

## Global well-posedness on the derivative nonlinear Schrödinger equation revisited

— [Source link](#) 

Yifei Wu

**Published on:** 21 Apr 2014 - [arXiv: Analysis of PDEs](#)

Related papers:

- [On the derivative nonlinear Schrödinger equation](#)
- [Global well-posedness on the derivative nonlinear Schrödinger equation](#)
- [A Refined Global Well-Posedness Result for Schrödinger Equations with Derivative](#)
- [Finite energy solutions of nonlinear Schrödinger equations of derivative type](#)
- [An exact solution for a derivative nonlinear Schrödinger equation](#)

Share this paper:    

View more about this paper here: <https://typeset.io/papers/global-well-posedness-on-the-derivative-nonlinear-45mjz3e98i>

# ANALYSIS & PDE

Volume 8

No. 5

2015

YIFEI WU

**GLOBAL WELL-POSEDNESS ON THE  
DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION**

# GLOBAL WELL-POSEDNESS ON THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

YIFEI WU

As a continuation of our previous work, we consider the global well-posedness for the derivative nonlinear Schrödinger equation. We prove that it is globally well posed in the energy space, provided that the initial data  $u_0 \in H^1(\mathbb{R})$  with  $\|u_0\|_{L^2} < 2\sqrt{\pi}$ .

## 1. Introduction

We study the following Cauchy problem of the nonlinear Schrödinger equation with derivative (DNLS):

$$\begin{cases} i \partial_t u + \partial_x^2 u = i \partial_x (|u|^2 u), & t \in \mathbb{R}, x \in \mathbb{R}, \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}). \end{cases} \quad (1-1)$$

It arises from studying the propagation of circularly polarized Alfvén waves in magnetized plasma with a constant magnetic field; see [Mio et al. 1976; Mjølhus 1976; Sulem and Sulem 1999] and the references therein. The equation in (1-1) is  $L^2$ -critical and completely integrable. The  $H^1$ -solution of (1-1) obeys the following mass, energy, and momentum conservation laws:

$$M(u(t)) := \int_{\mathbb{R}} |u(t, x)|^2 dx = M(u_0), \quad (1-2)$$

$$E_D(u(t)) := \int_{\mathbb{R}} (|u_x(t, x)|^2 + \frac{3}{2} \operatorname{Im} |u(t, x)|^2 u(t, x) \overline{u_x(t, x)} + \frac{1}{2} |u(t, x)|^6) dx = E_D(u_0), \quad (1-3)$$

$$P_D(u(t)) := \operatorname{Im} \int_{\mathbb{R}} \overline{u(t, x)} u_x(t, x) dx - \frac{1}{2} \int_{\mathbb{R}} |u(t, x)|^4 dx = P_D(u_0). \quad (1-4)$$

Local well-posedness for the Cauchy problem (1-1) is well understood. It was proved in the energy space  $H^1(\mathbb{R})$  in [Hayashi 1993; Hayashi and Ozawa 1992; 1994], and earlier by Guo and Tan [1991] and Tsutsumi and Fukuda [1980; 1981] in smooth spaces. See [Biagioni and Linares 2001; Takaoka 1999; 2001] for local well-posedness and ill-posedness results for rough data below the energy space.

The global well-posedness for (1-1) has also been widely studied. By using mass and energy conservation laws, and the gauge transformations, Hayashi and Ozawa [1994; Ozawa 1996] proved that (1-1) is globally well-posed in the energy space  $H^1(\mathbb{R})$  under the condition

$$\|u_0\|_{L^2} < \sqrt{2\pi}. \quad (1-5)$$

---

The author was partially supported by the NSF of China (No. 11101042), and the Fundamental Research Funds for the Central Universities of China.

*MSC2010:* primary 35Q55; secondary 35A01.

*Keywords:* nonlinear Schrödinger equation with derivative, global well-posedness, energy space.



Here  $2\pi$  is the mass of the ground state  $Q$ , which is the unique (up to some symmetries) positive solution of the elliptic equation

$$-Q_{xx} + Q - \frac{3}{16}Q^5 = 0. \tag{1-6}$$

As shown in [Weinstein 1983],  $Q = 2[\cosh(2x)]^{-1/2}$ . Since  $Q$  is an optimizer for the Gagliardo–Nirenberg inequality (1-12), any function with mass strictly less than the mass of  $Q$  has positive energy.

Condition (1-5) was improved recently in [Wu 2013]. We proved that there exists a small constant  $\varepsilon_* > 0$  such that (1-1) is still globally well-posed in the energy space when the initial data satisfies  $\|u_0\|_{L^2} < \sqrt{2\pi} + \varepsilon_*$ . The result implies that, for (1-1), the ground state mass  $2\pi$  is not the threshold of the global well-posedness and blow-up. This is different from the  $L^2$ -critical power-type Schrödinger equation (the nonlinearity  $i\partial_x(|u|^2u)$  in (1-1) is replaced by  $-\frac{3}{16}|u|^4u$ ); see [Wu 2013] for further discussion.

For related results on the well-posedness and stability theory for the derivative nonlinear Schrödinger equation (1-1), see [Colin and Ohta 2006; Colliander et al. 2001; 2002; Grünrock and Herr 2008; Guo and Wu 1995; Herr 2006; Miao et al. 2011; Nahmod et al. 2012; Takaoka 2001; Thomann and Tzvetkov 2010; Win 2010].

In this paper, we continue to consider the  $L^2$ -assumption on initial data and obtain the global well-posedness as follows:

**Theorem 1.1.** *For any  $u_0 \in H^1(\mathbb{R})$  with*

$$\int_{\mathbb{R}} |u_0(x)|^2 dx < 4\pi, \tag{1-7}$$

*the Cauchy problem (1-1) is globally well-posed in  $H^1(\mathbb{R})$  and the solution  $u$  satisfies*

$$\|u\|_{L_t^\infty H_x^1} \leq C(\|u_0\|_{H^1}).$$

As  $2\pi = \|Q\|_{L^2}^2$ , we notice that there is also a solitary wave solution whose mass is  $4\pi$ , given by

$$u(t, x) = e^{3i/4 \int_{-\infty}^{x+t} |W(y)|^2 dy} e^{-it/4 - ix/2} W(x+t), \tag{1-8}$$

where  $W$  is the ground state of the elliptic equation

$$-W_{xx} + \frac{1}{2}W^3 - \frac{3}{16}W^5 = 0. \tag{1-9}$$

Up to some symmetries,

$$W(x) = 2(x^2 + 1)^{-1/2}. \tag{1-10}$$

Therefore, Theorem 1.1 indicates that the Cauchy problem (1-1) is globally well-posed in  $H^1(\mathbb{R})$  when  $\|u_0\|_{L^2} < \|W\|_{L^2}$ .

Compared to  $Q$ ,  $W$  is polynomial decaying at infinity. Furthermore,  $W$  is an optimal function of the sharp Gagliardo–Nirenberg inequality (see [Aguah 2006])

$$\|f\|_{L^6} \leq C_{\text{GN}} \|f\|_{L^4}^{8/9} \|f_x\|_{L^2}^{1/9}, \tag{1-11}$$

where we wrote  $C_{GN}$  for the sharp constant  $C_{GN} = 3^{1/6}(2\pi)^{-1/9}$ . This inequality plays an important role in the proof of our main theorem. There is also a comparison with another sharp Gagliardo–Nirenberg inequality (see [Weinstein 1983]),

$$\|f\|_{L^6}^6 \leq \frac{4}{\pi^2} \|f\|_{L^2}^4 \|f_x\|_{L^2}^2, \tag{1-12}$$

in which the equality is attained by  $Q$ , which was applied previously to prove the global well-posedness when  $\|u_0\|_{L^2} < \sqrt{2\pi}$ .

So there is an interesting problem of whether  $\|W\|_{L^2}^2 = 4\pi$  is the mass threshold of the global well-posedness and blowup for (1-1). See Section 3 below for further discussion.

Now let us have a look at the strategy of the proof of Theorem 1.1. Developed by Hayashi and Ozawa, the gauge transformation is an important tool to study the derivative nonlinear Schrödinger equation. Let

$$v(t, x) := e^{-3i/4 \int_{-\infty}^x |u(t,y)|^2 dy} u(t, x); \tag{1-13}$$

then, from (1-1),  $v$  is the solution of

$$i\partial_t v + \partial_x^2 v = \frac{1}{2}i|v|^2 v_x - \frac{1}{2}i v^2 \bar{v}_x - \frac{3}{16}|v|^4 v \tag{1-14}$$

with the initial data  $v_0 = \exp(-\frac{3}{4}i \int_{-\infty}^x |u_0(y)|^2 dy)u_0$ . Moreover,  $v$  obeys the same mass conservation law as (1-2), the energy conservation law (1-3) becomes

$$E(v(t)) := \|v_x(t)\|_{L_x^2}^2 - \frac{1}{16}\|v(t)\|_{L_x^6}^6 = E(v_0), \tag{1-15}$$

and the momentum conservation law (1-4) becomes

$$P(v(t)) := \text{Im} \int_{\mathbb{R}} \overline{v(t, x)} v_x(t, x) dx + \frac{1}{4} \int_{\mathbb{R}} |v(t, x)|^4 dx = P(v_0). \tag{1-16}$$

From the argument used in [Wu 2013] to prove the global well-posedness for the DNLS, an important consideration is the usage of the momentum conservation law. We observe that the key point is to give a small control of the following term from (1-16):

$$\text{Im} \int_{\mathbb{R}} \overline{v(t, x)} v_x(t, x) dx. \tag{1-17}$$

To be more precise, one may prove that

$$-\text{Im} \int_{\mathbb{R}} \overline{v(t, x)} v_x(t, x) dx \leq c \|v_x(t)\|_{L^2} \|v(t)\|_{L^2}, \tag{1-18}$$

where  $c$  is a positive constant. This is trivial for  $c = 1$  by Hölder’s inequality. Suppose that one can obtain the inequality with a suitable small constant  $c$ . Then the global well-posedness will follow. In [Wu 2013], by using the rigidity of the ground state  $Q$ , we proved that, if the mass is larger but close to  $2\pi$  and there is a time sequence  $\{t_n\}$  such that  $\|v(t_n)\|_{H^1}$  tends to infinity, then  $v(t_n)$  is close to  $Q$  up to some symmetries. Since  $Q$  is real-valued, (1-18) can be given for small  $c > 0$ .

In this paper, we present a different argument to prove the bound (1-18) under the suitable but explicit assumption of  $L^2$ -norm of the initial data. Our method here does not need to use the property of the ground

state  $Q$  of (1-6). As was previously mentioned, it depends heavily on the sharp Gagliardo–Nirenberg inequality (1-11). This is to be expected, since the norms involved in the inequality (1-11) are strongly related to the energy and momentum conservation laws.

Let us expand our argument. If  $\|v(t)\|_{H^1}$  tends to infinity, then, by the momentum and energy conservation laws, (1-18) is approximately

$$\frac{1}{4} \|v(t)\|_{L^4}^4 \approx -\operatorname{Im} \int_{\mathbb{R}} \overline{v(t, x)} v_x(t, x) dx \leq c \|v_x(t)\|_{L^2} \|v(t)\|_{L^2} \approx c \|v_0\|_{L^2} \|v(t)\|_{L^6}^3.$$

So, to obtain the small bound  $c$ , we turn to consider the quantity

$$f(t) := \frac{\|v(t)\|_{L^4}^4}{\|v(t)\|_{L^6}^3}.$$

Indeed, we shall prove that  $f^2$  obeys some cubic inequality. Thus, the condition for global well-posedness is transformed to finding the solution to an elementary cubic equation.

This paper is organized as follows. In Section 2, we present the proof of Theorem 1.1. In Section 3, we discuss some related problems.

### 2. The proof of Theorem 1.1

Let  $v$  be the function in (1-13), which is the solution of the equation (1-14). Note that

$$u_x = e^{3i/4 \int_{-\infty}^x |v(t,y)|^2 dy} \left( \frac{3}{4} i |v|^2 v + v_x \right).$$

Therefore, by the sharp Gagliardo–Nirenberg inequality (1-12) and mass conservation law, for any  $t \in \mathbb{R}$ ,

$$\begin{aligned} \|u_x(t)\|_{L^2} &\leq \|v_x(t)\|_{L^2} + \frac{3}{4} \|v(t)\|_{L^6}^3 \leq \|v_x(t)\|_{L^2} + \frac{3}{2\pi} \|v(t)\|_{L^2}^2 \|v_x(t)\|_{L^2} \\ &\leq \left( 1 + \frac{3}{2\pi} \|u_0\|_{L^2}^2 \right) \|v_x(t)\|_{L^2}. \end{aligned}$$

That is, the boundedness of  $v$  in  $H^1$ -norm implies the boundedness of  $u$  in  $H^1$ -norm. Therefore, to prove the theorem, we may consider the function  $v$  in (1-13) instead. To simplify the notations, we set

$$E_0 = E(v_0), \quad P_0 = P(v_0), \quad m_0 = M(v_0).$$

Furthermore, we assume  $m_0 > 2\pi$ . Otherwise, the global well-posedness has been proved in [Hayashi and Ozawa 1994; Wu 2013].

Let  $(-T_-(v_0), T_+(v_0))$  be the maximal lifespan of the solution  $v$  of (1-14). To prove Theorem 1.1, it is sufficient to obtain the (indeed uniformly) a priori estimate of the solutions in  $H^1$ -norm. That is,

$$\sup_{t \in (-T_-(v_0), T_+(v_0))} \|v_x(t)\|_{L^2} < +\infty.$$

As in [Wu 2013], we argue by contradiction. Suppose that there exists a sequence  $\{t_n\}_{n=1}^\infty$  with limit  $-T_-(v_0)$  or  $T_+(v_0)$  such that

$$\|v_x(t_n)\|_{L^2} \rightarrow +\infty \quad \text{as } n \rightarrow \infty. \tag{2-1}$$

Then, from the energy conservation law, we also have

$$\|v(t_n)\|_{L^6} \rightarrow +\infty \quad \text{as } n \rightarrow \infty.$$

Let us define the sequence  $\{f_n\}_{n=1}^\infty$  by

$$f_n = \frac{\|v(t_n)\|_{L^4}^4}{\|v(t_n)\|_{L^6}^3};$$

then we have both the lower and upper bounds of  $f_n$  as follows:

**Lemma 2.1.** *There exists a sequence  $\varepsilon_n$ , with  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , such that*

$$2C_{\text{GN}}^{-9/2} + \varepsilon_n \leq f_n \leq \sqrt{m_0}. \tag{2-2}$$

*Proof of Lemma 2.1.* From Hölder’s inequality, we have

$$\|v(t_n)\|_{L^4}^4 \leq \|v(t_n)\|_{L^2} \|v(t_n)\|_{L^6}^3 = \sqrt{m_0} \|v(t_n)\|_{L^6}^3,$$

and thus

$$f_n \leq \sqrt{m_0}.$$

On the other hand, from the sharp Gagliardo–Nirenberg inequality (1-11) and the energy conservation law (1-15), we have

$$\begin{aligned} f_n &\geq \frac{(C_{\text{GN}}^{-6} \|v(t_n)\|_{L^6}^6 \|v_x(t_n)\|_{L^2}^{-2/3})^{3/4}}{\|v(t_n)\|_{L^6}^3} = C_{\text{GN}}^{-9/2} \frac{\|v(t_n)\|_{L^6}^{3/2}}{\|v_x(t_n)\|_{L^2}^{1/2}} \\ &= 2C_{\text{GN}}^{-9/2} \frac{\|v(t_n)\|_{L^6}^{3/2}}{(\|v(t_n)\|_{L^6}^6 + 16E_0)^{1/4}} \\ &= 2C_{\text{GN}}^{-9/2} + \varepsilon_n, \end{aligned}$$

where

$$\varepsilon_n := 2C_{\text{GN}}^{-9/2} \frac{\|v(t_n)\|_{L^6}^{3/2} - (\|v(t_n)\|_{L^6}^6 + 16E_0)^{1/4}}{(\|v(t_n)\|_{L^6}^6 + 16E_0)^{1/4}}.$$

By the mean value theorem, we have

$$\varepsilon_n = O(\|v(t_n)\|_{L^6}^{-6}) \rightarrow 0.$$

This proves the lemma. □

By Lemma 2.1, and  $\|v(t_n)\|_{L^4}^4 = f_n \|v(t_n)\|_{L^6}^3$ , we have

$$\|v(t_n)\|_{L^4} \rightarrow +\infty \quad \text{as } n \rightarrow \infty.$$

In the spirit of [Banica 2004], we define

$$\phi(t, x) = e^{i\alpha x} v(t, x),$$

where the parameter  $\alpha$  depends on  $t$  and is given below. Then  $\phi_x(t, x) = e^{i\alpha x}(i\alpha v(t, x) + v_x(t, x))$ , and thus

$$\|\phi_x\|_{L^2}^2 = \|v_x\|_{L^2}^2 + 2\alpha \operatorname{Im} \int \bar{v}v_x \, dx + \alpha^2 \|v\|_{L^2}^2.$$

Subtracting  $\frac{1}{16}\|\phi\|_{L^6}^6 = \frac{1}{16}\|v\|_{L^6}^6$  from both sides yields

$$E(\phi) = E(v) + 2\alpha \operatorname{Im} \int \bar{v}v_x \, dx + \alpha^2 \|v\|_{L^2}^2.$$

By the mass and energy conservation laws (1-2) and (1-15), this gives

$$-2\alpha \operatorname{Im} \int \overline{v(t, x)}v_x(t, x) \, dx = -E(\phi(t)) + \alpha^2 m_0 + E_0. \tag{2-3}$$

On the other hand, using (1-11), we have

$$\begin{aligned} E(\phi(t_n)) &= \|\phi_x(t_n)\|_{L^2}^2 - \frac{1}{16}\|\phi(t_n)\|_{L^6}^6 \\ &\geq C_{\text{GN}}^{-18}\|\phi(t_n)\|_{L^6}^{18}\|\phi(t_n)\|_{L^4}^{-16} - \frac{1}{16}\|\phi(t_n)\|_{L^6}^6 \\ &= (C_{\text{GN}}^{-18}\|v(t_n)\|_{L^6}^{12}\|v(t_n)\|_{L^4}^{-16} - \frac{1}{16})\|\phi(t_n)\|_{L^6}^6 \\ &= (C_{\text{GN}}^{-18}f_n^{-4} - \frac{1}{16})\|v(t_n)\|_{L^6}^6. \end{aligned}$$

Combining this with (2-3) gives

$$-2\alpha \operatorname{Im} \int \overline{v(t_n, x)}v_x(t_n, x) \, dx \leq (\frac{1}{16} - C_{\text{GN}}^{-18}f_n^{-4})\|v(t_n)\|_{L^6}^6 + \alpha^2 m_0 + E_0,$$

which implies, for  $\alpha > 0$ ,

$$-\operatorname{Im} \int \overline{v(t_n, x)}v_x(t_n, x) \, dx \leq \frac{1}{2\alpha}(\frac{1}{16} - C_{\text{GN}}^{-18}f_n^{-4})\|v(t_n)\|_{L^6}^6 + \frac{1}{2}\alpha m_0 + \frac{1}{2\alpha}E_0. \tag{2-4}$$

For convenience, we define  $\beta_n$  as

$$\beta_n := m_0^{-1}(\frac{1}{16} - C_{\text{GN}}^{-18}f_n^{-4})\|v(t_n)\|_{L^6}^6.$$

We split this into two cases:

*Case 1:*  $\beta_n < 1$  for infinitely many  $n$ . This implies that, for such  $n$ ,

$$(\frac{1}{16} - C_{\text{GN}}^{-18}f_n^{-4})\|v(t_n)\|_{L^6}^6 < m_0.$$

Therefore, from (2-4), we have

$$-\operatorname{Im} \int \overline{v(t_n, x)}v_x(t_n, x) \, dx \leq \frac{1}{2\alpha}m_0 + \frac{1}{2}\alpha m_0 + \frac{1}{2\alpha}E_0. \tag{2-5}$$

In particular, choosing  $\alpha = 1$ , we obtain

$$-\operatorname{Im} \int \overline{v(t_n, x)}v_x(t_n, x) \, dx \leq m_0 + \frac{1}{2}E_0. \tag{2-6}$$



By the momentum conservation law (1-16), we have

$$\frac{1}{4} \|v(t_n)\|_{L^4}^4 = -\operatorname{Im} \int \overline{v(t_n, x)} v_x(t_n, x) dx + P_0. \tag{2-7}$$

Hence, combining this with (2-6) and (2-7), we obtain

$$\|v(t_n)\|_{L^4}^4 \leq 2(2m_0 + E_0 + 2P_0).$$

This contradicts  $\|v(t_n)\|_{L^4} \rightarrow +\infty$ , and thus we can rule out this case.

*Case 2:  $\beta_n \geq 1$  for all sufficiently large  $n$ .* In this case, we set  $\alpha = \alpha(t_n) = \sqrt{\beta_n}$ . Then (2-4) becomes

$$-\operatorname{Im} \int \overline{v(t_n, x)} v_x(t_n, x) dx \leq \frac{1}{4} \sqrt{m_0(1 - 16C_{\text{GN}}^{-18} f_n^{-4})} \|v(t_n)\|_{L^6}^3 + \frac{1}{2} \beta_n^{-1/2} E_0. \tag{2-8}$$

By (2-7) and (2-8),

$$\|v(t_n)\|_{L^4}^4 \leq \sqrt{m_0(1 - 16C_{\text{GN}}^{-18} f_n^{-4})} \|v(t_n)\|_{L^6}^3 + 2\beta_n^{-1/2} E_0 + 4P_0,$$

which implies that

$$f_n \leq \sqrt{m_0(1 - 16C_{\text{GN}}^{-18} f_n^{-4})} + (2\beta_n^{-1/2} E_0 + 4P_0) \|v(t_n)\|_{L^6}^{-3}.$$

This provides the inequality

$$f_n^6 \leq m_0 f_n^4 - 16m_0 C_{\text{GN}}^{-18} + f_n^4 \mathcal{R}_n, \tag{2-9}$$

where

$$\mathcal{R}_n = 2\sqrt{m_0(1 - 16C_{\text{GN}}^{-18} f_n^{-4})} (2\beta_n^{-1/2} E_0 + 4P_0) \|v(t_n)\|_{L^6}^{-3} + (2\beta_n^{-1/2} E_0 + 4P_0)^2 \|v(t_n)\|_{L^6}^{-6}.$$

Since  $\beta_n \geq 1$  and  $0 \leq 1 - 16C_{\text{GN}}^{-18} f_n^{-4} \leq 1$ , we have

$$\mathcal{R}_n \leq 2\sqrt{m_0} (2E_0 + 4P_0) \|v(t_n)\|_{L^6}^{-3} + (2E_0 + 4P_0)^2 \|v(t_n)\|_{L^6}^{-6} = O(\|v(t_n)\|_{L^6}^{-3}).$$

From Lemma 2.1, we have

$$f_n^4 \mathcal{R}_n = O(\|v(t_n)\|_{L^6}^{-3}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, for any small fixed  $\epsilon > 0$ , by choosing  $n$  large enough we have  $f_n^4 \mathcal{R}_n \leq \epsilon$ . Hence (2-9) becomes

$$f_n^6 \leq m_0 f_n^4 - 16m_0 C_{\text{GN}}^{-18} + \epsilon. \tag{2-10}$$

Let  $X = f_n^2$ ; then (2-10) becomes the inequality

$$X^3 - m_0 X^2 + b \leq 0, \tag{2-11}$$

where  $b = 16m_0 C_{\text{GN}}^{-18} - \epsilon > 0$ . Let

$$F(X) = X^3 - m_0 X^2 + b;$$

then  $F(X)$  attains its minimum value at  $\frac{2}{3}m_0$  in the region  $[0, \infty)$ . Therefore, there are two positive solutions  $X_1$  and  $X_2$  of the equation

$$X^3 - m_0X^2 + b = 0 \tag{2-12}$$

if and only if  $F(\frac{2}{3}m_0) < 0$ . In other words, the inequality (2-11) has no solution in the region  $[0, +\infty)$  if and only if

$$F(\frac{2}{3}m_0) > 0. \tag{2-13}$$

Hence, this leads to a contradiction under the condition (2-13).

Condition (2-13) is equivalent to

$$\frac{8}{27}m_0^3 - \frac{4}{9}m_0^3 + b > 0.$$

Since  $\epsilon$  is arbitrarily small, this reduces to

$$\frac{8}{27}m_0^3 - \frac{4}{9}m_0^3 + 16m_0C_{GN}^{-18} > 0,$$

which yields

$$m_0 < 6\sqrt{3}C_{GN}^{-9} = 6\sqrt{3}\frac{1}{3^{3/2}(2\pi)^{-1}} = 4\pi.$$

Therefore, (1-14) is globally well-posed when  $m_0 < 4\pi$ . This proves the theorem.

One may expect to get some profit from the restriction  $X \in (4C_{GN}^{-9}, m_0)$  (rather than  $[0, +\infty)$ ) given by Lemma 2.1. However, we cannot get any more from it. To see this, we note that, in the case  $m_0 \geq 4\pi$ , (2-11) is solved in the region  $[0, +\infty)$  by

$$X_1 < X < X_2,$$

and we claim that

$$4C_{GN}^{-9} < X_1 < X_2 < m_0. \tag{2-14}$$

Indeed, when  $m_0 \geq 4\pi$ ,

$$\frac{2}{3}m_0 \geq \frac{8}{3}\pi > 4C_{GN}^{-9} = \frac{8}{3\sqrt{3}}\pi,$$

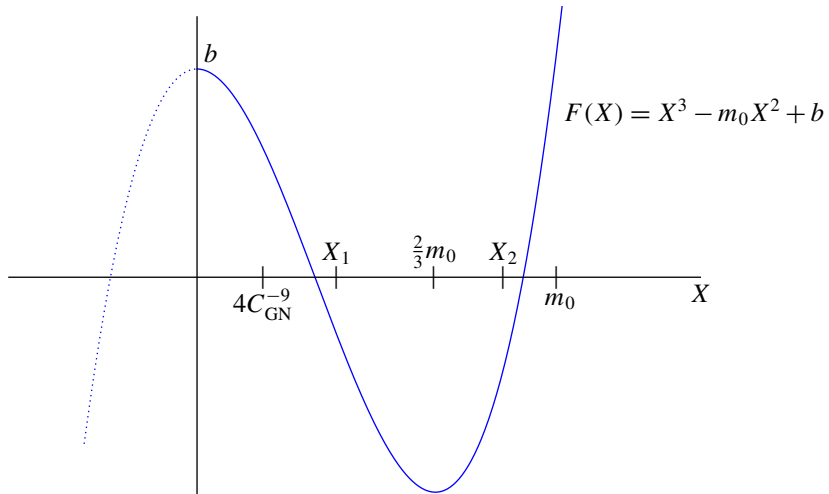
and, for small  $\epsilon$ , we have

$$F(4C_{GN}^{-9}) = 64C_{GN}^{-27} - \epsilon > 0,$$

which together imply that  $4C_{GN}^{-9} < X_1$ . Similarly, since

$$\frac{2}{3}m_0 < m_0 \quad \text{and} \quad F(m_0) = b > 0,$$

we have  $X_2 < m_0$ . In conclusion, we have (2-14). Therefore, the inequality (2-11) is always solvable in the region of  $(4C_{GN}^{-9}, m_0)$  when  $m_0 \geq 4\pi$ , and so we can not obtain the contradiction from the restriction of  $(4C_{GN}^{-9}, m_0)$ . We show this case graphically in Figure 1.



**Figure 1.** Graph of  $F(X)$ .

### 3. Further discussion

In this section, we would like to make a few remarks and indicate some related problems which remain open.<sup>1</sup>

First of all, whether or not the mass  $M(W) = 4\pi$  is the mass threshold for global well-posedness of (1-1) is not resolved in this paper. To understand the problem, we make some remarks on  $W$  and the equation (1-9) in the following.

As shown in [Colin and Ohta 2006; Guo and Wu 1995], (1-14) has a two-parameter family of solitary wave solutions,

$$v_{\omega,c} = \phi_{\omega,c}(x + ct)e^{i\omega t - (ic/2)(x+ct)}, \tag{3-1}$$

where  $(\omega, c) \in \mathbb{R}^2$  and  $\phi_{\omega,c}$  is a positive solution of the elliptic equation

$$-\partial_{xx}\phi + \left(\omega - \frac{1}{4}c^2\right)\phi + \frac{1}{2}c\phi^3 - \frac{3}{16}\phi^5 = 0. \tag{3-2}$$

When  $c^2 < 4\omega$ ,  $\phi_{\omega,c}$  can be written as

$$\phi_{\omega,c}(x) = \left\{ \frac{\sqrt{\omega}}{4\omega - c^2} \left[ \cosh(\sqrt{4\omega - c^2}x) - \frac{c}{2\sqrt{\omega}} \right] \right\}^{-\frac{1}{2}}.$$

Guo and Wu [1995] proved that the solitary wave solutions (3-1) are orbitally stable when  $c < 0$  and  $c^2 < 4\omega$ . This was extended by Colin and Ohta [2006], who proved the orbital stability for any  $c^2 < 4\omega$ .

Now we consider the other cases. From Pohožaev’s identity, there is no solution for (3-2) when  $4\omega \leq c^2$  and  $c \leq 0$ , and, from [Berestycki and Lions 1983] (see Section 6, Theorem 5), when  $c^2 > 4\omega$  (3-2) has no positive solution which vanishes at infinity. Hence, the only remaining case is the “zero mass” case,

<sup>1</sup>Part of the contents in this section are from discussions with Soonsik Kwon.

$c^2 = 4\omega$  and  $c > 0$ . Thus, the “zero mass” case can be regarded as the endpoint case in the family of the solitary wave solutions (3-1).

For the endpoint case  $c^2 = 4\omega$  and  $c > 0$ ,

$$-\partial_{xx}\phi + \frac{1}{2}c\phi^3 - \frac{3}{16}\phi^5 = 0$$

is exactly solved by

$$W_c(x) = c^{1/2}W(cx),$$

where  $W$  is as defined in (1-10). Moreover,

$$\|W_c\|_{L^2}^2 = \|W\|_{L^2}^2 = 4\pi.$$

So it is an interesting problem whether the solitary wave solution (1-8) is orbitally stable or unstable, which is not covered in [Colin and Ohta 2006; Guo and Wu 1995]. See [Ohta 2014] for related studies.

The existence of the finite-time blow-up solution is also an open problem for (1-1). There are some related results on the generalized derivative nonlinear Schrödinger equation,

$$i\partial_t u + \partial_x^2 u = i|u|^{2\sigma}\partial_x u, \quad \sigma > 1. \quad (3-3)$$

This is a mass supercritical equation. See [Ambrose and Simpson 2014; Hao 2007; Liu et al. 2013b] for local and stability theories. Numerical simulations by Liu, Simpson and Sulem [Liu et al. 2013a] suggest the existence of finite-time blow-up solutions for (3-3). However, a rigorous proof remains to be found.

### Acknowledgements

The author would like to thank Professor Martial Agueh for useful discussions on the sharp Gagliardo–Nirenberg inequality. The author is also grateful to Soonsik Kwon who kindly suggested incorporating some private discussions in this paper, and to the anonymous referee who carefully read the paper and gave many helpful comments and suggestions.

### References

- [Agueh 2006] M. Agueh, “Sharp Gagliardo–Nirenberg inequalities and mass transport theory”, *J. Dynam. Differential Equations* **18**:4 (2006), 1069–1093. MR 2008j:49108 Zbl 1155.35320
- [Ambrose and Simpson 2014] D. Ambrose and G. Simpson, “Local existence theory for derivative nonlinear Schrödinger equations with non-integer power nonlinearities”, preprint, 2014. To appear in *SIAM J. Math. Anal.* arXiv 1401.7060
- [Banica 2004] V. Banica, “Remarks on the blow-up for the Schrödinger equation with critical mass on a plane domain”, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **3**:1 (2004), 139–170. MR 2005e:35209 Zbl 1170.35528
- [Berestycki and Lions 1983] H. Berestycki and P.-L. Lions, “Nonlinear scalar field equations, I: Existence of a ground state”, *Arch. Rational Mech. Anal.* **82**:4 (1983), 313–345. MR 84h:35054a Zbl 0533.35029
- [Biagioni and Linares 2001] H. A. Biagioni and F. Linares, “Ill-posedness for the derivative Schrödinger and generalized Benjamin–Ono equations”, *Trans. Amer. Math. Soc.* **353**:9 (2001), 3649–3659. MR 2002e:35215 Zbl 0970.35154
- [Colin and Ohta 2006] M. Colin and M. Ohta, “Stability of solitary waves for derivative nonlinear Schrödinger equation”, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **23**:5 (2006), 753–764. MR 2007e:35255 Zbl 1104.35050
- [Colliander et al. 2001] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, “Global well-posedness for Schrödinger equations with derivative”, *SIAM J. Math. Anal.* **33**:3 (2001), 649–669. MR 2002j:35278 Zbl 1002.35113

- [Colliander et al. 2002] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, “A refined global well-posedness result for Schrödinger equations with derivative”, *SIAM J. Math. Anal.* **34**:1 (2002), 64–86. MR 2004c:35381 Zbl 1034.35120
- [Grünrock and Herr 2008] A. Grünrock and S. Herr, “Low regularity local well-posedness of the derivative nonlinear Schrödinger equation with periodic initial data”, *SIAM J. Math. Anal.* **39**:6 (2008), 1890–1920. MR 2009a:35233 Zbl 1156.35471
- [Guo and Tan 1991] B. L. Guo and S. B. Tan, “On smooth solutions to the initial value problem for the mixed nonlinear Schrödinger equations”, *Proc. Roy. Soc. Edinburgh Sect. A* **119**:1–2 (1991), 31–45. MR 92i:35114 Zbl 0766.35051
- [Guo and Wu 1995] B. L. Guo and Y. P. Wu, “Orbital stability of solitary waves for the nonlinear derivative Schrödinger equation”, *J. Differential Equations* **123**:1 (1995), 35–55. MR 96k:35166 Zbl 0844.35116
- [Hao 2007] C. Hao, “Well-posedness for one-dimensional derivative nonlinear Schrödinger equations”, *Commun. Pure Appl. Anal.* **6**:4 (2007), 997–1021. MR 2008k:35443 Zbl 1152.35104
- [Hayashi 1993] N. Hayashi, “The initial value problem for the derivative nonlinear Schrödinger equation in the energy space”, *Nonlinear Anal.* **20**:7 (1993), 823–833. MR 94c:35007 Zbl 0787.35099
- [Hayashi and Ozawa 1992] N. Hayashi and T. Ozawa, “On the derivative nonlinear Schrödinger equation”, *Phys. D* **55**:1–2 (1992), 14–36. MR 93h:35190 Zbl 0741.35081
- [Hayashi and Ozawa 1994] N. Hayashi and T. Ozawa, “Finite energy solutions of nonlinear Schrödinger equations of derivative type”, *SIAM J. Math. Anal.* **25**:6 (1994), 1488–1503. MR 95i:35272 Zbl 0809.35124
- [Herr 2006] S. Herr, “On the Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition”, *Int. Math. Res. Not.* **2006** (2006), Art. ID 96763, 33. MR 2007e:35258 Zbl 1149.35074
- [Liu et al. 2013a] X. Liu, G. Simpson, and C. Sulem, “Focusing singularity in a derivative nonlinear Schrödinger equation”, *Phys. D* **262** (2013), 48–58. MR 3144018
- [Liu et al. 2013b] X. Liu, G. Simpson, and C. Sulem, “Stability of solitary waves for a generalized derivative nonlinear Schrödinger equation”, *J. Nonlinear Sci.* **23**:4 (2013), 557–583. MR 3079669 Zbl 1271.35004
- [Miao et al. 2011] C. Miao, Y. Wu, and G. Xu, “Global well-posedness for Schrödinger equation with derivative in  $H^{1/2}(\mathbb{R})$ ”, *J. Differential Equations* **251**:8 (2011), 2164–2195. MR 2012i:35372 Zbl 1227.35236
- [Mio et al. 1976] K. Mio, T. Ogino, K. Minami, and S. Takeda, “Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas”, *J. Phys. Soc. Japan* **41**:1 (1976), 265–271. MR 57 #2116
- [Mjølhus 1976] E. Mjølhus, “On the modulational instability of hydromagnetic waves parallel to the magnetic field”, *J. Plasma Phys.* **16** (1976), 321–334.
- [Nahmod et al. 2012] A. R. Nahmod, T. Oh, L. Rey-Bellet, and G. Staffilani, “Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS”, *J. Eur. Math. Soc. (JEMS)* **14**:4 (2012), 1275–1330. MR 2928851 Zbl 1251.35151
- [Ohta 2014] M. Ohta, “Instability of solitary waves for nonlinear Schrödinger equations of derivative type”, preprint, 2014. arXiv 1408.5537
- [Ozawa 1996] T. Ozawa, “On the nonlinear Schrödinger equations of derivative type”, *Indiana Univ. Math. J.* **45**:1 (1996), 137–163. MR 98b:35186 Zbl 0859.35117
- [Sulem and Sulem 1999] C. Sulem and P.-L. Sulem, *The nonlinear Schrödinger equation: self-focusing and wave collapse*, Applied Mathematical Sciences **139**, Springer, New York, 1999. MR 2000f:35139 Zbl 0928.35157
- [Takaoka 1999] H. Takaoka, “Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity”, *Adv. Differential Equations* **4**:4 (1999), 561–580. MR 2000e:35221 Zbl 0951.35125
- [Takaoka 2001] H. Takaoka, “Global well-posedness for Schrödinger equations with derivative in a nonlinear term and data in low-order Sobolev spaces”, *Electron. J. Differential Equations* **42** (2001), 1–23. MR 2002f:35033
- [Thomann and Tzvetkov 2010] L. Thomann and N. Tzvetkov, “Gibbs measure for the periodic derivative nonlinear Schrödinger equation”, *Nonlinearity* **23**:11 (2010), 2771–2791. MR 2011j:35228 Zbl 1204.35154
- [Tsutsumi and Fukuda 1980] M. Tsutsumi and I. Fukuda, “On solutions of the derivative nonlinear Schrödinger equation: existence and uniqueness theorem”, *Funkcial. Ekvac.* **23**:3 (1980), 259–277. MR 83c:35108a Zbl 0478.35032
- [Tsutsumi and Fukuda 1981] M. Tsutsumi and I. Fukuda, “On solutions of the derivative nonlinear Schrödinger equation, II”, *Funkcial. Ekvac.* **24**:1 (1981), 85–94. MR 83c:35108b Zbl 0491.35016



- [Weinstein 1983] M. I. Weinstein, “Nonlinear Schrödinger equations and sharp interpolation estimates”, *Comm. Math. Phys.* **87**:4 (1983), 567–576. [MR 84d:35140](#) [Zbl 0527.35023](#)
- [Win 2010] Y. Y. S. Win, “Global well-posedness of the derivative nonlinear Schrödinger equations on  $T$ ”, *Funkcial. Ekvac.* **53**:1 (2010), 51–88. [MR 2011j:35229](#) [Zbl 1194.35433](#)
- [Wu 2013] Y. Wu, “Global well-posedness for the nonlinear Schrödinger equation with derivative in energy space”, *Anal. PDE* **6**:8 (2013), 1989–2002. [MR 3198590](#) [Zbl 1298.35207](#)

Received 14 Sep 2014. Revised 4 Feb 2015. Accepted 6 Mar 2015.

YIFEI WU: [yifei@bnu.edu.cn](mailto:yifei@bnu.edu.cn)

*Center for Applied Mathematics, Tianjin University, Tianjin 300072, China*

and

*School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Ministry of Education, Beijing, 100875, China*

# Analysis & PDE

[msp.org/apde](http://msp.org/apde)

## EDITORS

EDITOR-IN-CHIEF

Maciej Zworski  
[zworski@math.berkeley.edu](mailto:zworski@math.berkeley.edu)

University of California  
Berkeley, USA

## BOARD OF EDITORS

Nicolas Burq	Université Paris-Sud 11, France <a href="mailto:nicolas.burq@math.u-psud.fr">nicolas.burq@math.u-psud.fr</a>	Yuval Peres	University of California, Berkeley, USA <a href="mailto:peres@stat.berkeley.edu">peres@stat.berkeley.edu</a>
Sun-Yung Alice Chang	Princeton University, USA <a href="mailto:chang@math.princeton.edu">chang@math.princeton.edu</a>	Gilles Pisier	Texas A&M University, and Paris 6 <a href="mailto:pisier@math.tamu.edu">pisier@math.tamu.edu</a>
Michael Christ	University of California, Berkeley, USA <a href="mailto:mchrist@math.berkeley.edu">mchrist@math.berkeley.edu</a>	Tristan Rivière	ETH, Switzerland <a href="mailto:riviere@math.ethz.ch">riviere@math.ethz.ch</a>
Charles Fefferman	Princeton University, USA <a href="mailto:cf@math.princeton.edu">cf@math.princeton.edu</a>	Igor Rodnianski	Princeton University, USA <a href="mailto:irod@math.princeton.edu">irod@math.princeton.edu</a>
Ursula Hamenstaedt	Universität Bonn, Germany <a href="mailto:ursula@math.uni-bonn.de">ursula@math.uni-bonn.de</a>	Wilhelm Schlag	University of Chicago, USA <a href="mailto:schlag@math.uchicago.edu">schlag@math.uchicago.edu</a>
Vaughan Jones	U.C. Berkeley & Vanderbilt University <a href="mailto:vaughan.f.jones@vanderbilt.edu">vaughan.f.jones@vanderbilt.edu</a>	Sylvia Serfaty	New York University, USA <a href="mailto:serfaty@cims.nyu.edu">serfaty@cims.nyu.edu</a>
Herbert Koch	Universität Bonn, Germany <a href="mailto:koch@math.uni-bonn.de">koch@math.uni-bonn.de</a>	Yum-Tong Siu	Harvard University, USA <a href="mailto:siu@math.harvard.edu">siu@math.harvard.edu</a>
Izabella Laba	University of British Columbia, Canada <a href="mailto:ilaba@math.ubc.ca">ilaba@math.ubc.ca</a>	Terence Tao	University of California, Los Angeles, USA <a href="mailto:tao@math.ucla.edu">tao@math.ucla.edu</a>
Gilles Lebeau	Université de Nice Sophia Antipolis, France <a href="mailto:lebeau@unice.fr">lebeau@unice.fr</a>	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA <a href="mailto:met@math.unc.edu">met@math.unc.edu</a>
László Lempert	Purdue University, USA <a href="mailto:lempert@math.purdue.edu">lempert@math.purdue.edu</a>	Gunther Uhlmann	University of Washington, USA <a href="mailto:gunther@math.washington.edu">gunther@math.washington.edu</a>
Richard B. Melrose	Massachusetts Institute of Technology, USA <a href="mailto:rbm@math.mit.edu">rbm@math.mit.edu</a>	András Vasy	Stanford University, USA <a href="mailto:andras@math.stanford.edu">andras@math.stanford.edu</a>
Frank Merle	Université de Cergy-Pontoise, France <a href="mailto:Frank.Merle@u-cergy.fr">Frank.Merle@u-cergy.fr</a>	Dan Virgil Voiculescu	University of California, Berkeley, USA <a href="mailto:dvv@math.berkeley.edu">dvv@math.berkeley.edu</a>
William Minicozzi II	Johns Hopkins University, USA <a href="mailto:minicozz@math.jhu.edu">minicozz@math.jhu.edu</a>	Steven Zelditch	Northwestern University, USA <a href="mailto:zelditch@math.northwestern.edu">zelditch@math.northwestern.edu</a>
Werner Müller	Universität Bonn, Germany <a href="mailto:mueller@math.uni-bonn.de">mueller@math.uni-bonn.de</a>		

## PRODUCTION

[production@msp.org](mailto:production@msp.org)

Silvio Levy, Scientific Editor

---

See inside back cover or [msp.org/apde](http://msp.org/apde) for submission instructions.

---

The subscription price for 2015 is US \$205/year for the electronic version, and \$390/year (+\$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

---

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

---

APDE peer review and production are managed by EditFLOW<sup>®</sup> from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2015 Mathematical Sciences Publishers

# ANALYSIS & PDE

Volume 8 No. 5 2015

---

Partial collapsing and the spectrum of the Hodge–de Rham operator	1025
COLETTE ANNÉ and JUNYA TAKAHASHI	
Sharp $L^p$ bounds for the wave equation on groups of Heisenberg type	1051
DETLEF MÜLLER and ANDREAS SEEGER	
Global well-posedness on the derivative nonlinear Schrödinger equation	1101
YIFEI WU	
On the boundary value problem for the Schrödinger equation: compatibility conditions and global existence	1113
CORENTIN AUDIARD	
On estimates for fully nonlinear parabolic equations on Riemannian manifolds	1145
BO GUAN, SHUJUN SHI and ZHENAN SUI	
Concentration phenomena for the nonlocal Schrödinger equation with Dirichlet datum	1165
JUAN DÁVILA, MANUEL DEL PINO, SERENA DIPIERRO and ENRICO VALDINOCI	
Local spectral asymptotics for metric perturbations of the Landau Hamiltonian	1237
TOMÁS LUNGENSTRASS and GEORGI RAIKOV	
Hilbert transform along measurable vector fields constant on Lipschitz curves: $L^2$ boundedness	1263
SHAOMING GUO	