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GLOBAL WELL-POSEDNESS OF THE 3D PRIMITIVE EQUATIONS WITH HORIZONTAL VISCOSITY AND VERTICAL DIFFUSIVITY

CHONGSHENG CAO, JINKAI LI, AND EDRISS S. TITI

ABSTRACT. In this paper, we consider the 3D primitive equations of oceanic and atmospheric dynamics with only horizontal eddy viscosities in the horizontal momentum equations and only vertical diffusivity in the temperature equation. Global well-posedness of strong solutions is established for any initial data such that the initial horizontal velocity $v_0 \in H^2(\Omega)$ and the initial temperature $T_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ with $\nabla_H T_0 \in L^q(\Omega)$, for some $q \in (2, \infty)$. Moreover, the strong solutions enjoy correspondingly more regularities if the initial temperature belongs to $H^2(\Omega)$. The main difficulties are the absence of the vertical viscosity and the lack of the horizontal diffusivity, which, interact with each other, thus causing the “mismatching” of regularities between the horizontal momentum and temperature equations. To handle this “mismatching” of regularities, we introduce several auxiliary functions, i.e., η, θ, φ , and ψ in the paper, which are the horizontal curls or some appropriate combinations of the temperature with the horizontal divergences of the horizontal velocity v or its vertical derivative $\partial_z v$. To overcome the difficulties caused by the absence of the horizontal diffusivity, which leads to the requirement of some $L_t^1(W_x^{1,\infty})$ -type a priori estimates on v , we decompose the velocity into the “temperature-independent” and temperature-dependent parts and deal with them in different ways, by using the logarithmic Sobolev inequalities of the Brézis-Gallouet-Wainger and Beale-Kato-Majda types, respectively. Specifically, a logarithmic Sobolev inequality of the limiting type, introduced in our previous work [12], is used, and a new logarithmic type Gronwall inequality is exploited.

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1. INTRODUCTION

The incompressible primitive equations form a fundamental block in models of oceanic and atmospheric dynamics, see, e.g., the books Lewandowski [27], Majda [35], Pedlosky [36], Vallis [42], Washington–Parkinson [43], and Zeng [45]. The primitive equations are derived from the Navier–Stokes equations by applying the Boussinesq and hydrostatic approximations. The hydrostatic approximation is based on the fact that the vertical scale of the the ocean and atmosphere is much smaller than the horizontal ones, and its mathematical justification, by taking small aspect ratio limit, was carried out by Azérad–Guillén [1] in the framework of weak solutions and recently by Li–Titi [31] in the framework of strong solutions; moreover, the strong convergence rates were also obtained in [31]. In the oceanic and atmospheric dynamics, due to the strong horizontal turbulent mixing, the horizontal viscosity is much stronger than the vertical viscosity and the vertical viscosity is very weak and often neglected.

In this paper, we consider the following incompressible primitive equations, which have only horizontal viscosities and vertical diffusivity

$$\partial_t v + (v \cdot \nabla_H) v + w \partial_z v + \nabla_H p - \Delta_H v + f_0 \vec{k} \times v = 0, \quad (1.1)$$

$$\partial_z p + T = 0, \quad (1.2)$$

$$\nabla_H \cdot v + \partial_z w = 0, \quad (1.3)$$

$$\partial_t T + v \cdot \nabla_H T + w \partial_z T - \partial_z^2 T = 0, \quad (1.4)$$

where the horizontal velocity $v = (v^1, v^2)$, the vertical velocity w , the temperature T and the pressure p are the unknowns, and f_0 is the Coriolis parameter. In this paper, we use the notations $\nabla_H = (\partial_x, \partial_y)$ and $\Delta_H = \partial_x^2 + \partial_y^2$ to denote the horizontal gradient and the horizontal Laplacian, respectively. Here, the term $\vec{k} \times v$ is understood as the first two components of the vector product of $\vec{k} = (0, 0, 1)$ with $(v^1, v^2, 0)$, i.e., $\vec{k} \times v = (-v^2, v^1)$.

The first systematically mathematical studies of the primitive equations were carried out in 1990s by Lions–Temam–Wang [32–34], where they considered the systems with both full viscosities and full diffusivity, and established the global existence

of weak solutions; however, the uniqueness of weak solutions is still an open question, even for the two-dimensional case. Note that this is different from the incompressible Navier-Stokes equations, as it is well-known that the weak solutions to the two-dimensional incompressible Navier-Stokes equations are unique (see, e.g., Constantin–Foias [18], Ladyzhenskaya [26] and Temam [40], and even in the framework of the three-dimensional Navier-Stokes equations, see Bardos et al. [2]). However, we would like to point out that, though the general uniqueness of weak solutions to the primitive equations is still unknown, some particular cases have been solved, see [4, 24, 30, 37, 39], and in particular, it is proved in [30] that weak solutions, with bounded initial data, to the primitive equations are unique, as long as the discontinuity of the initial data is sufficiently small. Remarkably, different from the three-dimensional Navier-Stokes equations, global existence and uniqueness of strong solutions to the three-dimensional primitive equations has already been known since the breakthrough work by Cao–Titi [16]. This global existence of strong solutions to the primitive equations were also proved later by Kobelkov [23] and Kukavica–Ziane [25], by using some different approaches, see also Hieber–Kashiwabara [22] and Hieber–Hussien–Kashiwabara [21] for some generalizations in the L^p settings.

Note that in all the papers mentioned in the previous paragraph, the systems in question are assumed to have full viscosities in the horizontal momentum equations and full diffusivity in the temperature equation. As stated in the previous paragraph, the primitive equations with both full viscosities and full diffusivity have a unique global strong solution, which is smooth away from the initial time. However, on the other hand, it has already been proven that smooth solutions to the inviscid primitive equations, with or without coupling to the temperature equation, can develop singularities in finite time, see Cao et al. [8] and Wong [44]. Comparing these two kind results of the two extreme cases, i.e., global existence for the primitive equations with both full viscosities and full diffusivity and blowup in finite time for the inviscid primitive equations, it is natural for us to consider the intermediate cases, i.e., the primitive equations with partial viscosities or partial diffusivity, and to ask of whether the solutions exist globally in time or blow up in finite time for these intermediate cases.

There has been several works concerning the mathematical studies on the primitive equations with partial viscosities or partial diffusivity. It has been proved by Cao–Titi [17] and Cao–Li–Titi [10, 11] that the primitive equations with full viscosities and with either horizontal or vertical diffusivity have a unique global strong solution. It turns out that the vertical viscosity is even not necessary for the global well-posedness of the primitive equations. In fact, it was proved by Cao–Li–Titi [12] that strong solutions are unique and exist globally in time for the primitive equations with only horizontal viscosity and only horizontal diffusivity for any initial data in H^2 (see Cao–Li–Titi [13] for some generalization of the result in [12]). We would like to point out that there is a notable difference between the arguments for the primitive equations with full viscosities and those for the case of only horizontal viscosity: for

the primitive equations with full viscosities, the a priori $L^\infty(L^q)$ estimate on v for some $q \in (3, \infty)$ is sufficient for establishing higher order estimates, but it is not the case for the primitive equations with only horizontal viscosity. In fact, as pointed out in [12], due to the absence of the vertical viscosity, in order to obtain higher order energy estimates, one has in some sense to get the a priori $L^2(L^\infty)$ estimate on v . The idea used in [12] to overcome this difficulty is to carry out the precise growth with respect to q of the L^q norms of v for $q \in [4, \infty)$, and connect the L^∞ norm of v with such precise growth, by an N -dimensional logarithmic Sobolev embedding inequality, which states that the L^∞ norm can be dominated by some appropriate growth in q of estimates for the L^q norms, up to some logarithmic of the higher order norms.

In this paper, we continue to study the primitive equations with partial viscosities or partial diffusivity. Recall that the case with horizontal viscosity and horizontal diffusivity has been investigated in [12, 13], as a counterpart, we consider in the current paper the case with only horizontal viscosity, but with vertical diffusivity, i.e., system (1.1)–(1.4). The aim of this paper is to show that system (1.1)–(1.4), subject to appropriate boundary and initial conditions, is global well-posed.

We consider system (1.1)–(1.4) on the domain $\Omega := M \times (-h, h)$, with $M = (0, 1) \times (0, 1)$, and complement it with the following boundary and initial conditions

$$v, w, p, T \text{ are periodic in } x, y, z, \quad (1.5)$$

$$v \text{ and } p \text{ are even in } z, \quad w \text{ and } T \text{ are odd in } z, \quad (1.6)$$

$$(v, T)|_{t=0} = (v_0, T_0). \quad (1.7)$$

Note that condition (1.6) is preserved by system (1.1)–(1.4), as long as it is satisfied initially. Also, we remark that no initial condition is imposed on w . This is because there is no dynamical equation for w , and in fact, w is uniquely determined by the incompressibility condition (1.3).

Conspicuously, we observe that the periodic and symmetry boundary conditions (1.5)–(1.6) on the domain $M \times (-h, h)$ are equivalent to the physical boundary conditions of no-permeability and stress-free at the solid physical boundaries $z = -h$ and $z = 0$ in the sub-domain $M \times (-h, 0)$, namely:

$$v, w, p, T \text{ are periodic in } x \text{ and } y, \quad (1.8)$$

$$(\partial_z v, w)|_{z=-h, 0} = 0, \quad T|_{z=-h, 0} = 0. \quad (1.9)$$

This equivalence between the two problems can be easily achieved by suitable reflections and extensions of the solutions. More precisely, if (v, w, p, T) is a strong solution (see Definition 1.1, below, for the definition of strong solutions) to system (1.1)–(1.4) on the domain $M \times (-h, h)$, subject to (1.5)–(1.7), then the restriction of (v, w, p, T) to the sub-domain $M \times (-h, 0)$ is also a strong solution to the same system but on the sub-domain, subject to (1.7) and (1.8)–(1.9); and, conversely, if (v, w, p, T) is a strong solution to system (1.1)–(1.4) on the sub-domain $M \times (-h, 0)$, subject to (1.7)

and (1.8)–(1.9), then by extending v, w, p and T to the larger domain $M \times (-h, h)$, respectively, even, odd, even and odd with respect to z , (v, w, p, T) is also a strong solution to the same system but on the larger domain, subject to (1.5)–(1.7).

Using equation (1.2), the pressure p can be represented by

$$p(x, y, z, t) = p_s(x, y, t) - \int_{-h}^z T(x, y, \xi, t) d\xi,$$

for unknown “surface pressure” p_s . Using this representation, system (1.1)–(1.4) can be rewritten as

$$\begin{aligned} & \partial_t v + (v \cdot \nabla_H) v + w \partial_z v - \Delta_H v \\ & + f_0 \vec{k} \times v + \nabla_H \left(p_s(x, y, t) - \int_{-h}^z T(x, y, \xi, t) d\xi \right) = 0, \end{aligned} \quad (1.10)$$

$$\nabla_H \cdot v + \partial_z w = 0, \quad (1.11)$$

$$\partial_t T + v \cdot \nabla_H T + w \partial_z T - \partial_z^2 T = 0. \quad (1.12)$$

Concerning the boundary and initial conditions, we can now drop the boundary conditions for the pressure from (1.5)–(1.7), since it is hidden in the above formulation, in other words, the boundary and initial conditions now read as

$$v, w, T \text{ are periodic in } x, y, z, \quad (1.13)$$

$$v \text{ is even in } z, \quad w \text{ and } T \text{ are odd in } z, \quad (1.14)$$

$$(v, T)|_{t=0} = (v_0, T_0). \quad (1.15)$$

By the aid of the periodic boundary condition (1.13) and the divergence free condition (1.11), it is obviously that

$$\int_{-h}^h \nabla_H \cdot v(x, y, z, t) dz = 0, \quad (1.16)$$

for any $(x, y) \in M$. By the periodic and symmetry conditions (1.13) and (1.14), one has $w|_{z=-h} = w|_{z=h} = -w|_{z=-h} = 0$ and, as a result, using (1.11), w can be represented in v as

$$w(x, y, z, t) = - \int_{-h}^z \nabla_H \cdot v(x, y, \xi, t) d\xi. \quad (1.17)$$

On the other hand side, (1.17) obviously implies (1.11) and, furthermore, (1.16) and the conditions for v as stated in (1.13)–(1.14) imply those for w as stated in (1.13)–(1.14).

On account of what we stated in the previous paragraph, with the aid of the expression (1.17), one can replace (1.11) by (1.16) and drop the conditions for w in (1.13) and (1.14), without changing the system. In other words, system (1.10)–(1.12), subject to the boundary and initial conditions (1.13)–(1.15), is equivalent to the following system

$$\partial_t v + (v \cdot \nabla_H) v + w \partial_z v - \Delta_H v$$

$$+f_0 \vec{k} \times v + \nabla_H \left(p_s(x, y, t) - \int_{-h}^z T(x, y, \xi, t) d\xi \right) = 0, \quad (1.18)$$

$$\int_{-h}^h \nabla_H \cdot v(x, y, z, t) dz = 0, \quad (1.19)$$

$$\partial_t T + v \cdot \nabla_H T + w \partial_z T - \partial_z^2 T = 0, \quad (1.20)$$

with w given by (1.17), subject to the boundary and initial conditions

$$v, T \text{ are periodic in } x, y, z, \quad (1.21)$$

$$v \text{ and } T \text{ are even and odd in } z, \text{ respectively}, \quad (1.22)$$

$$(v, T)|_{t=0} = (v_0, T_0). \quad (1.23)$$

Applying the operator div_H to equation (1.18) and integrating the resulting equation with respect to z over $(-h, h)$, one can see that $p_s(x, y, t)$ satisfies the following (see Appendix A for the details)

$$\begin{cases} -\Delta_H p_s = \frac{1}{2h} \nabla_H \cdot \int_{-h}^h \left(\nabla_H \cdot (v \otimes v) + f_0 \vec{k} \times v - \int_{-h}^z \nabla_H T d\xi \right) dz, \\ \int_M p_s(x, y, t) dx dy = 0, \quad p_s \text{ is periodic in } x \text{ and } y. \end{cases} \quad (1.24)$$

Here the condition $\int_M p_s(x, y, t) dx dy = 0$ is imposed to guarantee the uniqueness of such p_s .

Before stating our main results, let's introduce some necessary notations and give the definitions of strong solutions. Throughout this paper, for $1 \leq q \leq \infty$, we use $L^q(\Omega)$, $L^q(M)$ and $W^{m,q}(\Omega)$, $W^{m,q}(M)$ to denote the standard Lebesgue and Sobolev spaces, respectively. For $q = 2$, we use H^m instead of $W^{m,2}$. For simplicity, we still use the notations L^p and H^m to denote the N -product spaces $(L^p)^N$ and $(H^m)^N$, respectively. We always use $\|u\|_p$ to denote the $L^p(\Omega)$ norm of u , while use $\|f\|_{p,M}$ to denote the $L^p(M)$ norm of f . For shortening the expressions, we sometimes use $\|(f_1, f_2, \dots, f_n)\|_X$ to denote the sum $\sum_{i=1}^n \|f_i\|_X$.

We introduce the following functions which will play crucial roles in this paper

$$u = \partial_z v, \quad \theta = \nabla_H^\perp \cdot v, \quad \eta = \nabla_H \cdot v + \int_{-h}^z T d\xi - \frac{1}{2h} \int_{-h}^h \int_{-h}^z T d\xi dz, \quad (1.25)$$

where $\nabla_H^\perp = (-\partial_y, \partial_x)$. As it will be seen later, these functions are introduced to overcome the ‘‘mismatching’’ of regularities between the horizontal momentum equations and the temperature equation.

Definition 1.1. *Given a positive time \mathcal{T} . Let $v_0 \in H^2(\Omega)$ and $T_0 \in H^1(\Omega)$, with $\int_{-h}^h \nabla_H \cdot v_0(x, y, z) dz = 0$ and $\nabla_H T_0 \in L^q(\Omega)$, for some $q \in (2, \infty)$, be two periodic functions, such that they are even and odd in z , respectively. A pair (v, T) is called a strong solution to system (1.18)–(1.23) on $\Omega \times (0, \mathcal{T})$ if*

- (i) v and T are periodic in x, y, z , and they are even and odd in z , respectively;
- (ii) v and T have the regularities

$$v \in L^\infty(0, \mathcal{T}; H^2(\Omega)) \cap C([0, \mathcal{T}]; H^1(\Omega)), \quad \partial_t v \in L^2(0, \mathcal{T}; H^1(\Omega)),$$

$$\begin{aligned} T &\in L^\infty(0, \mathcal{T}; H^1(\Omega) \cap L^\infty(\Omega)) \cap C([0, \mathcal{T}]; L^2(\Omega)), \quad \partial_t T \in L^2(0, \mathcal{T}; L^2(\Omega)), \\ (\nabla_H \partial_z v, \partial_z T) &\in L^2(0, \mathcal{T}; H^1(\Omega)), \quad \nabla_H T \in L^\infty(0, \mathcal{T}; L^q(\Omega)), \\ \eta &\in L^2(0, \mathcal{T}; H^2(\Omega)), \quad \theta \in L^2(0, \mathcal{T}; H^2(\Omega)); \end{aligned}$$

(iii) v and T satisfy equations (1.18)–(1.20) a.e. in $\Omega \times (0, \mathcal{T})$, with w and p_s given by (1.17) and (1.24), respectively, and satisfy the initial condition (1.23).

Remark 1.1. (i) The regularities in Definition 1.1 seem a little bit nonstandard. This is caused by the “mismatching” of regularities between the horizontal momentum equation (1.18) and the temperature equation (1.20): a term involving the horizontal derivatives of the temperature appears in the horizontal momentum equation, but it is only in the vertical direction that the temperature has dissipation. More precisely, though one can obtain the regularity that $\nabla_H \partial_z v \in L^2(0, \mathcal{T}; H^1(\Omega))$, which is included in Definition (1.1), we have no reason to ask for the regularity that $\nabla_H^2 v \in L^2(0, \mathcal{T}; H^1(\Omega))$, under the assumption on the initial data in Definition 1.1. In fact, recalling the regularity theory for parabolic system, and checking the horizontal momentum equation (1.18), the regularity that $\nabla_H^2 v \in L^2(0, \mathcal{T}; H^1(\Omega))$ appeals to somehow $\nabla_H^2 T \in L^2(\Omega \times (0, \mathcal{T}))$; however, this last requirement need not to be fulfilled, because we only have the smoothing effect in the vertical direction for the temperature.

(ii) The “mismatching” of regularities as stated in (i) does not occur in the system considered in our previous work [12], i.e., the system with only both horizontal viscosities and horizontal diffusivity, because, in this case, the horizontal diffusivity in the temperature equation provides the required regularity that $\nabla_H^2 T \in L^2(\Omega \times (0, \mathcal{T}))$.

(iii) As stated in (i), one can not expect such regularity that $\nabla_H^2 v \in L^2(0, \mathcal{T}; H^1(\Omega))$. However, with the help of η and θ in (1.25), one can expect that some appropriate combinations of $\nabla_H v$ and T can indeed have second order spatial derivatives, that is $(\eta, \theta) \in L^2(0, \mathcal{T}; H^2(\Omega))$, as included in Definition 1.1.

Definition 1.2. A pair (v, T) is called a global strong solution to system (1.18)–(1.20), subject to the boundary and initial conditions (1.21)–(1.23), if it is a strong solution on $\Omega \times (0, \mathcal{T})$ for any $\mathcal{T} \in (0, \infty)$.

The main result of this paper is the following global well-posedness result.

Theorem 1.1. Let $v_0 \in H^2(\Omega)$ and $T_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, with $\int_{-h}^h \nabla_H \cdot v_0(x, y, z) dz = 0$ and $\nabla_H T_0 \in L^q(\Omega)$, for some $q \in (2, \infty)$, be two periodic functions, such that they are even and odd in z , respectively. Then system (1.18)–(1.20), subject to the boundary and initial conditions (1.21)–(1.23), has a unique global strong solution (v, T) , which is continuously depending on the initial data.

If we assume, in addition, that $T_0 \in H^2(\Omega)$, then (v, T) obeys the following additional regularities

$$T \in L^\infty(0, \mathcal{T}; H^2(\Omega)) \cap C(0, \mathcal{T}; H^1(\Omega)), \quad \partial_t T \in L^2(0, \mathcal{T}; H^1(\Omega)),$$

$$\nabla_H v \in L^2(0, \mathcal{T}; H^2(\Omega)), \quad \partial_z T \in L^2(0, \mathcal{T}; H^2(\Omega)),$$

for any time $\mathcal{T} \in (0, \infty)$.

Remark 1.2. *Generally, if we imposed more regularities on the initial data, then one can expect more regularities of the strong solutions, and in particular, the strong solution will belong to $C^\infty(\overline{\Omega} \times [0, \infty))$, as long as the initial datum lies in $C^\infty(\overline{\Omega})$. However, one can not expect that the solutions have as high regularities as desired, if the initial data are not accordingly smooth enough.*

Remark 1.3. *Thanks to Theorem 1.1, and recalling the results in [10–13, 16, 17], one can conclude that the primitive equations are globally well-posed, as long as one has the horizontal viscosity and either horizontal or vertical diffusivity.*

The main difficulties for the mathematical analysis of system (1.18)–(1.20) come from three aspects: the absence of the vertical viscosity in the horizontal momentum equation, the lack of the horizontal diffusivity in the temperature equation, and the “mismatching” of regularities between the horizontal momentum equations and the temperature equation caused by the interaction between the lack of the vertical viscosity and the absence of the horizontal diffusivity. Concerning the difficulties caused by the absence of the vertical viscosity, [12, 13] provide us with some ideas. As indicated in [12, 13], the absence of the vertical viscosity forces us to estimate $\|v\|_\infty^2$, which appears as factors in the energy inequalities. To obtain this estimate, similar to [12, 13], we estimate the precise growth in q of the L^q norms of v (see Proposition 3.2), based on which, by applying a logarithmic type Sobolev inequality (see Lemma 2.4), we can control the L^∞ norm of v by logarithm of high order norms. However, in the current case, because of the “mismatching” of regularities between the horizontal momentum equations and the temperature equation (recall Remark 1.1 (i)), we are not able to obtain the appropriate estimates in the same way as in [12, 13]. Note that in [12, 13] all energy estimates for the derivatives of the velocity are carried out through multiplying the corresponding testing functions to the momentum equations directly; however, for the current case, when working on the energy estimates for the horizontal derivatives of the velocity, it is inappropriate to use the momentum equations as the tested ones. To see this, let’s take the $L_t^\infty(H_x^1)$ kind estimate as example: if trying to use the momentum equation to get the $L_t^\infty(L_x^2)$ estimate on $\nabla_H v$, one may multiply the momentum equation by $-\Delta_H v$ and, thus, requires the a priori $L_t^2(L_x^2)$ type estimate on $\nabla_H T$, which is obviously not guaranteed by the system, as we only have the vertical diffusivity in the temperature equation. To overcome this kind of difficulties, we consider the horizontal curl and some appropriate combination of the temperature with the horizontal divergence of v or its derivatives, which prove to have better regularities than the horizontal derivatives of v or its derivatives. In other words, the estimates on the horizontal derivatives of the velocity are achieved indirectly through the corresponding estimates on the horizontal curls and some appropriate combinations of the temperature with the horizontal divergences.

For the a priori $L_t^\infty(H_x^1)$ type estimate on v , recalling the ideas mentioned above, it is achieved by carrying out the $L_t^\infty(L_x^2)$ type energy estimates on (u, η, θ) , rather than on $(u, \nabla_H v)$ directly, and using the precise L^q estimates on v to dominate the main part of $\|v\|_\infty$. These are carried out in Proposition 3.4 and Corollary 3.1. Note that the following fact plays an important role in proving Corollary 3.1: inequality

$$A'(t) + B(t) \leq CA(t) \log B(t) + \text{“other terms”}$$

guarantees the boundness of $A(t)$ globally in time. The above inequality is a special case of the general logarithmic type Gronwall inequality stated in Lemma 2.5.

Based on the a priori $L_t^\infty(H_x^1)$ type estimate on v , one can obtain the a priori $L_t^\infty(H_x^1)$ type estimate on u . Again, because of the same reason as before, this a priori estimate is achieved through the $L_t^\infty(L_x^2)$ estimate on $(\partial_z u, \varphi, \psi)$, where

$$\varphi = \nabla_H \cdot u + T, \quad \psi = \nabla_H^\perp \cdot u, \quad (1.26)$$

rather than directly on $(\partial_z u, \nabla_H u)$.

Some higher order a priori estimates, especially those on the derivatives of the temperature, are still needed to ensure the global well-posedness. When working on the energy inequalities for the horizontal derivatives of T , caused by the absence of the horizontal diffusivity in the temperature equation, one has to appeal to somehow L^∞ estimate on $\nabla_H v$ to deal with the worst term $\int_\Omega |\nabla_H v| |\nabla_H T|^q dx dy dz$. To deal with this term, we decompose the velocity into a “temperature-independent” part and another temperature-dependent part and then deal with them in different ways, by using the logarithmic Sobolev inequalities of the Brézis-Gallouet-Wainger and Beale-Kato-Majda types, respectively. The resulting corresponding energy inequalities are of the type

$$A'(t) + B(t) \leq Cn(t)A(t) \log B(t) + \text{“other terms”},$$

where n is a locally integrable function on $[0, \infty)$. Note that this inequality does not necessary guarantee the boundness of the quantity A , in general; however, if it happens that the following additional relationship holds

$$n(t) \leq CA^\alpha(t)$$

for some positive number α , then it indeed implies the boundness of the quantity A , see Lemma 2.5. Fortunately, it is the case in our higher order energy inequality, and therefore, we are able to obtain the a priori higher order estimates, and furthermore the global existence of strong solutions. The additional regularities stated in the theorem follow from the energy inequality for the second order derivatives of T , which is somehow standard.

The rest of this paper is arranged as follows: in the next section, section 2, we collect some preliminaries which will be used throughout the paper. Section 3 is the main part of this paper, in which, by using the ideas obtained above, we establish several a priori estimates for a regularized system, and the a priori estimates are

independent of the regularization parameters. In section 4, based on the a priori estimates obtained in section 3, we give the proof of Theorem 1.1.

Throughout this paper, the letter C denotes a general positive constant, which may vary from line to line.

2. PRELIMINARIES

In this section, we collect some preliminary results which will be used in the rest of this paper.

Lemma 2.1 (see Lemma 2.1 in [12]). *The following inequality holds:*

$$\begin{aligned} & \int_M \left(\int_{-h}^h |\phi| dz \right) \left(\int_{-h}^h |\varphi \psi| dz \right) dx dy \\ & \leq C \|\phi\|_2 \|\varphi\|_2^{\frac{1}{2}} \left(\|\varphi\|_2^{\frac{1}{2}} + \|\nabla_H \varphi\|_2^{\frac{1}{2}} \right) \|\psi\|_2^{\frac{1}{2}} \left(\|\psi\|_2^{\frac{1}{2}} + \|\nabla_H \psi\|_2^{\frac{1}{2}} \right), \end{aligned}$$

for every ϕ, φ , and ψ such that the right hand sides make sense and are finite.

Lemma 2.2. *We have the following inequalities*

$$\int_M \left(\int_{-h}^h |\phi| dz \right) \left(\int_{-h}^h |\varphi| |\psi| dz \right) dx dy \leq \left(\int_{-h}^h \|\phi\|_{4,M} dz \right) \left(\int_{-h}^h \|\varphi\|_{4,M}^2 dz \right)^{\frac{1}{2}} \|\psi\|_2, \quad (2.1)$$

and

$$\begin{aligned} & \int_M \left(\int_{-h}^h |\phi| |\varphi| dz \right) \left(\int_{-h}^h |\psi| dz \right) dx dy \\ & \leq \left(\int_{-h}^h \|\phi\|_{4,M}^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h \|\varphi\|_{4,M}^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h \|\psi\|_{2,M} dz \right), \end{aligned} \quad (2.2)$$

for any functions ϕ, φ and ψ , such that the quantities on the right-hand sides make sense and are finite.

Proof. By the Hölder and Minkowski inequalities, we have

$$\begin{aligned} & \int_M \left(\int_{-h}^h |\phi| dz \right) \left(\int_{-h}^h |\varphi| |\psi| dz \right) dx dy \\ & \leq \int_M \left(\int_{-h}^h |\phi| dz \right) \left(\int_{-h}^h |\varphi|^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h |\psi|^2 dz \right)^{\frac{1}{2}} dx dy \\ & \leq \left[\int_M \left(\int_{-h}^h |\phi| dz \right)^4 dx dy \right]^{\frac{1}{4}} \left[\int_M \left(\int_{-h}^h |\varphi|^2 dz \right)^2 dx dy \right]^{\frac{1}{4}} \|\psi\|_2 \\ & \leq \left(\int_{-h}^h \|\phi\|_{4,M} dz \right) \left(\int_{-h}^h \|\varphi\|_{4,M}^2 dz \right)^{\frac{1}{2}} \|\psi\|_2, \end{aligned}$$

and

$$\begin{aligned}
& \int_M \left(\int_{-h}^h |\phi| |\varphi| dz \right) \left(\int_{-h}^h |\psi| dz \right) dx dy \\
& \leq \int_M \left(\int_{-h}^h |\phi|^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h |\varphi|^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h |\psi| dz \right) dx dy \\
& \leq \left[\int_M \left(\int_{-h}^h |\phi|^2 dz \right)^2 dx dy \right]^{\frac{1}{4}} \left[\int_M \left(\int_{-h}^h |\varphi|^2 dz \right)^2 dx dy \right]^{\frac{1}{4}} \\
& \quad \times \left[\int_M \left(\int_{-h}^h |\psi| dz \right)^2 dx dy \right]^{\frac{1}{2}} \\
& \leq \left(\int_{-h}^h \|\phi\|_{4,M}^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h \|\varphi\|_{4,M}^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h \|\psi\|_{2,M} dz \right),
\end{aligned}$$

proving (2.1) and (2.2). \square

Lemma 2.3. *The following inequalities hold*

$$\begin{aligned}
\left(\int_{-h}^h \|f\|_{4,M}^2 dz \right)^{\frac{1}{2}} & \leq C \left(\|f\|_{2}^{\frac{1}{2}} \|\nabla_H f\|_{2}^{\frac{1}{2}} + \|f\|_{2} \right), \\
\int_{-h}^h \|f\|_{4,M}^2 dz & \leq C\sqrt{h} \left(\|f\|_{2}^{\frac{1}{2}} \|\nabla_H f\|_{2}^{\frac{1}{2}} + \|f\|_{2} \right),
\end{aligned}$$

for any function f such that the right-hand sides make sense and are finite. As a consequence, by the Poincaré inequality, the following holds

$$\begin{aligned}
\left(\int_{-h}^h \|\nabla_H f\|_{4,M}^2 dz \right)^{\frac{1}{2}} & \leq C \|\nabla_H f\|_{2}^{\frac{1}{2}} \|\nabla_H^2 f\|_{2}^{\frac{1}{2}}, \\
\int_{-h}^h \|\nabla_H f\|_{4,M}^2 dz & \leq C\sqrt{h} \|\nabla_H f\|_{2}^{\frac{1}{2}} \|\nabla_H^2 f\|_{2}^{\frac{1}{2}},
\end{aligned}$$

if moreover f is periodic in (x, y) .

Proof. The conclusion follow easily from the Hölder and Ladyzhenskaya inequalities and, thus, the proofs are omitted here. \square

The following logarithmic Sobolev inequality, which links the L^∞ norm in terms of the L^q norms up to the logarithm of the high order norms. Some relevant inequalities can be found in [9, 14, 19], where the two dimensional case are considered.

Lemma 2.4 (Logarithmic Sobolev embedding inequality, see Lemma 2.2 in [12]). *Let $F \in W^{1,p}(\Omega)$, with $p > 3$, be a periodic function. Then the following inequality*

holds true

$$\|F\|_\infty \leq C_{p,\lambda} \max \left\{ 1, \sup_{r \geq 2} \frac{\|F\|_r}{r^\lambda} \right\} \log^\lambda(\|F\|_{W^{1,p}(\Omega)} + e),$$

for any $\lambda > 0$.

The logarithmic type Gronwall inequality stated and proved in the following lemma will be used in establishing the global a priori estimates with critical nonlinearities. The first logarithmic type Gronwall inequality in the same spirit as stated here was obtained by Li–Titi [28], see also Li–Titi [29] for some related inequalities.

Lemma 2.5 (Logarithmic Gronwall inequality). *Given $\mathcal{T} \in (0, \infty)$. Let A and B be two nonnegative measurable functions defined on $(0, \mathcal{T})$, with A is absolutely continuous on $(0, \mathcal{T})$ and is continuous on $[0, \mathcal{T})$, satisfying*

$$\frac{d}{dt}A + B \leq [\ell(t) + m(t) \log(A + e) + n(t) \log(A + B + e)](A + e) + f(t),$$

where ℓ, m, n , and f are all nonnegative functions on $(0, \mathcal{T})$ belonging to $L^1((0, \mathcal{T}))$. Assume further that there are two positive constants K and α , such that

$$n(t) \leq K(A(t) + e)^\alpha$$

for all $t \in (0, \mathcal{T})$. Then, we have the following estimate

$$A(t) + \int_0^t B(s)ds \leq (2Q(t) + 1)e^{Q(t)}$$

for all $t \in (0, \mathcal{T})$, where

$$Q(t) = e^{(\alpha+1) \int_0^t (m(s)+n(s))ds} \left(\log(A(0) + e) + \int_0^t (\ell(s) + f(s) + \log(2K)n(s))ds + t \right).$$

Proof. Setting $A_1 = A + e$ and $B_1 = B + A + e$, then

$$\begin{aligned} \frac{d}{dt}A_1 + B_1 &= \frac{d}{dt}A + B + A + e \\ &\leq [\ell(t) + 1 + m(t) \log(A + e) + n(t) \log(A + B + e)](A + e) + f(t) \\ &= (\ell(t) + 1 + m(t) \log A_1 + n(t) \log B_1)A_1 + f(t). \end{aligned}$$

Dividing both sides of the above inequality by A_1 yields

$$\begin{aligned} \frac{d}{dt} \log A_1 + \frac{B_1}{A_1} &\leq \ell(t) + 1 + \frac{f(t)}{A_1(t)} + m(t) \log A_1 + n(t) \log B_1 \\ &\leq \ell(t) + 1 + f(t) + m(t) \log A_1 + n(t) \log B_1. \end{aligned}$$

Noticing that $\log z \leq \log(z + 1) \leq z$ for any $z \in (0, \infty)$, and recalling that $n(t) \leq K(A + e)^\alpha = K A_1^\alpha$, we deduce

$$n(t) \log B_1 = n(t) \left(\log \frac{B_1}{2K A_1^{\alpha+1}} + (\alpha + 1) \log A_1 + \log(2K) \right)$$

$$\begin{aligned} &\leq n(t) \left(\frac{B_1}{2K A_1^{\alpha+1}} + (\alpha + 1) \log A_1 + \log(2K) \right) \\ &\leq \frac{B_1}{2A_1} + (\alpha + 1)n(t) \log A_1 + n(t) \log(2K). \end{aligned}$$

Therefore, one has

$$\frac{d}{dt} \log A_1 + \frac{B_1}{2A_1} \leq (m(t) + (\alpha + 1)n(t)) \log A_1 + \ell(t) + 1 + f(t) + n(t) \log(2K),$$

from which, by denoting $G(t) = \log A_1(t) + \int_0^t \frac{B_1(s)}{2A_1(s)} ds$, one obtains

$$G'(t) \leq (m(t) + (\alpha + 1)n(t))G(t) + \ell(t) + 1 + f(t) + n(t) \log(2K);$$

and, thus,

$$\begin{aligned} G(t) &\leq e^{\int_0^t (m(s) + (\alpha+1)n(s)) ds} \left(G(0) + \int_0^t (\ell(s) + f(s) + \log(2K)n(s) + 1) ds \right) \\ &\leq e^{(\alpha+1) \int_0^t (m(s) + n(s)) ds} \left(\int_0^t (\ell(s) + f(s) + \log(2K)n(s)) ds + t \right) \\ &\quad + e^{(\alpha+1) \int_0^t (m(s) + n(s)) ds} \log(A(0) + e) =: Q(t). \end{aligned}$$

Recalling the definition of $G(t)$, it follows from the above estimate that

$$A_1(t) \leq e^{G(t)} \leq e^{Q(t)},$$

and further that

$$\begin{aligned} \int_0^t B_1(s) ds &= 2 \int_0^t A_1(s) \frac{B_1(s)}{2A_1(s)} ds \leq 2 \sup_{0 \leq s \leq t} A_1(s) \int_0^t \frac{B_1(s)}{2A_1(s)} ds \\ &\leq 2e^{Q(t)} G(t) \leq 2Q(t)e^{Q(t)}. \end{aligned}$$

Thanks to the above estimates and recalling the definitions of A_1 and B_1 , the conclusion follows. \square

Remark 2.1. (i) A special form of the logarithmic Gronwall inequality in Lemma 2.5 reads as

$$\frac{d}{dt} A + B \leq A \log(A + B + e). \quad (2.3)$$

Note that this is essentially different from the classic logarithmic Gronwall inequality like $\frac{d}{dt} A + B \leq A \log(A + e)$. Noticing that the PDEs with dissipation, the quantities represented by B in the above inequality usually have higher order norms than those by A and, therefore, compared with the using of the usual logarithmic Gronwall inequality, by using (2.3), one can relax the regularity assumptions on the initial data and may need only to carry out some lower order energy estimates, see Li–Titi [28].

(ii) Another special form of the logarithmic Gronwall inequality in Lemma 2.5 is

$$\frac{d}{dt} A + B \leq n(t) A \log(A + B + e),$$

where $n \in L^1((0, \mathcal{T}))$. The above inequality does not necessarily imply the boundedness of A on $(0, \mathcal{T})$ in general; however, as stated in Lemma 2.5, if n satisfies, in addition, that $n(t) \leq K(A(t) + 1)^\alpha$, for some positive constants K and α , then it indeed implies the desired boundedness of A on $(0, \mathcal{T})$.

(iii) In the spirit of the system version of the (classic) Gronwall inequality exploited in Cao–Li–Titi [12], one can also exploit the corresponding system version of the logarithmic Gronwall inequality stated in Lemma 2.5.

We also need the following Aubin-Lions compactness lemma.

Lemma 2.6 (Aubin-Lions Lemma, see Simon [38] Corollary 4). *Assume that X, B and Y are three Banach spaces, with $X \hookrightarrow B \hookrightarrow Y$. Then it holds that*

(i) *If \mathcal{F} is a bounded subset of $L^p(0, T; X)$, where $1 \leq p < \infty$, and $\frac{\partial \mathcal{F}}{\partial t} = \{\frac{\partial f}{\partial t} | f \in \mathcal{F}\}$ is bounded in $L^1(0, T; Y)$, then \mathcal{F} is relatively compact in $L^p(0, T; B)$;*

(ii) *If \mathcal{F} is bounded in $L^\infty(0, T; X)$, and $\frac{\partial \mathcal{F}}{\partial t}$ is bounded in $L^r(0, T; Y)$, where $r > 1$, then \mathcal{F} is relatively compact in $C([0, T]; B)$.*

3. SYSTEM WITH FULL VISCOSITIES AND FULL DIFFUSIVITY

In this section, we are concerned with energy estimates for the strong solutions to the following regularized system, with both full viscosities and full diffusivity,

$$\begin{aligned} & \partial_t v + (v \cdot \nabla_H) v + w \partial_z v - \Delta_H v - \varepsilon \partial_z^2 v \\ & + f_0 \vec{k} \times v + \nabla_H \left(p_s(x, y, t) - \int_{-h}^z T(x, y, \xi, t) d\xi \right) = 0, \end{aligned} \quad (3.1)$$

$$\int_{-h}^h \nabla_H \cdot v(x, y, z, t) dz = 0, \quad (3.2)$$

$$\partial_t T + v \cdot \nabla_H T + w \partial_z T - \varepsilon \Delta_H T - \partial_z^2 T = 0, \quad (3.3)$$

with w given by (1.17), subject to the boundary and initial conditions (1.21)–(1.23).

For any periodic functions $v_0, T_0 \in H^2(\Omega)$, which are even and odd in z , respectively, there is a unique strong solution to the above system, subject to the boundary and initial conditions (1.21)–(1.23), and in fact, we have the following proposition.

Proposition 3.1. *Suppose that the periodic functions $v_0, T_0 \in H^2(\Omega)$ are even and odd in z , respectively, with $\int_{-h}^h \nabla_H \cdot v_0(x, y, z) dz = 0$. Then for any $\varepsilon > 0$, there is a unique global strong solution (v, T) to system (3.1)–(3.3), subject to the boundary and initial conditions (1.21)–(1.23), such that*

$$\begin{aligned} & (v, T) \in L_{loc}^\infty([0, \infty); H^2(\Omega)) \cap C([0, \infty); H^1(\Omega)), \\ & (v, T) \in L_{loc}^2([0, \infty); H^3(\Omega)), \quad (\partial_t v, \partial_t T) \in L_{loc}^2([0, \infty); H^1(\Omega)). \end{aligned}$$

Proof. The proof can be given in the same way as in [10] (see Proposition 2.1 there), and thus we omit it here. \square

The strong solutions satisfy the following estimates.

Proposition 3.2. *For any $0 < \mathcal{T} < \infty$, we have the following:*

(i) *Basic energy estimate:*

$$\sup_{0 \leq t \leq \mathcal{T}} \|(v, T)\|_2^2(t) + \int_0^{\mathcal{T}} \|(\nabla_H v, \partial_z T, \sqrt{\varepsilon} \nabla_H T)\|_2^2 dt \leq C e^{\mathcal{T}} (\|v_0\|_2^2 + \|T_0\|_2^2),$$

where C is a positive constant depending only on h ;

(ii) L^∞ estimate on T :

$$\sup_{0 \leq t \leq \mathcal{T}} \|T\|_\infty(t) \leq \|T_0\|_\infty;$$

(iii) L^q estimate on v :

$$\sup_{0 \leq t \leq \mathcal{T}} \|v\|_q(t) \leq C \sqrt{q}, \quad \text{for every } q \in [4, \infty),$$

for a positive constant C depending only on h , \mathcal{T} , and $\|(v_0, T_0)\|_\infty$.

Proof. (i) Multiplying equations (3.1) and (3.3) by v and T , respectively, summing the resulting equations, and integrating over Ω , it follows from integration by parts, using (3.2), and the Hölder and Young inequalities that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} (|v|^2 + |T|^2) dx dy dz \\ & + \int_{\Omega} \left(|\nabla_H v|^2 + \varepsilon |\partial_z v|^2 + \varepsilon |\nabla_H T|^2 + |\partial_z T|^2 \right) dx dy dz \\ & = - \int_{\Omega} \left(\int_{-h}^z T d\xi \right) \nabla_H \cdot v dx dy dz \\ & \leq C \|T\|_2 \|\nabla_H v\|_2 \leq \frac{1}{2} \|\nabla_H v\|_2^2 + C \|T\|_2^2, \end{aligned}$$

and thus

$$\frac{d}{dt} \|(v, T)\|_2^2 + \|(\nabla_H v, \partial_z T, \sqrt{\varepsilon} \partial_z v, \sqrt{\varepsilon} \nabla_H T)\|_2^2 \leq C \|T\|_2^2,$$

from which, by the Gronwall inequality, one obtains (i).

(ii) Multiplying equation (3.3) by $|T|^{q-2} T$, with $q \in [2, \infty)$, and integrating the resultant over Ω , it follows from integration by parts and using (3.2) that

$$\frac{1}{q} \frac{d}{dt} \|T\|_q^q \leq 0,$$

which implies $\sup_{0 \leq t \leq \mathcal{T}} \|T\|_q \leq \|T_0\|_q$. The conclusion follows by taking $q \rightarrow \infty$ and using the fact that $\|T\|_q \rightarrow \|T\|_\infty$, as $q \rightarrow \infty$.

(iii) This has been proven in (iii) of Proposition 3.1 in [12] (note that the diffusivity plays no role for the proof of (iii) there) and, thus, we omit the details here. \square

3.1. A priori $L_t^\infty(H_x^1)$ estimate on v . In this subsection, we establish the a priori estimates a priori $L_t^\infty(H_x^1)$ estimates of v . As it will be seen below, we achieve the a priori $L_t^\infty(H_x^1)$ on v not through performing directly the energy inequalities to v , but by carrying out the corresponding $L_t^\infty(L_x^2)$ estimates for u, η and θ defined below:

$$u := \partial_z v, \quad \eta := \nabla_H \cdot v + \Phi, \quad \theta := \nabla_H^\perp \cdot v, \quad (3.4)$$

where $\nabla_H^\perp = (-\partial_y, \partial_x)$ and Φ is given by

$$\Phi(x, y, z, t) = \int_{-h}^z T(x, y, \xi, t) d\xi - \frac{1}{2h} \int_{-h}^h \left(\int_{-h}^z T(x, y, \xi, t) d\xi \right) dz. \quad (3.5)$$

Differentiating equation (3.1) with respect to z yields

$$\begin{aligned} \partial_t u + (v \cdot \nabla_H) u + w \partial_z u - \Delta_H u - \varepsilon \partial_z^2 u \\ + f_0 k \times u + (u \cdot \nabla_H) v - (\nabla_H \cdot v) u - \nabla_H T = 0. \end{aligned} \quad (3.6)$$

The functions η and θ satisfy (see Appendix A for the derivation)

$$\begin{aligned} \partial_t \eta - \Delta_H \eta - \varepsilon \partial_z^2 \eta = -\nabla_H \cdot [(v \cdot \nabla_H) v + w \partial_z v + f_0 k \times v] + (1 - \varepsilon) \partial_z T \\ - w T - \int_{-h}^z (\nabla_H \cdot (v T) - \varepsilon \Delta_H T) d\xi + f(x, y, t) \end{aligned} \quad (3.7)$$

and

$$\partial_t \theta - \Delta_H \theta - \varepsilon \partial_z^2 \theta = -\nabla_H^\perp \cdot [(v \cdot \nabla_H) v + w \partial_z v + f_0 k \times v], \quad (3.8)$$

respectively, with the function $f = f(x, y, t)$ given by

$$\begin{aligned} f(x, y, t) = \frac{1}{2h} \int_{-h}^h \left(\int_{-h}^z (\nabla_H \cdot (v T) - \varepsilon \Delta_H T) d\xi + w T \right) dz \\ + \frac{1}{2h} \int_{-h}^h \nabla_H \cdot (\nabla_H \cdot (v \otimes v) + f_0 \vec{k} \times v) dz. \end{aligned} \quad (3.9)$$

For the convenience, we first prove the following proposition which will be used later:

Proposition 3.3. *Let η and θ be as in (3.4). The following estimates hold:*

$$\begin{aligned} \|\eta\|_2^2 + \|\theta\|_2^2 &\leq C(\|\nabla_H v\|_2^2 + 1), \\ \|\nabla_H v\|_2^2 &\leq C(\|\eta\|_2^2 + \|\theta\|_2^2 + 1), \\ \left(\int_{-h}^h \|\nabla_H v\|_{4,M}^2 dz \right)^{\frac{1}{2}} &\leq C \left(\|(\eta, \theta)\|_2^{\frac{1}{2}} \|\nabla_H(\eta, \theta)\|_2^{\frac{1}{2}} + \|(\eta, \theta)\|_2 + 1 \right), \\ \int_{-h}^h \|\nabla_H v\|_{4,M} dz &\leq C \left(\|(\eta, \theta)\|_2^{\frac{1}{2}} \|\nabla_H(\eta, \theta)\|_2^{\frac{1}{2}} + \|(\eta, \theta)\|_2 + 1 \right), \end{aligned}$$

where C is a positive constant depending only on $\|T_0\|_\infty$ and h .

Proof. By Proposition 3.2, and recalling the definition of Φ , we have $\|\Phi\|_\infty \leq C\|T\|_\infty \leq C\|T_0\|_\infty$. The first conclusion follows directly from the definitions of η, θ , and Φ . By the elliptic estimates, we have

$$\begin{aligned} \|\nabla_H v\|_2^2 &\leq C(\|\nabla_H \cdot v\|_2^2 + \|\nabla_H^\perp \cdot v\|_2^2) \leq C(\|\eta\|_2^2 + \|\theta\|_2^2 + \|\Phi\|_2^2) \\ &\leq C(\|\eta\|_2^2 + \|\theta\|_2^2 + 1), \end{aligned}$$

proving the second estimate. For the third estimate, by the elliptic estimates and Lemma 2.3, we have

$$\begin{aligned} \int_{-h}^h \|\nabla_H v\|_{4,M}^2 dz &\leq C \int_{-h}^h (\|\nabla_H \cdot v\|_{4,M}^2 + \|\nabla_H^\perp \cdot v\|_{4,M}^2) dz \\ &\leq C \int_{-h}^h (\|(\eta, \theta)\|_{4,M}^2 + \|\Phi\|_{4,M}^2) dz \\ &\leq C \left(\|(\eta, \theta)\|_2^{\frac{1}{2}} \|\nabla_H(\eta, \theta)\|_2^{\frac{1}{2}} + \|(\eta, \theta)\|_2 + 1 \right), \end{aligned}$$

while the last inequality follows by applying the Hölder inequality to the third one. \square

The energy inequality for (u, η, θ) is contained in the next proposition.

Proposition 3.4. *Given $\mathcal{T} \in (0, \infty)$ and assume that $\varepsilon \in (0, 1)$. Let η, θ and u be as in (3.4). We have the following energy inequality:*

$$\begin{aligned} &\frac{d}{dt} \left(\|(\theta, \eta, u)\|_2^2 + \frac{\|u\|_4^4}{2} \right) + \|\nabla_H(\theta, \eta, u)\|_2^2 + \| |u| \nabla_H u \|_2^2 + \|\sqrt{\varepsilon} \partial_z(\theta, \eta, u)\|_2^2 \\ &\leq C(\|v\|_\infty^2 + \|\nabla_H v\|_2^2 + 1)(\|(\theta, \eta)\|_2^2 + \|u\|_4^4 + 1) + C\|(\partial_z T, \sqrt{\varepsilon} \nabla_H T)\|_2^2 \end{aligned}$$

for any $t \in (0, \mathcal{T})$, where C is a positive constant depending only on h, \mathcal{T} , and $\|T_0\|_\infty$.

Proof. Multiplying equation (3.6) by $(|u|^2 + 1)u$ and integrating the resulting equation over Ω , it follows from integration by parts, Proposition 3.2, and using the Young inequality that

$$\begin{aligned} &\frac{d}{dt} \left(\frac{\|u\|_2^2}{2} + \frac{\|u\|_4^4}{4} \right) + \int_\Omega [|\nabla_H u|^2 + \varepsilon |\partial_z u|^2 \\ &\quad + |u|^2 (|\nabla_H u|^2 + 2|\nabla_H |u||^2 + \varepsilon |\partial_z u|^2 + 2\varepsilon |\partial_z |u|^2)] dx dy dz \\ &= \int_\Omega [\nabla_H T + (\nabla_H \cdot v)u - (u \cdot \nabla_H)v] \cdot (|u|^2 + 1)u dx dy dz \\ &\leq C \int_\Omega [|T|(|u|^2 + 1)|\nabla_H u| + |v|(|u|^2 + 1)|u||\nabla_H u|] dx dy dz \\ &\leq \frac{1}{2} \int_\Omega (|u|^2 + 1)|\nabla_H u|^2 dx dy dz + C \int_\Omega (|u|^2 + 1)(|T|^2 + |v|^2|u|^2) dx dy dz \end{aligned}$$

$$\leq \frac{1}{2} \int_{\Omega} (|u|^2 + 1) |\nabla_H u|^2 dx dy dz + C(1 + \|v\|_{\infty}^2) (\|u\|_4^4 + 1),$$

which gives

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\|u\|_2^2}{2} + \frac{\|u\|_4^4}{4} \right) + \|(\nabla_H u, |u| \nabla_H u, \sqrt{\varepsilon} \partial_z u)\|_2^2 \\ & \leq C(1 + \|v\|_{\infty}^2) (\|u\|_4^4 + 1) + \frac{1}{2} (\|\nabla_H u\|_2^2 + \| |u| \nabla_H u \|_2^2). \end{aligned} \quad (3.10)$$

Multiplying equation (3.8) by θ and integrating the resulting equation over Ω , it follows from integration by parts that

$$\frac{1}{2} \frac{d}{dt} \|\theta\|_2^2 + \|(\nabla_H \theta, \sqrt{\varepsilon} \partial_z \theta)\|_2^2 = \int_{\Omega} ((v \cdot \nabla_H) v + w \partial_z v + f_0 \vec{k} \times v) \cdot \nabla_H^{\perp} \theta dx dy dz. \quad (3.11)$$

By Propositions 3.2 and 3.3, it follows from the Young inequality that

$$\begin{aligned} \int_{\Omega} ((v \cdot \nabla_H) v + f_0 \vec{k} \times v) \cdot \nabla_H^{\perp} \theta dx dy dz & \leq \frac{1}{8} \|\nabla_H \theta\|_2^2 + C(\|v\|_{\infty}^2 \|\nabla_H v\|_2^2 + \|v\|_2^2) \\ & \leq \frac{1}{8} \|\nabla_H \theta\|_2^2 + C(\|v\|_{\infty}^2 + 1) (\|(\eta, \theta)\|_2^2 + 1). \end{aligned} \quad (3.12)$$

By Lemma 2.2, Proposition 3.3, (1.17), and the Hölder and Young inequalities, we have

$$\begin{aligned} & \int_{\Omega} w \partial_z v \cdot \nabla_H^{\perp} \theta dx dy dz \\ & \leq \int_M \left(\int_{-h}^h |\nabla_H v| dz \right) \left(\int_{-h}^h |u| |\nabla_H \theta| dz \right) dx dy \\ & \leq \left(\int_{-h}^h \|\nabla_H v\|_{4,M} dz \right) \left(\int_{-h}^h \|u\|_{4,M}^2 dz \right)^{\frac{1}{2}} \|\nabla_H \theta\|_2 \\ & \leq C \left(\|(\eta, \theta)\|_2^{\frac{1}{2}} \|\nabla_H(\eta, \theta)\|_2^{\frac{1}{2}} + \|(\eta, \theta)\|_2 + 1 \right) \|u\|_4 \|\nabla_H \theta\|_2 \\ & \leq \frac{1}{8} \|\nabla_H(\eta, \theta)\|_2^2 + C[\|(\eta, \theta)\|_2^2 \|u\|_4^4 + (\|(\eta, \theta)\|_2^2 + 1) \|u\|_4^2] \\ & \leq \frac{1}{8} \|\nabla_H(\eta, \theta)\|_2^2 + C(\|\nabla_H v\|_2^2 + 1) (\|u\|_4^4 + 1). \end{aligned} \quad (3.13)$$

Substituting (3.12) and (3.13) into (3.11) yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta\|_2^2 + \|(\nabla_H \theta, \sqrt{\varepsilon} \partial_z \theta)\|_2^2 & \leq C(\|v\|_{\infty}^2 + \|\nabla_H v\|_2^2 + 1) (\|(\eta, \theta)\|_2^2 + \|u\|_4^4 + 1) \\ & \quad + \frac{1}{4} (\|\nabla_H \eta\|_2^2 + \|\nabla_H \theta\|_2^2). \end{aligned} \quad (3.14)$$

Recalling the definitions of η and Φ and using (3.2), one has

$$\int_{-h}^h \eta(x, y, z, t) dz = \int_{-h}^h [(\nabla_H \cdot v) + \Phi] dz = 0.$$

On account of this, multiplying equation (3.7) by η , and integrating the resultant over Ω , it follows from integration by parts that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\eta\|_2^2 + \|\nabla_H \eta\|_2^2 + \varepsilon \|\partial_z \eta\|_2^2 \\ &= \int_{\Omega} \left[((1 - \varepsilon) \partial_z T - wT) \eta + \left(\int_{-h}^z (vT - \varepsilon \nabla_H T) d\xi \right) \cdot \nabla_H \eta \right] dx dy dz \\ & \quad + \int_{\Omega} [(v \cdot \nabla_H) v + w \partial_z v + f_0 \vec{k} \times v] \cdot \nabla_H \eta dx dy dz. \end{aligned} \quad (3.15)$$

Same arguments as for (3.12) and (3.13) yield

$$\begin{aligned} & \int_{\Omega} [(v \cdot \nabla_H) v + w \partial_z v + f_0 \vec{k} \times v] \cdot \nabla_H \eta dx dy dz \\ & \leq \frac{1}{8} \|\nabla_H(\eta, \theta)\|_2^2 + C(\|v\|_{\infty}^2 + \|\nabla_H v\|_2^2 + 1)(\|(\eta, \theta)\|_2^2 + \|u\|_4^4 + 1). \end{aligned}$$

By Propositions 3.2 and 3.3, it follows from the Hölder and Young inequalities that

$$\begin{aligned} & \int_{\Omega} \left[((1 - \varepsilon) \partial_z T - wT) \eta + \left(\int_{-h}^z (vT - \varepsilon \nabla_H T) d\xi \right) \cdot \nabla_H \eta \right] dx dy dz \\ & \leq (1 - \varepsilon) \|\partial_z T\|_2 + \|T\|_{\infty} \|w\|_2 \|\eta\|_2 + C(\|T\|_{\infty} \|v\|_2 + \varepsilon \|\nabla_H T\|_2) \|\nabla_H \eta\|_2 \\ & \leq C(\|\partial_z T\|_2 + \|\nabla_H v\|_2) \|\eta\|_2 + C(1 + \varepsilon \|\nabla_H T\|_2) \|\nabla_H \eta\|_2 \\ & \leq \frac{1}{8} \|\nabla_H \eta\|_2^2 + C(\|\partial_z T\|_2^2 + \|\nabla_H v\|_2^2 + \varepsilon \|\nabla_H T\|_2^2 + \|\eta\|_2^2 + 1) \\ & \leq \frac{1}{8} \|\nabla_H \eta\|_2^2 + C(\|\partial_z T\|_2^2 + \varepsilon \|\nabla_H T\|_2^2 + \|\eta\|_2^2 + \|\theta\|_2^2 + 1). \end{aligned}$$

Substituting the above two inequalities into (3.15) yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\eta\|_2^2 + \|(\nabla_H \eta, \sqrt{\varepsilon} \partial_z \eta)\|_2^2 & \leq C(\|\nabla_H v\|_2^2 + \|v\|_{\infty}^2 + 1)(\|(\eta, \theta)\|_2^2 + \|u\|_4^4 + 1) \\ & \quad + C(\|\partial_z T\|_2^2 + \varepsilon \|\nabla_H T\|_2^2) + \frac{1}{4} \|\nabla_H(\eta, \theta)\|_2^2, \end{aligned}$$

which, summed with (3.10) and (3.14), yields the conclusion. \square

Thanks to Proposition 3.4 and using the logarithmic type Gronwall inequality, i.e. Lemma 2.5, we can obtain the a priori $L_t^{\infty}(L_{\mathbf{x}}^2)$ estimate on (u, η, θ) . In fact, we have the following corollary:

Corollary 3.1. *Given $\mathcal{T} \in (0, \infty)$ and let $\varepsilon \in (0, 1)$. Let η, θ , and u be as in (3.4). The following a priori estimate holds:*

$$\begin{aligned} \sup_{0 \leq t \leq \mathcal{T}} (\|(\eta, \theta, u)\|_2^2(t) + \|u\|_4^4(t)) + \int_0^{\mathcal{T}} (\|\nabla_H(\eta, \theta, u)\|_2^2 + \|u\|\|\nabla_H u\|_2^2 \\ + \|\sqrt{\varepsilon}\partial_z(\eta, \theta, u)\|_2^2) dt \leq C \end{aligned}$$

for a positive constant C depending only on $h, \mathcal{T}, \|(v_0, T_0)\|_\infty$, and $\|\nabla_H v_0\|_2 + \|\partial_z v_0\|_4$; in particular, C is independent of $\varepsilon \in (0, 1)$.

Proof. Denoting

$$A_2 = \|\theta\|_2^2 + \|\eta\|_2^2 + \|u\|_2^2 + \frac{\|u\|_4^4}{2} + e, \quad (3.16)$$

$$B_2 = \|\nabla_H(\theta, \eta, u)\|_2^2 + \|\sqrt{\varepsilon}\partial_z(\eta, \theta, u)\|_2^2, \quad (3.17)$$

one obtains

$$\frac{d}{dt} A_2 + B_2 \leq C(\|v\|_\infty^2 + \|\nabla_H v\|_2^2 + 1)A_2 + C(\|\partial_z T\|_2^2 + \varepsilon\|\nabla_H T\|_2^2) \quad (3.18)$$

for $t \in (0, \mathcal{T})$ and for a positive constant C depending only on h, \mathcal{T} , and $\|T_0\|_\infty$.

By (iii) of Proposition 3.2 and applying Lemma 2.4, we have

$$\begin{aligned} \|v\|_\infty &\leq C \max \left\{ 1, \sup_{q \geq 2} \frac{\|v\|_q}{\sqrt{q}} \right\} \log^{\frac{1}{2}}(\|v\|_{W^{1,4}(\Omega)} + e) \\ &\leq C \log^{\frac{1}{2}}(\|v\|_{W^{1,4}(\Omega)} + e). \end{aligned} \quad (3.19)$$

Recalling the definitions of η and θ and using the elliptic estimate, it follows from the Sobolev embedding inequality that

$$\begin{aligned} \|\nabla_H v\|_4 &\leq C(\|\nabla_H \cdot v\|_4 + \|\nabla_H^\perp \cdot v\|_4) \leq C(\|\eta\|_4 + \|\theta\|_4 + \|\Phi\|_4) \\ &\leq C(\|\eta\|_4 + \|\theta\|_4 + 1) \leq C(\|\eta\|_{H^1(\Omega)} + \|\theta\|_{H^1(\Omega)} + e) \\ &\leq C(\|\eta\|_2 + \|\theta\|_2 + \|\nabla_H \eta\|_2 + \|\nabla_H \theta\|_2 + \|\partial_z \eta\|_2 + \|\partial_z \theta\|_2 + 1) \\ &\leq C(\|\eta\|_2 + \|\theta\|_2 + \|\nabla_H \eta\|_2 + \|\nabla_H \theta\|_2 + \|\nabla_H \cdot u + T\|_2 + \|\nabla_H^\perp \cdot u\|_2 + 1) \\ &\leq C(\|\eta\|_2 + \|\theta\|_2 + \|\nabla_H \eta\|_2 + \|\nabla_H \theta\|_2 + \|\nabla_H u\|_2 + 1). \end{aligned}$$

Therefore, by Propositions 3.1 and 3.3, one has

$$\begin{aligned} \|v\|_{W^{1,4}(\Omega)} &\leq C(\|v\|_4 + \|\partial_z v\|_4 + \|\nabla_H v\|_4) \leq C(\|v\|_{H^1(\Omega)} + \|\partial_z v\|_4 + \|\nabla_H v\|_4) \\ &\leq C(\|v\|_2 + \|\nabla_H v\|_2 + \|\partial_z v\|_2 + \|\partial_z v\|_4 + \|\nabla_H v\|_4) \\ &\leq C(\|v\|_2 + \|\eta\|_2 + \|\theta\|_2 + 1 + \|u\|_2 + \|u\|_4 + \|\nabla_H v\|_4) \\ &\leq C(1 + \|\eta\|_2 + \|\theta\|_2 + \|u\|_2 + \|u\|_4 + \|\nabla_H \eta\|_2 + \|\nabla_H \theta\|_2 + \|\nabla_H u\|_2) \\ &\leq C(A_2 + B_2). \end{aligned}$$

With the aid of the above inequality, it follows from (3.19) that

$$\|v\|_\infty^2 \leq C \log(A_2 + B_2) \quad (3.20)$$

and, consequently, by (3.18), we obtain

$$\frac{d}{dt}A_2 + B_2 \leq C(\|\nabla_H v\|_2^2 + 1 + \log(A_2 + B_2))A_2 + C(\|\partial_z T\|_2^2 + \varepsilon\|\nabla_H T\|_2^2).$$

Applying Lemma 2.5 to the above inequality and using Proposition 3.2, one obtains the conclusion. \square

3.2. A priori $L_t^\infty(H_x^1)$ estimate on $u = \partial_z v$. In this subsection, we perform the a priori $L_t^\infty(H_x^1)$ estimate on u . As it will be shown below, the a priori $L_t^\infty(L_x^2)$ estimate on $\partial_z u$ can be achieved through performing the energy estimates for u directly, while the desired estimate on $\nabla_H u$ is done by carrying out the corresponding estimates for (φ, ψ) defined, below, in (3.24). We first carry out the $L_t^\infty(L_x^2)$ estimate for $\partial_z u$.

Proposition 3.5. *Given $\mathcal{T} \in (0, \infty)$ and assume $\varepsilon \in (0, 1)$. Then, the following a priori estimate holds:*

$$\sup_{0 \leq t \leq \mathcal{T}} \|\partial_z u\|_2^2(t) + \int_0^{\mathcal{T}} \|(\nabla_H \partial_z u, \sqrt{\varepsilon} \partial_z^2 u)\|_2^2 dt \leq C$$

for a positive constant C depending only on $h, \mathcal{T}, \|(v_0, T_0)\|_\infty, \|\nabla_H v_0\|_2 + \|\partial_z v_0\|_4 + \|\partial_z^2 v_0\|_2$; in particular, C is independent of $\varepsilon \in (0, 1)$.

Proof. Multiplying equation (3.6) by $-\partial_z^2 u$ and integrating the resultant over Ω , it follows from integration by parts and the Hölder inequality that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_z u\|_2^2 + \|\nabla_H \partial_z u\|_2^2 + \varepsilon \|\partial_z^2 u\|_2^2 \\ &= \int_\Omega [(v \cdot \nabla_H)u + w \partial_z u + (u \cdot \nabla_H)v - (\nabla_H \cdot v)u - \nabla_H T] \cdot \partial_z^2 u dx dy dz \\ &= - \int_\Omega [2(u \cdot \nabla_H)u - 2(\nabla_H \cdot v) \partial_z u + \partial_z u \cdot \nabla_H v - (\nabla_H \cdot u)u] \cdot \partial_z u dx dy dz \\ &\quad - \int_\Omega \partial_z T \nabla_H \cdot \partial_z u dx dy dz \\ &\leq 3 \int_\Omega (|u| |\nabla_H u| |\partial_z u| + |\nabla_H v| |\partial_z u|^2) dx dy dz + \|\partial_z T\|_2^2 + \frac{1}{4} \|\nabla_H \partial_z u\|_2^2. \end{aligned} \quad (3.21)$$

We need to estimate the terms $\int_\Omega |u| |\nabla_H u| |\partial_z u| dx dy dz$ and $\int_\Omega |\nabla_H v| |\partial_z u|^2 dx dy dz$. Noticing that

$$|\nabla_H u(x, y, z, t)| \leq \int_{-h}^h |\nabla_H \partial_z u(x, y, z, t)| dz,$$

it follows from Lemmas 2.2 and 2.3 and the Hölder and Young inequalities that

$$3 \int_\Omega |u| |\nabla_H u| |\partial_z u| dx dy dz$$

$$\begin{aligned}
&\leq C \int_M \left(\int_{-h}^h |\nabla_H \partial_z u| dz \right) \left(\int_{-h}^h |u| |\partial_z u| dz \right) dx dy \\
&\leq C \left(\int_{-h}^h \|\nabla_H \partial_z u\|_{2,M} dz \right) \left(\int_{-h}^h \|u\|_{4,M}^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h \|\partial_z u\|_{4,M}^2 dz \right)^{\frac{1}{2}} \\
&\leq C \|\nabla_H \partial_z u\|_2 \|u\|_4 \left(\|\partial_z u\|_2 + \|\partial_z u\|_2^{\frac{1}{2}} \|\nabla_H \partial_z u\|_2^{\frac{1}{2}} \right) \\
&\leq \frac{1}{8} \|\nabla_H \partial_z u\|_2^2 + C(\|u\|_4^4 + 1) \|\partial_z u\|_2^2.
\end{aligned} \tag{3.22}$$

Noticing that

$$|\nabla_H v(x, y, z, t)| \leq \frac{1}{2h} \int_{-h}^h |\nabla_H v| dz + \int_{-h}^h |\nabla_H u| dz,$$

it follows from Lemmas 2.2 and 2.3 and the Young inequality that

$$\begin{aligned}
&3 \int_{\Omega} |\nabla_H v| |\partial_z u|^2 dx dy dz \\
&\leq 3 \int_M \left(\int_{-h}^h (|\nabla_H v| + |\nabla_H u|) dz \right) \left(\int_{-h}^h |\partial_z u|^2 dz \right) dx dy \\
&\leq 3(\|\nabla_H v\|_2 + \|\nabla_H u\|_2) \left(\int_{-h}^h \|\partial_z u\|_{4,M}^2 dz \right) \\
&\leq C(\|\nabla_H v\|_2 + \|\nabla_H u\|_2) (\|\partial_z u\|_2^2 + \|\partial_z u\|_2 \|\nabla_H \partial_z u\|_2) \\
&\leq \frac{1}{8} \|\nabla_H \partial_z u\|_2^2 + C(\|\nabla_H v\|_2^2 + \|\nabla_H u\|_2^2 + 1) \|\partial_z u\|_2^2.
\end{aligned} \tag{3.23}$$

Substituting (3.22) and (3.23) into (3.21), one obtains

$$\begin{aligned}
&\frac{d}{dt} \|\partial_z u\|_2^2 + \|\nabla_H \partial_z u\|_2^2 + \varepsilon \|\partial_z^2 u\|_2^2 \\
&\leq C(\|\nabla_H v\|_2^2 + \|\nabla_H u\|_2^2 + \|u\|_4^4 + 1) \|\partial_z u\|_2^2 + C \|\partial_z T\|_2^2.
\end{aligned}$$

Applying the Gronwall inequality to the above inequality and using Proposition 3.1 and Corollary 3.1, the conclusion follows. \square

Before proceeding to obtain estimate on $\nabla_H u$, we define

$$\varphi := \nabla_H \cdot u + T, \quad \psi := \nabla_H^\perp \cdot u. \tag{3.24}$$

Equations satisfied by (φ, ψ) are derived as follows. Applying the horizontal divergence operator div_H , or $\nabla_H \cdot$, to equation (3.6) and noticing that

$$\begin{aligned}
\nabla_H \cdot ((v \cdot \nabla_H)u) &= v \cdot \nabla_H (\nabla_H \cdot u) + \nabla_H v : (\nabla_H u)^T, \\
\nabla_H \cdot (w \partial_z u) &= w \partial_z (\nabla_H \cdot u) + \nabla_H w \cdot \partial_z u, \\
\nabla_H \cdot (\vec{k} \times u) &= \nabla_H \cdot u^\perp = -\nabla_H^\perp \cdot u = -\psi,
\end{aligned}$$

one has

$$\begin{aligned} & \partial_t(\nabla_H \cdot u) + v \cdot \nabla_H(\nabla_H \cdot u) + w\partial_z(\nabla_H \cdot u) - \Delta_H(\nabla_H \cdot u + T) - \varepsilon\partial_z^2\nabla_H \cdot u \\ & = f_0\psi - \nabla_H \cdot ((u \cdot \nabla_H)v - (\nabla_H \cdot v)u) - \nabla_H : (\nabla_H u)^T - \nabla_H w \cdot \partial_z u. \end{aligned}$$

Adding the above equation with (3.3) yields

$$\begin{aligned} & \partial_t\varphi + v \cdot \nabla_H\varphi + w\partial_z\varphi - \Delta_H\varphi - \varepsilon\partial_z^2\varphi \\ & = f_0\psi - \nabla_H \cdot ((u \cdot \nabla_H)v - (\nabla_H \cdot v)u) + \varepsilon\Delta_H T + (1 - \varepsilon)\partial_z^2 T \\ & \quad - \nabla_H v : (\nabla_H u)^T - \nabla_H w \cdot \partial_z u. \end{aligned} \tag{3.25}$$

Applying the operator $\nabla_H^\perp \cdot$ to equation (3.6) and noticing that

$$\begin{aligned} \nabla_H^\perp \cdot ((v \cdot \nabla_H)u) & = v \cdot \nabla_H(\nabla_H^\perp \cdot u) + \nabla_H^\perp v : (\nabla_H u)^T \\ & = v \cdot \nabla_H\psi + \nabla_H^\perp v : (\nabla_H u)^T, \\ \nabla_H^\perp \cdot (w\partial_z u) & = w\partial_z(\nabla_H^\perp \cdot u) + \nabla_H^\perp w \cdot \partial_z u = w\partial_z\psi + \nabla_H^\perp w \cdot \partial_z u, \\ \nabla_H^\perp \cdot (\vec{k} \times u) & = \nabla_H^\perp \cdot u^\perp = \nabla_H \cdot u, \end{aligned}$$

where $u^\perp = (-u^2, u^1)$, one obtains

$$\begin{aligned} & \partial_t\psi + v \cdot \nabla_H\psi + w\partial_z\psi - \Delta_H\psi - \varepsilon\partial_z^2\psi \\ & = -f_0\nabla_H \cdot u - \nabla_H^\perp \cdot ((u \cdot \nabla_H)v - (\nabla_H \cdot v)u) \\ & \quad - \nabla_H^\perp v : (\nabla_H u)^T - \nabla_H^\perp w \cdot \partial_z u. \end{aligned} \tag{3.26}$$

A priori $L_t^\infty(L_x^2)$ estimate on (φ, ψ) is stated in the next proposition.

Proposition 3.6. *Given $\mathcal{T} \in (0, \infty)$ and assume that $\varepsilon \in (0, 1)$. Let φ and ψ be given in (3.24). Then, the following a priori estimate holds:*

$$\sup_{0 \leq t \leq \mathcal{T}} \|(\varphi, \psi)\|_2^2(t) + \int_0^{\mathcal{T}} \|(\nabla_H\varphi, \nabla_H\psi, \sqrt{\varepsilon}\partial_z\varphi, \sqrt{\varepsilon}\partial_z\psi)\|_2^2 dt \leq C$$

for a positive constant C depending only on $h, \mathcal{T}, \|(v_0, T_0)\|_\infty, \|\nabla_H v_0\|_2 + \|\partial_z v_0\|_{H^1}$; in particular, C is independent of $\varepsilon \in (0, 1)$.

Proof. Multiplying equation (3.25) by φ and integrating the resulting equation over Ω , it follows from integration by parts that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\varphi\|_2^2 + \|\nabla_H\varphi\|_2^2 + \varepsilon\|\partial_z\varphi\|_2^2 \\ & = \int_\Omega [(f_0\psi + \varepsilon\Delta_H T + (1 - \varepsilon)\partial_z^2 T)\varphi + ((u \cdot \nabla_H)v - \nabla_H \cdot vu) \cdot \nabla_H\varphi] dx dy dz \\ & \quad - \int_\Omega [\nabla_H v : (\nabla_H u)^T + \nabla_H w \cdot \partial_z u] \varphi dx dy dz. \end{aligned} \tag{3.27}$$

Noticing that $\|\partial_z \varphi\|_2 \leq \|\nabla_H \partial_z u\|_2 + \|\partial_z T\|_2$ and $\|(\varphi, \psi)\|_2^2 \leq C(\|\nabla_H u\|_2^2 + 1)$, it follows from integrating by parts and the Hölder and Young inequalities that

$$\begin{aligned} & \int_{\Omega} [f_0 \psi + \varepsilon \Delta_H T + (1 - \varepsilon) \partial_z^2 T] \varphi dx dy dz \\ & \leq C[(\|\nabla_H u\|_2^2 + 1) + \varepsilon \|\nabla_H T\|_2 \|\nabla_H \varphi\|_2 + \|\partial_z T\|_2 (\|\nabla_H \partial_z u\|_2 + \|\partial_z T\|_2)] \\ & \leq \frac{1}{8} \|\nabla_H \varphi\|_2^2 + C(\varepsilon \|\nabla_H T\|_2^2 + \|\nabla_H \partial_z u\|_2^2 + \|\partial_z T\|_2^2 + \|\nabla_H u\|_2^2 + 1). \end{aligned} \quad (3.28)$$

Noticing that $|u(x, y, z, t)| \leq \int_{-h}^h |\partial_z u(x, y, z, t)| dz$, by Lemmas 2.2 and 2.3, Proposition 3.3, and using the Young inequality, we have

$$\begin{aligned} & \int_{\Omega} [(u \cdot \nabla_H) v - \nabla_H \cdot vu] \cdot \nabla_H \varphi dx dy dz \\ & \leq 2 \int_M \left(\int_{-h}^h |\partial_z u| dz \right) \left(\int_{-h}^h |\nabla_H v| |\nabla_H \varphi| dz \right) dx dy \\ & \leq 2 \left(\int_{-h}^h \|\partial_z u\|_{4,M} dz \right) \left(\int_{-h}^h \|\nabla_H v\|_{4,M}^2 dz \right)^{\frac{1}{2}} \|\nabla_H \varphi\|_2 \\ & \leq C \left(\|\partial_z u\|_2^{\frac{1}{2}} \|\nabla_H \partial_z u\|_2^{\frac{1}{2}} + \|\partial_z u\|_2 \right) \left(\|\eta\|_2 + \|\theta\|_2 \right. \\ & \quad \left. + (\|\eta\|_2 + \|\theta\|_2)^{\frac{1}{2}} (\|\nabla_H \eta\|_2 + \|\nabla_H \theta\|_2)^{\frac{1}{2}} + 1 \right) \|\nabla_H \varphi\|_2 \\ & \leq \frac{1}{8} \|\nabla_H \varphi\|_2^2 + C[(\|\partial_z u\|_2^2 + \|\eta\|_2^2 + \|\theta\|_2^2) (\|\nabla_H \partial_z u\|_2^2 \\ & \quad + \|\nabla_H \eta\|_2^2 + \|\nabla_H \theta\|_2^2) + \|(\partial_z u, \eta, \theta)\|_2^4 + 1] \end{aligned} \quad (3.29)$$

and

$$\begin{aligned} & - \int_{\Omega} \nabla_H v : (\nabla_H u)^T \varphi dx dy dz \\ & \leq \int_M \left(\int_{-h}^h |\nabla_H \partial_z u| dz \right) \left(\int_{-h}^h |\nabla_H v| |\varphi| dz \right) dx dy \\ & \leq \left(\int_{-h}^h \|\nabla_H \partial_z u\|_{2,M} dz \right) \left(\int_{-h}^h \|\nabla_H v\|_{4,M}^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h \|\varphi\|_{4,M}^2 dz \right)^{\frac{1}{2}} \\ & \leq C \|\nabla_H \partial_z u\|_2 \left(\|(\eta, \theta)\|_2^{\frac{1}{2}} \|\nabla_H(\eta, \theta)\|_2^{\frac{1}{2}} + \|(\eta, \theta)\|_2 + 1 \right) \left(\|\varphi\|_2^{\frac{1}{2}} \|\nabla_H \varphi\|_2^{\frac{1}{2}} + \|\varphi\|_2 \right) \\ & \leq \frac{1}{8} \|\nabla_H \varphi\|_2^2 + C[(\|(\eta, \theta)\|_2^2 \|\nabla_H(\eta, \theta)\|_2^2 + \|(\eta, \theta)\|_2^4 + 1) \|\varphi\|_2^2 + \|\nabla_H \partial_z u\|_2^2] \end{aligned} \quad (3.30)$$

Applying Lemmas 2.2 and 2.3 and Proposition 3.3, it follows from integrating by parts and the Young inequalities that

$$\begin{aligned}
& - \int_{\Omega} \nabla_H w \cdot \partial_z u \varphi dx dy dz \\
&= \int_{\Omega} w (\nabla_H \cdot \partial_z u \varphi + \partial_z u \cdot \nabla_H \varphi) dx dy dz \\
&\leq \int_M \left(\int_{-h}^h |\nabla_H v| dz \right) \left(\int_{-h}^h (|\nabla_H \partial_z u| |\varphi| + |\partial_z u| |\nabla_H \varphi|) dz \right) dx dy \\
&\leq \left(\int_{-h}^h \|\nabla_H v\|_{4,M} dz \right) \left(\int_{-h}^h \|\varphi\|_{4,M}^2 dz \right)^{\frac{1}{2}} \|\nabla_H \partial_z u\|_2 \\
&\quad + \left(\int_{-h}^h \|\nabla_H v\|_{4,M} dz \right) \left(\int_{-h}^h \|\partial_z u\|_{4,M}^2 dz \right)^{\frac{1}{2}} \|\nabla_H \varphi\|_2 \\
&\leq C \left(\|(\eta, \theta)\|_2^{\frac{1}{2}} \|\nabla_H(\eta, \theta)\|_2^{\frac{1}{2}} + \|(\eta, \theta)\|_2 + 1 \right) \left[\left(\|\varphi\|_2^{\frac{1}{2}} \|\nabla_H \varphi\|_2^{\frac{1}{2}} + \|\varphi\|_2 \right) \right. \\
&\quad \left. \times \|\nabla_H \partial_z u\|_2 + \left(\|\partial_z u\|_2^{\frac{1}{2}} \|\nabla_H \partial_z u\|_2^{\frac{1}{2}} + \|\partial_z u\|_2 \right) \|\nabla_H \varphi\|_2 \right] \\
&\leq C \left(\|(\partial_z u, \eta, \theta)\|_2^2 \|\nabla_H(\partial_z u, \eta, \theta)\|_2^2 + \|(\partial_z u, \eta, \theta)\|_2^4 + 1 \right) \left(\|\varphi\|_2^2 + 1 \right) \\
&\quad + \frac{1}{8} \|\nabla_H \varphi\|_2^2 + C \|\nabla_H \partial_z u\|_2^2. \tag{3.31}
\end{aligned}$$

Substituting (3.28)–(3.31) into (3.27) yields

$$\begin{aligned}
& \frac{d}{dt} \|\varphi\|_2^2 + \|\nabla_H \varphi\|_2^2 + \varepsilon \|\partial_z \varphi\|_2^2 \\
&\leq C \left(\|(\partial_z u, \eta, \theta)\|_2^2 \|\nabla_H(\partial_z u, \eta, \theta)\|_2^2 + \|(\partial_z u, \eta, \theta)\|_2^4 + 1 \right) \left(\|\varphi\|_2^2 + 1 \right) \\
&\quad + C \left(\|(\nabla_H u, \nabla_H \partial_z u, \partial_z T, \sqrt{\varepsilon} \nabla_H T)\|_2^2 + 1 \right). \tag{3.32}
\end{aligned}$$

Multiplying equation (3.26) by ψ and integrating the resultant over Ω , it follows from integration by parts that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\psi\|_2^2 + \|\nabla_H \psi\|_2^2 + \varepsilon \|\partial_z \psi\|_2^2 \\
&= \int_{\Omega} [-f_0 \nabla_H \cdot u \psi + (u \cdot \nabla_H v - \nabla_H \cdot v u) \cdot \nabla_H^{\perp} \psi] dx dy dz \\
&\quad - \int_{\Omega} (\nabla_H^{\perp} : (\nabla_H u)^T + \nabla_H^{\perp} w \cdot \partial_z u) \psi dx dy dz. \tag{3.33}
\end{aligned}$$

The same arguments as for (3.29)–(3.31) yield the estimates

$$\int_{\Omega} (u \cdot \nabla_H v - \nabla_H \cdot v u) \cdot \nabla_H^{\perp} \psi dx dy dz$$

$$\begin{aligned}
&\leq \frac{1}{6} \|\nabla_H \psi\|_2^2 + C(\|(\partial_z u, \eta, \theta)\|_2^2 \|\nabla_H(\partial_z u, \eta, \theta)\|_2^2 + \|(\partial_z u, \eta, \theta)\|_2^4 + 1), \\
- \int_{\Omega} \nabla_H^\perp : (\nabla_H u)^T \psi dx dy dz &\leq C(\|(\eta, \theta)\|_2^2 \|\nabla_H(\eta, \theta)\|_2^2 + \|(\eta, \theta)\|_2^4 + 1) \|\psi\|_2^2 \\
&\quad + \frac{1}{6} \|\nabla_H \psi\|_2^2 + C \|\nabla_H \partial_z u\|_2^2,
\end{aligned}$$

and

$$\begin{aligned}
&- \int_{\Omega} \nabla_H^\perp w \cdot \partial_z u \psi dx dy dz \\
&\leq C(\|(\partial_z u, \eta, \theta)\|_2^2 \|\nabla_H(\partial_z u, \eta, \theta)\|_2^2 + \|(\partial_z u, \eta, \theta)\|_2^4 + 1) (\|\psi\|_2^2 + 1) \\
&\quad + \frac{1}{6} \|\nabla_H \psi\|_2^2 + C \|\nabla_H \partial_z u\|_2^2
\end{aligned}$$

Thanks to the above estimates, we obtain from (3.33) that

$$\begin{aligned}
&\frac{d}{dt} \|\psi\|_2^2 + \|\nabla_H \psi\|_2^2 + \varepsilon \|\partial_z \psi\|_2^2 \\
&\leq C(\|(\partial_z u, \eta, \theta)\|_2^2 \|\nabla_H(\partial_z u, \eta, \theta)\|_2^2 + \|(\partial_z u, \eta, \theta)\|_2^4 + 1) (\|\psi\|_2^2 + 1) \\
&\quad + C(\|(\nabla_H u, \nabla_H \partial_z u)\|_2^2 + 1).
\end{aligned}$$

Summing the above inequality with (3.32) yields

$$\begin{aligned}
&\frac{d}{dt} \|(\varphi, \psi)\|_2^2 + \|(\nabla_H \varphi, \nabla_H \psi, \sqrt{\varepsilon} \partial_z \varphi, \sqrt{\varepsilon} \partial_z \psi)\|_2^2 \\
&\leq C(\|(\partial_z u, \eta, \theta)\|_2^2 \|\nabla_H(\partial_z u, \eta, \theta)\|_2^2 + \|(\partial_z u, \eta, \theta)\|_2^4 + 1) (\|(\varphi, \psi)\|_2^2 + 1) \\
&\quad + C(\|(\nabla_H u, \nabla_H \partial_z u, \partial_z T, \sqrt{\varepsilon} \nabla_H T)\|_2^2 + 1),
\end{aligned}$$

from which, by the Gronwall inequality, and using Proposition 3.1, Corollary 3.1, and Proposition 3.5, one obtains the conclusion. \square

3.3. Energy inequalities for $(\nabla_H \eta, \nabla_H \theta)$. In this subsection, we are concerned with deriving energy inequalities for $(\nabla_H \eta, \nabla_H \theta)$, where η and θ are given in (3.4). It should be noticed that the energy inequalities for $(\nabla_H \eta, \nabla_H \theta)$ do not yield the a priori estimates of themselves, without appealing to the energy inequalities for ∇T .

As a preparation, we prove the following:

Proposition 3.7. *Let η, θ , and u be given in (3.4), and φ and ψ as in (3.24). The following estimates hold:*

$$\begin{aligned}
\left(\int_{-h}^h \|\nabla_H u\|_{4,M}^2 dz \right)^{\frac{1}{2}} &\leq C \left(\|(\varphi, \psi)\|_2^{\frac{1}{2}} \|\nabla_H(\varphi, \psi)\|_2^{\frac{1}{2}} + \|(\varphi, \psi)\|_2 + 1 \right), \\
\int_{-h}^h \|\nabla_H u\|_{4,M} dz &\leq C \left(\|(\varphi, \psi)\|_2^{\frac{1}{2}} \|\nabla_H(\varphi, \psi)\|_2^{\frac{1}{2}} + \|(\varphi, \psi)\|_2 + 1 \right),
\end{aligned}$$

$$\left(\int_{-h}^h \|u\|_{\infty, M}^2 dz \right)^{\frac{1}{2}} \leq C \left(\|u\|_4 + \|(\varphi, \psi)\|_2^{\frac{1}{2}} \|\nabla_H(\varphi, \psi)\|_2^{\frac{1}{2}} + \|(\varphi, \psi)\|_2 + 1 \right)$$

for a positive constant C depending only on $\|T_0\|_\infty$ and h , in particular it is independent of $\varepsilon \in (0, 1)$.

Proof. By Proposition 3.2, it follows from the elliptic estimate that for any $z \in (-h, h)$

$$\begin{aligned} \|\nabla_H u(\cdot, z)\|_{4, M} &\leq C(\|\nabla_H \cdot u(\cdot, z)\|_{4, M} + \|\nabla_H^\perp \cdot u(\cdot, z)\|_{4, M}) \\ &\leq C(\|\varphi(\cdot, z)\|_{4, M} + \|T(\cdot, z)\|_{4, M} + \|\psi(\cdot, z)\|_{4, M}) \\ &\leq C(\|\varphi(\cdot, z)\|_{4, M} + \|\psi(\cdot, z)\|_{4, M} + 1) \end{aligned}$$

for a positive constant C depending only on $\|T_0\|_\infty$. Thanks to the above, it follows from Lemma 2.3 that

$$\begin{aligned} \left(\int_{-h}^h \|\nabla_H u\|_{4, M}^2 dz \right)^{\frac{1}{2}} &\leq C \left[\left(\int_{-h}^h \|(\varphi, \psi)\|_{4, M}^2 dz \right)^{\frac{1}{2}} + 1 \right] \\ &\leq C \left(\|(\varphi, \psi)\|_2^{\frac{1}{2}} \|\nabla_H(\varphi, \psi)\|_2^{\frac{1}{2}} + \|(\varphi, \psi)\|_2 + 1 \right), \end{aligned}$$

proving the first inequality, while the second one follows from the first one by applying the Hölder inequality. For the third inequality, by the Sobolev embedding inequality and the Hölder inequality, and using the first conclusion, we have

$$\begin{aligned} \left(\int_{-h}^h \|u\|_{\infty, M}^2 dz \right)^{\frac{1}{2}} &\leq C \left(\int_{-h}^h (\|u\|_{4, M}^2 + \|\nabla_H u\|_{4, M}^2) dz \right)^{\frac{1}{2}} \\ &\leq C \left(\|u\|_4 + \|(\varphi, \psi)\|_2^{\frac{1}{2}} \|\nabla_H(\varphi, \psi)\|_2^{\frac{1}{2}} + \|(\varphi, \psi)\|_2 + 1 \right), \end{aligned}$$

proving the third inequality. \square

We have the following proposition about the energy inequality for $(\nabla_H \eta, \nabla_H \theta)$:

Proposition 3.8. *We have the following estimate*

$$\begin{aligned} &\frac{d}{dt} \|\nabla_H(\eta, \theta)\|_2^2 + \|(\Delta_H \eta, \Delta_H \theta, \sqrt{\varepsilon} \nabla_H \partial_z \eta, \sqrt{\varepsilon} \nabla_H \partial_z \theta)\|_2^2 \\ &\leq C(\|v\|_\infty^2 + \|u\|_4^2 + \|(\varphi, \psi)\|_2 \|\nabla_H(\varphi, \psi)\|_2 + \|(\varphi, \psi)\|_2^2 + 1) \\ &\quad \times \|\nabla_H(\eta, \theta, T)\|_2^2 + C(\|(\eta, \theta, \varphi, \psi)\|_2^2 + \|u\|_4^4 + 1) \\ &\quad \times (\|\nabla_H(v, \eta, \theta, \varphi, \psi)\|_2^2 + \|\partial_z T\|_2^2 + 1) + C\varepsilon^2 \|\Delta_H T\|_2^2, \end{aligned}$$

where C is a positive constant depending only on h and $\|T_0\|_\infty$; in particular, C is independent of $\varepsilon \in (0, 1)$.

Proof. Multiplying equation (4.4) by $-\Delta_H \theta$ and integrating the resultant over Ω , it follows from integration by parts that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla_H \theta\|_2^2 + \|\Delta_H \theta\|_2 + \varepsilon \|\nabla_H \partial_z \theta\|_2^2 \\
&= \int_{\Omega} \nabla_H^{\perp} \cdot (v \cdot \nabla_H v + w \partial_z v + f_0 k \times v) \Delta_H \theta dx dy dz \\
&\leq \frac{1}{16} \|\Delta_H \theta\|_2^2 + f_0 \|\nabla_H v\|_2^2 + \int_{\Omega} (|v| |\nabla_H^2 v| + |\nabla_H v|^2 \\
&\quad + |w| |\nabla_H u| + |\nabla_H w| |u|) |\Delta_H \theta| dx dy dz.
\end{aligned} \tag{3.34}$$

We estimate the terms in (3.34) as follows. First, for $\int_{\Omega} |v| |\nabla_H^2 v| |\Delta_H \theta| dx dy dz$, by using (3.4) and the Young inequality, we have

$$\begin{aligned}
& \int_{\Omega} |v| |\nabla_H^2 v| |\Delta_H \theta| dx dy dz \leq \|v\|_{\infty} \|\Delta_H v\|_2 \|\Delta_H \theta\|_2 \\
&= \|v\|_{\infty} \|\nabla_H(\nabla_H \cdot v) - \nabla_H^{\perp}(\nabla_H^{\perp} \cdot v)\|_2 \|\Delta_H \theta\|_2 \\
&= \|v\|_{\infty} \|\nabla_H(\eta + \Phi) - \nabla_H^{\perp} \theta\|_2 \|\Delta_H \theta\|_2 \\
&\leq C \|v\|_{\infty} \|\nabla_H(\eta, \theta, T)\|_2 \|\Delta_H \theta\|_2 \\
&\leq \frac{1}{16} \|\Delta_H \theta\|_2^2 + C \|v\|_{\infty}^2 \|\nabla_H(\eta, \theta, T)\|_2^2.
\end{aligned} \tag{3.35}$$

Then, for the term $\int_{\Omega} |\nabla_H v|^2 |\Delta_H \theta| dx dy dz$, recalling that

$$|\nabla_H v(x, y, z, t)| \leq \frac{1}{2h} \int_{-h}^h |\nabla_H v| dz + \int_{-h}^h |\nabla_H u| dz,$$

we deduce by Lemma 2.2, Propositions 3.3 and 3.7, and the Young inequality that

$$\begin{aligned}
& \int_{\Omega} (|\nabla_H v|^2 |\Delta_H \theta| + |w| |\nabla_H u| |\Delta_H \theta|) dx dy dz \\
&\leq C \int_M \left(\int_{-h}^h (|\nabla_H v| + |\nabla_H u|) dz \right) \left(\int_{-h}^h |\nabla_H v| |\Delta_H \theta| dz \right) dx dy \\
&\quad + \int_M \left(\int_{-h}^h |\nabla_H v| dz \right) \left(\int_{-h}^h |\nabla_H u| |\Delta_H \theta| dz \right) dx dy \\
&\leq C \left(\int_{-h}^h \|\nabla_H(v, u)\|_{4,M} dz \right) \left(\int_{-h}^h \|\nabla_H v\|_{4,M}^2 dz \right)^{\frac{1}{2}} \|\Delta_H \theta\|_2 \\
&\quad + \left(\int_{-h}^h \|\nabla_H v\|_{4,M} dz \right) \left(\int_{-h}^h \|\nabla_H u\|_{4,M}^2 dz \right)^{\frac{1}{2}} \|\Delta_H \theta\|_2 \\
&\leq C (\|(\eta, \theta, \varphi, \psi)\|_2 \|\nabla_H(\eta, \theta, \varphi, \psi)\|_2 + \|(\eta, \theta, \varphi, \psi)\|_2^2 + 1) \|\Delta_H \theta\|_2
\end{aligned}$$

$$\leq \frac{1}{16} \|\Delta_H \theta\|_2^2 + C(\|(\eta, \theta, \varphi, \psi)\|_2^2 + 1)(\|\nabla_H(\eta, \theta, \varphi, \psi)\|_2^2 + 1). \quad (3.36)$$

Finally, for the term $\int_{\Omega} |\nabla_H w| |u| |\Delta_H \theta| dx dy dz$, thanks to (1.17), (3.4)–(3.5), we have

$$\begin{aligned} & \int_{\Omega} |\nabla_H w| |u| |\Delta_H \theta| dx dy dz \\ & \leq C \int_M \left(\int_{-h}^h (|\nabla_H \eta| + |\nabla_H T|) dz \right) \left(\int_{-h}^h |u| |\Delta_H \theta| dz \right) dx dy. \end{aligned} \quad (3.37)$$

For the term $C \int_M \left(\int_{-h}^h (|\nabla_H \eta| + |\nabla_H T|) dz \right) \left(\int_{-h}^h |u| |\Delta_H \theta| dz \right) dx dy$, by Lemmas 2.2 and 2.3 and using the Hölder and Young inequalities, we have

$$\begin{aligned} & C \int_M \left(\int_{-h}^h (|\nabla_H \eta| + |\nabla_H T|) dz \right) \left(\int_{-h}^h |u| |\Delta_H \theta| dz \right) dx dy \\ & \leq C \left(\int_{-h}^h \|\nabla_H \eta\|_{4,M} dz \right) \left(\int_{-h}^h \|u\|_{4,M}^2 dz \right)^{\frac{1}{2}} \|\Delta_H \theta\|_2 \\ & \leq C \|\nabla_H \eta\|_2^{\frac{1}{2}} \|\nabla_H^2 \eta\|_2^{\frac{1}{2}} \|u\|_4 \|\Delta_H \theta\|_2 \leq \frac{1}{32} \|\Delta_H(\eta, \theta)\|_2^2 + C \|\nabla_H \eta\|_2^2 \|u\|_4^4. \end{aligned} \quad (3.38)$$

For the term $\int_M \left(\int_{-h}^h |\nabla_H T| dz \right) \left(\int_{-h}^h |u| |\Delta_H \theta| dz \right) dx dy$, we estimate it as follows. Using the Hölder and Young inequalities and applying Proposition 3.7, we deduce

$$\begin{aligned} & C \int_M \left(\int_{-h}^h |\nabla_H T| dz \right) \left(\int_{-h}^h |u| |\Delta_H \theta| dz \right) dx dy \\ & \leq C \int_M \left(\int_{-h}^h |\nabla_H T| dz \right) \left(\int_{-h}^h |u|^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h |\Delta_H \theta|^2 dz \right)^{\frac{1}{2}} dx dy \\ & \leq C \left[\int_M \left(\int_{-h}^h |\nabla_H T| dz \right)^2 dx dy \right]^{\frac{1}{2}} \left(\int_{-h}^h \|u\|_{\infty,M}^2 dz \right)^{\frac{1}{2}} \|\Delta_H \theta\|_2 \\ & \leq C \left(\int_{-h}^h \|u\|_{\infty,M}^2 dz \right)^{\frac{1}{2}} \|\Delta_H \theta\|_2 \|\nabla_H T\|_2 \\ & \leq C \left(\|u\|_4 + \|(\varphi, \psi)\|_2^{\frac{1}{2}} \|\nabla_H(\varphi, \psi)\|_2^{\frac{1}{2}} + \|(\varphi, \psi)\|_2 + 1 \right) \|\Delta_H \theta\|_2 \|\nabla_H T\|_2 \\ & \leq \frac{1}{32} \|\Delta_H \theta\|_2^2 + C[\|u\|_4^2 + \|(\varphi, \psi)\|_2 (\|\nabla_H(\varphi, \psi)\|_2 + 1) + 1] \|\nabla_H T\|_2^2. \end{aligned} \quad (3.39)$$

Thanks to (3.38) and (3.39), we obtain from (3.37) that

$$\begin{aligned} & \int_{\Omega} |\nabla_H w| |u| |\Delta_H \theta| dx dy dz \\ & \leq \frac{1}{16} \|\Delta_H(\eta, \theta)\|_2^2 + C(\|u\|_4^2 + \|(\varphi, \psi)\|_2 \|\nabla_H(\varphi, \psi)\|_2) \end{aligned}$$

$$+ \|(\varphi, \psi)\|_2^2 + 1) \|\nabla_H T\|_2^2 + C \|\nabla_H \eta\|_2^2 \|u\|_4^4. \quad (3.40)$$

Combining (3.35), (3.36) and (3.40), we obtain

$$\begin{aligned} & \int_{\Omega} (|v| |\nabla_H^2 v| + |\nabla_H v|^2 + |w| |\nabla_H u| + |\nabla_H w| |u|) |\Delta_H \theta| dx dy dz \\ & \leq C (\|v\|_{\infty}^2 + \|u\|_4^2 + \|(\varphi, \psi)\|_2 \|\nabla_H(\varphi, \psi)\|_2 + \|(\varphi, \psi)\|_2^2 + 1) \|\nabla_H(\eta, \theta, T)\|_2^2 \\ & \quad + C (\|(\eta, \theta, \varphi, \psi)\|_2^2 + \|u\|_4^4 + 1) (\|\nabla_H(\eta, \theta, \varphi, \psi)\|_2^2 + 1) + \frac{3}{16} \|\Delta_H(\eta, \theta)\|_2^2. \end{aligned} \quad (3.41)$$

Therefore, recalling that $\|\nabla_H v\|_2^2 \leq C(\|\eta\|_2^2 + \|\theta\|_2^2 + 1)$, guaranteed by Proposition 3.3, it follows from (3.34) that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla_H \theta\|_2^2 + \|\Delta_H \theta\|_2 + \varepsilon \|\nabla_H \partial_z \theta\|_2^2 \\ & \leq \frac{1}{4} \|\Delta_H(\eta, \theta)\|_2^2 + C (\|v\|_{\infty}^2 + \|u\|_4^2 + \|(\varphi, \psi)\|_2 \|\nabla_H(\varphi, \psi)\|_2 \\ & \quad + \|(\varphi, \psi)\|_2^2 + 1) \|\nabla_H(\eta, \theta, T)\|_2^2 + C (\|(\eta, \theta, \varphi, \psi)\|_2^2 \\ & \quad + \|u\|_4^4 + 1) (\|\nabla_H(v, \eta, \theta, \varphi, \psi)\|_2^2 + 1). \end{aligned} \quad (3.42)$$

Recall that $\int_{-h}^h \eta(x, y, z, t) dy = 0$, which implies

$$\int_{\Omega} f(x, y, t) \Delta_H \eta(x, y, z, t) dx dy dz = 0,$$

where $f(x, y, t)$ is given by (3.9). Multiplying equation (3.7) by $-\Delta_H \eta$ and integrating the resultant over Ω , it follows from integration by parts, Proposition 3.2, and the Hölder inequality that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla_H \eta\|_2^2 + \|\Delta_H \eta\|_2^2 + \varepsilon \|\nabla_H \partial_z \eta\|_2^2 \\ & = \int_{\Omega} \left\{ \nabla_H \cdot [(v \cdot \nabla_H)v + w \partial_z v + f_0 \vec{k} \times v] - (1 - \varepsilon) \partial_z T \right. \\ & \quad \left. + wT + \left(\int_{-h}^z (\nabla_H \cdot (vT) - \varepsilon \Delta_H T) d\xi \right) + f(x, y, t) \right\} \Delta_H \eta dx dy dz \\ & = \int_{\Omega} \left\{ \nabla_H \cdot [(v \cdot \nabla_H)v + w \partial_z v] + f_0 \nabla_H \cdot (\vec{k} \times v) - (1 - \varepsilon) \partial_z T + wT \right. \\ & \quad \left. + \left(\int_{-h}^z ((\nabla_H \cdot v)T + v \cdot \nabla_H T - \varepsilon \Delta_H T) d\xi \right) \right\} \Delta_H \eta dx dy dz \\ & \leq C (\|\nabla_H v\|_2 + \|\partial_z T\|_2 + \|v\|_{\infty} \|\nabla_H T\|_2 + \varepsilon \|\Delta_H T\|_2) \|\Delta_H \eta\|_2 \\ & \quad + \int_{\Omega} \nabla_H \cdot [(v \cdot \nabla_H)v + w \partial_z v] \Delta_H \eta dx dy dz. \end{aligned} \quad (3.43)$$

Same arguments as for (3.41) yield

$$\begin{aligned}
& \int_{\Omega} \nabla_H \cdot [(v \cdot \nabla_H)v + w\partial_z v] \Delta_H \eta dx dy dz \\
& \leq \int_{\Omega} (|v| |\nabla_H^2 v| + |\nabla_H v|^2 + |w| |\nabla_H u| + |\nabla_H w| |u|) |\Delta_H \eta| dx dy dz \\
& \leq C(\|v\|_{\infty}^2 + \|u\|_4^2 + \|(\varphi, \psi)\|_2 \|\nabla_H(\varphi, \psi)\|_2 + \|(\varphi, \psi)\|_2^2 \\
& \quad + 1) \|\nabla_H(\eta, \theta, T)\|_2^2 + C(\|(\eta, \theta, \varphi, \psi)\|_2^2 + \|u\|_4^4 + 1) \\
& \quad \times (\|\nabla_H(\eta, \theta, \varphi, \psi)\|_2^2 + 1) + \frac{3}{16} \|\Delta_H(\eta, \theta)\|_2^2.
\end{aligned}$$

Thanks to this estimate, it follows from (3.43) and the Young inequality that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla_H \eta\|_2^2 + \|\Delta_H \eta\|_2^2 + \varepsilon \|\nabla_H \partial_z \eta\|_2^2 \\
& \leq \frac{1}{4} \|\Delta_H(\eta, \theta)\|_2^2 + C(\|v\|_{\infty}^2 + \|u\|_4^2 + \|(\varphi, \psi)\|_2 \|\nabla_H(\varphi, \psi)\|_2 \\
& \quad + \|(\varphi, \psi)\|_2^2 + 1) \|\nabla_H(\eta, \theta, T)\|_2^2 + C(\|(\eta, \theta, \varphi, \psi)\|_2^2 + \|u\|_4^4 \\
& \quad + 1) (\|\nabla_H(v, \eta, \theta, \varphi, \psi)\|_2^2 + \|\partial_z T\|_2^2 + 1) + C\varepsilon^2 \|\Delta_H T\|_2^2,
\end{aligned}$$

which, summed with (3.42), yields the conclusion. \square

Note that $\nabla_H T$ is involved in the energy inequality of Proposition 3.8 and, thus, it does not yield the a priori estimate for $(\nabla_H \eta, \nabla_H \theta)$. Therefore, we need to combine the energy inequalities for $(\nabla_H \eta, \nabla_H \theta)$, which have already been stated in Proposition 3.8, with those for $\nabla_H T$, which will be stated in the next subsection.

3.4. Energy inequality for ∇T . In this subsection, we are concerned with performing the energy inequalities for the first order derivatives of T .

Define the function $\varpi(x, y, z, t)$ as follows: for any $z \in (-h, h)$ and $t \in (0, \infty)$, $\varpi(\cdot, z, t)$ is the unique solution to the two-dimensional elliptic system subject to horizontal boundary conditions

$$\begin{cases} \nabla_H \cdot \varpi(x, y, z, t) = \Phi(x, y, z, t) - \frac{1}{|M|} \int_M \Phi(x, y, z, t) dx dy, & \text{in } \Omega, \\ \nabla_H^{\perp} \cdot \varpi(x, y, z, t) = 0, & \text{in } \Omega, \quad \int_M \varpi(x, y, z, t) dx dy = 0, \end{cases} \quad (3.44)$$

where Φ is the function given by (3.5). Define a function ζ as

$$\zeta(x, y, z, t) = v(x, y, z, t) + \varpi(x, y, z, t), \quad (3.45)$$

then, recalling the definitions of η and θ , one can easily check that

$$\nabla_H \cdot \zeta = \eta - \frac{1}{|M|} \int_M \Phi dx dy, \quad \nabla_H^{\perp} \cdot \zeta = \theta. \quad (3.46)$$

The following proposition will be used later.

Proposition 3.9. *Let η and θ as in (3.4), ϖ as in (3.44), and ζ as in (3.45). Then, the following inequalities hold:*

$$\begin{aligned} \int_{-h}^h \|\nabla_H \zeta(\cdot, z, t)\|_{\infty, M} dz &\leq C(\|\nabla_H \eta\|_2(t) + \|\nabla_H \theta\|_2(t) + 1) \\ &\quad \times \log^{\frac{1}{2}}(e + \|\Delta_H \eta\|_2(t) + \|\Delta_H \theta\|_2(t)), \\ \sup_{-h \leq z \leq h} \|\nabla_H \varpi(\cdot, z, t)\|_{\infty, M} &\leq C \log(e + \|\nabla_H T\|_q(t)), \quad q \in (2, \infty), \end{aligned}$$

for a positive constant C depending only on h, q , and $\|T_0\|_{\infty}$.

Proof. Recall the two-dimensional version of the Brézis-Gallouet-Wainger inequality (see, e.g., [5, 6])

$$\|g\|_{\infty, M} \leq C(1 + \|g\|_{H^1(M)}) \log^{\frac{1}{2}}(e + \|g\|_{H^2(M)})$$

for any $g \in H^2(M)$. By the aid of this, recalling (3.46), it follows from the two-dimensional elliptic estimates, the Poincaré, Hölder and Jensen inequalities that

$$\begin{aligned} \int_{-h}^h \|\nabla_H \zeta\|_{\infty, M} dz &\leq C \int_{-h}^h (\|\nabla_H \zeta\|_{H^1(M)} + 1) \log^{\frac{1}{2}}(e + \|\nabla_H \zeta\|_{H^2(M)}) dz \\ &\leq C \int_{-h}^h (\|(\nabla_H^{\perp} \cdot \zeta, \nabla_H \cdot \zeta)\|_{H^1(M)} + 1) \log^{\frac{1}{2}}(e + \|(\nabla_H^{\perp} \cdot \zeta, \nabla_H \cdot \zeta)\|_{H^2(M)}) dz \\ &\leq C \int_{-h}^h (\|(\nabla_H \nabla_H^{\perp} \cdot \zeta, \nabla_H \nabla_H \cdot \zeta)\|_{2, M} + 1) \log^{\frac{1}{2}}(e + \|\Delta_H(\nabla_H^{\perp} \cdot \zeta, \nabla_H \cdot \zeta)\|_{2, M}) dz \\ &\leq C \int_{-h}^h (\|\nabla_H \eta\|_{2, M} + \|\nabla_H \theta\|_{2, M} + 1) \log^{\frac{1}{2}}(e + \|\Delta_H \eta\|_{2, M} + \|\Delta_H \theta\|_{2, M}) dz \\ &\leq C \left(\int_{-h}^h \|\nabla_H(\eta, \theta)\|_{2, M}^2 dz + 1 \right)^{\frac{1}{2}} \left(\int_{-h}^h \log(e + \|\Delta_H(\eta, \theta)\|_{2, M}) \frac{dz}{2h} \right)^{\frac{1}{2}} \\ &\leq C(\|\nabla_H \eta\|_2 + \|\nabla_H \theta\|_2 + 1) \left[\log \left(\int_{-h}^h (e + \|\Delta_H \eta\|_{2, M} + \|\Delta_H \theta\|_{2, M}) \frac{dz}{2h} \right) \right]^{\frac{1}{2}} \\ &\leq C(\|\nabla_H \eta\|_2 + \|\nabla_H \theta\|_2 + 1) \log^{\frac{1}{2}}(e + \|\Delta_H \eta\|_2 + \|\Delta_H \theta\|_2), \end{aligned}$$

proving the first conclusion.

Recall the following logarithmic Sobolev type inequality (see, e.g., [3]), for any function $g = (g^1, g^2) \in W^{1, q}(M)$, $q \in (2, \infty)$,

$$\|\nabla_H g\|_{\infty, M} \leq C(\|\nabla_H^{\perp} \cdot g\|_{\infty, M} + \|\nabla_H \cdot g\|_{\infty, M} + 1) \log(e + \|g\|_{W^{1, q}(M)}).$$

By the aid of this, recalling (3.44) and $\|T\|_{\infty} \leq \|T_0\|_{\infty}$, and applying the elliptic estimates, one has

$$\sup_{-h \leq z \leq h} \|\nabla_H \varpi(\cdot, z, t)\|_{\infty, M}$$

$$\begin{aligned}
&\leq C \sup_{-h \leq z \leq h} (\|\nabla_H^\perp \cdot \varpi(\cdot, z, t)\|_{\infty, M} + \|\nabla_H \cdot \varpi(\cdot, z, t)\|_{\infty, M} + 1) \\
&\quad \times \log(e + \|\nabla_H \varpi(\cdot, z, t)\|_{W^{1,q}(M)}) \\
&\leq C \sup_{-h \leq z \leq h} \log(e + \|\nabla_H^\perp \cdot \varpi(\cdot, z, t)\|_{W^{1,q}(M)} + \|\nabla_H \cdot \varpi(\cdot, z, t)\|_{W^{1,q}(M)}) \\
&= C \sup_{-h \leq z \leq h} \log(e + \|\nabla_H \cdot \varpi(\cdot, z, t)\|_{W^{1,q}(M)}) \\
&\leq C \sup_{-h \leq z \leq h} \log(e + \|\nabla_H \nabla_H \cdot \varpi(\cdot, z, t)\|_{q, M}) \\
&= C \sup_{-h \leq z \leq h} \log(e + \|\nabla_H \Phi(\cdot, z, t)\|_{q, M}).
\end{aligned}$$

Recalling the definition of Φ , (3.5), one can easily check that

$$\|\nabla_H \Phi(\cdot, z, t)\|_{q, M} \leq C \|\nabla_H T(\cdot, t)\|_q$$

and, therefore, we have

$$\sup_{-h \leq z \leq h} \|\nabla_H \varpi(\cdot, z, t)\|_{\infty, M} \leq C \log(e + \|\nabla_H T\|_q),$$

proving the second conclusion. \square

Energy inequality for ∇T is stated in the next proposition.

Proposition 3.10. *Let η and θ be as in (3.4). Then, the following energy inequalities hold:*

$$\begin{aligned}
&\frac{d}{dt} \left(\frac{\|\nabla T\|_2^2}{2} + \frac{\|\nabla_H T\|_q^q}{q} \right) + \|(\partial_z^2 T, \nabla_H \partial_z T, \sqrt{\varepsilon} \Delta_H T)\|_2^2 \\
&\leq C_\sigma (\|\nabla_H(\eta, \theta)\|_2^2 + 1) (\|\nabla_H T\|_q^q + 1) \log(e + \|\Delta_H(\eta, \theta)\|_2 + \|\nabla_H T\|_q) \\
&\quad + C_\sigma (\|v\|_\infty^2 + 1) (\|\nabla T\|_2^2 + \|\eta\|_2^2 + 1) + \sigma (\|\Delta_H \eta\|_2^2 + \|\nabla_H \partial_z T\|_2^2),
\end{aligned}$$

for any $\sigma > 0$, if $q \in (2, 4]$, where C_σ is a positive number depending only on $h, q, \|T_0\|_\infty$, and σ ; and

$$\begin{aligned}
&\frac{d}{dt} \left(\frac{2}{q} \|\nabla_H T\|_q^q + \|\nabla_H T\|_2^2 \right) + \|\nabla_H \partial_z T\|_2^2 + \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2^2 \\
&\leq C (\|\nabla_H(\eta, \theta)\|_2^2 + 1) (\|\nabla_H T\|_q^q + 1) \log(e + \|\Delta_H \eta\|_2 + \|\Delta_H \theta\|_2 + \|\nabla_H T\|_q) \\
&\quad + C (\|\Delta_H \eta\|_2^2 + \|\nabla_H \eta\|_2^2 + 1) (\|\nabla_H T\|_q^q + 1),
\end{aligned}$$

if $q \in (4, \infty)$, where C is a positive number depending only on h, q , and $\|T_0\|_\infty$.

Proof. Integration by parts and using the Hölder inequality yield

$$\int_\Omega |\partial_z T|^4 dx dy dz = - \int_\Omega \partial_z (|\partial_z T|^2 \partial_z T) T dx dy dz \leq 3 \|T\|_\infty \|\partial_z^2 T\|_2 \|\partial_z T\|_4^2,$$

which implies

$$\|\partial_z T\|_4^2 \leq 3 \|T\|_\infty \|\partial_z^2 T\|_2. \quad (3.47)$$

Multiplying equation (3.3) by $-\partial_z^2 T$ and integrating the resultant over Ω , it follows from integration by parts, Proposition 3.2, (3.47), and the Hölder and Young inequalities that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\partial_z T\|_2^2 + \|\partial_z^2 T\|_2^2 + \varepsilon \|\nabla_H \partial_z T\|_2^2 \\
&= \int_{\Omega} (v \cdot \nabla_H T + w \partial_z T) \partial_z^2 T dx dy dz \\
&= \int_{\Omega} (v \cdot \nabla_H T \partial_z^2 T + \frac{1}{2} (\nabla_H \cdot v) |\partial_z T|^2) dx dy dz \\
&= \int_{\Omega} (v \cdot \nabla_H T \partial_z^2 T + \frac{1}{2} (\eta - \Phi) |\partial_z T|^2) dx dy dz \\
&\leq C (\|v\|_{\infty} \|\nabla_H T\|_2 \|\partial_z^2 T\|_2 + \|\Phi\|_{\infty} \|\partial_z T\|_2^2 + \|\eta\|_2 \|\partial_z T\|_4^2) \\
&\leq C (\|v\|_{\infty} \|\nabla_H T\|_2 \|\partial_z^2 T\|_2 + \|\Phi\|_{\infty} \|\partial_z T\|_2^2 + \|\eta\|_2 \|T\|_{\infty} \|\partial_z^2 T\|_2) \\
&\leq \sigma \|\partial_z^2 T\|_2^2 + C_{\sigma} (\|v\|_{\infty}^2 \|\nabla_H T\|_2^2 + \|\partial_z T\|_2^2 + \|\eta\|_2^2) \\
&\leq \sigma \|\partial_z^2 T\|_2^2 + C_{\sigma} (\|v\|_{\infty}^2 + 1) (\|\nabla_H T\|_2^2 + \|\partial_z T\|_2^2 + \|\eta\|_2^2 + 1), \tag{3.48}
\end{aligned}$$

for any $\sigma > 0$ (to be chosen later) and for some $C_{\sigma} > 0$.

Recalling the definitions of η and Φ , (3.4) and (3.5), respectively, then by the Hölder inequality, one can easily check

$$\|\nabla_H \partial_z w\|_2, \|\nabla_H w\|_2 \leq C (\|\nabla_H \eta\|_2 + \|\nabla_H T\|_2).$$

Thanks to the above inequality, multiplying equation (3.3) by $-\Delta_H T$, and integrating the resultant over Ω , it follows from integration by parts, Proposition 3.2, and the Hölder and Young inequalities that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla_H T\|_2^2 + \|\nabla_H \partial_z T\|_2^2 + \varepsilon \|\Delta_H T\|_2^2 \\
&= \int_{\Omega} (v \cdot \nabla_H T + w \partial_z T) \Delta_H T dx dy dz \\
&= - \int_{\Omega} (\nabla_H T \cdot \nabla_H v + \partial_z T \nabla_H w) \cdot \nabla_H T dx dy dz \\
&= - \int_{\Omega} [\nabla_H T \cdot \nabla_H v \cdot \nabla_H T - (\nabla_H \partial_z w \cdot \nabla_H T + \nabla_H w \cdot \partial_z \nabla_H T) T] dx dy dz \\
&\leq \int_{\Omega} |\nabla_H v| |\nabla_H T|^2 dx dy dz + C (\|\nabla_H \partial_z w\|_2 \|\nabla_H T\|_2 + \|\nabla_H w\|_2 \|\nabla_H \partial_z T\|_2) \\
&\leq \int_{\Omega} |\nabla_H v| |\nabla_H T|^2 dx dy dz + C (\|\nabla_H \eta\|_2 + \|\nabla_H T\|_2) (\|\nabla_H T\|_2 + \|\nabla_H \partial_z T\|_2) \\
&\leq \sigma \|\nabla_H \partial_z T\|_2^2 + C_{\sigma} (\|\nabla_H \eta\|_2^2 + \|\nabla_H T\|_2^2) + \int_{\Omega} |\nabla_H v| |\nabla_H T|^2 dx dy dz, \tag{3.49}
\end{aligned}$$

for any $\sigma > 0$ (to be chosen later) and for some $C_\sigma > 0$.

Multiplying equation (3.3) by $-\operatorname{div}_H(|\nabla_H T|^{q-2}\nabla_H T)$, for $q \in (2, \infty)$, integrating the resultant over Ω , and noticing that $\|T\|_\infty \leq \|T_0\|_\infty$, it follows from integration by parts, (1.17), (3.4), and (3.5), that

$$\begin{aligned}
& \frac{1}{q} \frac{d}{dt} \|\nabla_H T\|_q^q + \int_\Omega |\nabla_H T|^{q-2} (|\nabla_H \partial_z T|^2 + (q-2)|\partial_z |\nabla_H T||^2 \\
& \quad + \varepsilon |\nabla_H^2 T|^2 + (q-2)\varepsilon |\nabla_H |\nabla_H T||^2) dx dy dz \\
& = - \int_\Omega |\nabla_H T|^{q-2} (\nabla_H T \cdot \nabla_H v + \partial_z T \nabla_H w) \cdot \nabla_H T dx dy dz \\
& = - \int_\Omega [|\nabla_H T|^{q-2} \nabla_H T \cdot \nabla_H v \cdot \nabla_H T - T |\nabla_H T|^{q-2} \nabla_H \partial_z w \cdot \nabla_H T \\
& \quad + T \nabla_H w \cdot \partial_z (|\nabla_H T|^{q-2} \nabla_H T)] dx dy dz \\
& \leq C \int_M \left(\int_{-h}^h |\nabla_H (\eta - \Phi)| dz \right) \left(\int_{-h}^h |\nabla_H T|^{q-2} |\nabla_H \partial_z T| dz \right) dx dy \\
& \quad + C \int_\Omega (|\nabla_H v| |\nabla_H T|^q + |\nabla_H (\eta - \Phi)| |\nabla_H T|^{q-1}) dx dy dz,
\end{aligned}$$

which, summed with (3.49) and using the Young inequality, gives

$$\begin{aligned}
& \frac{d}{dt} \left(\frac{\|\nabla_H T\|_q^q}{q} + \frac{\|\nabla_H T\|_2^2}{2} \right) + \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2^2 + \|\nabla_H \partial_z T\|_2^2 + \varepsilon \|\Delta_H T\|_2^2 \\
& \leq \sigma \|\nabla_H \partial_z T\|_2^2 + C_\sigma (\|\nabla_H \eta\|_2^2 + \|\nabla_H T\|_2^2) \\
& \quad + C \int_\Omega |\nabla_H v| (|\nabla_H T|^q + 1) dx dy dz + C \int_\Omega |\nabla_H (\eta - \Phi)| |\nabla_H T|^{q-1} dx dy dz \\
& \quad + C \int_M \left(\int_{-h}^h |\nabla_H (\eta - \Phi)| dz \right) \left(\int_{-h}^h |\nabla_H T|^{q-2} |\nabla_H \partial_z T| dz \right) dx dy \\
& =: \sigma \|\nabla_H \partial_z T\|_2^2 + C_\sigma (\|\nabla_H \eta\|_2^2 + \|\nabla_H T\|_2^2) + C(J_1 + J_2 + J_3). \tag{3.50}
\end{aligned}$$

We have to estimate the terms J_1 , J_2 , and J_3 in (3.50). First, we show that $J_2 \leq (q-1)J_3$. In fact, since T is odd and periodic with respect to z , one has $T|_{z=-h} = T|_{z=h} = -T|_{z=-h} = 0$. Therefore, we have

$$|\nabla_H T(x, y, z, t)|^{q-1} \leq (q-1) \int_{-h}^h |\nabla_H T(x, y, z, t)|^{q-2} |\nabla_H \partial_z T(x, y, z, t)| dz$$

and, thus,

$$\begin{aligned}
J_2 & = \int_\Omega |\nabla_H (\eta - \Phi)| |\nabla_H T|^{q-1} dx dy dz \\
& \leq (q-1) \int_M \int_{-h}^h |\nabla_H (\eta - \Phi)| dz \int_{-h}^h |\nabla_H T|^{q-2} |\nabla_H \partial_z T| dz dx dy = (q-1)J_3.
\end{aligned}$$

Next, we estimate J_3 . By the Hölder and Minkowski inequalities, we have

$$\begin{aligned}
J_{3,1} &:= \int_M \left(\int_{-h}^h |\nabla_H \eta| dz \right) \left(\int_{-h}^h |\nabla_H T|^{q-2} |\nabla_H \partial_z T| dz \right) dx dy \\
&\leq \int_M \left(\int_{-h}^h |\nabla_H \eta| dz \right) \left(\int_{-h}^h |\nabla_H T|^{q-2} dz \right)^{\frac{1}{2}} \left(\int_{-h}^h |\nabla_H T|^{q-2} |\nabla_H \partial_z T|^2 dz \right)^{\frac{1}{2}} dx dy \\
&\leq \left\| \int_{-h}^h |\nabla_H \eta| dz \right\|_{q,M} \left\| \left(\int_{-h}^h |\nabla_H T|^{q-2} dz \right)^{\frac{1}{2}} \right\|_{\frac{2q}{q-2}, M} \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2 \\
&\leq \left(\int_{-h}^h \|\nabla_H \eta\|_{q,M} dz \right) \left(\int_{-h}^h \|\nabla_H T\|_{q,M}^{q-2} dz \right)^{\frac{1}{2}} \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2 \\
&\leq C \left(\int_{-h}^h \|\nabla_H \eta\|_{q,M} dz \right) \|\nabla_H T\|_q^{\frac{q-2}{2}} \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2
\end{aligned}$$

and, similarly,

$$\begin{aligned}
J_{3,2} &:= \int_M \left(\int_{-h}^h |\nabla_H T| dz \right) \left(\int_{-h}^h |\nabla_H T|^{q-2} |\nabla_H \partial_z T| dz \right) dx dy \\
&\leq C \left(\int_{-h}^h \|\nabla_H T\|_{q,M} dz \right) \|\nabla_H T\|_q^{\frac{q-2}{2}} \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2.
\end{aligned}$$

Therefore, we have by the Hölder and Young inequalities that

$$\begin{aligned}
J_{3,2} &\leq \|\nabla_H T\|_q^{\frac{q}{2}} \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2 \\
&\leq \sigma \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2^2 + C_\sigma \|\nabla_H T\|_q^q
\end{aligned} \tag{3.51}$$

for any $\sigma > 0$ (to be chosen later) and for some $C_\sigma > 0$. Moreover, from the above and by the Gagliardo-Nirenberg and Hölder inequalities we have

$$\begin{aligned}
J_{3,1} &\leq C \left(\int_{-h}^h \|\nabla_H \eta\|_{q,M} dz \right) \|\nabla_H T\|_q^{\frac{q-2}{2}} \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2 \\
&\leq C \left(\int_{-h}^h \|\nabla_H \eta\|_{2,M}^{\frac{2}{q}} \|\Delta_H \eta\|_{2,M}^{1-\frac{2}{q}} dz \right) \|\nabla_H T\|_q^{\frac{q-2}{2}} \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2 \\
&\leq C \|\nabla_H \eta\|_2^{\frac{2}{q}} \|\Delta_H \eta\|_2^{1-\frac{2}{q}} \|\nabla_H T\|_q^{\frac{q-2}{2}} \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2.
\end{aligned}$$

We further estimate $J_{3,1}$, in accordance with two different ranges, by the Young inequality as follows:

$$J_{3,1} \leq \sigma \left(\left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2^2 + \|\Delta_H \eta\|_2^2 \right) + C_\sigma \|\nabla_H \eta\|_2^2 \|\nabla_H T\|_q^{\frac{q-2}{2}q}$$

$$\leq \sigma \left(\left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2^2 + \|\Delta_H \eta\|_2^2 \right) + C_\sigma \|\nabla_H \eta\|_2^2 (\|\nabla_H T\|_q^q + 1),$$

if $q \in (2, 4]$, and

$$J_{3,1} \leq \sigma \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2^2 + C_\sigma (\|\Delta_H \eta\|_2^2 \|\nabla_H T\|_q^q + \|\nabla_H \eta\|_2^2),$$

if $q \in (4, \infty)$, for any $\sigma > 0$ and for some $C_\sigma > 0$. Recalling (3.51) and noticing that $J_3 \leq J_{3,1} + J_{3,2}$, we have

$$J_3 \leq \sigma \left(\left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2^2 + \|\Delta_H \eta\|_2^2 \right) + C_\sigma (\|\nabla_H \eta\|_2^2 + 1) (\|\nabla_H T\|_q^q + 1), \quad (3.52)$$

if $q \in (2, 4]$, and

$$J_3 \leq \sigma \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2^2 + C_\sigma (\|\Delta_H \eta\|_2^2 + \|\nabla_H \eta\|_2^2 + 1) (\|\nabla_H T\|_q^q + 1), \quad (3.53)$$

if $q \in (4, \infty)$, for any $\sigma > 0$ and for some $C_\sigma > 0$.

Finally, we estimate the term J_1 . Recalling the decomposition $v = \zeta - \varpi$, we have

$$\begin{aligned} J_1 &= \int_{\Omega} |\nabla_H v| (1 + |\nabla_H T|^q) dx dy dz \\ &\leq \int_{\Omega} (|\nabla_H \zeta| + |\nabla_H \varpi|) (1 + |\nabla_H T|^q) dx dy dz \\ &\leq \left(\int_{-h}^h \|\nabla_H \zeta\|_{\infty, M} dz \right) \left(\sup_{-h \leq z \leq h} \|\nabla_H T(\cdot, z, t)\|_{q, M}^q + 1 \right) \\ &\quad + \left(\sup_{-h \leq z \leq h} \|\nabla_H \varpi(\cdot, z, t)\|_{\infty, M} \right) (\|\nabla_H T\|_q^q + 1). \end{aligned}$$

Recalling that $T|_{z=-h} = 0$, it follows from the Hölder inequality that

$$\begin{aligned} \sup_{-h \leq z \leq h} \|\nabla_H T(\cdot, z, t)\|_{q, M}^q &= \sup_{-h \leq z \leq h} \int_M |\nabla_H T(\cdot, z)|^q dx dy \\ &= q \sup_{-h \leq z \leq h} \int_M \left(\int_{-h}^z |\nabla_H T|^{q-2} \nabla_H T \cdot \nabla_H \partial_z T d\xi \right) dx dy \\ &\leq q \int_M \int_{-h}^h |\nabla_H T|^{q-1} |\nabla_H \partial_z T| dz dx dy \\ &\leq q \|\nabla_H T\|_q^{\frac{q}{2}} \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2. \end{aligned}$$

With the aid of the above inequality, applying Proposition 3.9, and using the Young inequality, we obtain

$$\left(\int_{-h}^h \|\nabla_H \zeta\|_{\infty, M} dz \right) \left(\sup_{-h \leq z \leq h} \|\nabla_H T\|_{q, M}^q + 1 \right)$$

$$\begin{aligned}
&\leq C(\|\nabla_H \eta\|_2 + \|\nabla_H \theta\|_2 + 1) \log^{\frac{1}{2}}(e + \|\Delta_H \eta\|_2 + \|\Delta_H \theta\|_2) \\
&\quad \times \left(\|\nabla_H T\|_q^{\frac{q}{2}} \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2 + 1 \right) \\
&\leq \sigma \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2^2 + C_\sigma (\|\nabla_H \eta\|_2^2 + \|\nabla_H \theta\|_2^2 + 1) \\
&\quad \times (\|\nabla_H T\|_q^q + 1) \log(e + \|\Delta_H \eta\|_2 + \|\Delta_H \theta\|_2)
\end{aligned}$$

for any $\sigma > 0$ and for some $C_\sigma > 0$, and

$$\left(\sup_{-h \leq z \leq h} \|\nabla_H \varpi(\cdot, z, t)\|_{\infty, M} \right) (\|\nabla_H T\|_q^q + 1) \leq C(\|\nabla_H T\|_q^q + 1) \log(e + \|\nabla_H T\|_q).$$

Therefore, we have

$$\begin{aligned}
J_1 &\leq \sigma \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2^2 + C_\sigma (\|\nabla_H \eta\|_2^2 + \|\nabla_H \theta\|_2^2 + 1) (\|\nabla_H T\|_q^q + 1) \\
&\quad \times \log(e + \|\Delta_H \eta\|_2 + \|\Delta_H \theta\|_2 + \|\nabla_H T\|_q)
\end{aligned} \tag{3.54}$$

for any $\sigma > 0$ and for some $C_\sigma > 0$.

Thanks to the estimates for J_1 and J_3 , i.e. (3.52)–(3.54), and recalling that $J_2 \leq (q-1)J_3$, it follows from (3.50) that

$$\begin{aligned}
&\frac{d}{dt} \left(\frac{\|\nabla_H T\|_q^q}{q} + \frac{\|\nabla_H T\|_2^2}{2} \right) + \|\nabla_H \partial_z T\|_2^2 + \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2^2 + \varepsilon \|\Delta_H T\|_2^2 \\
&\leq \sigma \left(\|\nabla_H \partial_z T\|_2^2 + \|\Delta_H \eta\|_2^2 + \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2^2 \right) + C_\sigma (\|\nabla_H(\eta, \theta)\|_2^2 + 1) \\
&\quad \times (\|\nabla_H T\|_q^q + 1) \log(e + \|\Delta_H \eta\|_2 + \|\Delta_H \theta\|_2 + \|\nabla_H T\|_q)
\end{aligned} \tag{3.55}$$

for any $\sigma > 0$ and for some $C_\sigma > 0$, if $q \in (2, 4]$, and

$$\begin{aligned}
&\frac{d}{dt} \left(\frac{2}{q} \|\nabla_H T\|_q^q + \|\nabla_H T\|_2^2 \right) + \|\nabla_H \partial_z T\|_2^2 + \left\| |\nabla_H T|^{\frac{q}{2}-1} \nabla_H \partial_z T \right\|_2^2 \\
&\leq C(\|\nabla_H(\eta, \theta)\|_2^2 + 1) (\|\nabla_H T\|_q^q + 1) \log(e + \|\Delta_H \eta\|_2 + \|\Delta_H \theta\|_2 + \|\nabla_H T\|_q) \\
&\quad + C(\|\Delta_H \eta\|_2^2 + \|\nabla_H \eta\|_2^2 + 1) (\|\nabla_H T\|_q^q + 1),
\end{aligned} \tag{3.56}$$

if $q \in (4, \infty)$.

The first conclusion follows from summing (3.48) and (3.55), and the second one follows from (3.56). \square

3.5. A priori estimates on $(\nabla_H \eta, \nabla_H \theta, \nabla T)$. Combining the energy inequalities established in the previous two subsections and applying the logarithmic type Gronwall inequality, i.e., Lemma 2.5, we are able to obtain the required a priori estimates on $\nabla_H \eta$, $\nabla_H \theta$, and ∇T . In fact, we have the following proposition.

Proposition 3.11. *Given $\mathcal{T} \in (0, \infty)$. There is a positive number $\varepsilon_0 \in (0, 1)$ depending only on h and $\|T_0\|_\infty$, such that, for any $\varepsilon \in (0, \varepsilon_0)$ and any $q \in (2, \infty)$, we have the following estimate:*

$$\begin{aligned} \sup_{0 \leq t \leq \mathcal{T}} (\|\nabla_H(\eta, \theta)\|_2^2 + \|\nabla T\|_2^2 + \|\nabla_H T\|_q^q) + \int_0^{\mathcal{T}} (\|\Delta_H(\eta, \theta)\|_2^2 \\ + \|(\partial_z^2 T, \nabla_H \partial_z T)\|_2^2 + \varepsilon \|(\nabla_H \partial_z \eta, \nabla_H \partial_z \theta, \Delta_H T) dt \leq C, \end{aligned}$$

where C is a positive constant depending only on h, \mathcal{T} , and $\|v_0\|_{H^2} + \|T_0\|_{H^1 \cap L^\infty} + \|\nabla_H T_0\|_q$; in particular, C is independent of $\varepsilon \in (0, \varepsilon_0)$.

Proof. We first consider the case that $q \in (2, 4]$. By Proposition 3.8 and Proposition 3.10, where we choose $\sigma = \frac{1}{4}$, we have

$$\begin{aligned} & \frac{d}{dt} \|\nabla_H(\eta, \theta)\|_2^2 + \|(\Delta_H \eta, \Delta_H \theta, \sqrt{\varepsilon} \nabla_H \partial_z \eta, \sqrt{\varepsilon} \nabla_H \partial_z \theta)\|_2^2 \\ & \leq C(\|v\|_\infty^2 + \|u\|_4^2 + \|(\varphi, \psi)\|_2 \|\nabla_H(\varphi, \psi)\|_2 + \|(\varphi, \psi)\|_2^2 + 1) \\ & \quad \times \|\nabla_H(\eta, \theta, T)\|_2^2 + C(\|(\eta, \theta, \varphi, \psi)\|_2^2 + \|u\|_4^4 + 1) \\ & \quad \times (\|\nabla_H(v, \eta, \theta, \varphi, \psi)\|_2^2 + \|\partial_z T\|_2^2 + 1) + C\varepsilon^2 \|\Delta_H T\|_2^2, \end{aligned}$$

where C is a positive constant depending only on h and $\|T_0\|_\infty$, and

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\|\nabla T\|_2^2}{2} + \frac{\|\nabla_H T\|_q^q}{q} \right) + \|(\partial_z^2 T, \nabla_H \partial_z T, \sqrt{\varepsilon} \Delta_H T)\|_2^2 \\ & \leq C(\|\nabla_H(\eta, \theta)\|_2^2 + 1)(\|\nabla_H T\|_q^q + 1) \log(e + \|\Delta_H(\eta, \theta)\|_2 + \|\nabla_H T\|_q) \\ & \quad + C(\|v\|_\infty^2 + 1)\|\nabla T\|_2^2 + C(\|v\|_\infty^2 + 1)(\|\eta\|_2^2 + 1) + \frac{1}{4} (\|\Delta_H \eta\|_2^2 + \|\nabla_H \partial_z T\|_2^2) \end{aligned}$$

provided $q \in (2, 4]$, where C is a positive number depending only on h, q , and $\|T_0\|_\infty$. Choose a small positive number $\varepsilon_0 \in (0, 1)$ depending only on h and $\|T_0\|_\infty$ and let $\varepsilon \in (0, \varepsilon_0)$. Summing the above two inequalities and denoting

$$\begin{aligned} A_3 &= \|\nabla_H(\eta, \theta)\|_2^2 + \frac{\|\nabla T\|_2^2}{2} + \frac{\|\nabla_H T\|_q^q}{q}, \\ B_3 &= \frac{1}{2} \|(\Delta_H \eta, \Delta_H \theta, \sqrt{\varepsilon} \nabla_H \partial_z \eta, \sqrt{\varepsilon} \nabla_H \partial_z \theta)\|_2^2 + \|(\partial_z^2 T, \nabla_H \partial_z T, \sqrt{\varepsilon} \Delta_H T)\|_2^2, \\ \ell_3(t) &= (\|v\|_\infty^2 + \|u\|_4^2 + \|(\varphi, \psi)\|_2 \|\nabla_H(\varphi, \psi)\|_2 + \|(\varphi, \psi)\|_2^2)(t) + 1, \\ n_3(t) &= \|\nabla_H(\eta, \theta)\|_2^2(t) + 1, \quad f_3(t) = (\|v\|_\infty^2(t) + 1)(\|\eta\|_2^2(t) + 1), \end{aligned}$$

one obtains

$$A_3' + B_3 \leq C(\ell_3(t) + n_3(t) \log(A_3 + B_3 + e))A_3 + C f_3(t). \quad (3.57)$$

Recalling (3.20), i.e., $\|v\|_\infty^2 \leq \log(A_2 + B_2)$, where A_2 and B_2 are given by (3.16) and (3.17), respectively, and noticing that $\log z \leq \log(1 + z) \leq z$, for $z > 0$, we have by

Corollary 3.1 that

$$\int_0^{\mathcal{T}} \|v\|_{\infty}^2 dt \leq C \int_0^{\mathcal{T}} \log(A_2 + B_2) dt \leq C \int_0^{\mathcal{T}} (A_2 + B_2) dt \leq C. \quad (3.58)$$

With the aid of this, and applying Corollary 3.1 and Proposition 3.6, we have

$$\int_0^{\mathcal{T}} (\ell_3(t) + n_3(t) + f_3(t)) dt \leq C$$

for a positive constant C depending only on $h, \mathcal{T}, \|(v_0, T_0)\|_{\infty}$, and $\|\nabla_H v_0\|_2 + \|\partial_z v_0\|_{H^1}$. Thanks to the above and noticing that $n_3 \leq A_3$, one can apply Lemma 2.5 to (3.57) and obtains

$$\sup_{0 \leq t \leq \mathcal{T}} A_3(t) + \int_0^{\mathcal{T}} B_3(t) dt \leq C, \quad (3.59)$$

where C is a positive constant depending only on h, \mathcal{T} , and $\|v_0\|_{H^2} + \|T_0\|_{H^1 \cap L^{\infty}} + \|\nabla_H T_0\|_q$. This proves the conclusion for the case $q \in (2, 4]$.

We now consider the case when $q \in (4, \infty)$. Thanks to what we have proven in (3.59), we have

$$\begin{aligned} & \sup_{0 \leq t \leq \mathcal{T}} (\|\nabla_H(\eta, \theta)\|_2^2(t) + \|\nabla T\|_2^2(t) + \|\nabla_H T\|_4^4(t)) \\ & + \int_0^{\mathcal{T}} (\|\Delta_H(\eta, \theta)\|_2^2 + \|(\partial_z^2 T, \nabla_H \partial_z T)\|_2^2) dt \leq C, \end{aligned} \quad (3.60)$$

where C is a positive constant depending only on h, \mathcal{T} , and $\|v_0\|_{H^2} + \|T_0\|_{H^1 \cap L^{\infty}} + \|\nabla_H T_0\|_4$. One still need to show the a priori $L^{\infty}(0, \mathcal{T}; L^q)$ estimate on $\nabla_H T$, for $q \in (4, \infty)$. By Proposition 3.10 and noticing that

$$\begin{aligned} & \log(e + \|\Delta_H \eta\|_2 + \|\Delta_H \theta\|_2 + \|\nabla_H T\|_q) \\ & \leq \log[(e + \|\nabla_H T\|_q)(1 + \|\Delta_H \eta\|_2 + \|\Delta_H \theta\|_2)] \\ & \leq \log(e + \|\nabla_H T\|_q) + \log(1 + \|\Delta_H \eta\|_2 + \|\Delta_H \theta\|_2) \\ & \leq \log(e + \|\nabla_H T\|_q) + \|\Delta_H \eta\|_2 + \|\Delta_H \theta\|_2, \end{aligned}$$

we have, for $q \in (4, \infty)$,

$$\begin{aligned} & \frac{d}{dt} \left(\frac{2}{q} \|\nabla_H T\|_q^q + \|\nabla_H T\|_2^2 \right) + \|\nabla_H \partial_z T\|_2^2 \\ & \leq C(\|\nabla_H(\eta, \theta)\|_2^2 + 1)(\|\nabla_H T\|_q^q + 1) \log(e + \|\nabla_H T\|_q) \\ & \quad + C(\|\nabla_H(\eta, \theta)\|_2^2 + 1)(\|\Delta_H \eta\|_2 + \|\Delta_H \theta\|_2)(\|\nabla_H T\|_q^q + 1) \\ & \quad + C(\|\Delta_H \eta\|_2^2 + \|\nabla_H \eta\|_2^2 + 1)(\|\nabla_H T\|_q^q + 1), \end{aligned}$$

from which, denoting

$$\begin{aligned} A_4 &= \frac{2}{q} \|\nabla_H T\|_q^q + \|\nabla_H T\|_2^2, \quad B_4 = \|\nabla_H \partial_z T\|_2^2, \\ \ell_4(t) &= (\|\nabla_H(\eta, \theta)\|_2^2 + 1)(\|\Delta_H \eta\|_2 + \|\Delta_H \theta\|_2) + \|\Delta_H \eta\|_2^2 + \|\nabla_H \eta\|_2^2 + 1, \end{aligned}$$

$$m_4(t) = \|\nabla_H(\eta, \theta)\|_2^2 + 1,$$

one obtains

$$A'_4 + B_4 \leq C(\ell_4(t) + m_4(t) \log A_4)A_4.$$

Thanks to (3.60) and applying Lemma 2.5 to the above inequality, we have

$$\sup_{0 \leq t \leq \mathcal{T}} \|\nabla_H T\|_q(t) \leq C,$$

where C is a positive constant depending only on h, \mathcal{T} , and $\|v_0\|_{H^2} + \|T_0\|_{H^1 \cap L^\infty} + \|\nabla_H T_0\|_q$. Thus, this proves the case that $q \in (4, \infty)$. \square

3.6. A priori estimates on $\nabla^2 T$. This subsection is devoted to establishing the a priori estimates on the second order spatial derivatives of T , which is stated in the following proposition:

Proposition 3.12. *Given a positive time $\mathcal{T} \in (0, \infty)$, let $\varepsilon_0 \in (0, 1)$ be the constant given in Proposition 3.11, and assume that $\varepsilon \in (0, \varepsilon_0)$. The following a priori estimate holds:*

$$\sup_{0 \leq t \leq \mathcal{T}} \|\nabla^2 T\|_2^2(t) + \int_0^{\mathcal{T}} (\|\partial_z \nabla^2 T\|_2^2 + \varepsilon \|\nabla_H \nabla^2 T\|_2^2) dt \leq C$$

for a positive constant C depending only on h, \mathcal{T} , and $\|(v_0, T_0)\|_{H^2}$; in particular, C is independent of $\varepsilon \in (0, \varepsilon_0)$.

Proof. By virtue of (3.2), one can easily check that

$$|\nabla_H \cdot v(x, y, z, t)| \leq \int_{-h}^h |\nabla_H \cdot u(x, y, \xi, t)| d\xi.$$

By the aid of this inequality, differentiating equation (3.3) with respect to z , multiplying the resulting equation by $-\partial_z^3 T$, and integrating over Ω , it follows from integration by parts and using the Hölder inequality that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_z^2 T\|_2^2 + \|\partial_z^3 T\|_2^2 + \varepsilon \|\nabla_H \partial_z^2 T\|_2^2 \\ &= \int_{\Omega} \partial_z (v \cdot \nabla_H T + w \partial_z T) \partial_z^3 T dx dy dz \\ &= \int_{\Omega} (v \cdot \nabla_H \partial_z T + \partial_z v \cdot \nabla_H T - (\nabla_H \cdot v) \partial_z T + w \partial_z^2 T) \partial_z^3 T dx dy dz \\ &= \int_{\Omega} [(v \cdot \nabla_H \partial_z T + u \cdot \nabla_H T - (\nabla_H \cdot v) \partial_z T) \partial_z^3 T + \frac{1}{2} (\nabla_H \cdot v) |\partial_z^2 T|^2] dx dy dz \\ &= \int_{\Omega} [(v \cdot \nabla_H \partial_z T + u \cdot \nabla_H T - (\nabla_H \cdot v) \partial_z T) \partial_z^3 T - v \cdot \nabla_H \partial_z^2 T \partial_z^2 T] dx dy dz \\ &\leq (\|v\|_{\infty} \|\nabla_H \partial_z T\|_2 + \|u\|_4 \|\nabla_H T\|_4) \|\partial_z^3 T\|_2 + \|v\|_{\infty} \|\nabla_H \partial_z^2 T\|_2 \|\partial_z^2 T\|_2 \end{aligned}$$

$$+ \int_M \left(\int_{-h}^h |\nabla_H \cdot u| dz \right) \left(\int_{-h}^h |\partial_z T| |\partial_z^3 T| dz \right) dx dy. \quad (3.61)$$

By Lemma 2.2 and Lemma 2.3, and recalling that $\|T\|_\infty \leq \|T_0\|_\infty$, guaranteed by Proposition 3.2, it follows from the Hölder inequality that

$$\begin{aligned} & \int_M \left(\int_{-h}^h |\nabla_H \cdot u| dz \right) \left(\int_{-h}^h |\partial_z T| |\partial_z^3 T| dz \right) dx dy \\ & \leq \int_M \left(\int_{-h}^h (|\varphi| + |T|) dz \right) \left(\int_{-h}^h |\partial_z T| |\partial_z^3 T| dz \right) dx dy \\ & \leq \left(\int_{-h}^h \|(\varphi, T)\|_{4,M} dz \right) \left(\int_{-h}^h \|\partial_z T\|_{4,M}^2 dz \right)^{\frac{1}{2}} \|\partial_z^3 T\|_2 \\ & \leq C \left(\|\varphi\|_2^{\frac{1}{2}} \|\nabla_H \varphi\|_2^{\frac{1}{2}} + \|\varphi\|_2 + 1 \right) \|\partial_z T\|_4 \|\partial_z^3 T\|_2. \end{aligned} \quad (3.62)$$

Recalling (3.47), i.e.,

$$\|\partial_z T\|_4^2 \leq 3\|T\|_\infty \|\partial_z^2 T\|_2. \quad (3.63)$$

Similarly, we have

$$\|\nabla_H T\|_4^2 \leq 3\|T\|_\infty \|\Delta_H T\|_2. \quad (3.64)$$

It follows from integration by parts and the Hölder and Cauchy-Schwarz inequalities that

$$\begin{aligned} \int_\Omega |\nabla_H \partial_z T|^2 dx dy dz &= \int_\Omega \Delta_H T \partial_z^2 T dx dy dz \\ &\leq \|\Delta_H T\|_2 \|\partial_z^2 T\|_2 \leq \frac{1}{2} (\|\Delta_H T\|_2^2 + \|\partial_z^2 T\|_2^2) \end{aligned}$$

and, similarly, $\|\nabla_H \partial_z^2 T\|_2^2 \leq \frac{1}{2} (\|\partial_z^3 T\|_2^2 + \|\Delta_H \partial_z T\|_2^2)$. On account of these facts and recalling that $\|T\|_\infty \leq \|T_0\|_\infty$, guaranteed by Proposition 3.2, it follows from (3.61)–(3.64) and using the Young inequality that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_z^2 T\|_2^2 + \|\partial_z^3 T\|_2^2 + \varepsilon \|\nabla_H \partial_z^2 T\|_2^2 \\ & \leq (\|v\|_\infty \|\nabla_H \partial_z T\|_2 + \sqrt{3} \|u\|_4 \|T\|_\infty^{\frac{1}{2}} \|\Delta_H T\|_2^{\frac{1}{2}}) \|\partial_z^3 T\|_2 \\ & \quad + \|v\|_\infty \|\nabla_H \partial_z^2 T\|_2 \|\partial_z^2 T\|_2 + C \left(\|\varphi\|_2^{\frac{1}{2}} \|\nabla_H \varphi\|_2^{\frac{1}{2}} + \|\varphi\|_2 + 1 \right) \|\partial_z^2 T\|_2^{\frac{1}{2}} \|\partial_z^3 T\|_2 \\ & \leq \frac{1}{4} (\|\partial_z^3 T\|_2^2 + \|\Delta_H \partial_z T\|_2^2) + C [\|\varphi\|_2^2 \|\nabla_H \varphi\|_2^2 + \|\varphi\|_2^4 + 1 \\ & \quad + (\|v\|_\infty^2 + \|u\|_4^2) (\|\Delta_H T\|_2^2 + \|\partial_z^2 T\|_2^2 + 1)] \end{aligned}$$

and, thus,

$$\frac{1}{2} \frac{d}{dt} \|\partial_z^2 T\|_2^2 + \frac{3}{4} \|\partial_z^3 T\|_2^2 + \varepsilon \|\nabla_H \partial_z^2 T\|_2^2$$

$$\begin{aligned} &\leq \frac{1}{4} \|\Delta_H \partial_z T\|_2^2 + C[\|\varphi\|_2^2 \|\nabla_H \varphi\|_2^2 + \|\varphi\|_2^4 + 1 \\ &\quad + (\|v\|_\infty^2 + \|u\|_4^2)(\|\Delta_H T\|_2^2 + \|\partial_z^2 T\|_2^2 + 1)]. \end{aligned} \quad (3.65)$$

Applying the horizontal gradient ∇_H to equation (3.3), multiplying the resulting equation by $-\nabla_H \Delta_H T$, and integrating over Ω , it follows from integrating by parts and the Hölder inequality that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\Delta_H T\|_2^2 + \|\Delta_H \partial_z T\|_2^2 + \varepsilon \|\nabla_H \Delta_H T\|_2^2 \\ &= - \int_{\Omega} \Delta_H (v \cdot \nabla_H T + w \partial_z T) \Delta_H T \, dx dy dz \\ &= - \int_{\Omega} (2 \nabla_H v \cdot \nabla_H^2 T + \Delta_H v \cdot \nabla_H T + 2 \nabla_H w \cdot \nabla_H \partial_z T + \Delta_H w \partial_z T) \Delta_H T \, dx dy dz \\ &= - \int_{\Omega} (2 \nabla_H v \cdot \nabla_H^2 T + \Delta_H v \cdot \nabla_H T) \Delta_H T \, dx dy dz \\ &\quad + 2 \int_{\Omega} (-\nabla_H (\nabla_H \cdot v) \cdot \nabla_H T \Delta_H T + \nabla_H w \cdot \nabla_H T \Delta_H \partial_z T) \, dx dy dz \\ &\quad + \int_{\Omega} (-\Delta_H (\nabla_H \cdot v) T \Delta_H T + \Delta_H w T \Delta_H \partial_z T) \, dx dy dz \\ &\leq 2 \int_{\Omega} |\nabla_H v| |\nabla_H^2 T|^2 \, dx dy dz + 3 \int_{\Omega} |\nabla_H^2 v| |\nabla_H T| |\Delta_H T| \, dx dy dz \\ &\quad + 2 \int_{\Omega} \left(\int_{-h}^z |\nabla_H (\nabla_H \cdot v)| \, d\xi \right) |\nabla_H T| |\Delta_H \partial_z T| \, dx dy dz \\ &\quad + \|T\|_\infty (\|\Delta_H (\nabla_H \cdot v)\|_2 \|\Delta_H T\|_2 + \|\Delta_H w\|_2 \|\Delta_H \partial_z T\|_2). \end{aligned} \quad (3.66)$$

Next, we are going to estimate the terms on the right-hand side of the above inequality. Recalling that $T|_{z=-h} = 0$, we have

$$|\Delta_H T(x, y, z, t)| \leq \int_{-h}^z |\Delta_H T(x, y, \xi, t)| \, d\xi.$$

Recalling the definitions of η, θ and Φ , (3.4) and (3.5), it follows from the two-dimensional horizontal elliptic estimates, the Ladyzhenskaya and Poincaré inequalities that

$$\begin{aligned} \|\nabla_H^2 v\|_{4,M}^2 &\leq C \|\Delta_H v\|_{4,M}^2 = C \|\nabla_H \nabla_H \cdot v - \nabla_H^\perp \nabla_H^\perp \cdot v\|_{4,M}^2 \\ &\leq C (\|\nabla_H (\nabla_H \cdot v)\|_{4,M}^2 + \|\nabla_H (\nabla_H^\perp \cdot v)\|_{4,M}^2) \\ &\leq C (\|\nabla_H \eta\|_{4,M}^2 + \|\nabla_H \Phi\|_{4,M}^2 + \|\nabla_H \theta\|_{4,M}^2) \\ &\leq C (\|\nabla_H (\eta, \theta)\|_{2,M} \|\Delta_H (\eta, \theta)\|_{2,M} + \|\nabla_H \Phi\|_{4,M}^2). \end{aligned}$$

On account of the above inequality, applying Lemma 2.2, recalling that $\|T\|_\infty \leq \|T_0\|_\infty$, and using (3.64), it follows from the Hölder and Young inequalities that

$$\begin{aligned}
& 3 \int_{\Omega} |\nabla_H^2 v| |\nabla_H T| |\Delta_H T| dx dy dz \\
& \leq C \int_M \left(\int_{-h}^h |\nabla_H^2 v| |\nabla_H T| dz \right) \left(\int_{-h}^h |\Delta_H \partial_z T| dz \right) dx dy \\
& \leq C \left(\int_{-h}^h \|\nabla_H T\|_{4,M}^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h \|\nabla_H^2 v\|_{4,M}^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h \|\Delta_H \partial_z T\|_{2,M} dz \right) \\
& \leq C \|\nabla_H T\|_4 \left[\int_{-h}^h (\|\nabla_H(\eta, \theta)\|_{2,M} \|\Delta_H(\eta, \theta)\|_{2,M} + \|\nabla_H \Phi\|_{4,M}^2) dz \right]^{\frac{1}{2}} \|\Delta_H \partial_z T\|_2 \\
& \leq C \|T\|_\infty^{\frac{1}{2}} \|\Delta_H T\|_2^{\frac{1}{2}} (\|\nabla_H(\eta, \theta)\|_2^{\frac{1}{2}} \|\Delta_H(\eta, \theta)\|_2^{\frac{1}{2}} + \|\nabla_H T\|_4) \|\Delta_H \partial_z T\|_2 \\
& \leq C \|T\|_\infty^{\frac{1}{2}} \|\Delta_H T\|_2^{\frac{1}{2}} (\|\nabla_H(\eta, \theta)\|_2^{\frac{1}{2}} \|\Delta_H(\eta, \theta)\|_2^{\frac{1}{2}} + \|T\|_\infty^{\frac{1}{2}} \|\Delta_H T\|_2^{\frac{1}{2}}) \|\Delta_H \partial_z T\|_2 \\
& \leq \frac{1}{16} \|\Delta_H \partial_z T\|_2^2 + C(\|\nabla_H(\eta, \theta)\|_2^2 \|\Delta_H(\eta, \theta)\|_2^2 + \|\Delta_H T\|_2^2). \tag{3.67}
\end{aligned}$$

By Lemma 2.2 and Lemma 2.3, it follows from (3.64) and the Hölder and Young inequalities that

$$\begin{aligned}
& 2 \int_{\Omega} \left(\int_{-h}^z |\nabla_H(\nabla_H \cdot v)| d\xi \right) |\nabla_H T| |\Delta_H \partial_z T| dx dy dz \\
& \leq 4 \int_M \left(\int_{-h}^h (|\nabla_H \eta| + |\nabla_H T|) dz \right) \left(\int_{-h}^h |\nabla_H T| |\Delta_H \partial_z T| dz \right) dx dy \\
& \leq 4 \left(\int_{-h}^h (\|\nabla_H \eta\|_{4,M} + \|\nabla_H T\|_{4,M}) dz \right) \left(\int_{-h}^h \|\nabla_H T\|_{4,M}^2 dz \right)^{\frac{1}{2}} \|\Delta_H \partial_z T\|_2 \\
& \leq C(\|\nabla_H \eta\|_2^{\frac{1}{2}} \|\Delta_H \eta\|_2^{\frac{1}{2}} + \|\nabla_H T\|_4) \|\nabla_H T\|_4 \|\Delta_H \partial_z T\|_2 \\
& \leq \frac{1}{16} \|\Delta_H \partial_z T\|_2^2 + C(\|\nabla_H T\|_4^4 + \|\nabla_H \eta\|_2^2 \|\Delta_H \eta\|_2^2) \\
& \leq \frac{1}{16} \|\Delta_H \partial_z T\|_2^2 + C(\|\Delta_H T\|_2^2 + \|\nabla_H \eta\|_2^2 \|\Delta_H \eta\|_2^2). \tag{3.68}
\end{aligned}$$

Recalling the definitions of η and Φ , (3.4) and (3.5), respectively, and using the Young inequality, one has

$$\begin{aligned}
& \|T\|_\infty (\|\Delta_H(\nabla_H \cdot v)\|_2 \|\Delta_H T\|_2 + \|\Delta_H w\|_2 \|\Delta_H \partial_z T\|_2) \\
& \leq C[(\|\Delta_H \eta\|_2 + \|\Delta_H T\|_2) \|\Delta_H T\|_2 + (\|\Delta_H \eta\|_2 + \|\Delta_H T\|_2) \|\Delta_H \partial_z T\|_2] \\
& \leq \frac{1}{16} \|\Delta_H \partial_z T\|_2^2 + C(\|\Delta_H \eta\|_2^2 + \|\Delta_H T\|_2^2). \tag{3.69}
\end{aligned}$$

Recalling that $\Delta_H T|_{z=-h} = 0$, we have

$$\begin{aligned} & \sup_{-h \leq z \leq h} \|\Delta_H T(\cdot, z, t)\|_{2,M}^2 = \sup_{-h \leq z \leq h} \int_M |\Delta_H T(x, y, z, t)|^2 dx dy \\ & = 2 \sup_{-h \leq z \leq h} \int_{-h}^z \int_M \Delta_H T \Delta_H \partial_z T dx dy d\xi \leq 2 \|\Delta_H T\|_2 \|\Delta_H \partial_z T\|_2. \end{aligned}$$

Thanks to the above, recalling the decomposition of v , i.e. (3.45), it follows from Proposition 3.9 and the Young inequality that

$$\begin{aligned} & 2 \int_{\Omega} |\nabla_H v| |\nabla_H^2 T|^2 dx dy dz \\ & \leq 2 \int_{-h}^h (\|\nabla_H \zeta\|_{\infty, M} + \|\nabla_H \varpi\|_{\infty, M}) \|\nabla_H^2 T\|_{2, M}^2 dz \\ & \leq 2 \left(\int_{-h}^h \|\nabla_H \zeta\|_{\infty, M} dz \right) \left(\sup_{-h \leq z \leq h} \|\Delta_H T(\cdot, z, t)\|_{2, M}^2 \right) \\ & \quad + 2 \left(\sup_{-h \leq z \leq h} \|\nabla_H \varpi(\cdot, z, t)\|_{\infty, M} \right) \|\Delta_H T\|_2^2 \\ & \leq C \|\Delta_H T\|_2 \|\Delta_H \partial_z T\|_2 (\|\nabla_H(\eta, \theta)\|_2 + 1) \log^{\frac{1}{2}}(e + \|\Delta_H(\eta, \theta)\|_2) \\ & \quad + C \|\Delta_H T\|_2^2 \log(e + \|\nabla_H T\|_q) \\ & \leq \frac{1}{16} \|\Delta_H \partial_z T\|_2^2 + C(1 + \|\nabla_H(\eta, \theta)\|_2^2) \|\Delta_H T\|_2^2 \\ & \quad \times \log(e + \|\Delta_H(\eta, \theta)\|_2 + \|\nabla_H T\|_4), \end{aligned}$$

from which, noticing that $\log z \leq \log(1+z) \leq z$, for $z > 0$, and using the Young inequality, one obtains

$$\begin{aligned} \int_{\Omega} |\nabla_H v| |\nabla_H^2 T|^2 dx dy dz & \leq \frac{1}{16} \|\Delta_H \partial_z T\|_2^2 + C(1 + \|\nabla_H(\eta, \theta)\|_2^2) \|\Delta_H T\|_2^2 \\ & \quad \times (1 + \|\Delta_H(\eta, \theta)\|_2^2 + \|\nabla_H T\|_4). \end{aligned} \quad (3.70)$$

Substituting (3.67)–(3.70) into (3.66) yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta_H T\|_2^2 + \frac{3}{4} \|\partial_z \Delta_H T\|_2^2 + \varepsilon \|\nabla_H \partial_z^2 T\|_2^2 \\ & \leq C(1 + \|\nabla_H(\eta, \theta)\|_2^2) (1 + \|\Delta_H(\eta, \theta)\|_2^2 + \|\nabla_H T\|_4) (\|\Delta_H T\|_2^2 + 1). \end{aligned}$$

Summing the above with (3.65) yields

$$\begin{aligned} & \frac{d}{dt} \|(\partial_z^2 T, \Delta_H T)\|_2^2 + \|(\partial_z^3 T, \partial_z \Delta_H T, \sqrt{\varepsilon} \nabla_H \partial_z^2 T, \sqrt{\varepsilon} \nabla_H \Delta_H T)\|_2^2 \\ & \leq C[(1 + \|\nabla_H(\eta, \theta)\|_2^2) (1 + \|\Delta_H(\eta, \theta)\|_2 + \|\nabla_H T\|_4) + \|v\|_{\infty}^2 + \|u\|_4^2] \\ & \quad \times (\|\Delta_H T\|_2^2 + \|\partial_z^2 T\|_2^2 + 1) + C(\|\varphi\|_2^2 \|\nabla_H \varphi\|_2^2 + \|\varphi\|_2^4 + 1), \end{aligned}$$

from which, by Corollary 3.1, Proposition 3.6, Proposition 3.11, recalling (3.58), and using the Gronwall inequality, one obtains

$$\sup_{0 \leq t \leq \mathcal{T}} \|(\partial_z^2 T, \Delta_H T)\|_2^2(t) + \int_0^{\mathcal{T}} \|(\partial_z^3 T, \partial_z \Delta_H T, \sqrt{\varepsilon} \nabla_H \partial_z^2 T, \sqrt{\varepsilon} \nabla_H \Delta_H T)\|_2^2 dt \leq C$$

for a positive constant C depending only on h, \mathcal{T} , and $\|(v_0, T_0)\|_{H^2}$. The conclusion follows from the above estimates by the elliptic estimates. \square

3.7. Uniform a priori estimates. With the aid of the energy inequalities established in the previous subsections, we can obtain the uniform estimates, which are independent of the regularization parameter ε , stated in the following proposition.

Proposition 3.13. *Given a positive time $\mathcal{T} \in (0, \infty)$ and let $\varepsilon_0 \in (0, 1)$ be as in Proposition 3.11. Suppose that $(v_0, T_0) \in H^2(\Omega)$ and $\varepsilon \in (0, \varepsilon_0)$. Let (v, T) be the unique global strong solution to system (3.1)–(3.3), subject to the boundary and initial conditions (1.21)–(1.23), and u, η and θ the functions defined by (3.4). Define two quantities \mathcal{Q}_1 and \mathcal{Q}_2 as follows*

$$\mathcal{Q}_1 := \|v_0\|_{H^2}^2 + \|T_0\|_{H^1}^2 + \|\nabla_H T_0\|_q^q + \|T_0\|_\infty^2, \quad \mathcal{Q}_2 := \|v_0\|_{H^2}^2 + \|T_0\|_{H^2}^2,$$

where $q \in (2, \infty)$.

Then, for any $\varepsilon \in (0, \varepsilon_0)$, we have the following a priori estimate:

$$\begin{aligned} & \sup_{0 \leq t \leq \mathcal{T}} (\|v\|_{H^2}^2(t) + \|T\|_{H^1}^2(t) + \|\nabla_H T\|_q^q(t) + \|p_s\|_{H^1(M)(t)}^2 + \|T\|_\infty^2(t)) \\ & + \int_0^{\mathcal{T}} (\|\nabla_H v\|_{H^1}^2 + \|\partial_z T\|_{H^1}^2 + \|\partial_t v\|_{H^1}^2 + \|\partial_t T\|_2^2 \\ & + \|\eta\|_{H^2}^2 + \|\theta\|_{H^2}^2 + \|\partial_t \eta\|_2^2 + \|\partial_t \theta\|_2^2) dt \leq C_1, \end{aligned}$$

for a positive constant C_1 depending only on h, \mathcal{T} , and the upper bound of \mathcal{Q}_1 , but is independent of $\varepsilon \in (0, \varepsilon_0)$, and

$$\sup_{0 \leq t \leq \mathcal{T}} \|T\|_{H^2}^2(t) + \int_0^{\mathcal{T}} (\|\nabla_H v\|_{H^2}^2 + \|\partial_z T\|_{H^2}^2 + \|\partial_t T\|_{H^1}^2) dt \leq C_2,$$

for a positive constant C_2 , depending only on h, \mathcal{T} and the upper bound of \mathcal{Q}_2 , but is independent of ε .

Proof. By Proposition 3.2, Corollary 3.1, Proposition 3.5, Proposition 3.6, and Proposition 3.11, we have the following

$$\|v\|_{L^\infty(0, \mathcal{T}; L^2)} + \|T\|_{L^\infty(0, \mathcal{T}; L^\infty)} + \|(\nabla_H v, \partial_z T)\|_{L^2(0, \mathcal{T}; L^2)} \leq C, \quad (3.71)$$

$$\begin{aligned} & \|(\eta, \theta)\|_{L^\infty(0, \mathcal{T}; L^2)} + \|u\|_{L^\infty(0, \mathcal{T}; L^4)} \\ & + \|(\nabla_H \eta, \nabla_H \theta, \nabla_H u, \sqrt{\varepsilon} \partial_z u)\|_{L^2(0, \mathcal{T}; L^2)} \leq C, \end{aligned} \quad (3.72)$$

$$\|\partial_z u\|_{L^\infty(0, \mathcal{T}; L^2)} + \|(\nabla_H \partial_z u, \sqrt{\varepsilon} \partial_z^2 u)\|_{L^2(0, \mathcal{T}; L^2)} \leq C, \quad (3.73)$$

$$\|(\varphi, \psi)\|_{L^\infty(0, \mathcal{T}; L^2)} + \|(\nabla_H \varphi, \nabla_H \psi)\|_{L^2(0, \mathcal{T}; L^2)} \leq C, \quad (3.74)$$

$$\begin{aligned} & \|\nabla T\|_{L^\infty(0,\mathcal{T};L^2)} + \|\nabla_H T\|_{L^\infty(0,\mathcal{T};L^q)} \\ & + \|(\partial_z^2 T, \nabla_H \partial_z T, \sqrt{\varepsilon} \Delta_H T)\|_{L^2(0,\mathcal{T};L^2)} \leq C, \end{aligned} \quad (3.75)$$

$$\|(\nabla_H \eta, \nabla_H \theta)\|_{L^\infty(0,\mathcal{T};L^2)} + \|(\Delta_H \eta, \Delta_H \theta, \sqrt{\varepsilon} \partial_z^2 \eta, \sqrt{\varepsilon} \partial_z^2 \theta)\|_{L^2(0,\mathcal{T};L^2)} \leq C, \quad (3.76)$$

where the constant C depending only on h, q, \mathcal{T} , and \mathcal{Q}_1 , but is independent of ε .

Thanks to (3.71)–(3.76), and recalling the definitions of η, θ, φ , and ψ , it follows from the two-dimensional horizontal elliptic estimate that

$$\begin{aligned} \|v\|_{L^\infty(0,\mathcal{T};H^2)} & \leq C(\|v\|_{L^\infty(0,\mathcal{T};L^2)} + \|\Delta_H v\|_{L^\infty(0,\mathcal{T};L^2)} + \|\partial_z u\|_{L^\infty(0,\mathcal{T};L^2)}) \\ & \leq C(1 + \|\Delta_H v\|_{L^\infty(0,\mathcal{T};L^2)}) \\ & = C(1 + \|\nabla_H \nabla_H \cdot v - \nabla_H^\perp \nabla_H^\perp \cdot v\|_{L^\infty(0,\mathcal{T};L^2)}) \\ & \leq C(1 + \|\nabla_H(\eta, \theta, T)\|_{L^\infty(0,\mathcal{T};L^2)}) \leq C, \end{aligned} \quad (3.77)$$

$$\begin{aligned} \|\nabla_H u\|_{L^2(0,\mathcal{T};H^1)}^2 & = \|\nabla_H u\|_{L^2(0,\mathcal{T};L^2)}^2 + \|\nabla_H \partial_z u\|_{L^2(0,\mathcal{T};L^2)}^2 + \|\nabla_H^2 u\|_{L^2(0,\mathcal{T};L^2)}^2 \\ & = \|\nabla_H u\|_{L^2(0,\mathcal{T};L^2)}^2 + \|\nabla_H \partial_z u\|_{L^2(0,\mathcal{T};L^2)}^2 + \|\Delta_H u\|_{L^2(0,\mathcal{T};L^2)}^2 \\ & = \|\nabla_H(u, \partial_z u)\|_{L^2(0,\mathcal{T};L^2)}^2 + \|\nabla_H \nabla_H \cdot u - \nabla_H^\perp \nabla_H^\perp \cdot u\|_{L^2(0,\mathcal{T};L^2)}^2 \\ & \leq \|\nabla_H(u, \partial_z u)\|_{L^2(0,\mathcal{T};L^2)}^2 + C\|\nabla_H(\varphi, \psi, T)\|_{L^2(0,\mathcal{T};L^2)}^2 \leq C, \end{aligned} \quad (3.78)$$

and

$$\begin{aligned} \|(\eta, \theta)\|_{L^2(0,\mathcal{T};H^2)} & \leq C(\|(\eta, \theta)\|_{L^2(0,\mathcal{T};L^2)} + \|\Delta_H(\eta, \theta)\|_{L^2(0,\mathcal{T};L^2)} + \|\partial_z^2(\eta, \theta)\|_{L^2(0,\mathcal{T};L^2)}) \\ & \leq C(1 + \|\partial_z \nabla_H u\|_{L^2(0,\mathcal{T};L^2)} + \|\partial_z T\|_{L^2(0,\mathcal{T};L^2)}) \leq C, \end{aligned} \quad (3.79)$$

where the constant C depending only on h, q, \mathcal{T} , and \mathcal{Q}_1 , but is independent of ε .

With the aid of (1.17), (3.4), (3.5), (3.71), and (3.72), it follows from the Sobolev inequality, two-dimensional horizontal elliptic estimates, and the Hölder inequality that

$$\begin{aligned} \|w\|_\infty & = \left\| \int_h^z (\nabla_H \cdot v) d\xi \right\|_\infty \leq \int_{-h}^h \|\nabla_H \cdot v\|_{\infty, M} dz \\ & \leq \int_{-h}^h (\|\eta\|_{\infty, M} + \|\Phi\|_{\infty, M}) dz \leq C \int_{-h}^h (\|\eta\|_{H^2(M)} + \|\Phi\|_{\infty, M}) dz \\ & \leq C \int_{-h}^h (\|\eta\|_{2, M} + \|\Delta_H \eta\|_{2, M} + \|\Phi\|_{\infty, M}) dz \\ & \leq C(\|\eta\|_2 + \|\Delta_H \eta\|_2 + \|\Phi\|_\infty) \leq C(1 + \|\Delta_H \eta\|_2), \end{aligned} \quad (3.80)$$

where the constant C depending only on h, q, \mathcal{T} , and \mathcal{Q}_1 , but is independent of ε .

Thanks to (3.75), (3.77), and (3.80), it follows from equation (3.3) and using the Sobolev embedding inequality that

$$\begin{aligned} \|\partial_t T\|_2^2 & \leq (\|v\|_\infty^2 \|\nabla_H T\|_2^2 + \|w\|_\infty^2 \|\partial_z T\|_2^2 + \|\partial_z^2 T\|_2^2 + \varepsilon^2 \|\Delta_H T\|_2^2) \\ & \leq C[\|v\|_{H^2}^2 \|\nabla_H T\|_2^2 + (1 + \|\Delta_H \eta\|_2^2) \|\partial_z T\|_2^2 + \|\partial_z^2 T\|_2^2 + \varepsilon^2 \|\Delta_H T\|_2^2] \end{aligned}$$

$$\leq C(1 + \|\Delta_H \eta\|_2^2 + \|\partial_z^2 T\|_2^2 + \varepsilon^2 \|\Delta_H T\|_2^2)$$

and, thus, recalling (3.75) and (3.79), we have

$$\|\partial_t T\|_{L^2(0, \mathcal{T}; L^2)} \leq C, \quad (3.81)$$

where the constant C depending only on h, q, \mathcal{T} , and \mathcal{Q}_1 , but is independent of ε . Recalling that $p_s(x, y, t)$ satisfies (see Appendix A (5.3))

$$\begin{cases} -\Delta_H p_s = \frac{1}{2h} \nabla_H \cdot \int_{-h}^h \left(\nabla_H \cdot (v \otimes v) + f_0 \vec{k} \times v - \int_{-h}^z \nabla_H T d\xi \right) dz, \\ \int_M p_s(x, y, t) dx dy = 0, \quad p_s \text{ is periodic in } x, y. \end{cases}$$

By elliptic estimates, Poincaré inequality, and recalling (3.75) and (3.77), we have

$$\begin{aligned} \|p_s\|_{H^1(M)}^2 &= (\|p_s\|_{2, M}^2 + \|\nabla_H p_s\|_{2, M}^2) \\ &\leq C \|\nabla_H p_s\|_{2, M}^2 \leq C (\|\nabla_H T\|_2^2 + \|\nabla_H \cdot (v \otimes v)\|_2^2 + \|v\|_2^2) \\ &\leq C (\|\nabla_H T\|_2^2 + \|v\|_\infty^2 \|\nabla_H v\|_2^2 + \|v\|_2^2) \leq C(1 + \|v\|_{H^2}^2) \leq C, \end{aligned} \quad (3.82)$$

where the constant C depending only on h, q, \mathcal{T} , and \mathcal{Q}_1 , but is independent of ε . Therefore, recalling (3.75), (3.77), (3.80), it follows from (3.1) and the Sobolev inequality that

$$\begin{aligned} \|\partial_t v\|_2^2 &\leq C (\|v\|_\infty^2 \|\nabla_H v\|_2^2 + \|w\|_\infty^2 \|\partial_z v\|_2^2 + \|\Delta_H v\|_2^2 + \varepsilon^2 \|\partial_z^2 v\|_2^2 \\ &\quad + \|v\|_2^2 + \|\nabla_H p_s\|_2^2 + \|\nabla_H T\|_2^2) \\ &\leq C (\|v\|_{H^2}^2 + 1 + \|\Delta_H \eta\|_2) \leq C(1 + \|\Delta_H \eta\|_2^2), \end{aligned}$$

which, recalling (3.79), gives

$$\|\partial_t v\|_{L^2(0, \mathcal{T}; L^2)} \leq C, \quad (3.83)$$

where the constant C depending only on h, q, \mathcal{T} , and \mathcal{Q}_1 , but is independent of ε .

For simplifying the notations, we introduce S and R as follows

$$S = (v \cdot \nabla_H) v + w \partial_z v + f_0 \vec{k} \times v, \quad R = \int_{-h}^z (\nabla_H \cdot (vT) - \varepsilon \Delta_H T) d\xi.$$

By the Hölder and Sobolev embedding inequalities, we have

$$\begin{aligned} &\int_\Omega |\nabla_H w|^2 |\partial_z v|^2 dx dy dz \\ &\leq C \int_M \left(\int_{-h}^h (|\nabla_H \eta| + |\nabla_H T|) dz \right)^2 \left(\int_{-h}^h |u|^2 dz \right) dx dy \\ &\leq C \left\| \int_{-h}^h |u|^2 dz \right\|_{\infty, M} (\|\nabla_H \eta\|_2^2 + \|\nabla_H T\|_2^2) \leq C \int_{-h}^h \|u\|_{\infty, M}^2 dz \\ &\leq C \int_{-h}^h (\|u\|_{2, M}^2 + \|\Delta_H u\|_{2, M}^2) dz \leq C \|(u, \Delta_H u)\|_2^2 \end{aligned}$$

$$\leq C(1 + \|\nabla_H u\|_{H^1}^2).$$

Thanks to the above and recalling (3.75), (3.77), and (3.80), it follows from the Sobolev inequality that

$$\begin{aligned} \|\nabla_H S\|_2^2 &\leq \int_{\Omega} (|v|^2 |\nabla_H^2 v|^2 + |\nabla_H v|^4 + |w|^2 |\nabla_H \partial_z v|^2 \\ &\quad + |\nabla_H w|^2 |\partial_z v|^2 + f_0^2 |\nabla_H v|^2) dx dy dz \\ &\leq C(\|v\|_{\infty}^2 \|\nabla_H^2 v\|_2^2 + \|\nabla_H v\|_4^4 + \|w\|_{\infty}^2 \|\nabla_H \partial_z v\|_2^2 + 1 + \|\nabla_H u\|_{H^1}^2) \\ &\leq C(\|v\|_{H^2}^4 + \|\Delta_H \eta\|_2^2 + 1 + \|\nabla_H u\|_{H^1}^2) \\ &\leq C(1 + \|\Delta_H \eta\|_2^2 + \|\nabla_H u\|_{H^1}^2) \end{aligned} \quad (3.84)$$

and

$$\begin{aligned} \|R\|_2^2 &\leq C(\|\nabla_H \cdot (vT)\|_2^2 + \varepsilon^2 \|\Delta_H T\|_2^2) \\ &\leq C(\|T\|_{\infty}^2 \|\nabla_H v\|_2^2 + \|v\|_{\infty}^2 \|\nabla_H T\|_2^2 + \varepsilon^2 \|\Delta_H T\|_2^2) \\ &\leq C(1 + \|v\|_{H^2}^2 + \varepsilon^2 \|\Delta_H T\|_2^2) \leq C(1 + \|\sqrt{\varepsilon} \Delta_H T\|_2^2), \end{aligned} \quad (3.85)$$

where the constant C depending only on h, q, \mathcal{T} , and \mathcal{Q}_1 , but is independent of ε . Recalling the expression of $f(x, y, t)$ in (3.9), one can easily check that

$$f = \frac{1}{2h} \int_{-h}^h (\nabla_H \cdot S + R + wT) dz,$$

and thus

$$\begin{aligned} \|f\|_2^2 &\leq C(\|\nabla_H S\|_2^2 + \|R\|_2^2 + \|\nabla_H v\|_2^2) \\ &\leq C(1 + \|(\Delta_H \eta, \sqrt{\varepsilon} \Delta_H T)\|_2^2 + \|\nabla_H u\|_{H^1}^2), \end{aligned} \quad (3.86)$$

where the constant C depending only on h, q, \mathcal{T} , and \mathcal{Q}_1 , but is independent of ε .

Thanks to (3.73), (3.75), (3.77), and (3.80), it follows from equation (3.6) and the Sobolev and Hölder inequalities that

$$\begin{aligned} \|\partial_t u\|_2^2 &\leq C(\|v\|_{\infty}^2 \|\nabla_H u\|_2^2 + \|w\|_{\infty}^2 \|\partial_z u\|_2^2 + \|\Delta_H u\|_2^2 + \varepsilon^2 \|\partial_z^2 u\|_2^2 \\ &\quad + \|u\|_2^2 + \|u\|_4^2 \|\nabla_H v\|_4^2 + \|\nabla_H T\|_2^2) \\ &\leq C[\|v\|_{H^2}^4 + (1 + \|\Delta_H \eta\|_2^2) \|v\|_{H^2}^2 + \|\Delta_H u\|_2^2 + \varepsilon^2 \|\partial_z^2 u\|_2^2 + 1] \\ &\leq C(1 + \|\Delta_H \eta\|_2^2 + \|\nabla_H u\|_{H^1}^2 + \|\sqrt{\varepsilon} \partial_z^2 u\|_2^2), \end{aligned} \quad (3.87)$$

where the constant C depending only on h, q, \mathcal{T} , and \mathcal{Q}_1 , but is independent of ε . Using (3.84)–(3.86), and recalling (3.75) and (3.77), it follows from equations (3.7)–(3.8) and the Sobolev and Hölder inequalities that

$$\begin{aligned} \|\partial_t \theta\|_2^2 &\leq \|\Delta_H \theta\|_2^2 + \varepsilon^2 \|\partial_z^2 \theta\|_2^2 + \|\nabla_H S\|_2^2 \\ &\leq C(\|(\Delta_H \theta, \sqrt{\varepsilon} \partial_z^2 \theta)\|_2^2 + 1 + \|\Delta_H \eta\|_2^2 + \|\nabla_H u\|_{H^1}^2) \end{aligned} \quad (3.88)$$

and

$$\begin{aligned} \|\partial_t \eta\|_2^2 &\leq \|\Delta_H \eta\|_2^2 + \varepsilon^2 \|\partial_z^2 \eta\|_2^2 + \|\nabla_H S\|_2^2 + (1 - \varepsilon)^2 \|\partial_z T\|_2^2 + \|wT\|_2^2 + \|R\|_2^2 + \|f\|_2^2 \\ &\leq C(\|(\Delta_H \eta, \sqrt{\varepsilon} \partial_z^2 \eta, \sqrt{\varepsilon} \Delta_H T)\|_2^2 + 1 + \|\nabla_H u\|_{H^1}^2) \end{aligned} \quad (3.89)$$

for a positive constant C depending only on h, q, \mathcal{T} , and \mathcal{Q}_1 , but is independent of ε .

Thanks to (3.73), (3.75), (3.76), (3.78), (3.81), (3.83), and using the elliptic estimates, it follows from (3.87)–(3.89) that

$$\begin{aligned} \|\partial_t T\|_{L^2(0, \mathcal{T}; L^2)}^2 + \|\partial_t v\|_{L^2(0, \mathcal{T}; H^1)}^2 &= \|(\partial_t T, \partial_t v, \partial_t u, \partial_t \nabla_H v)\|_{L^2(0, \mathcal{T}; L^2)}^2 \\ &\leq \|(\partial_t T, \partial_t v, \partial_t u)\|_{L^2(0, \mathcal{T}; L^2)}^2 + C(\|\nabla_H \cdot \partial_t v\|_2^2 + \|\nabla_H^\perp \cdot \partial_t v\|_2^2) \\ &\leq \|(\partial_t T, \partial_t v, \partial_t u)\|_{L^2(0, \mathcal{T}; L^2)}^2 + C\|(\partial_t \eta, \partial_t T, \partial_t \theta)\|_{L^2(0, \mathcal{T}; L^2)}^2 \leq C \end{aligned}$$

for a positive constant C depending only on h, q, \mathcal{T} , and \mathcal{Q}_1 , but is independent of ε . The first conclusion follows from the above inequality, (3.71), (3.75), (3.77)–(3.79), and (3.82).

We now prove the second conclusion. By Proposition 3.12, one has

$$\sup_{0 \leq t \leq \mathcal{T}} \|\nabla^2 T\|_2^2(t) + \int_0^{\mathcal{T}} (\|\partial_z \nabla^2 T\|_2^2 + \varepsilon \|\nabla_H \nabla^2 T\|_2^2) dt \leq C \quad (3.90)$$

for a positive constant C depending only on h, \mathcal{T} , and $\|(v_0, T_0)\|_{H^2}$, but is independent of ε . Recalling the expressions of η and θ , it follows from the elliptic estimates that

$$\|\nabla_H v\|_{H^2}^2 \leq C(\|\nabla_H \cdot v\|_{H^2}^2 + \|\nabla_H^\perp \cdot v\|_{H^2}^2) \leq C(\|\theta\|_{H^2}^2 + \|\eta\|_{H^2}^2 + \|T\|_{H^2}^2) \quad (3.91)$$

for a positive constant C depending only on h , but is independent of ε . Recalling (3.80) and the first conclusion, it follows from equation (3.3), the estimate (3.90), and the Hölder and Sobolev inequalities that

$$\begin{aligned} \|\nabla \partial_t T\|_2^2 &\leq \int_{\Omega} (|v|^2 |\nabla_H^2 T|^2 + |\nabla_H v|^2 |\nabla_H T|^2 + |w|^2 |\nabla_H \partial_z T|^2 \\ &\quad + |\nabla_H w|^2 |\partial_z T|^2 + |\nabla \partial_z^2 T|^2 + \varepsilon^2 |\nabla \Delta_H T|^2) dx dy dz \\ &\leq C(\|v\|_{\infty}^2 \|\nabla_H^2 T\|_2^2 + \|\nabla_H v\|_4^2 \|\nabla_H T\|_4^2 + \|w\|_{\infty}^2 \|\nabla_H \partial_z T\|_2^2 \\ &\quad + \|\nabla_H w\|_4^2 \|\partial_z T\|_4^2 + \|\nabla \partial_z^2 T\|_2^2 + \varepsilon^2 \|\nabla \Delta_H T\|_2^2) \\ &\leq C[\|v\|_{H^2}^2 \|T\|_{H^2}^2 + (1 + \|\Delta_H \eta\|_2^2) \|T\|_{H^2}^2 \\ &\quad + \|\nabla_H v\|_{H^2}^2 \|T\|_{H^2}^2 + \|\partial_z T\|_{H^2}^2 + \varepsilon^2 \|\nabla_H \Delta_H T\|_2^2] \\ &\leq C(1 + \|\Delta_H \eta\|_2^2 + \|\nabla_H v\|_{H^2}^2 + \|\partial_z T\|_{H^2}^2 + \varepsilon \|\nabla_H \Delta_H T\|_2^2) \end{aligned} \quad (3.92)$$

for a positive constant C depending only on h, \mathcal{T} , and $\|(v_0, T_0)\|_{H^2}$, but is independent of ε . Combining this inequality with (3.90)–(3.91), as well as the first conclusion, yields the second conclusion. \square

4. PROOF OF THEOREM 1.1

In this section, we consider the system with only horizontal viscosities and only vertical diffusivity, i.e., system (1.18)–(1.20), subject to the boundary and initial conditions (1.21)–(1.23), and establish the global well-posedness of strong solutions. In other words, we prove our main result, Theorem 1.1.

Proof of Theorem 1.1. Global existence. As in Proposition 3.13, we set

$$\mathcal{Q}_1 = \|v_0\|_{H^2}^2 + \|T_0\|_{H^1}^2 + \|\nabla_H T_0\|_q^q + \|T_0\|_\infty^2,$$

with $q \in (2, \infty)$. Thanks to the regularities and spatial symmetries of v_0 and T_0 , one can choose periodic functions $v_{0\varepsilon}$ and $T_{0\varepsilon}$, which are even and odd in z , respectively, such that $v_{0\varepsilon} \in H^2(\Omega)$, $T_{0\varepsilon} \in H^2(\Omega)$,

$$\int_{-h}^h \nabla_H \cdot v_{0\varepsilon}(x, y, z) dz = 0, \quad \|T_{0\varepsilon}\|_\infty \leq \|T_0\|_\infty,$$

and

$$v_{0\varepsilon} \rightarrow v_0 \text{ in } H^2(\Omega), \quad T_{0\varepsilon} \rightarrow T_0 \text{ in } H^1(\Omega), \quad \nabla_H T_{0\varepsilon} \rightarrow \nabla_H T_0 \text{ in } L^q(\Omega).$$

Note that such $v_{0\varepsilon}$ and $T_{0\varepsilon}$ can be chosen as the standard mollification of v_0 and T_0 , respectively. Set

$$\mathcal{Q}_{1\varepsilon} = \|v_{0\varepsilon}\|_{H^2}^2 + \|T_{0\varepsilon}\|_{H^1}^2 + \|\nabla_H T_{0\varepsilon}\|_q^q + \|T_{0\varepsilon}\|_\infty^2$$

then $\mathcal{Q}_{1\varepsilon} \leq 2\mathcal{Q}_1$, for sufficiently small ε . By Proposition 3.1, there is a unique global strong solution $(v_\varepsilon, T_\varepsilon)$ to system (3.1)–(3.3), subject to the boundary conditions (1.21)–(1.22) and the initial condition

$$(v_\varepsilon, T_\varepsilon)|_{t=0} = (v_{0\varepsilon}, T_{0\varepsilon}).$$

By Proposition 3.13, the following uniform estimate

$$\begin{aligned} & \sup_{0 \leq t \leq \mathcal{T}} (\|v_\varepsilon\|_{H^2}^2 + \|T_\varepsilon\|_{H^1}^2 + \|\nabla_H T_\varepsilon\|_q^q + \|\nabla_H p_\varepsilon\|_2^2 + \|T_\varepsilon\|_\infty^2) \\ & + \int_0^\mathcal{T} (\|\nabla_H u_\varepsilon\|_{H^1}^2 + \|\theta_\varepsilon\|_{H^2}^2 + \|\eta_\varepsilon\|_{H^2}^2 + \|\partial_z T_\varepsilon\|_{H^1}^2 + \|\partial_t \eta_\varepsilon\|_2^2 \\ & + \|\partial_t \theta_\varepsilon\|_2^2 + \|\partial_t v_\varepsilon\|_{H^1}^2 + \|\partial_t T_\varepsilon\|_{H^1}^2) \leq C, \end{aligned} \quad (4.1)$$

for a positive constant C depending only on h, \mathcal{T} , and \mathcal{Q}_1 and, thus, is independent of ε , here $u_\varepsilon, \eta_\varepsilon, \theta_\varepsilon$ are the associated functions defined by (3.4) and $p_\varepsilon = p_\varepsilon(x, y, t)$ is the associated pressure function determined by (1.24).

On account of the above a priori estimates, by the Aubin-Lions lemma, i.e. Lemma 2.6, there is a subsequence, still denoted by $(v_\varepsilon, T_\varepsilon)$, and (v, T) , such that

$$\begin{aligned} v_\varepsilon & \rightarrow v \text{ in } C([0, \mathcal{T}]; H^1(\Omega)), & T_\varepsilon & \rightarrow T \text{ in } C([0, \mathcal{T}]; L^2(\Omega)), \\ v_\varepsilon & \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0, \mathcal{T}; H^2(\Omega)), & \partial_t v_\varepsilon & \rightharpoonup \partial_t v \text{ in } L^2(0, \mathcal{T}; H^1(\Omega)), \\ T_\varepsilon & \overset{*}{\rightharpoonup} T \text{ in } L^\infty(0, \mathcal{T}; H^1(\Omega)), & \partial_t T_\varepsilon & \rightharpoonup \partial_t T \text{ in } L^2(0, \mathcal{T}; L^2(\Omega)), \end{aligned}$$

$$\begin{aligned}
\nabla_H u_\varepsilon &\rightharpoonup \nabla_H u \text{ in } L^2(0, \mathcal{T}; H^1(\Omega)), & \partial_z T_\varepsilon &\rightharpoonup \partial_z T \text{ in } L^2(0, \mathcal{T}; H^1(\Omega)), \\
\nabla_H T_\varepsilon &\overset{*}{\rightharpoonup} \nabla_H T \text{ in } L^\infty(0, \mathcal{T}; L^q(\Omega)), & p_\varepsilon &\rightharpoonup p_s \text{ in } L^2(0, \mathcal{T}; H^1(M)), \\
\theta_\varepsilon &\rightharpoonup \theta \text{ in } L^2(0, \mathcal{T}; H^2(\Omega)), & \partial_t \theta_\varepsilon &\rightharpoonup \partial_t \theta \text{ in } L^2(0, \mathcal{T}; L^2(\Omega)), \\
\eta_\varepsilon &\rightharpoonup \eta \text{ in } L^2(0, \mathcal{T}; H^2(\Omega)), & \partial_t \eta_\varepsilon &\rightharpoonup \partial_t \eta \text{ in } L^2(0, \mathcal{T}; L^2(\Omega)),
\end{aligned}$$

where \rightharpoonup and $\overset{*}{\rightharpoonup}$ denote the weak and weak-* convergences, respectively. Due to these convergences, one can take the limit $\varepsilon \rightarrow 0$ in systems (3.1)–(3.3) and (3.6)–(3.8), to show that (v, T) satisfies system (1.18)–(1.20), and (u, η, θ) , defined by (3.4), satisfies

$$\begin{aligned}
\partial_t u + (v \cdot \nabla_H)u + w \partial_z u - \Delta_H u + f_0 k \times u \\
+ (u \cdot \nabla_H)v - (\nabla_H \cdot v)u - \nabla_H T = 0,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\partial_t \eta - \Delta_H \eta = -\nabla_H \cdot [(v \cdot \nabla_H)v + w \partial_z v + f_0 k \times v] \\
+ \partial_z T - wT - \int_{-h}^z \nabla_H \cdot (vT) d\xi + f(x, y, t),
\end{aligned} \tag{4.3}$$

$$\partial_t \theta - \Delta_H \theta = -\nabla_H^\perp \cdot [(v \cdot \nabla_H)v + w \partial_z v + f_0 \vec{k} \times v], \tag{4.4}$$

in the sense of distribution, where the function $f = f(x, y, t)$ is now given by

$$f = \frac{1}{2h} \int_{-h}^h \left(\int_{-h}^z \nabla_H \cdot (vT) d\xi + wT + \nabla_H \cdot (\nabla_H \cdot (v \otimes v) + f_0 \vec{k} \times v) \right) dz. \tag{4.5}$$

Moreover, by the weakly lower semi-continuity of the norms and recalling that $(v_\varepsilon, T_\varepsilon)$ satisfies the a priori estimate (4.1), we can see that (v, T) still satisfies the same a priori estimate as (4.1). This implies the regularity properties stated in Definition 1.1 and, as a result, systems (1.18)–(1.20) and (4.2)–(4.4) are satisfied a.e. in $\Omega \times (0, \mathcal{T})$. Furthermore, recalling the first line of the previous convergences, one can easily show that (v, T) satisfies the initial condition (1.23) and, therefore, (v, T) is a strong solution to system (1.18)–(1.20), subject to the boundary and initial conditions (1.21)–(1.23).

Now, if we assume, in addition, that $T_0 \in H^2(\Omega)$, then the mollification $T_{0\varepsilon}$ converges strongly to T_0 in $H^2(\Omega)$. As a result, the quantity $\mathcal{Q}_{2\varepsilon} := \|v_{0\varepsilon}\|_{H^2}^2 + \|T_{0\varepsilon}\|_{H^2}^2$ is bounded by $2\mathcal{Q}_2 = 2(\|v_0\|_{H^2}^2 + \|T_0\|_{H^2}^2)$ for small ε . By Proposition 3.13, for small ε , we have the following uniform estimate

$$\sup_{0 \leq t \leq \mathcal{T}} \|T_\varepsilon\|_{H^2}^2(t) + \int_0^\mathcal{T} (\|\nabla_H v_\varepsilon\|_{H^2}^2 + \|\partial_z T_\varepsilon\|_{H^2}^2 + \|\partial_t T_\varepsilon\|_{H^1}^2) dt \leq C$$

for a positive constant C depending only on h, \mathcal{T} , and \mathcal{Q}_2 , but is independent of ε . This a priori estimate, by the weakly lower semi-continuity of the norms and the Aubin-Lions lemma, implies the additional regularities as stated in the theorem. This completes the proof of the existence part of the theorem.

Continuous dependence on the initial data. Let (v_1, T_1) and (v_2, T_2) be two strong solutions to the same system with initial data (v_{01}, T_{01}) and (v_{02}, T_{02}) ,

respectively. Denote $v = v_1 - v_2$, $w = w_1 - w_2$, and $T = T_1 - T_2$. Then, (v, T) satisfies

$$\begin{aligned} & \partial_t v + (v_1 \cdot \nabla_H) v + w_1 \partial_z v - \Delta_H v + f_0 \vec{k} \times v + \nabla_H p_s(x, y, t) \\ &= \int_{-h}^z \nabla_H T(x, y, \xi, t) d\xi - (v \cdot \nabla_H) v_2 - w \partial_z v_2, \end{aligned} \quad (4.6)$$

$$\partial_t T + v_1 \cdot \nabla_H T + w_1 \partial_z T - \partial_z^2 T = -v \cdot \nabla_H T_2 - w \partial_z T_2, \quad (4.7)$$

with initial data $(v_0, T_0) = (v_{01} - v_{02}, T_{01} - T_{02})$. Recalling the regularities of strong solutions stated in Definition 1.1, it is clear that all terms the above equations are well defined pointwisely.

Note that

$$|\nabla_H v_2(x, y, z, t)| \leq \frac{1}{2h} \int_{-h}^h |\nabla_H v_2(x, y, z, t)| dz + \int_{-h}^h |\nabla_H \partial_z v_2(x, y, z, t)| dz.$$

Recalling the regularities of (v, T) , multiplying equation (4.6) by v and integrating the resultant over Ω , it follows from integration by parts, the Hölder inequality, and Lemma 2.1 that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|v\|_2^2 + \|\nabla_H v\|_2^2 \\ &= \int_{\Omega} \left[\nabla_H \left(\int_{-h}^z T d\xi \right) - (v \cdot \nabla_H) v_2 - w \partial_z v_2 \right] \cdot v dx dy dz \\ &= - \int_{\Omega} \left[\left(\int_{-h}^z T d\xi \right) (\nabla_H \cdot v) + ((v \cdot \nabla_H) v_2 + w \partial_z v_2) \cdot v \right] dx dy dz \\ &\leq C \|T\|_2 \|\nabla_H v\|_2 + C \int_{\Omega} \left[|v|^2 \left(\int_{-h}^h (|\nabla_H \partial_z v_2| + |\nabla_H v_2|) dz \right) \right. \\ &\quad \left. + \left(\int_{-h}^h |\nabla_H \cdot v| dz \right) |\partial_z v_2| |v| \right] dx dy dz \\ &\leq C \|T\|_2 \|\nabla_H v\|_2 + C \int_M \left(\int_{-h}^h |v|^2 dz \right) \left(\int_{-h}^h (|\nabla_H v_2| + |\nabla_H \partial_z v_2|) dz \right) dx dy \\ &\quad + C \int_M \left(\int_{-h}^h |\nabla_H v| dz \right) \left(\int_{-h}^h |\partial_z v_2| |v| dz \right) dx dy \\ &\leq C \|T\|_2 \|\nabla_H v\|_2 + C \|v\|_2 (\|v\|_2 + \|\nabla_H v\|_2) (\|\nabla_H v_2\|_2 + \|\nabla_H \partial_z v_2\|_2) \\ &\quad + C \|\nabla_H v\|_2 \|\partial_z v_2\|_2^{\frac{1}{2}} (\|\partial_z v_2\|_2^{\frac{1}{2}} + \|\nabla_H \partial_z v_2\|_2^{\frac{1}{2}}) \|v\|_2^{\frac{1}{2}} (\|v\|_2^{\frac{1}{2}} + \|\nabla_H v\|_2^{\frac{1}{2}}) \\ &\leq \frac{1}{2} \|\nabla_H v\|_2^2 + C [\|T\|_2^2 + (1 + \|\nabla_H v_2\|_2^2 + \|\nabla_H \partial_z v_2\|_2^2) \|v\|_2^2 \\ &\quad + \|\partial_z v_2\|_2^2 (\|\partial_z v_2\|_2^2 + \|\nabla_H \partial_z v_2\|_2^2) \|v\|_2^2] \end{aligned}$$

$$\leq \frac{1}{2} \|\nabla_H v\|_2^2 + C(1 + \|\nabla v_2\|_2^2)^2 (1 + \|\nabla_H \partial_z v_2\|_2^2) (\|T\|_2^2 + \|v\|_2^2)$$

and, thus,

$$\frac{d}{dt} \|v\|_2^2 + \|\nabla_H v\|_2^2 \leq C(1 + \|\nabla v_2\|_2^2)^2 (1 + \|\nabla_H \partial_z v_2\|_2^2) (\|T\|_2^2 + \|v\|_2^2) \quad (4.8)$$

for $t \in (0, \mathcal{T})$.

Recalling the regularities of (v, T) , multiplying equation (1.20) by T , integrating the resulting equation over Ω , and noticing that $\|T_2\|_\infty \leq \|T_{02}\|_\infty$, it follows from integration by parts, (2.1), and the Ladyzhenskaya inequality that

$$\frac{1}{2} \frac{d}{dt} \|T\|_2^2 + \|\partial_z T\|_2^2 = - \int_\Omega (v \cdot \nabla_H T_2 + w \partial_z T_2) T dx dy dz \quad (4.9)$$

Noticing that $\|T_2\|_\infty \leq \|T_{02}\|_\infty$, it follows from integration by parts and the Young inequalities that

$$\begin{aligned} & \int_\Omega w \partial_z T_2 T dx dy dz = - \int_\Omega T_2 (\partial_z w T + w \partial_z T) dx dy dz \\ & = \int_\Omega ((\nabla_H \cdot v) T_2 T - w \partial_z T T_2) dx dy dz \leq \|T_2\|_\infty (\|\nabla_H v\|_2 \|T\|_2 + \|\partial_z T\|_2 \|w\|_2) \\ & \leq C \|T_2\|_\infty \|\nabla_H v\|_2 (\|T\|_2 + \|\partial_z T\|_2) \leq \frac{1}{4} \|\partial_z T\|_2^2 + C(\|\nabla_H v\|_2^2 + \|T\|_2^2). \end{aligned} \quad (4.10)$$

Note that $T|_{z=-h} = 0$, we have $|T| \leq \int_{-h}^h |\partial_z T| dz$. With the aid of this, by the Hölder, Minkowski, Gagaliardo-Nirenberg, and Young inequalities, we deduce

$$\begin{aligned} & - \int_\Omega v \cdot \nabla_H T_2 T dx dy dz \leq \int_M \left(\int_{-h}^h |v| |\nabla_H v_2| dz \right) \left(\int_{-h}^h |\partial_z T| dz \right) dx dy \\ & \leq \int_M \left(\int_{-h}^h |v|^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h |\nabla_H T_2|^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h |\partial_z T| dz \right) dx dy \\ & \leq \left\| \left(\int_{-h}^h |v|^2 dz \right)^{\frac{1}{2}} \right\|_{\frac{2q}{q-2}, M} \left\| \left(\int_{-h}^h |\nabla_H T_2|^2 dz \right)^{\frac{1}{2}} \right\|_{q, M} \left\| \int_{-h}^h |\partial_z T| dz \right\|_{2, M} \\ & \leq \left(\int_{-h}^h \|v\|_{\frac{2q}{q-2}, M}^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h \|\nabla_H T_2\|_{q, M}^2 dz \right)^{\frac{1}{2}} \left(\int_{-h}^h \|\partial_z T\|_{2, M} dz \right) \\ & \leq C \left[\int_{-h}^h \left(\|v\|_{2, M}^2 + \|v\|_{2, M}^{\frac{2}{q}(q-2)} \|\nabla_H v\|_{2, M}^{\frac{4}{q}} \right) dz \right]^{\frac{1}{2}} \|\nabla_H T_2\|_q \|\partial_z T\|_2 \\ & \leq C \left(\|v\|_2 + \|v\|_2^{\frac{q-2}{q}} \|\nabla_H v\|_2^{\frac{2}{q}} \right) \|\nabla_H T_2\|_q \|\partial_z T\|_2 \end{aligned}$$

$$\leq \frac{1}{4}(\|\partial_z T\|_2^2 + \|\nabla_H v\|_2^2) + C \left(\|\nabla_H T_2\|_q^2 + \|\nabla_H T_2\|_q^{\frac{2q}{q-2}} \right) \|v\|_2^2. \quad (4.11)$$

Substituting (4.10) and (4.11) into (4.9) yields

$$\frac{d}{dt} \|T\|_2^2 + \|\partial_z T\|_2^2 \leq C \|\nabla_H v\|_2^2 + C \left(1 + \|\nabla_H T_2\|_q^2 + \|\nabla_H T_2\|_q^{\frac{2q}{q-2}} \right) (\|v\|_2^2 + \|T\|_2^2), \quad (4.12)$$

for $t \in (0, \mathcal{T})$.

Multiplying (4.8) by a sufficiently large positive constant A , and summing the resulting inequality with (4.12) up yields

$$\begin{aligned} & \frac{d}{dt} (A\|v\|_2^2 + \|T\|_2^2) + \frac{1}{2} (A\|\nabla_H v\|_2^2 + \|T\|_2^2) \\ & \leq C \left[(1 + \|\nabla v_2\|_2^2)^2 (1 + \|\nabla_H \partial_z v_2\|_2^2) + \left(\|\nabla_H T_2\|_q^2 + \|\nabla_H T_2\|_q^{\frac{2q}{q-2}} \right) \right] (\|T\|_2^2 + \|v\|_2^2), \end{aligned}$$

from which, by the Gronwall inequality, one obtains

$$\begin{aligned} & \sup_{0 \leq s \leq t} (\|v\|_2^2(s) + \|T\|_2^2(s)) + \int_0^t (\|\nabla_H v\|_2^2 + \|\partial_z T\|_2^2) ds \\ & \leq C e^{\int_0^t \left[(1 + \|\nabla v_2\|_2^2)^2 (1 + \|\nabla_H \partial_z v_2\|_2^2) + \left(\|\nabla_H T_2\|_q^2 + \|\nabla_H T_2\|_q^{\frac{2q}{q-2}} \right) \right] ds} (\|v_0\|_2^2 + \|T_0\|_2^2) \end{aligned}$$

for any $t \in (0, \mathcal{T})$. This proves the continuous dependence of the strong solutions on the initial data, in particular the uniqueness. This completes the proof of the theorem. \square

5. APPENDIX A: EQUATIONS FOR η AND θ

In this appendix, we present the details of the derivation of the equations for η and θ , where η and θ are the same functions as defined by (3.4), i.e.,

$$\begin{aligned} \eta &= \nabla_H \cdot v + \int_{-h}^z T(x, y, \xi, t) d\xi - \frac{1}{2h} \int_{-h}^h \left(\int_{-h}^z T(x, y, \xi, t) d\xi \right) dz, \\ \theta &= \nabla_H^\perp \cdot v, \quad \nabla_H^\perp = (-\partial_y, \partial_x), \end{aligned}$$

with (v, T) being a strong solution to system (3.1)–(3.3), subject to the boundary and initial conditions (1.21)–(1.23).

Applying the operator $\nabla_H^\perp \cdot$ to equation (3.1), and noticing that $\nabla_H^\perp \cdot \nabla_H p_s = 0$, one obtains

$$\partial_t \theta - \Delta_H \theta - \varepsilon \partial_z^2 \theta = -\nabla_H^\perp \cdot [(v \cdot \nabla_H) v + w \partial_z v + f_0 k \times v], \quad (5.1)$$

obtaining the equation for θ .

Applying the operator $\nabla_H \cdot$, i.e., div_H , to equation (3.1), one gets

$$\begin{aligned} \partial_t(\nabla_H \cdot v) - \Delta_H \left(\nabla_H \cdot v + \int_{-h}^z T(x, y, \xi, t) d\xi - p_s(x, y, t) \right) - \varepsilon \partial_z^2(\nabla_H \cdot v) \\ = -\nabla_H \cdot [(v \cdot \nabla_H)v + w \partial_z v + f_0 k \times v]. \end{aligned} \quad (5.2)$$

Integrating the above equation with respect to z over the interval $(-h, h)$, and noticing

$$\int_{-h}^h [(v \cdot \nabla_H)v + w \partial_z v] dz = \int_{-h}^h [(v \cdot \nabla_H)v + (\nabla_H \cdot v)v] dz = \int_{-h}^h \nabla_H \cdot (v \otimes v) dz,$$

and (recalling $\int_{-h}^h \nabla_H \cdot v dz = 0$)

$$\int_{-h}^h [\partial_t(\nabla_H \cdot v) - \Delta_H(\nabla_H \cdot v) - \partial_z^2(\nabla_H \cdot v)] dz = 0,$$

we obtain

$$-\Delta_H p_s = \frac{1}{2h} \nabla_H \cdot \int_{-h}^h \left(\nabla_H \cdot (v \otimes v) + f_0 k \times v - \int_{-h}^z \nabla_H T d\xi \right) dz. \quad (5.3)$$

Substituting (5.3) into (5.2), one has

$$\begin{aligned} \partial_t(\nabla_H \cdot v) - \Delta_H \left(\nabla_H \cdot v + \int_{-h}^z T d\xi - \frac{1}{2h} \int_{-h}^h \int_{-h}^z T d\xi dz \right) - \varepsilon \partial_z^2(\nabla_H \cdot v) \\ = -\nabla_H \cdot [(v \cdot \nabla_H)v + w \partial_z v + f_0 k \times v] + \frac{1}{2h} \int_{-h}^h \nabla_H \cdot (\nabla_H \cdot (v \otimes v) + f_0 k \times v) dz, \end{aligned}$$

from which, recalling the definition of η , one arrives at

$$\begin{aligned} \partial_t \eta - \Delta_H \eta - \varepsilon \partial_z^2 \eta = -\nabla_H \cdot [(v \cdot \nabla_H)v + w \partial_z v + f_0 k \times v] - \varepsilon \partial_z T \\ + \frac{1}{2h} \int_{-h}^h \nabla_H \cdot (\nabla_H \cdot (v \otimes v) + f_0 k \times v) + \partial_t \left(\int_{-h}^z T d\xi - \frac{1}{2h} \int_{-h}^h \int_{-h}^z T d\xi dz \right). \end{aligned} \quad (5.4)$$

We compute the last term on the right-hand side of (5.4) as follows. On account of equation (3.3), we have

$$\begin{aligned} \int_{-h}^z \partial_t T d\xi &= - \int_{-h}^z (\nabla_H \cdot (vT) + \partial_z(wT) - \varepsilon \Delta_H T - \partial_z^2 T) d\xi \\ &= - \int_{-h}^z (\nabla_H \cdot (vT) - \varepsilon \Delta_H T) d\xi - wT + \partial_z T + (wT - \partial_z T)|_{z=-h}, \end{aligned}$$

and thus

$$\begin{aligned} \int_{-h}^z \partial_t T d\xi - \frac{1}{2h} \int_{-h}^h \int_{-h}^z \partial_t T d\xi dz \\ = - \int_{-h}^z (\nabla_H \cdot (vT) - \varepsilon \Delta_H T) d\xi - wT + \partial_z T \end{aligned}$$

$$+ \frac{1}{2h} \int_{-h}^h \left(\int_{-h}^z (\nabla_H \cdot (vT) - \varepsilon \Delta_H T) d\xi \right) dz + \frac{1}{2h} \int_{-h}^h wT dz.$$

Substituting the above equality into (5.4) yields

$$\begin{aligned} \partial_t \eta - \Delta_H \eta - \varepsilon \partial_z^2 \eta = & - \nabla_H \cdot [(v \cdot \nabla_H)v + w \partial_z v + f_0 k \times v] + (1 - \varepsilon) \partial_z T - wT \\ & - \int_{-h}^z (\nabla_H \cdot (vT) - \varepsilon \Delta_H T) d\xi + f(x, y, t), \end{aligned} \quad (5.5)$$

with function $f = f(x, y, t)$ given by

$$\begin{aligned} f = & \frac{1}{2h} \int_{-h}^h \left(\int_{-h}^z (\nabla_H \cdot (vT) - \varepsilon \Delta_H T) d\xi + wT \right) dz \\ & + \frac{1}{2h} \int_{-h}^h \nabla_H \cdot (\nabla_H \cdot (v \otimes v) + f_0 k \times v) dz. \end{aligned}$$

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