssim \langle (t,x)-(s,y)\rangle^{-N}, \qquad N \sim \sigma^{-1}$$

Near the cone we need to be more careful. Denoting $T = |t-s|+|x-y|$ and $R = |t-s|-|x-y|$, in suitable (polar) coordinates this takes the form

$$K^a(t,x,s,y) = \int e^{-i(\psi_\pm(t,x,\xi')-\psi_\pm(s,y,\xi'))}e^{iR\xi_1}e^{iT\xi'^2}\tilde{a}(\xi)d\xi$$

In ξ_1 (the former radial variable) this is a straight Fourier transform. Given the bound (102), we can use stationary phase in ξ'. While the ξ derivatives of the $\psi_\pm$ part of the phase are not bounded, they only bring factors of T^σ, which is small enough not to affect the stationary phase (this works up to $\sigma = \frac{1}{2}$). We obtain

$$|K^a(t,x,s,y)| \lesssim T^{-\frac{3}{2}}(1+R)^{-N}$$

b) Away from the cone the estimate follows easily as above since the phase is nondegenerate. Near the cone we use again polar coordinates to express our oscillatory integral as above,

$$K^C(t,x,s,y) = \int e^{-i(\psi_\pm(t,x,\xi')-\psi_\pm(s,y,\xi'))}e^{iR\xi_1}e^{iT\xi'^2}\tilde{a}_C(\xi)d\xi$$

where a_C is a bump function in a rectangle on the 2^k scale in the radial variable ξ_1 and on the 2^{k+l} scale in the angular variable ξ'. Then we can separate variables in (ξ_1,ξ'). We note that this rectangle need not be centered at $\xi'=0$, though this is the worst case. In ξ_1 this is again a Fourier transform, so we get the factor

$$2^k\langle 2^k R\rangle^{-N}$$

In ξ' we can use stationary phase to get the factor

$$2^{3(k+l)}\langle 2^{2(k+l)T}\rangle^{-\frac{3}{2}}$$

The bound (106) follows by multiplying these two factors.

Finally, the estimate (107) corresponds to the case when a_C is supported in $|\xi'| > 2^l$ in the above representation. If $T < 2^{-2(k+l)}$ then there are no oscillations in ξ' on the 2^{k+l} scale, and we just use the brute force estimate. For $T > 2^{-2(k+l)}$ the phase is nonstationary in ξ', and we obtain the factor

$$2^{3(k+l)}(1+2^{2(k+l)}T)^{-N}$$

□

While the above proposition contains all the oscillatory integral estimates which are needed, it does not apply directly to the frequency localized operators $e^{-i\psi_\pm}_{<0}(t,x,D)$ and $e^{i\psi_\pm}_{<0}(D,y,s)$. For that we need to produce similar estimates for the kernels $K_{a,<0}$ of the operators

$$T^a_{<0} = e^{-i\psi_\pm}_{<0}(t,x,D)a(D)e^{\pm i(t-s)|D|}e^{i\psi_\pm}_{<0}(D,y,s)$$

The transition to such operators is made in the next

Proposition 7. *a) Assume that a is a smooth bump on the unit scale. Then the kernel $K^a_{<0}$ satisfies*

$$|K^a_{<0}(t,x;s,y)| \lesssim \langle t-s\rangle^{-\frac{3}{2}} \langle |t-s| - |x-y|\rangle^{-N} \tag{108}$$

In addition, the following fixed time bound holds:

$$|K^a_{<0}(t,x;t,y) - \check{a}(x-y)| \leq \epsilon |\log \epsilon| \tag{109}$$

b) Let $a = a_C$ be a bump function on a rectangular region C of size $2^k \times (2^{k+l})^3$ with $k \leq l \leq 0$. Then

$$|K^a_{<0}(t,x;s,y)| \lesssim 2^{4k+3l} \langle 2^{2(k+l)}(t-s)\rangle^{-\frac{3}{2}} \langle 2^k(|t-s| - |x-y|)\rangle^{-N} \tag{110}$$

c) Let $a = a_C$ be a bump function on a rectangular region C of size $1 \times (2^l)^3$ with $l \leq 0$. Let $\omega \in \mathbb{S}^3$ be at angle l from C. Then we have the characteristic kernel bound

$$\begin{aligned} |K^a_{<0}(t,x;s,y)| &\lesssim 2^{3l} \langle 2^{2l}|t-s|\rangle^{-N} \langle 2^l |x'-y'|\rangle^{-N} \\ t-s &= (x-y)\cdot\omega \end{aligned} \tag{111}$$

Proof. a) We represent the symbol $e^{\pm i\psi_\pm}_{<0}$ as

$$e^{\pm i\psi_\pm}_{<0} = \int m(z) e^{\pm i T_z \psi_\pm}\, dz \tag{112}$$

where $m(z)$ is an integrable bump function on the unit scale and T_z denotes translation in the direction z, with z representing space-time coordinates. Since the wave equation is invariant to translations, the symbol $e^{\pm i T_z \psi_\pm}$ is of the same type as $e^{\pm i\psi_\pm}$. Using this representation for both $\psi_\pm$ exponentials, the kernel $K^a_{<0}$ can be expressed in the form

$$\begin{aligned} K^a_{<0}(t,x,s,y) &= \int\int e^{-iT_z\psi_\pm}(t,x,\xi) a(\xi) e^{i(\pm|\xi|,\xi)\cdot(t-s,x-y)} e^{iT_w\psi_\pm}(s,y,\xi)\, d\xi\ m(z)m(w)\ dzdw \\ &= \int T_z T_w \int e^{-i\psi_\pm}(t,x,\xi) a(\xi) e^{i(\pm|\xi|,\xi)\cdot(z-w)} e^{i\psi_\pm}(s,y,\xi)\, d\xi\ m(z)m(w)\ dzdw \end{aligned}$$

Denoting $a(z,w)(\xi) = a(\xi)e^{i(\pm|\xi|,\xi)\cdot(z-w)}$, we can express the above kernel in terms of the kernels K^a in the previous proposition, namely

$$K^a_{<0}(t,x,s,y) = \int T_z T_w K^{a(z,w)}(t,x,s,y)\ m(z)m(w)\ dzdw \tag{113}$$

To prove the bound (108) we use (105), together with the additional observation that the implicit constant in (105) depends on finitely many seminorms of a (at most 8, to be precise) which we denote by $|||a|||$. Then

$$|||a(z,w)||| \lesssim (1 + |z| + |w|)^N$$

However, this growth is compensated by the rapid decay of m, therefore the bound (105) for K^a transfers directly to $K^a_{<0}$ in (108).

To prove (109) we use the same representation as above to write

$$K^a_{<0}(t,x,t,y) - \check{a}(x-y) = \int\int [e^{-i(T_z\psi_\pm(t,x,\xi) - T_w\psi_\pm(t,y,\xi))} - 1] a(\xi) e^{i\xi(x-y)}\ d\xi\ m(z)m(w)\ dzdw$$

By (101) we have

$$|T_z\psi_\pm(t,x,\xi) - T_w\psi_\pm(t,y,\xi))| \lesssim \epsilon \log(1+|z|+|w|+|x-y|)$$

which yields

$$\begin{aligned}|K^a_{<0}(t,x,t,y) - \check{a}(x-y)| &\lesssim \epsilon \int \log(1+|z|+|w|+|x-y|)|m(z)||m(w)|dzdw \\ &\lesssim \epsilon \log(2+|x-y|)\end{aligned}$$

This suffices if $\log(2+|x-y|) \lesssim |\log\epsilon|$. But for larger $|x-y|$ we can use (108) directly.

b) Using the representation (113), the bound (110) follows from (106) exactly by the same argument as in case (a).

c) Using the representation (113), the same argument also yields the bound (105) provided we have the following estimate for K^a:

$$|K^a(t,x,s,y)| \lesssim 2^{3l}\langle 2^{2l}|t-s|\rangle^{-N}\langle 2^l|x'-y'|\rangle^{-N}(1+|(t-s)-(x-y)\cdot\omega|)^{10N}$$

To see that this is true, we consider three cases:

(i) If $|t-s| \lesssim 2^{-2l}$ then (106) applies directly.

(ii) If $|t-s| \gg 2^{-2l}$ but $||x-y|-|t-s|| \gtrsim 2^l|x'-y'| + 2^{2l}|t-s|$ then (106) still suffices.

(iii) If $|t-s| \gg 2^{-2l}$ and $|(t-s)-(x-y)\cdot\omega|)| \gtrsim 2^l|x'-y'|+2^{2l}|t-s|$ then (106) also applies.

(iv) Finally, if $|t-s| \gg 2^{-2l}$, but $||x-y|-|t-s|| \ll 2^l|x'-y'|+2^{2l}|t-s|$ and $|(t-s)-(x-y)\cdot\omega|)| \ll 2^l|x'-y'|+2^{2l}|t-s|$ then we must have $\angle(x-y,\omega) \ll 2^l$, which implies that $\angle(x-y,C) \approx 2^l$. Then (107) applies.

□

8.2. **Fixed-time L^2 estimates.** The following is an important application of the previous theorem which is at the heart of our parametrix construction: it proves the L^2-part of (79), (80) as well as that of (81).

Proposition 8. *The following fixed time L^2 estimates hold for functions localized at frequency* 1*:*

$$e^{\pm i\psi_\pm}_{<k}(t,x,D): \quad L^2 \to L^2, \tag{114}$$

$$e^{-i\psi_\pm}_{<0}(t,x,D)e^{i\psi_\pm}_{<0}(D,y,s) - I: \quad L^2 \to \epsilon^{\frac{N-4}{N}}\log\epsilon\ L^2 \tag{115}$$

$$\partial_{x,t}e^{\pm i\psi_\pm}_{<0}(t,x,D): \quad L^2 \to L^2 \tag{116}$$

Proof. a) By the estimate (105) with $s=t$, the TT^* type operator

$$e^{\pm i\psi_\pm}(t,x,D)P_0^2 e^{\mp i\psi_\pm}(D,y,t)$$

has an integrable kernel, so it is L^2 bounded. Therefore $e^{\pm i\psi_\pm}(t,x,D)P_0$ and its adjoint are L^2 bounded. To accomodate symbol localizations we observe that

$$e^{\pm i\psi_\pm}_{<k} = \int m_k(z)e^{\pm iT_z\psi_\pm}_{<0}\,dz$$

where $m(z)$ is an integrable bump function on the 2^{-k} scale and T_z denotes translation in the direction z, with z representing space-time coordinates. Since the wave equation is invariant to translations, the symbol $e^{\pm iT_z\psi_\pm}$ is of the same type as $e^{\pm i\psi_\pm}$ and its left and right quantizations are also L^2 bounded. Thus the bound (114) follows by integration with respect to z.

b) For the estimate (115) we note that the kernel of $e^{-i\psi_\pm}_{<0}(t,x,D)a(D)e^{i\psi_\pm}_{<0}(D,y,t) - a(D)$ is given by $K^a_{<0}(t,x,t,y) - \check{a}(x-y)$. Combining (105) and (109) we get

$$|K^a_{<0}(t,x,t,y) - \check{a}(x-y)| \lesssim \min\{\epsilon|\log\epsilon|, |x-y|^{-N}\}$$

The integral of the expression on the right is about $\epsilon^{\frac{N-4}{N}}|\log\epsilon|$, therefore the conclusion follows.

c) By translation invariance we discard the < 0 symbol localization, and show that $\partial_{x,t}e^{\pm i\psi_\pm}(t,x,D)P_0$ is L^2 bounded. We have

$$\partial_{x,t}e^{\pm i\psi_\pm} = \pm\partial_{x,t}\psi_\pm e^{\pm i\psi_\pm}.$$

By (99) we have $\partial_{x,t}\psi_\pm \in DL^\infty(L^\infty)$ therefore we can dispose of it and use the L^2 boundedness of $e^{\pm i\psi_\pm}(t,x,D)P_0$.

□

8.3. **Modulation localized estimates.** The next order of business is to show that the fixed time L^2 bounds for $e^{\pm i\Psi}_{<0}$ drastically improve to space-times $L^2(L^2)$ bounds if one selects a fixed frequency in the symbol. Precisely,

Proposition 9. *For $l \leqslant k' \pm O(1)$ one has the fixed frequency estimate:*

$$\| Q_l e^{\pm i\Psi}_{k'} Q_{<0} P_0 \|_{N^* \to X_1^{0,\frac{1}{2}}} \lesssim 2^{\delta(l-k')}\epsilon \ . \tag{117}$$

In particular summing over all (l,k') with $l \leqslant k$ and $k - O(1) \leqslant k'$ for a fixed $k \leqslant 0$ yields:

$$\| Q_{<k}(e^{\pm i\Psi}_{<0} - e^{\pm i\Psi}_{<k-C}) Q_{<0} P_0 \|_{N^* \to X_1^{0,\frac{1}{2}}} \lesssim \epsilon \ . \tag{118}$$

A key step in the proof of the proposition is the following result, which we state separately since it is of independent interest:

Lemma 10. *Assume that $1 \leqslant q \leqslant p \leqslant \infty$. Then for $k + C \leq l \leq 0$ we have :*

$$\| (e^{\pm i\psi_{<k}}_l)(t,x;D) \|_{L^p(L^2)\to L^q(L^2)} \lesssim \epsilon 2^{(\frac{1}{p}-\frac{1}{q})k} 2^{5(k-l)} \ , \tag{119}$$

This holds for both left and right quantizations.

Proof. For the symbol we iteratively write:

$$\begin{aligned} S_l e^{\pm i\psi_{<k}} &= \pm i 2^{-l} S_l\big(\partial_t\psi_{<k} \cdot e^{\pm i\psi_{<k}}\big) \ , \\ &= \ \ldots \ = (\pm i)^5 2^{-5l} \prod_{j=1}^{5} [S_l^{(j)}\partial_t\psi_{<k}] \cdot e^{\pm i\psi_{<k}} \ , \end{aligned}$$

where the product denotes a nested (repeated) application of multiplication by $S_l\partial_t\psi_{<k}$, for a series of frequency cutoffs $S_l^{(j+1)}S_l^{(j)} = S_l^{(j)} \approx S_l$ with expanding widths. Disposing of these translation invariant cutoffs we see that (119) follows directly from (98).

□

Proof of Proposition 9. We proceed in a series of steps.
Step 1:(*High modulation input*) First we estimate the contribution of $Q_k e^{\pm i\Psi}_{k'} Q_{\geqslant k-C} P_0$ to line (117). Using the $X^{0,\frac{1}{2}}_\infty$ bounds for the input, it suffices to prove the estimate:

$$\| Q_k e^{\pm i\Psi}_{k'} P_0 \|_{L^2(L^2)\to L^2(L^2)} \lesssim 2^{\frac{1}{5}(k-k')}\epsilon \ .$$

By Sobolev estimates in $|\tau| \pm |\xi|$, this reduces to the bound:

$$\| e^{\pm i\Psi}_{k'} P_0 \|_{L^2(L^2)\to L^{\frac{10}{7}}(L^2)} \lesssim 2^{-\frac{1}{5}k'}\epsilon \ .$$

Using continuous Littlewood-Paley resolutions to decompose the group element we have:

$$\begin{aligned} e^{\pm i\Psi}_{k'} \ &= \ \pm i \int_{k''>k'-C} S_{k'}(\Psi_{k''} e^{\pm i\Psi_{<k''}}) dk'' + S_{k'} e^{\pm i\Psi_{<k'-C}} \ , \\ &= \ \mathcal{L} + \mathcal{R} \ . \end{aligned} \tag{120}$$

We'll treat these two terms separately. In the case of $\mathcal{L}$ we use estimate (98) with $p = 5$. To estimate the contribution of $\mathcal{R}$ we use estimate (119).

Step 2:(*Main decomposition for low modulation input*) Now we estimate the expression $Q_k e^{\pm i\Psi}_{k'} Q_{<k-C} P_0 u$. First expand the untruncated group elements as follows:

$$\begin{aligned} e^{\pm i\Psi} \ &= \ e^{\pm i\Psi_{<k-C}} \pm i \int_{l>k-C} \Psi_l e^{\pm i\Psi_{<k-C}} dl - \iint_{l,l'>k-C} \Psi_l \Psi_{l'} e^{\pm i\Psi_{<k-C}} dl dl' \\ &\quad \mp i \iiint_{l,l',l''>k-C} \Psi_l \Psi_{l'} \Psi_{l''} e^{\pm i\Psi_{<l'''}} dl dl' dl'' \ , \\ &= \ \mathcal{Z} + \mathcal{L} + \mathcal{Q} + \mathcal{C} \ . \end{aligned}$$

We will estimate the effect of each of these terms separately.
Step 3:(*Estimating the zero order term* $\mathcal{Z}$) After localization via $S_{k'}$ the desired estimate follows directly from (119).

Step 4:(*Estimating the linear term* $\mathcal{L}$) First split the integral as follows:

$$\mathcal{L} \ = \ \int_{l>k-C} \Psi_l S_{<k-\frac{1}{2}C} e^{\pm i\Psi_{<k-C}} dl + \int_{l>k-C} \Psi_l S_{\geqslant k-\frac{1}{2}C} e^{\pm i\Psi_{<k-C}} dl \ . \tag{121}$$

Step 4a:(*Estimating the principal linear term in* $\mathcal{L}$) For the first term on RHS of line (121) it suffices to show the general estimate:

$$\begin{aligned} &\| Q_k \cdot (\psi_l b_{<k-\frac{1}{2}C})(t,x;D) \cdot Q_{<k-C} P_0 \|_{L^\infty(L^2)\to L^2(L^2)} \\ &\qquad \lesssim \epsilon\, 2^{-\frac{1}{2}k} 2^{\frac{1}{4}(k-l)} \sup_t \| B_{<k-\frac{1}{2}C}(t) \|_{L^2\to L^2} \ , \end{aligned} \tag{122}$$

for $l \geqslant k$, and for symbols $b(x,\xi)_{<k-\frac{1}{2}C}$ with either the left or right quantization. In this case the modulation of the output determines the angle between the spatial frequencies of $\psi_l(x,\xi)$ and the spatial frequency of the input, which is $\theta \sim 2^{\frac{1}{2}(k-l)}$. Since this is also the angle with ξ, we may restrict the symbol of Ψ_l to $\sum_{\theta \gtrsim 2^{\frac{1}{2}(k-l)}} \psi^{(\theta)}_l$ for which the estimate (122) follows immediately from (92) and summing over (97).

Step 4b:(*Estimating the frequency truncation error in* $\mathcal{L}$) For the second term on RHS of line (121) we use (98) for ψ_l with $p=6$ combined with (119) with $(p_2,q)=(6,3)$. If $l<k'-C$ the $S_{k'}$ localization lands on the exponential and we gain $2^{-\frac{1}{2}k}2^{\frac{1}{6}(k-l)}2^{5(k-k')}$. If $l\geqslant k'-C$ we can disregard $S_{k'}$ and directly get $2^{-\frac{1}{2}k}2^{\frac{1}{6}(k-l)}$.

Step 5:(*Estimating the quadratic term* $\mathcal{Q}$) We follow a similar procedure to **Step 4** above. First split $S_{<k-\frac{1}{2}C}e^{\pm i\psi_{<k-C}}+S_{\geqslant k-\frac{1}{2}C}e^{\pm i\psi_{<k-C}}$. For the second term one can proceed as in **Step 4b** above using (98), (119), and (90). Therefore we only need to consider the effect of the first term, for which we'll show the trilinear bound:

$$\| Q_k\cdot(\psi_l\psi_{l'}b_{<k-\frac{1}{2}C})(t,x;D)\cdot Q_{<k-C}P_0\|_{L^\infty(L^2)\to L^2(L^2)} \lesssim \epsilon^2\, 2^{-\frac{1}{2}k}2^{\frac{1}{4}(k-l)}2^{\frac{1}{6}(k-l')}\sup_t\| B_{<k-\frac{1}{2}C}(t)\|_{L^2\to L^2}\ , \tag{123}$$

for $l'\geqslant l\geqslant k$. By the localization of the output, the symbol $\psi_l\psi_{l'}$ is localized to the sum:

$$\sum_{\theta\gtrsim 2^{\frac{1}{2}(k-l)}}\psi_l^{(\theta)}\psi_{l'}+\sum_{\substack{\theta'\gtrsim 2^{\frac{1}{2}(k-l')}\\ \theta\ll 2^{\frac{1}{2}(k-l')}}}\psi_l^{(\theta)}\psi_{l'}^{(\theta')}\ =\ T_1+T_2\ .$$

For the term T_1 put the first factor in $DL^3(L^\infty)$ and the second in $DL^6(L^\infty)$. This gives us dyadic terms in LHS(123)(T_1) $\sim 2^{-\frac{1}{2}k}2^{\frac{1}{4}(k-l)}2^{\frac{1}{6}(k-l')}$. For the term T_2 do the opposite, which yields a similar bound.

Step 6:(*Estimating the cubic term* $\mathcal{C}$) In this case we can gain $2^{\frac{1}{6}(k-k')}$ directly through the use of (98) and three $DL^6(L^\infty)$ unless $l,l',l''<k'-C$. In the latter case $S_{k'}$ localizes the exponential and we can use a (119) instead. Further details are left to the reader.

□

9. Proof of the $N_0\to N_0$ bounds in (79) and (80)

Because of use in the sequel, we'll prove a somewhat more general and symmetric version:

Proposition 9.1 (Symmetric N Estimates)**.** *One has the following operator bounds for either right or left quantizations:*

$$\| e^{\pm i\Psi}_{<0}\|_{N\to N}\ \lesssim\ 1\ . \tag{124}$$

as well as

$$\| \partial_t e^{\pm i\Psi}_{<0}\|_{N\to N}\ \lesssim\ 1\ . \tag{125}$$

In particular, by duality one also has:

$$\| e^{\pm i\Psi}_{<0}\|_{N^*\to N^*}\ \lesssim\ 1\ , \tag{126}$$

for either right or left quantizations.

Proof. Applying $e^{\pm i\Psi}_{<0}$ to $F\in N_0$ we need to consider the following cases:
Case 1:(F *is an* $L^1(L^2)$ *atom*) This follows at once from the fixed time $L^2\to L^2$ mapping property of $e^{\pm i\Psi}_{<0}$.

Case 2:(*F is an $X_1^{0,-\frac{1}{2}}$ atom*) Fixing a modulation it suffices to consider Q_kP_0F, and show that:

$$\| e^{\pm i\Psi}_{<0} Q_k P_0 F \|_N \lesssim 2^{-\frac{1}{2}k} \| F \|_{L^2(L^2)} ,$$

which by duality reduces to showing:

$$\| Q_k e^{\pm i\Psi}_{<0} \widetilde{P}_0 G \|_{L^2(L^2)} \lesssim 2^{-\frac{1}{2}k} \| G \|_{N^*} ,$$

for a slightly larger cutoff $\widetilde{P}_0$. At this point the estimate follows the usual game of splitting $e^{\pm i\Psi}_{<0} = e^{\pm i\Psi}_{<k-C} + (e^{\pm i\Psi}_{<0} - e^{\pm i\Psi}_{<k-C})$. For the first term there is no modulation interference and the estimate is direct. For the second use (118). □

10. Proof of the Renormalization Error Estimates

10.1. **The N and N^* Space Recovery Estimate** (81)**.** We prove (81) for N^*. The $L^\infty L^2$ part of the estimate (81) for N^* is a direct consequence of the similar fixed time L^2 bound (115). It remains to consider the $X_\infty^{0,\frac{1}{2}}$ estimate. Denote

$$R_k = e^{-i\psi_\pm}_{<k}(x,D) e^{i\psi_\pm}_{<k}(D,y)$$

Fixing the modulation we write:

$$Q_k(R_0 - I) = Q_k(R_0 - I)Q_{>k-C} + Q_k R_0 Q_{<k-C}$$

For the first term we bound $Q_{>k-C}$ in L^2 and use again (115). For the second we can freely subtract the low frequencies in the symbol

$$\begin{aligned} Q_k R_0 Q_{<k-C} &= Q_k(R_0 - R_{<k-C})Q_{<k-C} \\ &= Q_k(e^{-i\psi_\pm}_{<0}(x,D) - e^{i\psi_\pm}_{<k-C}(x,D)) e^{i\Psi_\pm}_{<0}(D,y) Q_{<k-C} \\ &\quad + Q_k e^{-i\psi_\pm}_{<k-C}(x,D)(e^{i\psi_\pm}_{<0}(D,y) - e^{i\psi_\pm}_{<k-C}(D,y))Q_{<k-C} \end{aligned}$$

We estimate the two lines separately as operators from N^* to $X_\infty^{0,\frac{1}{2}}$. In the first line we can use (126) to discard $e^{i\Psi_\pm}_{<0}(y,D)Q_{<k-C}$, and then (118) for the remaining factor. In the second line we first write

$$Q_k e^{-i\psi_\pm}_{<k-C}(x,D) = Q_k e^{-i\psi_\pm}_{<k-C}(x,D)\tilde{Q}_k$$

and use (114) to discard the two left factors. Then we are reduced again to (118).

10.2. **Proof of the conjugation estimate** (82)**.** Recall the following product formula for left quantizations:

$$a(x,D)b(x,D) = c(x,D) , \quad c(x,\xi) \sim e^{-i\langle D_y, D_\xi\rangle}\big(a(x,\xi)b(y,\eta)\big)|_{\substack{x=y \\ \xi=\eta}} .$$

We are interested in the case where the first factor is the d'Alembertian. Then the above formula contains only finitely many terms and is exact and we'll denote it by $\#_l$. Using this to compute both terms on line (82) we have:

$$\begin{aligned} (-\partial_t^2 - |\xi|^2)\#_l e^{-i\psi_\pm}_{<0} =& e^{-i\psi_\pm}_{<0}(-\partial_t^2 - |\xi|^2) + 2[i\partial_t e^{-i\psi_\pm}_{<0}\partial_t - \partial_x e^{-i\psi_\pm}_{<0}\xi] - \Box e^{-i\psi}_{<0} \\ =& e^{-i\psi_\pm}_{<0}(-\partial_t^2 - |\xi|^2) + 2\partial_t e^{-i\psi_\pm}_{<0}(i\partial_t \pm |\xi|) \\ &- 2i[(\pm\partial_t\psi_\pm|\xi| - \partial_x\psi_\pm\xi)e^{-i\psi_\pm}]_{<0} \\ &- [(|\partial_t\psi_\pm|^2 - |\partial_x\psi_\pm|^2)e^{-i\psi_\pm}]_{<0} , \end{aligned}$$

On the other hand we can write

$$iA^j\xi_j \#_l e_{<0}^{-i\psi_\pm} = i[A^j e^{-i\psi_\pm}]_{<0}\xi_j + A^j\partial_j e_{<0}^{-i\psi_\pm} + i[A_j, S_{<0}]e^{-i\psi_\pm}\xi_j$$

Hence combining all terms we can write down the exact symbol for the conjugation operator

$$\begin{aligned}
\mathrm{Diff} &:= (-\partial_t^2 - |\xi|^2 + 2iA^j\xi_j)\#_l e_{<0}^{-i\psi_\pm} - e_{<0}^{-i\psi_\pm}(-\partial_t^2 - |\xi|^2) \\
&= -2i[(\pm\partial_t\psi_\pm|\xi| - \partial_x\psi_\pm\xi + A^j\xi_j)e^{-i\psi_\pm}]_{<0} \\
&\quad + 2\partial_t e_{<0}^{-i\psi_\pm}(i\partial_t \pm |\xi|) \\
&\quad - [(|\partial_t\psi_\pm|^2 - |\partial_x\psi_\pm|^2)e^{-i\psi_\pm}]_{<0} \\
&\quad - 2iA^j[\partial_j\psi_\pm e^{-i\psi_\pm}]_{<0} \\
&\quad + 2i[A_j, S_{<0}]e^{-i\psi_\pm}\xi_j \\
&:= \mathrm{Diff}_1 + \mathrm{Diff}_2 + \mathrm{Diff}_3 + \mathrm{Diff}_4 + \mathrm{Diff}_5
\end{aligned}$$

It remains to estimate all five terms as pseudodifferential operators from $S^\sharp_\pm \subset N^*_\pm$ to $N_\pm$.

The estimate for Diff$_1$: We recall that $\psi_\pm$ was chosen exactly so that the high angle interaction part of Diff$_1$ cancels. Using the definition of $\psi_\pm$ we rewrite this term as

$$\mathrm{Diff}_1 = [(I - \Pi^\sigma)A\cdot\xi)e^{-i\psi_\pm}]_{<0}$$

We can replace this by

$$\mathrm{Diff}_1 = [(I - \Pi^\sigma)A\cdot\xi)e_{<0}^{-i\psi_\pm}]_{<0}$$

where the inner < 0 truncation is on a slightly larger ball. By translation invariance we can discard the outer < 0 truncation. Also here we do not need the $S^\sharp$ structure, and it suffices to work with N^*. Then we can also use the bound (126) to discard $e_{<0}^{-i\psi_\pm}$. Hence we are left with having to prove a bilinear estimate, namely

$$(I - \Pi^\sigma)A^j\partial_j : N^* \to N \tag{127}$$

After a frequency and angular expansion of A_j, we obtain a sum

$$\sum_{k<0}\sum_{l<\sigma k}\sum_{\omega} P_l^\omega A_k^j P_l^{\omega'}\partial_j u_0$$

where the multipliers P_l^ω and $P_l^{\omega'}$ select 2^l sectors which are 2^l separated. To estimate the summands in N^* we split the into low and high modulation,

$$P_l^\omega A_k^j P_l^{\omega'} u_0 = P_l^\omega A_k^j P_l^{\omega'} Q_{>k+2l-C}\partial_j u_0 + P_l^\omega A_k^j P_l^{\omega'} Q_{<k+2l-C}\partial_j u_0$$

For the first term on the right, we place it into $L_t^1 L_x^2$ by using $L_t^2 L_x^\infty$ for the factor $P_l^\omega A_k^j$ and $l_{t,x}^2$ for the second factor $Q_{>k+2l-C}\partial_j u_0$. The Coulomb condition results in a gain 2^l (due to the operator ∂_j), and Bernstein's inequality via $P_k P_l^\omega L_t^2 L_x^6 \subset L_t^2 L_x^\infty$ yields a gain $2^{\frac{k+l}{2}}$, whence recalling the definition of N^*, we obtain a net gain of $2^{\frac{l}{2}}$, which suffices for summation both over l and k due to the condition $l < \sigma k$.

For the second term on the right, we place the output into $\dot X_1^{0,-\frac{1}{2}}$, which is possible since its modulation is of size $\sim 2^{k+2l}$ from elementary geometry. Placing the second input $\partial_j u_0$, the numerology is the same as for the first term.

The estimate for Diff$_2$: By the definition of $S^\sharp_\pm$ we have

$$i\partial_t \pm |D| : S^\sharp_\pm : N$$

Then we can use directly the bound (125).

The estimates for Diff$_3$ and Diff$_4$: Here we use decomposability. Precisely, by (97) we have $\partial_{t,x}\psi_\pm \in DL^2(L^\infty)$. It is also easy to see that A belongs to the same space. Then by (90) we can place the output into the energy space $L^1_t L^2_x$.

The estimate for Diff$_5$: This term is handled exactly like Diff$_1$; this time we do not have the small angular separation condition between the factors, but summation over k becomes possible since we essentially gain an extra low-frequency derivative:

$$[A_j, S_{<0}]\partial_j u_0 = L(\nabla_{t,x} A^j, \partial_j u_0)$$

where L is a translation invariant bilinear operator with integrable kernel.

11. Proof of the Dispersive Estimates

In this section we prove the mapping property (83) in a series of steps.

11.1. **Proof of the basic Strichartz estimates.** First we show:

$$e^{-i\psi_\pm}_{<0}(t,x,D) : S^\sharp_{0,\pm} \ \to \ S^{str} \ . \tag{128}$$

The first step is to reduce the problem to the case of homogeneous waves. This is done by foliating with respect to t for the $L^1 L^2$ part, and with respect to $\{\tau - |\xi| = \sigma\}$ slices for the $X_1^{0,-\frac{1}{2}}$ part,

$$u^\pm(t) \ = \ \int e^{it\sigma} e^{\pm it|D|} P_0 f^\pm(\tau) d\mu^\pm(\tau) \ , \quad \int \| f^\pm(\tau) \|_{L^2} d\mu^\pm(\tau) \ \lesssim \ \| F \|_{X_1^{0,-\frac{1}{2}}} \ .$$

Then it remains to show that

$$e^{-i\psi_\pm}_{<0}(t,x,D) e^{\pm it|D|} P_0 : L^2 \to S^{str}$$

By a TT^* argument this is equivalent to

$$e^{-i\psi_\pm}_{<0}(t,x,D) e^{\pm i(t-s)|D|} P_0 e^{i\psi_\pm}_{<0}(D,y,s) : L^{p'} L^{q'} \to L^p L^q$$

for all pairs of Strichartz exponents (p,q). In the non-endpoint case $p > 2$ his follows in a standard manner from an $L^2 \to L^2$ fixed time bound (see (114))and and the dispersive estimate (108). In the endpoint case $L^2 L^6$ we can use Theorem 4 in [28], which asserts that the endpoint case follows from the non-endpoint case plus the dispersive estimate.

11.2. **Proof of the square summed Strichartz estimates.** Fixing an angular scale 2^l and the corresponding modulation scale 2^{2l} as well as a rectangle scale $2^{k'} \times (2^{k'+l'})^3$ with $k' \le k, l' \le 0$ and $2l \le k' + l' \le l$ we need to show that

$$\sum_{C \in \mathcal{C}_{k',l'}} \| P_C Q_{<2l} e^{-i\psi_\pm}_{<0}(t,x,D) u \|^2_{S^{str}} \lesssim \| u \|^2_{S^\sharp}$$

For this we write

$$Q_{<2l} e^{-i\psi_\pm}_{<0}(t,x,D) u \ = \ Q_{<2l}(e^{-i\psi_\pm}_{<0} - e^{-i\psi_\pm}_{<2l-C}) u + Q_{<2l} e^{-i\psi_\pm}_{<2l-C} Q_{<2l+C} u$$

The first term is estimated in $X_1^{0,\frac{1}{2}}$ using (118). For the second one, the symbol of $e^{-i\psi_\pm}_{<2l-C}$ is strongly localized so that non adiacent rectangles do not interact. Then the square summation with respect to cubes is inherited from $S^\sharp$, and it remains to prove that for a single cube C we have

$$\|e^{-i\psi_\pm}_{<2l-C}(t,x,D)P_C u\|_{S^{str}} \lesssim \|u\|_{S^\sharp_\pm} \tag{129}$$

We can freely discard P_C. Then the last bound follows by translation invariance from (128).

11.3. **Proof of the square summed L^2L^∞ estimates.** The setting and the argument is identical to the one above up to the counterpart of (129), which now reads

$$\|e^{-i\psi_\pm}_{<2l-C}(t,x,D)Q_{<2l+C}P_C u\|_{L^2L^\infty} \lesssim 2^{k'}\|u\|_{S^\sharp_\pm} \tag{130}$$

We can no longer discard P_C due to the presence of the scale factor $2^{k'}$. Instead we repeat the argument in the proof of (128). irs the problem reduces to an estimate for free waves,

$$\|e^{-i\psi_\pm}_{<2l-C}(t,x,D)e^{\pm it|D|}P_C u\|_{L^2L^\infty} \lesssim 2^{k'}\|u_0\|_{L^2_x}$$

Then by a TT^* argument this is equivalent to

$$\|e^{-i\psi_\pm}_{<2l-C}(t,x,D)e^{\pm i(t-s)|D|}P_C e^{i\psi_\pm}_{<0}(D,y,s)f\|_{L^2L^\infty} \lesssim 2^{2k'}\|f\|_{L^2L^1}$$

This in turns follows from the dispersive estimate (110), which shows that kernel of the operator above is bounded by

$$2^{4k'+3l'}\langle 2^{2(k'+l')}(t-s)\rangle^{-\frac{3}{2}}$$

which integrates to $2^{2k'+l'} \leq 2^{2k'}$.

11.4. **Proof of the plane wave norm estimates.** Again using the nesting (5), at modulation $< 2^{2l}$ it suffices to consider expressions involving $Q_{<2l}e^{-i\psi_\pm}_{<2l-C}(t,x,D)$. By foliating as in the proof of (128), we further reduce considerations to expressions $e^{-i\psi_\pm}_{<2l-C}(t,x,D)e^{\pm it|D|}P_0 f$ for $\| f \|_{L^2} \leqslant 1$. Localizing $e^{i\Psi}_{<k-C}e^{\pm it|D|}P_0 f$ by $P_{\mathcal{C}}$, we employ polar coordinates in Fourier space to write:

$$e^{-i\psi_\pm}_{<2l-C}(t,x,D)e^{\pm it|D|}P_{\mathcal{C}}f \ = \ \frac{1}{(2\pi)^4}\int_{\mathbb{S}^3} e^{-i\psi(t,x;\omega)_\pm}_{<2l-C} f^\omega_\pm(t,x)d\omega \ ,$$

where:

$$f^\omega_\pm(t,x) \ = \ \int e^{i(\pm t + x\cdot\omega)\lambda}p_{\mathcal{C}}(\lambda\omega)\widehat{f}(\lambda\omega)\lambda^3 d\lambda \ .$$

Then computation of the square sum $PW^\mp$ norms follows immediately from these expressions, Hölder's inequality, and the Plancherel's theorem.

11.5. **Proof of the null energy estimates.** As in the case of the square summed Strichartz estimates, the problem reduces to proving that

$$e^{-i\psi_\pm}_{<0}(t,x,D) : S^\sharp \to NE$$

We fix a null direction ω and prove the null energy estimate in its associated frame,

$$\nabla\!\!\!\!/\, e^{-i\psi_\pm}_{<0}(t,x,D) : S^\sharp \to L^\infty_\omega L^2_{\omega^\perp}$$

For simplicity we take $\omega = (1,1,0,0,0)$. The symbol of its associated tangential gradient $\nabla\!\!\!\!/$ then consists of $\tau - \xi_1$ and ξ'. Depending on whether we are near the + cone or near the − cone we can replace

$$\tau - \xi_1 = (\tau \pm |\xi|) - (\xi_1 \pm |\xi|)$$

The contribution of the first term to the output is easily estimated via the N^* norm in L^2 and then in $L^\infty_\omega L^2_{\omega^\perp}$ by Bernstein. It remains to bound

$$(D_1 \pm |D|, D')e^{-i\psi_\pm}_{<0}(t,x,D) : S^\sharp_\pm \to L^\infty_\omega L^2_{\omega^\perp}$$

We fix the signs to $+$. Since no time multipliers are involved any more in the above operator, the reduction to the case of homogeneous waves still applies, and we are left with proving that

$$(D_1 \pm |D|, D')e^{-i\psi_\pm}_{<0}(t,x,D)e^{it|D|} : L^2 \to L^\infty_\omega L^2_{\omega^\perp}$$

The obvious course of action at this point is to use disposability arguments to commute the derivatives inside, and then use a TT^* argument. This unfortunately borderline fails due to the limited decay of the wave kernel along null cones. Instead we take an extra step, and perform a dyadic decomposition with respect to the angle between ξ and $\omega' = (1,0,0,0)$. There are two cases we need to consider separately.

For large $O(1)$ angles we can simply discard the multipliers, and show that

$$P^{\omega'}_1 e^{-i\psi_\pm}_{<0}(t,x,D)e^{it|D|} : L^2 \to L^\infty_\omega L^2_{\omega^\perp}$$

For this we commute $P^{\omega'}_1$ inside, using disposability to estimate the error in L^2. Then, by translation invariance, it remains to show that

$$e^{-i\psi_\pm}_{<0}(t,x,D)e^{it|D|}P^{\omega'}_1 : L^2 \to L^2(\Sigma), \qquad \Sigma = \{t = \omega' \cdot x\}$$

By a TT^* argument this is reduced to an estimate of the form

$$e^{-i\psi_\pm}_{<0}(t,x,D)e^{i(t-s)|D|}P^{\omega'}_1 e^{i\psi_\pm}_{<0}(D,y,s) : L^2(\Sigma) \to L^2(\Sigma)$$

Due to the angular separation, we can use (107) with $l = 0$ to conclude that the kernel of the above operator decays rapidly off diagonal, and the L^2 boundedness easily follows.

For small $\ll 1$ angles we can write $D_1 \pm |D| = a(D)D'$ and then discard a, and we are left with proving that

$$D' e^{-i\psi_\pm}_{<0}(t,x,D)e^{it|D|} : L^2 \to L^\infty_\omega L^2_{\omega^\perp}$$

The advantage of this is that multipliers which depend only on D' are compatible with the $L^2_{\omega^\perp}$. Hence via a dyadic decomposition in D', we need to show that

$$2^k P_k(D')e^{-i\psi_\pm}_{<0}(t,x,D)e^{it|D|} : L^2 \to l^2_k L^2(\Sigma), \qquad k < -C$$

We commute the multiplier inside,

$$\begin{aligned} 2^k P_k(D')e^{-i\psi_\pm}_{<0}(t,x,D)e^{it|D|} &= \tilde{P}_k(D')[2^k P_k(D'), e^{-i\psi_\pm}_{<0}(t,x,D)]e^{it|D|} \\ &\quad + \tilde{P}_k(D')e^{-i\psi_\pm}_{<0}(t,x,D)e^{it|D|}P_k(D') \end{aligned}$$

By the commutator estimate (94) the first term satisfies the same bounds as the operator

$$\tilde{P}_k(D')(\nabla' e^{-i\psi_\pm}_{<0})(t,x,D)e^{it|D|} = -i\tilde{P}_k(D')(\nabla'\psi_\pm e^{-i\psi_\pm})_{<0}(t,x,D)e^{it|D|}$$

Since $\nabla\psi_\pm \in DL^2L^r$ for some $r < \infty$, by disposability we have

$$(\nabla'\psi_\pm e^{-i\psi_\pm})_{<0}(t,x,D) : L^\infty L^2 \to L^2 L^{\frac{2r}{r+2}}$$

Hence by Bernstein,

$$\|\tilde{P}_k(D')(\nabla' e^{-i\psi_\pm}_{<0})(t,x,D)e^{it|D|}\|_{L^2_x to L^2_{x,t}} \lesssim 2^{-\frac{4}{r}k}$$

where the point is that there is some gain in k in order to guarantee summability.

It remains to show that

$$\tilde{P}_k(D')e^{-i\psi_\pm}_{<0}(t,x,D)e^{it|D|}P_k(D') : L^2 \to l^2_k L^2(\Sigma)$$

We harmlessly discard $\tilde{P}_k(D')$. The square summability with respect to k is now provided by the multiplier on the right, so it suffices to fix k. By a TT^* argument this reduces to

$$e^{-i\psi_\pm}_{<0}(t,x,D)e^{i(t-s)|D|}2^{2k}P_k^2 e^{i\psi_\pm}_{<0}(D,y,s) : L^2(\Sigma) \to L^2(\Sigma)$$

For the above operator we use the kernel bound provided by (111) with $l = k$, namely

$$2^{5l}\langle 2^{2l}|t-s|\rangle^{-N}\langle 2^l|x-y|\rangle^{-N}$$

This is an integrable kernel, so the $L^2 \to L^2$ bound easily follows.

12. Statements and proofs of the core multilinear estimates

It remains to prove (31), (32), (42), (53), (54), (55), (58), (59), and finally (60) for each of the expressions (61)–(63). This set of estimates can be roughly grouped into two rough categories. The first involves "null-form" type estimates, and the second involves product estimates without a microlocal gain from angular separation. The first category of bounds boils down to:

Theorem 12.1 (Core null form estimates)**.** *The following hold:*

$$\| P_{k_1}\mathcal{N}(\phi^{(2)}_{k_2},\phi^{(3)}_{k_3}) \|_N \ \lesssim\ 2^{k_1}2^{\delta(k_1-\max\{k_2,k_3\})}2^{-\delta|k_2-k_3|}\| \phi^{(2)}_{k_2} \|_{S^1}\| \phi^{(3)}_{k_3} \|_{S^1} \ , \tag{131}$$

$$\| (I-\mathcal{H}^*_{k_1})\mathcal{N}(\phi^{(1)}_{k_1},\phi^{(2)}_{k_2}) \|_N \lesssim 2^{k_1}\| \phi^{(1)}_{k_1} \|_{S^1}\| \phi^{(2)}_{k_2} \|_{S^1},\ k_1 < k_2 - C \ , \tag{132}$$

$$\| \mathcal{H}^*_{k_1}\mathcal{N}(\phi^{(1)}_{k_1},\phi^{(2)}_{k_2}) \|_{L^1(L^2)} \lesssim 2^{k_1}\| \phi^{(1)}_{k_1} \|_{Z^{hyp}}\| \phi^{(2)}_{k_2} \|_{S^1},\ k_1 < k_2 - C \ , \tag{133}$$

$$\begin{aligned} &\| (I-\mathcal{H}_{k_1})P_{k_1}\mathcal{N}(\phi^{(2)}_{k_2},\phi^{(3)}_{k_3}) \|_{\Box Z^{hyp}} \\ &\qquad \lesssim 2^{k_1}2^{\delta(k_1-\max\{k_2,k_3\})}2^{-\delta|k_2-k_3|}\| \phi^{(2)}_{k_2} \|_{S^1}\| \phi^{(3)}_{k_3} \|_{S^1} \ , \end{aligned} \tag{134}$$

$$\begin{aligned} &\| \mathcal{H}_{k_1}\mathcal{N}(\phi^{(2)}_{k_2},\phi^{(3)}_{k_3}) \|_{\Box Z^{hyp}} \\ &\qquad \lesssim\ 2^{k_1}2^{-\delta|k_2-k_3|}\| \phi^{(2)}_{k_2} \|_{S^1}\| \phi^{(3)}_{k_3} \|_{S^1},\ k_1 > \max\{k_2,k_3\} - C \ . \end{aligned} \tag{135}$$

In addition one has the following quadrilinear form bounds, which hold under the condition $k < k_i - C$:

$$\begin{aligned} &\big|\big\langle \Box^{-1}\mathcal{H}_k(\phi^{(1)}_{k_1}\cdot\partial_\alpha\phi^{(2)}_{k_2}),\mathcal{H}_k(\partial^\alpha\phi^{(3)}_{k_3}\cdot\psi_{k_4})\big\rangle\big| \\ &\qquad \lesssim\ 2^{\delta(k-\min\{k_i\})}\| \phi^{(1)}_{k_1} \|_{S^1}\| \phi^{(2)}_{k_2} \|_{S^1}\| \phi^{(3)}_{k_3} \|_{S^1}\| \psi_{k_4} \|_{N^*} \ , \end{aligned} \tag{136}$$

$$\begin{aligned} &\big|\big\langle (\Box\Delta)^{-1}\mathcal{H}_k\partial_\alpha(\phi^{(1)}_{k_1}\cdot\partial^\alpha\phi^{(2)}_{k_2}),\partial_t\mathcal{H}_k(\partial_t\phi^{(3)}_{k_3}\cdot\psi_{k_4})\big\rangle\big| \\ &\qquad \lesssim\ 2^{\delta(k-\min\{k_i\})}\| \phi^{(1)}_{k_1} \|_{S^1}\| \phi^{(2)}_{k_2} \|_{S^1}\| \phi^{(3)}_{k_3} \|_{S^1}\| \psi_{k_4} \|_{N^*} \ , \end{aligned} \tag{137}$$

$$\begin{aligned} &\big|\big\langle (\Box\Delta)^{-1}\nabla_x\mathcal{H}_k(\phi^{(1)}_{k_1}\cdot\nabla_x\phi^{(2)}_{k_2}),\mathcal{H}_k\partial_\alpha(\partial^\alpha\phi^{(3)}_{k_3}\cdot\psi_{k_4})\big\rangle\big| \\ &\qquad \lesssim\ 2^{\delta(k-\min\{k_i\})}\| \phi^{(1)}_{k_1} \|_{S^1}\| \phi^{(2)}_{k_2} \|_{S^1}\| \phi^{(3)}_{k_3} \|_{S^1}\| \psi_{k_4} \|_{N^*} \ . \end{aligned} \tag{138}$$

The second category of bounds is covered by:

Theorem 12.2 (Additional core product estimates)**.** *The following hold:*

(139) $\| (I-\mathcal{H}^*_{k_1})(Q_{<k_2-C}A_{k_1}\partial_t\phi_{k_2}) \|_N \lesssim 2^{\frac{3}{2}k_1}\| A_{k_1} \|_{L^2(L^2)}\| \phi_{k_2} \|_{S^1}, \quad k_1<k_2-C \ ,$

(140) $\| \mathcal{H}^*_{k_1}(A_{k_1}\partial_t\phi_{k_2}) \|_{L^1(L^2)} \lesssim \| A_{k_1} \|_{Z^{ell}}\| \phi_{k_2} \|_{S^1}, \quad k_1<k_2-C \ ,$

(141) $\| (I-\mathcal{H}_{k_1})P_{k_1}(\phi^{(2)}_{k_2}\partial_t\phi^{(3)}_{k_3}) \|_{\Delta Z^{ell}} \lesssim 2^{\delta(k_1-k_2)}\| \phi^{(2)}_{k_2} \|_{S^1}\| \phi^{(3)}_{k_3} \|_{S^1} \ , \quad k_1\leqslant k_2-C \ ,$

We conclude this Section with the application of Theorems 12.1 and 12.2 to the estimates listed above. This is for the most part straight forward and left to the reader.

Proof that Theorem 12.1 implies estimates (31)*,* (32)*,* (53)*,* (54)*,* (55)*, and* (60)*.* For (31) we use (131), for (32) use (134), for (53) use (131) with $k_2\geq k_1+O(1)$, for (54) use (132), for (55) use (133), and for (60) use (136) - (138). □

Proof that Theorem 12.2 implies estimates (42)*,* (58)*, and* (59)*.* For (42) use (141), for (58) use (139), and for (59) use (140).

□

For the remainder of this Section we prove Theorems 12.1 and 12.2. We begin with a simple calculation that will be used a number of times:

Lemma 12.3 (Square summed $L^2(L^6)\to L^2(L^\infty)$ estimate)**.** *Let* $j-C\leqslant k'+l'\leqslant \frac{1}{2}(j+k)+C$ *with* $l'<C$ *and* $k'\leqslant k+C$*. Then one has the following uniform estimate:*

(142) $$\Big(\sum_{\mathcal{C}_{k'}(l')}\| P_{\mathcal{C}_{k'}(l')}Q_{<j}\phi_k \|^2_{L^2(L^\infty)}\Big)^{\frac{1}{2}} \lesssim 2^{\frac{2}{3}k'}2^{\frac{1}{2}l'}2^{\frac{5}{6}k}\| \phi_k \|_{S_k[L^2(L^6)]} \ ,$$

where $\mathcal{C}_{k'}(l')$ *is a finitely overlapping collection of radially oriented rectangles of dimensions* $(2^{k'+l'})^3\times 2^{k'}$*. Here* $S_{k_2}[L^2(L^6)]$ *refers to the* $L^2(L^6)$ *portion (including square sums) of the norm from lines* (6) *and* (8)*.*

Proof of estimate (142)*.* This follows immediately from the $L^2(L^6)$ estimate on line (8) and Bernstein's inequality in the form $P_{\mathcal{C}_{k'}(l')}L^6\subseteq 2^{\frac{2}{3}k'}2^{\frac{1}{2}l'}L^\infty$. □

12.1. **Proof of the null form estimates.** We begin with estimates (131)–(133). These can be boiled down to an even more atomic form which is the following:

Lemma 12.4 (Core modulation estimates)**.** *The following estimate holds uniformly in the indices* j_i,k_i*, where* $j_2,j_3=j_1+O(1)$*:*

(143) $$\big|\big\langle Q_{j_1}\phi^{(1)}_{k_1},\mathcal{N}(Q_{<j_2}\phi^{(2)}_{k_2},Q_{<j_3}\phi^{(3)}_{k_3})\big\rangle\big| \lesssim 2^{-\delta|j_1-k_2|}2^{-\delta|k_1-k_3|}2^{\min\{k_1,k_3\}}2^{2k_2}\| \phi^{(1)}_{k_1} \|_{X^{0,\frac{1}{2}}_\infty}\| \phi^{(2)}_{k_2} \|_{S_{k_2}}\| \phi^{(3)}_{k_3} \|_{L^\infty(L^2)} \ ,$$

In addition, when $j>k_{min}+C$ *one has the improved bound:*

(144) $$\big|\big\langle Q_j\phi^{(1)}_{k_1},\mathcal{N}(\phi^{(2)}_{k_2},\phi^{(3)}_{k_3})\big\rangle\big|\lesssim 2^{-\delta(j-k_{min})}2^{2k_{min}}2^{k_{max}}\|\phi^{(1)}_{k_1}\|_{X^{0,\frac{1}{2}}_\infty}\|\phi^{(2)}_{k_2}\|_{S_{k_2}}\|\phi^{(3)}_{k_3}\|_{N^*} \ .$$

Proof of estimate (143)*.* There are three cases depending on the relative separation of spatial frequencies. Each of these is further split into low and high modulation subcases.

Case 1a:($k_2=k_3+O(1)$ *and* $j_1<k_1$) Here the frequency angle separation between the second two factors is $\angle(\phi^2,\phi^3)\lesssim 2^{-k_2}2^{\frac{1}{2}(k_1+j_1)}:=2^l$. The null form $\mathcal{N}$ saves one power of this angle. On the other hand, the angle of interaction with the output is $\angle(\phi^1,\phi^2)\lesssim$

$2^{\frac{1}{2}(j_1-k_1)} := 2^{l'}$. Breaking up the high frequency factors in the null form with respect to the 2^{k_1} radial scale and the 2^l sector scale it remains to estimate the expression $Q_{j_1}F$ in L^2 where

$$F := P_{k_1} \sum_{\mathcal{C}_{k_1}(l')} \mathcal{N}\big(P_{\mathcal{C}_{k_1}(l')} Q_{<j_2}\phi^{(2)}_{k_2}, P_{-\mathcal{C}_{k_1}(l')} Q_{<j_3}\phi^{(3)}_{k_3}\big)$$

Disposing of $\mathcal{N}$ we start with the fixed time estimate

$$\begin{aligned}\|F(t)\|^2_{L^2} &\lesssim 2^{2(k_2+k_3+l)}\big(\sum_{\mathcal{C}_{k_1}(l')} \|P_{\mathcal{C}_{k_1}(l')} Q_{<j_2}\phi^{(2)}_{k_2}(t)\|_{L^\infty} \|P_{-\mathcal{C}_{k_1}(l')} Q_{<j_3}\phi^{(3)}_{k_3}(t)\|_{L^2}\big)^2 \\ &\lesssim 2^{2k_3+k_1+j_1} \left(\sum_{\mathcal{C}_{k_1}(l')} \|P_{\mathcal{C}_{k_1}(l')} Q_{<j_2}\phi^{(2)}_{k_2}(t)\|^2_{L^\infty}\right)\left(\sum_{\mathcal{C}_{k_1}(l')} \|P_{-\mathcal{C}_{k_1}(l')} Q_{<j_3}\phi^{(3)}_{k_3}(t)\|^2_{L^2}\right) \\ &\lesssim 2^{2k_3+k_1+j_1} \left(\sum_{\mathcal{C}_{k_1}(l')} \|P_{\mathcal{C}_{k_1}(l')} Q_{<j_2}\phi^{(2)}_{k_2}(t)\|^2_{L^\infty}\right) \|\phi^{(3)}_{k_3}\|^2_{L^\infty L^2}\end{aligned}$$

where at the last step we have used orthogonality in frequency and then the $L^\infty L^2$ boundedness of $Q_{<j_3}$. Hence integrating in time we arrive at

$$\|F\|^2_{L^2} \lesssim \; 2^{2k_3+k_1+j_1}\|\phi^{(3)}_{k_3}\|^2_{L^\infty L^2} \sum_{\mathcal{C}_{k_1}(l)} \|P_{\mathcal{C}_{k_1}(l')} Q_{<j_2}\phi^{(2)}_{k_2}(t)\|^2_{L^2 L^\infty} \tag{145}$$

Then applying Lemma 12.3 to put $\phi^{(2)}_{k_2}$ in $L^2(L^\infty)$, we see that:

$$\begin{aligned}\|Q_{j_1}F\|_{L^2} &\lesssim 2^{\frac{2}{3}k_1} 2^{\frac{1}{2}l'} 2^{\frac{5}{6}k_3} 2^{k_3+\frac{1}{2}(k_1+j_1)} \| \phi^{(2)}_{k_2} \|_{S_{k_2}[L^2(L^6)]} \| \phi^{(3)}_{k_3} \|_{L^\infty(L^2)} \\ &= 2^{\frac{1}{2}j_1} 2^{\frac{1}{4}(j_1-k_1)} 2^{\frac{1}{6}(k_1-k_2)} 2^{k_1}2^{k_2}2^{k_3} \| \phi^{(2)}_{k_2} \|_{S_{k_2}[L^2(L^6)]} \| \phi^{(3)}_{k_3} \|_{L^\infty(L^2)}\end{aligned}$$

concluding the proof in this case.

Case 1b:($k_2 = k_3 + O(1)$ *and* $j_1 \geqslant k_1$) Here integrating by parts we get a factor of $2^{k_1+k_2}$ from the null form. Then we localize $\phi^{(2)}_{k_2}$ and $\phi^{(3)}_{k_3}$ with respect to 2^{k_1} sized frequency cubes, but without any modulation localization. For $\phi^{(2)}_{k_2}$ we use the bound (9), and for $\phi^{(3)}_{k_3}$ we use the $L^\infty L^2$ norm. Then the same computation as above for

$$F = \sum_{\mathcal{C}_{k_1}} \mathcal{N}\big(P_{\mathcal{C}_{k_1}} Q_{<j_2}\phi^{(2)}_{k_2}, P_{-\mathcal{C}_{k_1}} Q_{<j_3}\phi^{(3)}_{k_3}\big)$$

yields

$$\|F\|_{L^2} \lesssim 2^{2k_1+\frac{3}{2}k_2}\|\phi^{(2)}_{k_2}\|_{S_{k_2}}\|\phi^{(3)}_{k_3}\|_{L^\infty L^2}$$

which suffices.

Case 2a:($k_1 = k_2+O(1)$ *and* $j_1 < k_3$) This case is mostly analogous to **Case 1a**. Integrating by parts one may place the null form between $\phi^{(1)}_{k_1}$ and $\phi^{(2)}_{k_2}$. Then the angular decompositions between the inputs of these two terms, the angle of the output (frequency 2^{k_3} now) is the same as **Case 1** but with k_1 and k_3 transposed. With

$$F := P_{k_3} \sum_{\mathcal{C}_{k_3}(l')} \mathcal{N}\big(P_{\mathcal{C}_{k_3}(l')} Q_{j_1}\phi^{(1)}_{k_1}, P_{-\mathcal{C}_{k_3}(l')} Q_{<j_2}\phi^{(2)}_{k_2}\big)$$

the relevant computation becomes:

$$\|F\|_{L^1L^2} \lesssim 2^{k_1+\frac{1}{2}(k_3+j_1)} \sum_{\mathcal{C}_{k_3}(l')} \|P_{\mathcal{C}_{k_3}(l')} Q_{j_1}\phi^{(1)}_{k_1}\|_{L^2} \|P_{-\mathcal{C}_{k_3}(l')} Q_{<j_2}\phi^{(2)}_{k_2}\|_{L^2L^\infty}$$

$$\lesssim 2^{k_1+\frac{1}{2}(k_3+j_1)} \left(\sum_{\mathcal{C}_{k_3}(l')} \|P_{\mathcal{C}_{k_3}(l')} Q_{j_1}\phi^{(1)}_{k_1}\|_{L^2}\right)^{\frac{1}{2}} \left(\sum_{\mathcal{C}_{k_3}(l')} \|P_{-\mathcal{C}_{k_3}(l')} Q_{<j_2}\phi^{(2)}_{k_2}\|_{L^2L^\infty}\right)^{\frac{1}{2}}$$

$$\lesssim 2^{\frac{1}{2}j_1} 2^{\frac{1}{4}(j_1-k_3)} 2^{\frac{1}{6}(k_3-k_2)} 2^{k_1} 2^{k_2} 2^{k_3} \| Q_{j_1}\phi^{(1)}_{k_1} \|_{L^2(L^2)} \| \phi^{(2)}_{k_2} \|_{S_{k_2}[L^2(L^6)]},$$

which suffices.

Case 2b:($k_1 = k_2 + O(1)$ *and* $j_1 > k_3$) The same argument as in **Case 2a** applies, with the only difference that $\phi^{(1)}_{k_1}$ and $\phi^{(2)}_{k_2}$ are now decomposed on the frequency scale 2^{k_3}, and there is no modulation cutoff. Thus the bound (9) has to be used for the latter.

Case 3a:($k_1 = k_3 + O(1)$ *and* $j < k_2$) The computation here is essentially the same as above, but with slightly different numerology. The main difference is now that $\angle(\phi^2, \phi^3) \lesssim 2^{\frac{1}{2}(j_1-k_2)} := 2^l$, and we sum over sectors of size $\sim 2^{k_2} \times (2^{k_2+l})^3$. Then a computation similar to **Case 1** gives us:

$$\text{LHS}(143) \lesssim 2^{-\frac{1}{2}j_1} 2^{\frac{3}{2}l} 2^{\frac{5}{2}k_2} 2^{k_3} \| \phi^{(1)}_{k_1} \|_{X^{0,\frac{1}{2}}_\infty} \| \phi^{(2)}_{k_2} \|_{S_{k_2}[L^2(L^6)]} \| \phi^{(3)}_{k_3} \|_{L^\infty(L^2)} ,$$

$$= 2^{\frac{1}{4}(j_1-k_2)} 2^{2k_2} 2^{k_3} \| \phi^{(1)}_{k_1} \|_{X^{0,\frac{1}{2}}_\infty} \| \phi^{(2)}_{k_2} \|_{S_{k_2}[L^2(L^6)]} \| \phi^{(3)}_{k_3} \|_{L^\infty(L^2)} .$$

Case 3b:($k_1 = k_3 + O(1)$ *and* $j > k_2$) Here we directly use the easy product estimate:

$$\text{LHS}(143) \lesssim 2^{-\frac{1}{2}j_1} 2^{\frac{5}{2}k_2} 2^{k_3} \| \phi^{(1)}_{k_1} \|_{X^{0,\frac{1}{2}}_\infty} \| \phi^{(2)}_{k_2} \|_{S_{k_2}} \| \phi^{(3)}_{k_3} \|_{L^\infty(L^2)} .$$

□

Proof of estimate (144)*.* There are two cases depending on the relation of the modulations of $\phi^{(2)}_{k_2}$ and $\phi^{(3)}_{k_3}$ to 2^j.

Case 1:(*One of* $\phi^{(2)}_{k_2}$ *or* $\phi^{(3)}_{k_3}$ *has modulation comparable to* 2^j) This situation is symmetric. It suffices to show the single estimate:

$$\| P_{k_1}\mathcal{N}(Q_{>j-C}\phi^{(2)}_{k_2}, \phi^{(3)}_{k_3}) \|_{L^2(L^2)} \lesssim 2^{-\frac{1}{2}j} 2^{3\min\{k_i\}} 2^{\max\{k_i\}} \| \phi^{(2)}_{k_2} \|_{X^{0,\frac{1}{2}}_\infty} \| \phi^{(3)}_{k_3} \|_{L^\infty(L^2)} ,$$

which follows immediately from putting a derivative of $\mathcal{N}$ on the lowest frequency and $L^2 \to L^\infty$ Bernstein's inequality for the lowest frequency as well.

Case 2:(*Both* $\phi^{(2)}_{k_2}$ *and* $\phi^{(3)}_{k_3}$ *have modulation* $\ll 2^j$) This can happen only when $k_1 < k_2 - C$, in which case it is a $(++)$ or $(--)$ type $High \times High \Rightarrow Low$ interaction between $\phi^{(2)}_{k_2}$ and $\phi^{(3)}_{k_3}$. Thus, $j = k_2 + O(1)$ and it suffices to prove:

$$\| P_{k_1}\mathcal{N}(Q_{<j-C}\phi^{(2)}_{k_2}, Q_{<j-C}\phi^{(3)}_{k_3}) \|_{L^2(L^2)} \lesssim 2^{2k_1} 2^{\frac{3}{2}k_2} \| \phi^{(2)}_{k_2} \|_{S_{k_2}} \| \phi^{(3)}_{k_3} \|_{L^\infty(L^2)} .$$

This follows at once by putting a derivative on the low frequency output, and breaking the product into antipodal blocks $\mathcal{C}_{k_1}$ of scale 2^{k_1} while using the special $L^2(L^\infty)$ square sum bound (9) for the first factor. Note that the multiplier $Q_{<j-C}P_{\mathcal{C}_{k_1}}$ is disposable thanks to $j > k_1 + C$. □

Proof of estimate (131). This follows from the next two bounds:

$$\| (I - \mathcal{H}_{k_1} - Q_{>k_1+C}) P_{k_1} \mathcal{N}(\phi^{(2)}_{k_2}, \phi^{(3)}_{k_3}) \|_{L^1(L^2)} \lesssim \text{LHS}(131) \ , \tag{146}$$

$$\| (Q_{>k_1+C} + \mathcal{H}_{k_1}) P_{k_1} \mathcal{N}(\phi^{(2)}_{k_2}, \phi^{(3)}_{k_3}) \|_{X_1^{0,-\frac{1}{2}}} \lesssim \text{LHS}(131) \ . \tag{147}$$

Case 1:(*Estimate* (146)) The restriction induced by $(I - \mathcal{H}_{k_1} - Q_{>k_1+C})$ means that one of the two input factors always has the leading modulation. First permute notation so the output frequency is k_3 and the two inputs are k_1, k_2. Then by duality (note that $\mathcal{N}$ is skew-adjoint as a form) and symmetry of the estimate, it suffices to show bounds of the form:

$$\begin{aligned} &\big|\big\langle Q_j \phi^{(1)}_{k_1}, \mathcal{N}(Q_{<j+O(1)}\phi^{(2)}_{k_2}, Q_{<j+O(1)}\psi_{k_3})\big\rangle\big| \\ &\qquad \lesssim 2^{-\delta|j-k_2|} 2^{k_3} 2^{\delta(k_3 - \max\{k_1,k_2\})} 2^{-\delta|k_1-k_2|} 2^{k_1} 2^{k_2} \|\phi^{(1)}_{k_1}\|_{S_{k_1}} \|\phi^{(2)}_{k_2}\|_{S_{k_2}} \| \psi_{k_3} \|_{L^\infty(L^2)} \ , \end{aligned}$$

which follows directly from (143).

Case 2a:(*Estimate* (147) *for low modulations*) Freezing the output modulation it suffices to show:

$$\begin{aligned} &\| Q_j S_{k_1} \mathcal{N}(Q_{<j-C}\phi^{(2)}_{k_2}, Q_{<j-C}\phi^{(3)}_{k_3}) \|_{X_1^{0,-\frac{1}{2}}} \\ &\qquad \lesssim 2^{-\delta|j-k_2|} 2^{k_1} 2^{\delta(k_1 - \max\{k_2,k_3\})} 2^{-\delta|k_2-k_3|} 2^{k_2} 2^{k_3} \|\phi^{(2)}_{k_2}\|_{S_{k_2}} \|\phi^{(3)}_{k_3}\|_{S_{k_3}} \ , \end{aligned}$$

which again is a direct consequence of (143).

Case 2b:(*Estimate* (147) *for high modulations*) In this case we use:

$$\| Q_j S_{k_1} \mathcal{N}(\phi^{(2)}_{k_2}, \phi^{(3)}_{k_3}) \|_{X_1^{0,-\frac{1}{2}}} \lesssim 2^{-\delta|j-k_2|} 2^{k_1} 2^{\delta(k_1 - \max\{k_2,k_3\})} 2^{-\delta|k_2-k_3|} 2^{k_2} 2^{k_3} \|\phi^{(2)}_{k_2}\|_{S_{k_2}} \|\phi^{(3)}_{k_3}\|_{S_{k_3}} \ ,$$

follows immediately from (144) in the case $j > k_1 + C$. □

Proof of estimate (132). There are two main cases:

Case 1:($\phi^{(1)}_{k_1}$ *with highest modulation*) Due to the restriction of $\mathcal{H}^*$, this can only occur when the second factor is $Q_{>k_1+C}\phi^{(1)}_{k_1}$. Then we use (144) which directly implies for this case:

$$\| \mathcal{N}(Q_j \phi^{(1)}_{k_1}, \phi^{(2)}_{k_2}) \|_N \lesssim (2^{\delta(k_1-j)} + 2^{-\delta|j-k_2|}) 2^{2k_1} 2^{k_2} \| \phi^{(1)}_{k_1} \|_{S_{k_1}} \| \phi^{(2)}_{k_2} \|_{S_{k_2}} \ .$$

Case 2:(*Output or* $\phi^{(2)}_{k_2}$ *have the leading modulation*) In this case we end up needing bounds of the form:

$$\| Q_j \mathcal{N}(Q_{<j+O(1)}\phi^{(1)}_{k_1}, Q_{<j+O(1)}\phi^{(2)}_{k_2}) \|_{X_1^{0,-\frac{1}{2}}} \lesssim 2^{-\delta|j-k_1|} 2^{2k_1} 2^{k_2} \| \phi^{(1)}_{k_1} \|_{S_{k_1}} \| \phi^{(2)}_{k_2} \|_{S_{k_2}} \ .$$

$$\| Q_{<j+O(1)} \mathcal{N}(Q_{<j+O(1)}\phi^{(1)}_{k_1}, Q_j\phi^{(2)}_{k_2}) \|_{L^1(L^2)} \lesssim 2^{-\delta|j-k_1|} 2^{2k_1} 2^{k_2} \| \phi^{(1)}_{k_1} \|_{S_{k_1}} \| \phi^{(2)}_{k_2} \|_{S_{k_2}} \ ,$$

which are both immediate from (143). □

Proof of estimate (133). This is a direct computation using angular decompositions. At a fixed modulation 2^j the angle of interaction is $\angle(\phi^1, \phi^2) \lesssim 2^{\frac{1}{2}(j-k_1)} := 2^l$, and disposing of the null form we have:

$$\| Q_{<j-C}\mathcal{N}(Q_j\phi^{(1)}_{k_1}, Q_{<j-C}\phi^{(2)}_{k_2}) \|_{L^1(L^2)} \lesssim 2^l 2^{k_1}2^{k_2}\sum_\omega \| P^\omega_l Q_{k_1+2l}\phi^{(1)}_{k_1} \|_{L^1(L^\infty)} \cdot \| P^\omega_l Q_{<j-C}\phi^{(2)}_{k_2} \|_{L^\infty(L^2)} \ .$$

Using Cauchy-Schwarz and:

$$\| P^\omega_l Q_{<j-C}\phi^{(2)}_{k_2} \|^2_{L^\infty(L^2)} \lesssim \sum_{\omega'\subseteq\omega} \| P^{\omega'}_{2^{\frac{1}{2}(j-k_2)}} Q_{<j-C}\phi^{(2)}_{k_2} \|^2_{L^\infty(L^2)} \ ,$$

we have:

$$\| Q_{<j-C}\mathcal{N}(Q_j\phi^{(1)}_{k_1}, Q_{<j-C}\phi^{(2)}_{k_2}) \|_{L^1(L^2)} \lesssim 2^{\frac{1}{4}(j-k_1)}2^{k_1}\| \phi^{(1)}_{k_1} \|_{Z^{hyp}_{k_1}}\| \phi^{(2)}_{k_2} \|_{S^1_{k_2}} \ ,$$

which suffices. □

Proof of estimate (134). This follows immediately from (141) and (10). Notice that the lack of an $L^1(L^2)$ estimate for $Q_{>k_1+C}P_{k_1}\mathcal{N}(\phi^{(2)}_{k_2}, \phi^{(3)}_{k_3})$ is irrelevant because definition of Z^{hyp} limits modulations to $Q_{<k_1+C}P_{k_1}$. □

Proof of estimate (135). By symmetry we may assume $k_2 < k_3 + O(1)$, in which case $k_1 = k_3 + O(1)$. There are two subcases depending on the size of the output modulation 2^j:
Case 1:($k_2 > j-C$) In this case the angle of interaction between the two inputs is $\angle(\phi^2, \phi^3) \lesssim 2^{\frac{1}{2}(j-k_2)} := 2^{l'}$. On the other hand, the output sector localization of $2^l := 2^{\frac{1}{2}(j-k_1)} \approx 2^{\frac{1}{2}(j-k_3)}$ is passed to the high frequency factor, so using Lemma 12.3 we have:

$$\begin{aligned}
&\sum_\omega 2^l\| P^\omega_l Q_j S_{k_1}\mathcal{N}(Q_{<j-C}\phi^{(2)}_{k_2}, Q_{<j-C}\phi^{(3)}_{k_3}) \|^2_{L^1(L^\infty)}\\
&\lesssim 2^l 2^{2l'}2^{2k_2}2^{2k_3}\sum_{\omega'}\sum_{\substack{\omega:\\ \omega\subseteq\omega'}} \| P^{\omega'}_{l'} Q_{<j-C}\phi^{(2)}_{k_2} \|^2_{L^2(L^\infty)}\| P^\omega_l Q_{<j-C}\phi^{(3)}_{k_3} \|^2_{L^2(L^\infty)} \ ,\\
&\lesssim 2^{2l}2^{3l'}2^{5k_2}2^{5k_3}\| \phi^{(2)}_{k_2} \|^2_{S_{k_2}[L^2(L^6)]}\| \phi^{(3)}_{k_3} \|^2_{S_{k_3}[L^2(L^6)]} \ ,\\
&\lesssim (2^{k_1+j})^2 \cdot 2^{2k_1}2^{2(k_2-k_3)}\| \phi^{(2)}_{k_2} \|^2_{S^1_{k_2}}\| \phi^{(3)}_{k_3} \|^2_{S^1_{k_3}} \ .
\end{aligned}$$

Case 2:($k_2 < j-C$) Here the calculation is essentially the same as above, except that $l' = 0$. This again gives:

$$\| Q_j S_{k_1}\mathcal{N}(Q_{<j-C}\phi^{(2)}_{k_2}, Q_{<j-C}\phi^{(3)}_{k_3}) \|_{Z^{hyp}} \lesssim (2^{j+k_1}) \cdot 2^{k_1}2^{(k_2-k_3)}\| \phi^{(2)}_{k_2} \|_{S^1_{k_2}}\| \phi^{(3)}_{k_3} \|_{S^1_{k_3}} \ .$$

□

Proof of estimate (136). Freezing the output modulation, we will show:

$$(148)\quad \big|\big\langle \Box^{-1}Q_jP_k(Q_{<j-C}\phi^{(1)}_{k_1}\cdot\partial_\alpha Q_{<j-C}\phi^{(2)}_{k_2}), Q_jP_k(\partial^\alpha Q_{<j-C}\phi^{(3)}_{k_3}\cdot Q_{<j-C}\psi_{k_4})\big\rangle\big| \lesssim 2^{\frac{1}{4}(j-k)}2^{\frac{1}{2}(k-\min\{k_i\})}\| \phi^{(1)}_{k_1} \|_{S^1}\| \phi^{(2)}_{k_2} \|_{S^1}\| \phi^{(3)}_{k_3} \|_{S^1}\| \psi_{k_4} \|_{N^*} \ .$$

Here we are in the configuration $k_1 = k_2 + O(1)$, $k_3 = k_4 + O(1)$, and $k < k_i - C$. Thus, the left and right products are summed over diametrically opposite angular sectors of size

$2^k \times (2^{k+l})^3$, where $2^l := 2^{\frac{1}{2}(j-k)}$. On the other hand, the null form between the second and third terms gains us $\angle(\phi^2, \phi^3)^2$, where this angle mod π cannot exceed 2^{l+C}. Therefore we group the product of the two diagonal sums into dyadic values of $\angle(\phi^2, \phi^3)$ mod π and break into three cases:
Case 1:($\angle(\phi^2, \phi^3)$ mod $\pi \lesssim 2^l 2^{k-k_2}$) A little care is needed to use orthogonality in space. To gain this, at first keep the second diagonal sum under the time integral as follows:

$$
\begin{aligned}
&\text{LHS(148)}\big|_{\angle(\phi^2,\phi^3)\text{mod}\,\pi\lesssim 2^l 2^{k-k_2}} \\
&\lesssim\ 2^{-k_2}2^{k_3} \sum_{\mathcal{C}_k(l)} \| P_{\mathcal{C}_k(l)} Q_{<j-C}\phi^{(1)}_{k_1} \|_{L^2(L^\infty)} \| P_{-\mathcal{C}_k(l)} Q_{<j-C}\phi^{(2)}_{k_2} \|_{L^2(L^\infty)} \\
&\qquad \times \sup_t \sum_{\mathcal{C}'_k(l)} \| P_{\mathcal{C}'_k(l)} Q_{<j-C}\phi^{(3)}_{k_3}(t) \|_{L^2_x} \| P_{-\mathcal{C}'_k(l)} Q_{<j-C}\psi_{k_4}(t) \|_{L^2_x} \ .
\end{aligned}
$$

The inner sum of the second factor on the RHS can easily be reconstructed after Cauchy-Schwarz by spatial orthogonality. On the other hand, for the two $L^2(L^\infty)$ norms we use Lemma 12.3. This gives us:

$$
\begin{aligned}
&\text{LHS(148)}\big|_{\angle(\phi^2,\phi^3)\text{mod}\,\pi\lesssim 2^l 2^{k-k_2}} \\
&\lesssim 2^{\frac{4}{3}k}2^l 2^{\frac{5}{6}k_1}2^{-\frac{1}{6}k_2}2^{k_3} \| \phi^{(1)}_{k_1} \|_{S_{k_1}[L^2(L^6)]} \| \phi^{(2)}_{k_2} \|_{S_{k_2}[L^2(L^6)]} \| \phi^{(3)}_{k_3} \|_{S_{k_3}} \| \psi_{k_4} \|_{N^*} \ , \\
&\lesssim 2^{\frac{1}{2}(j-k)}2^{\frac{4}{3}(k-\min\{k_i\})} \| \phi^{(1)}_{k_1} \|_{S^1} \| \phi^{(2)}_{k_2} \|_{S^1} \| \phi^{(3)}_{k_3} \|_{S^1} \| \psi_{k_4} \|_{N^*} \ ,
\end{aligned}
$$

which is even better than (148).
Case 2:($\angle(\phi^2, \phi^3)$ mod $\pi \lesssim 2^l 2^{k-k_3}$) This is essentially the same as **Case 1** above, but since the angular gain is in frequency 2^{k_3} we put $\phi^{(3)}_{k_3}$ in a square summed $L^2(L^\infty)$ via Lemma 12.3 and $\phi^{(1)}_{k_1}$ in $L^\infty(L^2)$ instead. This gives:

$$
\begin{aligned}
&\text{LHS(148)}\big|_{\angle(\phi^2,\phi^3)\text{mod}\,\pi\lesssim 2^l 2^{k-k_3}} \\
&\lesssim 2^{\frac{4}{3}k}2^l 2^{k_1}2^{\frac{5}{6}k_2}2^{-\frac{1}{6}k_3} \| \phi^{(1)}_{k_1} \|_{L^\infty(L^2)} \| \phi^{(2)}_{k_2} \|_{S_{k_2}[L^2(L^6)]} \| \phi^{(3)}_{k_3} \|_{S_{k_3}[L^2(L^6)]} \| \psi_{k_4} \|_{N^*} \ , \\
&\lesssim 2^{\frac{1}{2}(j-k)}2^{\frac{4}{3}(k-\min\{k_i\})} \| \phi^{(1)}_{k_1} \|_{S^1} \| \phi^{(2)}_{k_2} \|_{S^1} \| \phi^{(3)}_{k_3} \|_{S^1} \| \psi_{k_4} \|_{N^*} \ ,
\end{aligned}
$$

which suffices. Note that a similar square summation for fixed time procedure as in **Case 1** was used here.
Case 3:($2^l 2^{k-\min\{k_2,k_3\}} \ll \angle(\phi^2, \phi^3)$ mod $\pi \lesssim 2^l$) In this case there is a definite angle between spatial frequencies in the blocks $\mathcal{C}_k(l), \mathcal{C}'_k(l)$ for fixed dyadic $2^{l'} = \angle(\phi^2, \phi^3)$. Thus, we can use one power of $\angle(\phi^2, \phi^3)$ to put the product of $\phi^{(2)}_{k_2}$ and $\phi^{(3)}_{k_3}$ in $L^2(L^2)$ using null frames. A little more care is needed here to gain spatial orthogonality for ψ_{k_4}. Thus, we first fix time

and compute:

$$\begin{aligned}
&\mathrm{LHS}(148)\big|_{\angle(\phi^2,\phi^3)\,\mathrm{mod}\,\pi\sim 2^{l'},t=const}\\
&\lesssim 2^{-2(k+l)}2^{k_2}2^{k_3}2^{2l'}\sum_{\substack{\mathcal{C}_k(l),\mathcal{C}_k'(l):\\ \angle(\mathcal{C}_k(l),\pm\mathcal{C}_k(l)')\sim 2^{l'}}} \| P_{\mathcal{C}_k(l)}Q_{<j-C}\phi^{(2)}_{k_2}(t)\cdot P_{\pm\mathcal{C}_k'(l)}Q_{<j-C}\phi^{(3)}_{k_3}(t)\|_{L^2_x}\\
&\qquad \times \| P_{\mathcal{C}_k(l)}Q_{<j-C}\phi^{(1)}_{k_1}(t)\|_{L^\infty}\| P_{\mp\mathcal{C}_k'(l)}Q_{<j-C}\psi_{k_4}(t)\|_{L^2_x} \ ,\\
&\lesssim 2^{-2(k+l)}2^{k_2}2^{k_3}2^{2l'}\| Q_{<j-C}\psi_{k_4}(t)\|_{L^2_x}\cdot \big(\sum_{\mathcal{C}_k(l)}\| P_{\mathcal{C}_k(l)}Q_{<j-C}\phi^{(1)}_{k_1}(t)\|^2_{L^\infty}\big)^{\frac{1}{2}}\\
&\qquad \times \big(\sum_{\substack{\mathcal{C}_k'(l):\\ \angle(\mathcal{C}_k(l),\pm\mathcal{C}_k(l)')\sim 2^{l'}}} \| P_{\mathcal{C}_k(l)}Q_{<j-C}\phi^{(2)}_{k_2}(t)\cdot P_{\pm\mathcal{C}_k'(l)}Q_{<j-C}\phi^{(3)}_{k_3}(t)\|^2_{L^2_x}\big)^{\frac{1}{2}} \ .
\end{aligned}$$

Integrating and using Cauchy-Schwarz in time, and then using the special microlocalized $L^2(L^\infty)$ block norm from line (8) for $\phi^{(1)}_{k_1}$, we get:

$$\mathrm{LHS}(148)\big|_{\angle(\phi^2,\phi^3)\,\mathrm{mod}\,\pi\sim 2^{l'}} \ \lesssim \ 2^{-k}2^{-2l}2^{l'}2^{\frac{1}{2}k_1}2^{k_2}2^{k_3}I_{23}(l')\| \phi^{(1)}_{k_1}\|_{S_{k_1}}\| \psi_{k_4}\|_{N^*} \ .$$

where:

$$I_{23}(l')^2 \ = \sum_{\substack{\mathcal{C}_k(l),\mathcal{C}_k'(l):\\ \angle(\mathcal{C}_k(l),\pm\mathcal{C}_k(l)')\sim 2^{l'}}} 2^{2l'}\| P_{\mathcal{C}_k(l)}Q_{<j-C}\phi^{(2)}_{k_2}\cdot P_{\pm\mathcal{C}_k'(l)}Q_{<j-C}\phi^{(3)}_{k_3}\|^2_{L^2(L^2)} \ .$$

Since we already spent 2^{k_3} we put the second factor in $L^\infty_\omega(L^2_{\omega^\perp})$ and use $L^2_\omega(L^\infty_{\omega^\perp})$ for the first. This gives us:

$$I_{23}(l') \ \lesssim \ 2^{\frac{3}{2}(k+l)}\| \phi^{(2)}_{k_2}\|_{S_{k_2}}\| \phi^{(3)}_{k_3}\|_{S_{k_3}} \ ,$$

and so:

$$\mathrm{LHS}(148)\big|_{\angle(\phi^2,\phi^3)\,\mathrm{mod}\,\pi\sim 2^{l'}} \ \lesssim \ 2^{\frac{1}{2}l'}2^{\frac{1}{2}(k-\min\{k_i\})}\| \phi^{(1)}_{k_1}\|_{S^1}\| \phi^{(2)}_{k_2}\|_{S^1}\| \phi^{(3)}_{k_3}\|_{S^1}\| \psi_{k_4}\|_{N^*} \ .$$

Summing this over all $l' < l + C$ gives (148) for this case. □

Proof of estimate (137). Freezing the output modulation, we'll show:

$$\begin{aligned}
&\big|\big\langle (\Box\Delta)^{-1}Q_jP_k\partial_t\partial_\alpha(Q_{<j-C}\phi^{(1)}_{k_1}\cdot\partial^\alpha Q_{<j-C}\phi^{(2)}_{k_2}), (\partial_tQ_{<j-C}\phi^{(3)}_{k_3}\cdot Q_{<j-C}\psi_{k_4})\big\rangle\big|\\
&\qquad\qquad \lesssim \ 2^{\frac{1}{2}(j-k)}2^{\frac{1}{6}(k-k_1)}\| \phi^{(1)}_{k_1}\|_{S^1}\| \phi^{(2)}_{k_2}\|_{S^1}\| \phi^{(3)}_{k_3}\|_{S^1}\| \psi_{k_4}\|_{N^*} \ .
\end{aligned}$$

We will use an $L^\infty(L^2)$ estimate for both the third and fourth factors. Thus, via Hölder's inequality, a simple weight calculation, and the Leibniz rule we have reduced matters to:

(149) $\| Q_jP_k(\partial_\alpha Q_{<j-C}\phi^{(1)}_{k_1}\cdot\partial^\alpha Q_{<j-C}\phi^{(2)}_{k_2})\|_{L^1(L^\infty)} \lesssim 2^{j+2k}2^{\frac{1}{2}(j-k)}2^{\frac{1}{3}(k-k_1)}\| \phi^{(1)}_{k_1}\|_{S^1}\| \phi^{(2)}_{k_2}\|_{S^1}$,

(150) $\| Q_jP_k(Q_{<j-C}\phi^{(1)}_{k_1}\cdot\Box Q_{<j-C}\phi^{(2)}_{k_2})\|_{L^1(L^\infty)} \lesssim 2^{j+2k}2^{\frac{3}{2}(j-k)}2^{\frac{1}{6}(k-k_1)}\| \phi^{(1)}_{k_1}\|_{S^1}\| \phi^{(2)}_{k_2}\|_{S^1}$.

To prove (149), note that the null form gains two powers of the angle $\angle(\phi^1,\phi^2) \lesssim 2^{\frac{1}{2}(k+j)}2^{-k_1} := 2^l$. Therefore, breaking the product into a sum over antipodal radially directed blocks of dimension $2^k\times(2^{k+l'})^3$, where $l' = \frac{1}{2}(j-k)$, and using (142) for both factors we have:

$$\mathrm{LHS}(149) \lesssim 2^{2l}2^{l'}2^{\frac{4}{3}k}2^{\frac{11}{6}k_1}2^{\frac{11}{6}k_2}\| \phi^{(1)}_{k_1} \|_{S_{k_1}[L^2(L^6)]}\| \phi^{(2)}_{k_2} \|_{S_{k_2}[L^2(L^6)]} ,$$
$$\lesssim 2^{j+2k}2^{\frac{1}{2}(j-k)}2^{\frac{1}{3}(k-k_1)}2^{k_1}2^{k_2}\| \phi^{(1)}_{k_1} \|_{S_{k_1}}\| \phi^{(2)}_{k_2} \|_{S_{k_2}} .$$

To prove (150) we use the same calculations as above, except for the second factor we trade (142) for:

$$(151) \qquad \Big(\sum_{\mathcal{C}_k(l')}\| P_{\mathcal{C}_k(l')}\Box Q_{<j-C}\phi^{(2)}_{k_2} \|^2_{L^2(L^\infty)}\Big)^{\frac{1}{2}} \lesssim 2^{2k}2^{\frac{3}{2}l'}2^{\frac{1}{2}j}2^{k_2}\| \phi^{(2)}_{k_2} \|_{X^{0,\frac{1}{2}}_\infty} ,$$

which is an immediate consequence or Bernstein's inequality and $\Box Q_{<j-C}P_{k_2}X^{0,\frac{1}{2}}_\infty \subseteq 2^{\frac{1}{2}j}2^{k_2}L^2(L^2)$. This gives:

$$\mathrm{LHS}(150) \lesssim 2^{j+2k}2^{\frac{3}{2}(j-k)}2^{\frac{1}{6}(k-k_1)}2^{k_1}2^{k_2}\| \phi^{(1)}_{k_1} \|_{S_{k_1}}\| \phi^{(2)}_{k_2} \|_{S_{k_2}} .$$

□

Proof of estimate (138). Freezing the output modulation, our goal here is to show:

$$\big|\big\langle(\Box\Delta)^{-1}\nabla_x Q_j P_k(Q_{<j-C}\phi^{(1)}_{k_1}\cdot\nabla_x Q_{<j-C}\phi^{(2)}_{k_2}), Q_jP_k\partial_\alpha(\partial^\alpha Q_{<j-C}\phi^{(3)}_{k_3}\cdot Q_{<j-C}\psi_{k_4})\big\rangle\big|$$
$$\lesssim 2^{\frac{1}{2}(j-k)}2^{\frac{1}{6}(k-\min\{k_i\})}\| \phi^{(1)}_{k_1} \|_{S^1}\| \phi^{(2)}_{k_2} \|_{S^1}\| \phi^{(3)}_{k_3} \|_{S^1}\| \psi_{k_4} \|_{N^*} .$$

Expanding the null form and computing the weights, it suffices to have:

$$(152) \quad \| Q_jP_k(Q_{<j-C}\phi^{(1)}_{k_1}\cdot\nabla_x Q_{<j-C}\phi^{(2)}_{k_2}) \|_{L^2(L^2)} \lesssim 2^{\frac{1}{2}k}2^{\frac{1}{4}(j-k)}2^{\frac{1}{6}(k-k_1)}\| \phi^{(1)}_{k_1} \|_{S^1}\| \phi^{(2)}_{k_2} \|_{S^1} ,$$

(153)

$$\| Q_jP_k(\partial_\alpha Q_{<j-C}\phi^{(3)}_{k_3}\cdot\partial^\alpha Q_{<j-C}\psi_{k_4}) \|_{L^2(L^2)} \lesssim 2^j2^{\frac{3}{2}k}2^{\frac{1}{4}(j-k)}2^{\frac{1}{6}(k-k_3)}\| \phi^{(3)}_{k_3} \|_{S^1}\| \psi_{k_4} \|_{N^*} ,$$

$$(154) \quad \| Q_jP_k(\Box Q_{<j-C}\phi^{(3)}_{k_3}\cdot Q_{<j-C}\psi_{k_4}) \|_{L^2(L^2)} \lesssim 2^j2^{\frac{3}{2}k}2^{\frac{1}{4}(j-k)}\| \phi^{(3)}_{k_3} \|_{S^1}\| \psi_{k_4} \|_{N^*} .$$

The proof of these three estimates follows from the same angular decompositions and antipodal block sums as in the last proof. Estimate (152) follows by using (142) for the first factor and $L^\infty(L^2)$ for the second. Estimate (153) follows similarly once once the null form is taken into account. Note that here the antipodal block sum for ψ_{k_4} needs to be reconstructed for fixed time using orthogonality. Finally, estimate (154) is the same as (153) but uses (151) instead for $\phi^{(3)}_{k_3}$. Further details are left to the reader. □

12.2. **Proof of the additional product estimates.** These estimates use the same kind of modulation/angular-sum decompositions as in previous proofs, so we leave more of the details to the reader.

Proof of estimate (139). There are three main cases:
Case 1:(A_{k_1} *with highest modulation*) Due to the restriction of $\mathcal{H}^*$, this can only occur when the second factor is $Q_{k_1+C<\cdot<k_2-C}A_{k_1}$. Then there are two subcases:
Case 1a:(*Contribution of* $\partial_t Q_{>k_1-C}\phi_{k_2}$) Here we use a product of the two bounds:

$$\| Q_{k_1+C<\cdot<k_2-C}A_{k_1} \|_{L^2(L^\infty)} \lesssim 2^{2k_1}\| A_{k_1} \|_{L^2(L^2)} , \quad \| \partial_t Q_{>k_1-C}\phi_{k_2} \|_{L^2(L^2)} \lesssim 2^{-\frac{1}{2}k_1}\| \partial_t\phi_{k_2} \|_{X^{0,\frac{1}{2}}_\infty} .$$

Case 1b:(*Contribution of* $\partial_t Q_{<k_1-C}\phi_{k_2}$, *and output modulation* $\gtrsim 2^{k_1}$) Note that this is the remaining case because an output modulation of $\ll 2^{k_1}$ is impossible due to the restrictions on

$Q_{k_1+C<\cdot<k_2-C}A_{k_1}$ and $\partial_t Q_{<k_1-C}\phi_{k_2}$. Here we use a product of the previous $L^2(L^\infty)$ estimate for the first and $L^\infty(L^2)$ for the latter.
Case 2:(*Output or ϕ_{k_2} have the leading modulation and $j<k_1+C$*) In this case we can reduce things to the bounds:

$$\| Q_j(Q_{<j+O(1)}A_{k_1}\partial_t Q_{<j+O(1)}\phi_{k_2})\|_{X_1^{0,-\frac{1}{2}}} \lesssim 2^{\frac{1}{4}(j-k_1)}2^{\frac{3}{2}k_1}2^{k_2}\| A_{k_1}\|_{L^2(L^2)}\|\phi_{k_2}\|_{S_{k_2}} .$$

$$\| Q_{<j+O(1)}(Q_{<j+O(1)}\phi^{(1)}_{k_1}\partial_t Q_j\phi^{(2)}_{k_2})\|_{L^1(L^2)} \lesssim 2^{\frac{1}{4}(j-k_1)}2^{\frac{3}{2}k_1}2^{k_2}\| A_{k_1}\|_{L^2(L^2)}\|\phi_{k_2}\|_{S_{k_2}} .$$

Both of these bounds follow from the usual angular sum decompositions and the Bernstein type estimate:

$$\Big(\sum_{\mathcal{C}_{k_1}(\frac{1}{2}(j-k_1))}\| P_{\mathcal{C}_{k_1}(\frac{1}{2}(j-k_1))}Q_{<j-C}A_{k_1}\|^2_{L^2(L^\infty)}\Big)^{\frac{1}{2}} \lesssim 2^{\frac{3}{4}j}2^{\frac{5}{4}k_1}\| A_{k_1}\|_{L^2(L^2)} .$$

Case 3:(*Output or ϕ_{k_2} have the leading modulation and $j\geqslant k_1+C$*) This is analogous to the last case except there are no angular decompositions and instead we simply use:

$$\| Q_{<j-C}A_{k_1}\|_{L^2(L^\infty)} \lesssim 2^{2k_1}\| A_{k_1}\|_{L^2(L^2)} ,$$

instead of the previous square sum bound. □

Proof of estimate (140). This follows at one from the definition of Z^{ell}, which allows one to sum over the modulation localizations enforced by $\mathcal{H}^*_{k_k}$. □

Proof of estimate (141). Freezing the output modulation of the product, and getting rid of the contribution of $\mathcal{H}_{k_1}$, it suffices to show:

$$\| Q_jP_{k_1}T_a\|_{L^1(L^\infty)} \lesssim 2^{2k_1}2^{\frac{1}{2}(j-k_1)}2^{\frac{1}{6}(k_1-k_2)}\|\phi^{(2)}_{k_2}\|_{S^1}\|\phi^{(3)}_{k_3}\|_{S^1} ,$$

for either one of:

$$T_1 = Q_{\geqslant j-C}\phi^{(2)}_{k_2}\partial_t\phi^{(3)}_{k_3} , \qquad T_2 = Q_{<j-C}\phi^{(2)}_{k_2}Q_{\geqslant j-C}\partial_t\phi^{(3)}_{k_3} .$$

Both cases are essentially the same due to the matching $k_2=k_3+O(1)$. After breaking the sum into angular sectors one uses:

$$\Big(\sum_{\mathcal{C}_{k_1}(l')}\| P_{\mathcal{C}_{k_1}(l')}Q_{\geqslant j-C}\phi^{(2)}_{k_2}\|^2_{L^2(L^\infty)}\Big)^{\frac{1}{2}} \lesssim 2^{2k_1}2^{\frac{3}{2}l'}2^{-\frac{1}{2}j}\|\phi^{(2)}_{k_2}\|_{X_\infty^{0,\frac{1}{2}}} ,$$

for the high modulation factor and (142) for the low modulation factor, both with $l'=\frac{1}{2}(j-k_1)$. Note that $\partial_t\phi^{(3)}_{k_3}$ can be further broken into high and low modulation pieces, and the product of two high modulations is even more favorable. □

Appendix: A multilinear null form for MKG-CG

Schematically we have:

$$\Box A_i = -\mathcal{P}_i\Im(\phi\overline{\nabla\phi}) , \qquad \Delta A_0 = -\Im(\phi\overline{\partial_t\phi}) ,$$

where $\mathcal{P}$ is the Hodge projection. In 3D one can write this operator conveniently in terms of the vector product $\nabla\times$. In higher dimensions it is easier to use the simple formula:

$$\mathcal{P} = I-\nabla\Delta^{-1}\nabla ,$$

where the first ∇ is grad and the second is div.

Now investigate the expression:

$$\mathcal{N}(A,\phi) \ = \ A^\alpha\partial_\alpha\phi \ .$$

Directly plugging things in we have:

$$\mathcal{N}(A,\phi) \ = \ \Delta^{-1}\Im(\phi\overline{\partial_t\phi})\cdot\partial_t\phi - \Box^{-1}\Im(\phi\overline{\partial_i\phi})\cdot\partial^i\phi + \frac{\partial^i\partial^j}{\Delta\Box}\Im(\phi\overline{\partial_i\phi})\cdot\partial_j\phi \ .$$

To uncover the null structure, we add and subtract the expression $\Box^{-1}\Im(\phi\overline{\partial_t\phi})\partial_t\phi$. This gives us:

$$\mathcal{N}(A,\phi) \ = \ -\mathcal{Q}_1(A,\phi) + \mathcal{N}_2(A,\phi) \ ,$$

where:

$$\mathcal{Q}_1(A,\phi) \ = \ \Box^{-1}\Im(\phi\overline{\partial_\alpha\phi})\cdot\partial^\alpha\phi \ ,$$

$$\mathcal{N}_2(A,\phi) \ = \ \Delta^{-1}\Im(\phi\overline{\partial_t\phi})\cdot\partial_t\phi - \Box^{-1}\Im(\phi\overline{\partial_t\phi})\cdot\partial_t\phi + \frac{\partial^i\partial^j}{\Delta\Box}\Im(\phi\overline{\partial_i\phi})\cdot\partial_j\phi \ .$$

To continue the computation we use $\Delta^{-1} - \Box^{-1} = -\partial_t^2\Delta^{-1}\Box^{-1}$ so the second term simplifies to:

$$\mathcal{N}_2(A,\phi) \ = \ -\Delta^{-1}\Box^{-1}\partial_t^2\Im(\phi\overline{\partial_t\phi})\cdot\partial_t\phi + \Delta^{-1}\Box^{-1}\partial^i\partial^j\Im(\phi\overline{\partial_i\phi})\cdot\partial_j\phi \ .$$

This can be further reduced to the sum of Q_0 null structures by adding and subtracting (for instance) $\partial^i\partial_t\Im(\phi\overline{\partial_i\phi})\partial_t\phi$, which gives:

$$\mathcal{N}_1(A,\phi) \ = \ \mathcal{Q}_2(A,\phi) + \mathcal{Q}_3(A,\phi) \ ,$$

where:

$$\mathcal{Q}_2(A,\phi) \ = \ \Delta^{-1}\Box^{-1}\partial_t\partial_\alpha\Im(\phi\overline{\partial^\alpha\phi})\cdot\partial_t\phi \ ,$$

$$\mathcal{Q}_3(A,\phi) \ = \ \Delta^{-1}\Box^{-1}\partial_\alpha\partial^i\Im(\phi\overline{\partial_i\phi})\cdot\partial^\alpha\phi \ .$$

There is a conceptual way to visualize this by duality. The expression $A^\alpha\partial_\alpha\phi$ is a sum of three Q_0 null structures: The interaction is between the first two factors, the second two factors, or mixed between ϕ^2 and ϕ^3. The latter is the main one which involves null frames.

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