Forum Math. **19** (2007), 633–675
DOI 10.1515/FORUM.2007.025

Forum Mathematicum

Global wellposedness and scattering for 3D energy critical Schrödinger equation with repulsive potential and radial data

Zhang Xiaoyi

(Communicated by Christopher D. Sogge)

Abstract. In this paper, we show that the Cauchy problem of the 3D nonlinear Schrödinger equation with repulsive potential is globally wellposed if the initial data u_0 is spherically symmetric and $u_0 \in \Sigma = \{f, f \in H^1, xf \in L^2\}$. We also prove that the scattering operator is holomorphic from the radial functions in Σ to themselves. In order to preclude the possible energy concentration, we first show the energy concentration may occur only at finite time by using the decay estimate of potential energy $\|u(t)\|_6$, then we preclude the possible finite time energy concentration by inductive arguments.

2000 Mathematics Subject Classification: 35Q55; 35A05, 35B30, 35B35.

1 Introduction

In this paper, we consider the Cauchy problem of defocusing energy critical equation with repulsive potential,

$$\left(i\partial_t + \frac{\Delta}{2}\right)u = -\frac{|x|^2}{2}u + |u|^4 u, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^3, \tag{1.2}$$

where $u(t,x)$ is a complex function on $\mathbb{R}^3 \times \mathbb{R}$, $u_0(x)$ is a complex function on $\mathbb{R}^3$ satisfying

$$u_0 \in \Sigma = \{v; \|v\|_\Sigma = \|v\|_{H^1} + \|xv\|_2 < \infty\}^1.$$

We are concerned with the global existence and long time behavior of the global solution.

The author is supported in part by N.S.F. grant No. 10426029 and the Morningside Center.

[1] In three and higher dimension, the facts that $xu \in L^2$ and $\nabla u \in L^2$ imply automatically that $u \in L^2$, and of course imply that $u \in \Sigma$ by Hardy and Hölder inequality.

Nonlinear Schrödinger equation without potential has been extensively studied. Here, we restrict our discussion to the Cauchy problem of the equation with power-like nonlinearity:

$$(1.3)\qquad \left(i\partial_t + \frac{\Delta}{2}\right)v = \mu|v|^p v, \quad \mu \neq 0,$$

$$(1.4)\qquad v(0) = v_0 \in H_x^1,$$

μ is a constant, the cases $\mu > 0$ and $\mu < 0$ correspond to defocusing and focusing respectively. (1.3) has energy conservation,

$$E(t) = \frac{1}{2}\|\nabla v(t)\|_2^2 + \frac{\mu}{p+2}\|v(t)\|_{p+2}^{p+2} = const.$$

The local theory of the Cauchy problem (1.3)–(1.4) can be roughly summarized as follows: (1.3) has a unique local solution in $C([0,T);H^1)$, when p is smaller than or equals to a certain exponent $p_c = \frac{4}{n-2}$ ($p_c = 4$ when $n = 3$) which is called energy critical for the reason that the natural scale invariance

$$v(x,t) \to \lambda^{-2/p}v(\lambda^{-1}x, \lambda^{-2}t)$$

of the equation leaves the energy invariant. In the supercritical case $p > p_c$, (1.3) is locally illposed in the sense that the solution $v(x,t)$ does not depend continuously on the initial data $v_0(x)$ in H^1 space. To get more details, see [4], [5], [10], [6] for instance. In the defocusing and subcritical case, the global existence is a direct consequence of the energy conservation. In the focusing case $\mu < 0$, blow up in finite time may occur, especially when the influence of potential energy $\|v\|_{p+2}^{p+2}$ surpasses that of kinetic energy $\|\nabla v\|_2^2$, see [9], [4] to get more details. The case of energy critical becomes rather difficult because the pure energy conservation is not enough to ensure the finite energy solution to exist globally. In other words, even the total energy of the solution is finite, part of it may possibly concentrate somewhere in space, so that the solution blows up in finite time.

The first important work in this area is due to J. Bourgain [1] and M. Grillakis [12]. They pointed out that the solution will concentrate somewhere in $\mathbb{R}^d$ unless the solution exists globally. To preclude this possibility, they use a Morawetz type estimate,

$$(1.5)\qquad \int_I \int_{K|I|^{1/2}} \frac{|v|^{p_c+2}(x,t)}{|x|}\,dx\,dt \le CKE(v)|I|^{1/2}, \quad K \ge 1.$$

The estimates of this type is originally due to J. Lin and W. Strauss in [14], and is adapted here to better suit the energy critical problem of NLS. Actually, (1.5) is useful for preventing the concentration of $v(t,x)$ at origin $x = 0$. This is especially helpful when the solution is radially symmetric since in this case, one can use the bounded energy estimate to show that v will not concentrate at any other location than the origin. Their results are restricted to $d = 3, 4$. Recently in [16], T. Tao proved the energy critical Cauchy problem with radial data is globally wellposed and scattering result holds in all dimensions $d \geq 3$ by a sightly simple approach.

A great breakthrough was made by J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao in [7], where they obtained the global wellposedness and scattering for 3D energy critical NLS with arbitrary initial data. The results were extended to 4D by E. Ryckman and M. Visan in [15] and were extended to all dimensions $n \geq 5$ by M. Visan [18].

Schrödinger equation with harmonic potential and power-like nonlinearity can be written in the form,

$$(1.6) \qquad \left(i\partial_t + \frac{\Delta}{2}\right)u = \frac{\omega|x|^2}{2}u + \mu|u|^p u.$$

Being similar with the NLS equation without potential, we call (1.6) is energy subcritical and critical if $p < p_c$ and $p = p_c$, with the same p_c defined before. In both subcritical and critical case, because of the presence of the potential term, it's natural to seek for the solution of the Cauchy problem for this equation with suitable decay in space, ie,

$$(1.7) \qquad u \in C_t(\Sigma).$$

Indeed, recently in [2], [3], R. Carles systematically studied the Cauchy problem for equation (1.6) with subcritical nonlinearity. He found that in the defocusing case and when a repulsive potential (ie., $\omega = -1$) is involved, the solution will be global and a scattering theory is available. In the focusing case, a sufficient strong repulsive potential will even prevent blow up in finite time. By "strong repulsive" potential, he means that $\omega < 0$ and $|\omega|$ is sufficiently large.

The problem remains open as to what will happen when the nonlinearity is energy critical, that is, $p = p_c = \frac{4}{n-2}$, $n \geq 3$. In this paper, we restrict our attention to the case $n = 3$, $\omega = -1$, $\mu = 1$ and the radial solution. The higher dimensional case will be considered elsewhere. We will show that the Cauchy problem (1.1)–(1.2) is globally wellposed, and the scattering operator is holomorphic from the radial function in Σ to itself. In the remaining part of this introduction, we will sketch the proofs.

To begin with, we see that the equation (1.1) has mass and energy conservation,

$$(1.8) \qquad \|u(t)\|_2 = const,$$

$$(1.9) \qquad E(u(t)) = \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{2}\|xu(t)\|_2^2 + \frac{1}{3}\|u(t)\|_6^6 = const,$$

the energy (1.9) is non-positive, so we split it into two positive parts:

$$
(1.10)\qquad \begin{aligned} E_1(u(t)) &= \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{3}\|u(t)\|_6^6, \\ E_2(u(t)) &= \frac{1}{2}\|xu(t)\|_2^2, \end{aligned}
$$

and consider the Cauchy problem with $E_1(u(0)) = E_1$, $E_2(u(0)) = E_2$ for some fixed constant $E_1 > 0$, $E_2 > 0$. Occasionally, we may drop u from the notation $E_i(u(t))$ for simplicity.

Although $E_1(t)$ and $E_2(t)$ are nonnegative all the time, we don't have any easily obtained information about their evolution. For this reason, we introduce another way that is provided by R. Carles [2], [3] to split the energy:

$$
(1.11)\qquad \begin{aligned} \mathscr{E}_1(t) &= \frac{1}{2}\|J(t)u(t)\|_2^2 + \frac{1}{3}\cosh^2 t\|u(t)\|_6^6, \\ \mathscr{E}_2(t) &= \frac{1}{2}\|H(t)u(t)\|_2^2 + \frac{1}{3}\sinh^2 t\|u(t)\|_6^6, \end{aligned}
$$

$\mathscr{E}_1(t) - \mathscr{E}_2(t) = E(t)$ and they coincide with $E_1(t)$ and $E_2(t)$ only at $t = 0$. The benefit of this decomposition is that neither of $\mathscr{E}_1(t)$, $\mathscr{E}_2(t)$ increases in time, also, the potential energy decays fast. (See Section 2 for details). Using these facts in addition to Strichartz estimate, it's not difficult to get global solution in the subcritical case. However, in the critical case, the decay estimates of $\mathscr{E}_1(t)$ and $\mathscr{E}_2(t)$ can't prevent immediately the occurrence of the finite time blow up.

Next, we will establish the local wellposedness, give as well the blow up criterion which says that the local solution will blow up in finite time unless it has certain spacetime bound. In principle, there are many choices about the spacetime bound, we prefer to working on the L^{10} spacetime bound for the simplicity of the exposition (L^{10} bound here will be defined clearly later). In addition, we get the global wellposedness provided the kinetic energy of the initial data $\|\nabla u_0\|_2$ is sufficiently small.

To extend the local solution to a global one, we will prove the "good local wellposedness" which means the time length of the local solution will be proved to depend on the Σ norm of the initial data only. On the other hand, since the quantities $E_1(t)$, $E_2(t)$ have at most exponentially growth by using the boundedness of $\mathscr{E}_1(t)$, $\mathscr{E}_2(t)$ and the relationship between them, we conclude that the solution is global and has finite spacetime control on any spacetime slab $I \times \mathbb{R}^n$, $|I| < \infty$.

Once the global solution is obtained, the scattering theory is easily available by the fast decay of the potential decay.

Now, let's make precisely the "good local wellposedness". It means that: There exists a small constant $\eta_1 = \eta_1(E_1, E_2) > 0$ such that the Cauchy problem (1.1), (1.2) is at least wellposed on $[-\eta_1^4, \eta_1^4]$ and

$$
\|u\|_{L^{10}([-\eta_1^4,\eta_1^4];L^{10})} \le C(E_1, E_2).
$$

Thanks to the local solution theory, we need only to prove the above estimate by assuming *apriori* that the solution has existed on the interval $[-\eta_1^4, \eta_1^4]$. And we will prove this by adopting the idea in [1], [16].

Fix the small constant η_1 such that it satisfies all the conditions that will appear in the proof, we subdivide $[0, \eta_1^4]$ into J_1 intervals and $[-\eta_1^4, 0]$ into J_2 intervals such that on each subinterval I_j, $\|u\|_{L^{10}(I_j;L^{10})}$ is comparable with η_1. We do analysis forward in time and only give the estimate of J_1 for simplicity. By some technical computation and the radial assumption, we get energy localization: there exists a time in each subinterval such that the local kinetic energy and the local mass near the origin is not small, we often refer this as a "bubble" at origin when it occurs. If the volume of every bubble is sizeable by the length of the corresponding time interval, then the solution is soliton-like and J_1 can be estimated by using Morawetz estimate. Otherwise, there is concentration for $E_1(u(t_*))$ at some $t_* \in (0, \eta_1^4)$. Now we need to estimate J_1 in this case.

By removing the small bubble (because of the concentration), we get a new function $w(t_*)$ for which $E_1(w(t_*)) \le E_1(u(t_*)) - c\eta_1^3$. The difference in size between $E_1(u(t_*))$ and $E_1(u(0))$ can be controlled roughly by $C\eta_1^4$, therefore, we get $E_1(w(t_*)) \le E_1 - C\eta_1^4$. At the same time, the size $E_2(w(t_*))$ will have slight increment $C\eta_1^4$ comparing with $E_2(u(0))$, therefore, we get $E_2(w(t_*)) \le E_2 + C\eta_1^4$.

Now, we are almost near the end of the argument if we can make an inductive assumption like following: the Cauchy problem of (1.1) is wellposed at least on $[-\eta_1^4, \eta_1^4]$, if the initial data satisfies that $E_1(u_0) \le E_1 - C\eta_1^4$, $E_2(u_0) \le E_2 + C\eta_1^4$.

An obvious flaw of this assumption is that the constant may not be uniform in the process of the induction because of the slight increment of the size of $E_2(\cdot)$, however, since the total increment of the process is at most E_1, (when the size of $E_1(\cdot)$ becomes very tiny), we can choose a constant η_1 depending on E_1, $E_1 + E_2$ to avoid such problem. Now, let's clarify the inductive process: we begin with a Cauchy problem with initial $E_1(0) < \epsilon$ and $E_2(0) < E_1 + E_2$, which is globally wellposed by small global theory; then we claim the Cauchy problem with initial $E_1(0) < \epsilon + \eta_1^4$, $E_2(0) < E_1 + E_2 - \eta_1^4$ is at least wellposed on $[-\eta_1^4, \eta_1^4]$ by the same concentration analysis; after finite steps, we can cover the case for which the initial $E_1(0) \le E_1$ and $E_2(0) \le E_2$.

Combining the inductive assumption and the perturbation arguments, we finally give the control of J_1 and J_2, therefore concluding the proof of the main theorems.

The remaining part of this paper is arranged as follows: In Section 2, we give some notations and some basic estimates. They include: Littlewood-Paley decomposition, Galilean operators, Strichartz estimates for the linear operator with potential, basic properties of Galilean operator, etc. In the first part of Section 3, we give the local wellposedness and global small solution theory. In the second part, we use the decay estimate to reduce the problem to proving a "good local wellposedness" result. Section 4 through Section 8 are devoted to prove this good local wellposedness. In Section 4, we prove Morawetz estimate of the solution of (1.1). In Section 5, we use Littlewood-Paley and paraproduct decomposition to prove the existence of a sequence of bubbles. In Section 6, we control J_1 and J_2 in the case of solitonlike solution. In Section 7, we control J_1 and J_2 if there is concentration by using the inductive assumption and close the induction by a perturbation analysis in Section 8.

2 Notations and basic estimates

Notations:

Let η_1, η_2, η_3 be small constants satisfying $0 < \eta_3 \ll \eta_2 \ll \eta_1 \ll 1$ and to be defined in the proof, $c(\eta_1)$, $c(\eta_2)$, $c(\eta_3)$ be small constants satisfying $0 < c(\eta_3) \ll c(\eta_2) \ll c(\eta_1) \ll 1$; $C(\eta_1)$, $C(\eta_2)$, $C(\eta_3)$ be large constants such that $1 \ll C(\eta_1) \ll C(\eta_2) \ll C(\eta_3)$. C, c are absolute constants and may be different from one line to another.

For any time slab I, we define the mixed spacetime Lebesgue space

$$L^q(I;L^r) = \left\{u(t,x), \|u\|_{L^q(I;L^r)} = \left(\int_I \left(\int_{\mathbb{R}^n} |u(t,x)|^r\,dx\right)^{q/r} dt\right)^{1/q} < \infty.\right\}$$

with the usual modification when $q = \infty$.

We also define admissible pairs corresponding to linear Schrödinger operator with repulsive potentials as follows,

Definition 2.1. A pair (q,r) is *admissible* if $2 \le r \le 6$ and $\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$.

Occasionally, we will use $S^0(I)$ to denote the Banach space $\bigcap_{(q,r)\text{ admissible}} L^q(I;L^r)$, and I is dropped when $I = \mathbb{R}$.

Next, we give the definition of Littlewood-Paley projection. Let $\{\phi_j(\xi)\}_{j=-\infty}^{j=\infty}$ be a sequence of smooth functions and each supported in an annulus $\{\xi; 2^{j-1} \le |\xi| \le 2^{j+1}\}$, and

$$\sum_{j=-\infty}^{\infty} \phi_j(\xi) = 1, \quad \forall \xi \neq 0.$$

For any $N = 2^j$, we define Littlewood-Paley projection as follows:

$$P_N = P_{2^j} = \mathscr{F}^{-1}(\phi_j) * \cdot,$$

$$P_{\le N} = P_{\le 2^j} = \mathscr{F}^{-1}\left(\sum_{j' \le j} \phi_{j'}\right) * \cdot,$$

$$P_{>N} = I - P_{\le N}.$$

In the subsequent chapter, we may use ϕ_N to denote ϕ_j when $N = 2^j$. Following is part of the properties of the projection.

- For any $1 \le p \le \infty$, and $s \ge 0$,

$$\| |\nabla|^s P_N f\|_p \sim N^s \|P_N f\|_p,$$

$$\| |\nabla|^s P_{\leq N}\|_p \leq C N^s \|P_{\leq N}\|_p.$$

- Bernstein estimate: For any $1 \leq q \leq p \leq \infty$,

$$\|P_N f\|_p \leq C N^{n(1/q-1/p)} \|P_N f\|_q,$$

$$\|P_{\leq N} f\|_p \leq C N^{n(1/q-1/p)} \|P_{\leq N} f\|_q.$$

Let $u(t,x)$ be the solution of 3-d free Schrödinger equation with repulsive potential:

$$\left(i\partial_t + \frac{\Delta}{2}\right)u = -\frac{|x|^2}{2}u, \qquad u(0) = u_0, \tag{2.1}$$

then it can be expressed through Mehler's formula (see [8]),

$$\begin{aligned} u(t,x) &= U(t)u_0 = e^{-(it/2)(-\Delta-|x|^2)}u_0 \\ &= e^{-(i3\pi/4)\operatorname{sgn} t}\left|\frac{1}{2\pi \sinh t}\right|^{3/2} \int_{\mathbb{R}^3} e^{(i/\sinh t)(((x^2+y^2)/2)\cosh t - x\cdot y)} u_0(y)\,dy, \end{aligned} \tag{2.2}$$

one sees from the above that the kernel of $U(t)$ has the better dispersive estimate than the kernel of Schrödinger operator without potential. By using Mehler's formula (2.2), and noting that $U(\cdot)$ is unitary on L^2, one has the following decay estimate

$$\|U(t)u_0\|_\infty \leq C|t|^{-3/2}\|u_0\|_1, \tag{2.3}$$

$$\|U(t)u_0\|_p \leq C|t|^{(3/2)(1/p-1/p')}\|u_0\|_{p'}, \quad 2 \leq p \leq \infty, \tag{2.4}$$

which, by [13], imply,

Lemma 2.2. *For any admissible pair (q,r), there exists $C_r > 0$ such that*

$$\|U(\cdot)\phi\|_{L^q(\mathbb{R};L^r)} \leq C_r \|\phi\|_2.$$

For any admissible pairs (q_1,r_1), (q_2,r_2) and any time interval I, there exists some constant C_{r_1,r_2}, such that

$$\left\| \int_{I\cap\{s<t\}} U(t-s)F(s)\,ds \right\|_{L^{q_1}(I;L^{r_1})} \leq C_{r_1 r_2}\|F\|_{L^{q_2'}(I;L^{r_2'})}.$$

There are Galilean type operators associated with the equation (2.1),

$$J(t) = x \sinh t + i \cosh t \nabla_x, \quad H(t) = x \cosh t + i \sinh t \nabla_x, \tag{2.5}$$

from which x and ∇_x can be recovered,

$$x = \cosh t H(t) - \sinh t J(t), \quad i\nabla_x = \cosh t J(t) - \sinh t H(t). \tag{2.6}$$

In addition, $J(t)$ and $H(t)$ enjoy the following properties,

Lemma 2.3. 1. *They are Heisenberg observables and consequently commute with the linear operator,*

$$J(t) = U(t) i \nabla_x U(-t), \quad H(t) = U(t) x U(-t),$$

$$\left[i\partial_t + \frac{\Delta}{2} + \frac{|x|^2}{2}, J(t) \right] = \left[i\partial_t + \frac{\Delta}{2} + \frac{|x|^2}{2}, H(t) \right] = 0.$$

2. *They can be factorized as follows, for $t \neq 0$,*

$$J(t) = i \cosh t e^{i(|x|^2/2) \tanh t} \nabla_x (e^{-i(|x|^2/2) \tanh t} \cdot),$$

$$H(t) = i \sinh t e^{i(|x|^2/2) \coth t} \nabla_x (e^{-i(|x|^2/2) \coth t} \cdot).$$

3. *Let $F \in C^1(\mathbf{C}, \mathbf{C})$ and $F(z) = G(|z|^2) z$, then,*

$$J(t) F(u) = \partial_z F(u) J(t) u - \partial_{\bar{z}} F(u) \overline{J(t) u},$$

$$H(t) F(u) = \partial_z F(u) H(t) u - \partial_{\bar{z}} F(u) \overline{H(t) u}.$$

4. *There are embeddings (for instance),*

$$\|f\|_{\rho^*} \le \|J(t) f\|_{\dot{H}^{1,\rho}}, \quad \frac{1}{\rho} - \frac{1}{n} = \frac{1}{\rho^*}, \quad \rho^* < \infty.$$

Proof. The first point is easily checked thanks to (2.5). The second one holds by direct computation, and implies the last two ones. □

Formally, the solution of (1.1)–(1.2) satisfies the following two conservation laws,

$$\text{Mass:} \quad M = \|u(t)\|_2 = \|u_0\|_2,$$

$$\text{Energy:} \quad E(t) = \frac{1}{2} \|\nabla u(t)\|_2^2 - \frac{1}{2} \|xu\|_2^2 + \frac{1}{3} \|u(t)\|_6^6 = const.$$

As mentioned in the introduction, we split $E(t)$ by two ways. First, define

$$E_1(u(t)) = \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{3}\|u(t)\|_6^6, \quad E_2(u(t)) = \frac{1}{2}\|xu\|_2^2,$$

it follows easily that,

$$E(u(t)) = E_1(u(t)) - E_2(u(t)).$$

Next, we define

$$\begin{aligned} \mathscr{E}_1(t) &:= \frac{1}{2}\|J(t)u(t)\|_{L^2}^2 + \frac{1}{3}\cosh^2 t\|u(t)\|_6^6, \\ \mathscr{E}_2(t) &:= \frac{1}{2}\|H(t)u(t)\|_{L^2}^2 + \frac{1}{3}\sinh^2 t\|u(t)\|_6^6, \end{aligned} \tag{2.7}$$

then $\mathscr{E}_1(t)$ and $\mathscr{E}_2(t)$ coincide with $E_1(t)$ and $E_2(t)$ only at $t = 0$. Furthermore, we have,

Lemma 2.4. 1. *$\mathscr{E}_1(t)$ and $\mathscr{E}_2(t)$ satisfy,*

$$\mathscr{E}_1(t) - \mathscr{E}_2(t) = E(t), \tag{2.8}$$

$$\frac{d\mathscr{E}_1(t)}{dt} = \frac{d\mathscr{E}_2(t)}{dt} = -\frac{2}{3}\sinh(2t)\|u(t)\|_6^6. \tag{2.9}$$

2. *The potential energy $\|u(t)\|_6^6$ has exponentially decay in time:*

$$\|u(t)\|_6^6 \leq 3E_1(0)\cosh^{-6} t, \quad \forall t \in \mathbb{R}.$$

3. *$\forall t \in \mathbb{R}$,*

$$\mathscr{E}_1(t) \leq \mathscr{E}_1(0) = E_1(0), \tag{2.10}$$

$$\|H(t)u(t)\|_2^2 \leq \|H(0)u(0)\|_2^2 = \|xu_0\|_2^2 = 2E_2(0). \tag{2.11}$$

Proof. The first point can be verified by (2.5) and the equation (1.1), see [2] for details. Now let us prove the second point. Integrating in time from 0 to t, we see from (2.9) that

$$\mathscr{E}_1(t) = \mathscr{E}_1(0) - \frac{2}{3}\int_0^t \sinh(2s)\|u(s)\|_6^6\, ds.$$

By (2.7), we have

$$\cosh^2 t\|u(t)\|_6^6 \le 3\mathscr{E}_1(0) - 2\int_0^t \sinh(2s)\|u(s)\|_6^6\,ds$$

$$= 3\mathscr{E}_1(0) - 2\int_0^t \frac{\sinh(2s)}{\cosh^2 s}\cosh^2 s\|u(s)\|_6^6\,ds.$$

Applying the Gronwall inequality yields:

$$\cosh^2 t\|u(t)\|_6^6 \le 3\mathscr{E}_1(0)\exp\left[-2\int_0^t \frac{\sinh(2s)}{\cosh^2 s}\,ds\right].$$

Noting by direct computation,

$$\int_0^t \frac{\sinh(2s)}{\cosh^2 s}\,ds = \ln\cosh t,$$

we have

$$\cosh^2 t\|u(t)\|_6^6 \le 3\mathscr{E}_1(0)\cosh^{-4} t,$$

and

$$\|u(t)\|_6^6 \le 3\mathscr{E}_1(0)\cosh^{-6} t.$$

Now, let's prove the third point. First, (2.10) is easily verified by using (2.9). Next, noting (2.7), (2.6) and energy conservation, we see that,

$$\frac{1}{2}\|H(t)u(t)\|_2^2 + \frac{1}{3}\sinh^2 t\|u(t)\|_6^6$$

$$= \mathscr{E}_1(t) - E(t)$$

$$\le \mathscr{E}_1(0) - E(0)$$

$$= \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{3}\|u_0\|_6^6 - \left(\frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{2}\|xu_0\|_2^2 + \frac{1}{3}\|u_0\|_6^6\right)$$

$$= \frac{1}{2}\|xu_0\|_2^2.$$

Thus we get

$$\|H(t)u(t)\|_2 \le \|xu_0\|_2,$$

which is exactly (2.11). □

Before ending this section, we give the main theorems of this paper.

Theorem 2.5. *Let $u_0 \in \Sigma$ be radial, then the Cauchy problem* (1.1)–(1.2) *has a unique global solution in $C_{\text{loc}}(\mathbb{R};\Sigma) \cap L^{10}(\mathbb{R};L^{10})$ which satisfies*

$$\|u\|_{L^{10}(\mathbb{R};L^{10})} \le C(\|\nabla u_0\|_2, \|xu_0\|_2), \tag{2.12}$$

$$\max_{A\in\{J,H,I\}} \|A(\cdot)u\|_{S^0} \le C(\|u_0\|_\Sigma), \tag{2.13}$$

where $A(t)$ denotes any of the operators $J(t)$, $H(t)$, or the identity I. For any compact interval $0 \in I \subset \mathbb{R}$, the data-solution map $u_0 \in \Sigma \to u \in C(I;\Sigma)$ is Lip continuous. Furthermore, there exits a unique $u_\pm \in \Sigma$ such that

$$\|U(-t)u(t) - u_\pm\|_\Sigma \to 0, \quad \text{as } t \to \pm\infty.$$

Theorem 2.6. *Let $u_\pm \in \Sigma$ be radial, then there exists a unique solution $u(x,t)$ of equation* (1.1) *satisfying*

$$\|u\|_{L^{10}(\mathbb{R};L^{10})} \le C(E_1(u_+), E_2(u_+)),$$

$$\max_{A\in\{J,H,I\}} \|A(\cdot)u\|_{S^0} \le C(\|u_+\|_\Sigma),$$

and

$$\|U(-t)u(t) - u_\pm\|_\Sigma \to 0, \quad \text{as } t \to \pm\infty.$$

Remark 2.7. In view of the two theorems, we can define scattering operator $\Omega_\pm$ from the radial functions in Σ to themselves by $\Omega_\pm u_- = u_+$, where u_- and u_+ are associated through the unique solution of $u(t,x)$ to (1.1) such that

$$\|U(-t)u(t) - u_\pm\|_\Sigma \to 0, \quad t \to \pm\infty. \tag{2.14}$$

Furthermore, it's also not hard to verify through Theorem 2.5 and Theorem 2.6 as well as (2.14) that Ω is one to one, and continuous in Σ, therefore is a holomorphic from the radial functions in Σ to themselves. The proof of non-potential NLS counterpart can be found in, for instance, [4].

3 Local wellposedness and global small solution

The goal of this section is to get local solution and small global solution to (1.1), (1.2), which by Duhamel, satisfies the integral equation

$$(3.1) \qquad u(t) = U(t-t_0)u(t_0) - i\int_{t_0}^{t} U(t-s)|u|^4 u(s)\,ds.$$

To begin with, let's introduce a lemma which will be used throughout the paper.

Lemma 3.1. *Let I be a time slab, u be the solution of* (1.1), (1.2) *in the sense of* (3.1) *such that*

$$(3.2) \qquad \|u\|_{L^{10}(I;L^{10})} \le C_1,$$

then we have

$$\|Au\|_{S^0(I)} \le C(C_1, \|u_0\|_\Sigma), \quad \textit{for } A \in \{J, H, I\}.$$

Remark 3.2. It should be noticed that the L^{10} norm control in condition (3.2) is not specially chosen and can be replaced by other spacetime bounds. For example, by assuming the boundedness of $\|u\|_{L^6(I;L^{18})}$ or $\|Ju\|_{L^2(I;L^6)}$, one can get the same results by some minor changes of the following proof.

Proof. Divide I into subintervals $I = \bigcup_{j=1}^{J} I_j$ such that $\|u\|_{L^{10}(I_j;L^{10})} \sim \eta$, then on $I_j = [t_{j-1}, t_j]$, u satisfies the equation,

$$u(t) = U(t-t_{j-1})u(t_{j-1}) - i\int_{t_{j-1}}^{t} U(t-s)|u|^4 u(s)\,ds,$$

by Strichartz, we have

$$\begin{aligned}\|Au\|_{S^0(I_j)} &\le C\|A(t_{j-1})u(t_{j-1})\|_2 + C\|u\|^4_{L^{10}(I_j;L^{10})}\|Au\|_{S^0(I_j)} \\ &\le C\|A(t_{j-1})u(t_{j-1})\|_2 + C\eta^4\|Au\|_{S^0(I_j)}\end{aligned}$$

which implies that

$$\|Au\|_{S^0(I_j)} \le 2C\|A(t_{j-1})u(t_{j-1})\|_2,$$

if η is small. This finally gives Lemma 3.1 by the boundedness of $\|A(t)u(t)\|_2$ and inductive arguments. □

By time reversal symmetry, we state the following result only in the positive time direction.

For I a time slab, define the space

$$X(I) = L^{10/3}(I; L^{10/3}) \cap L^{10}(I; L^{30/13}),$$

then we have

Proposition 3.3. *Let $u_0 \in \Sigma$, then there exists a $\eta_0 > 0$ such that when the linear flow satisfies*

$$\|JUu_0\|_{X([0,T])} \le \eta_0,$$

(1.1)–(1.2) *has a unique solution $u(x,t)$ satisfying*

$$\|Au\|_{S^0([0,T])} < C(\|u_0\|_\Sigma), \quad \forall A \in \{J, H, I\}. \tag{3.3}$$

Let $T^* = \sup_{T>0}\{(1.1) - (1.2)$ has a unique solution on $[0,T]\}$, and if $T^* < \infty$, then

$$\|u\|_{L^{10}([0,T^*); L^{10})} = \infty.$$

Proof. Define the solution map by

$$\Phi(u)(t) = U(t)u_0 - i\int_0^t U(t-s)|u|^4 u(s)\, ds,$$

and denote $I = [0,T]$, then we show Φ is contractive on the compact set

$$\mathscr{B} = \{u(x,t), \|Ju\|_{X(I)} \le 2\eta_0; \|Hu\|_{X(I)} \le 2C\|xu_0\|_2; \|u\|_{X(I)} \le 2C\|u_0\|_2\}$$

under the weak topology $X(I)$ if η_0 is small enough. Taking $u \in \mathscr{B}$, by Strichartz estimate, we have that

$$\begin{aligned}\|J\Phi u\|_{X(I)} &\le \|JUu_0\|_{X(I)} + C\|u\|^4_{L^{10}(I;L^{10})}\|Ju\|_{X(I)} \\ &\le \eta_0 + C\|Ju\|^5_{X(I)} \le \eta_0 + C(2\eta_0)^5 \le 2\eta_0,\end{aligned}$$

if η_0 is such that $C(2\eta_0)^5 \le \frac{1}{8}\eta_0$. By the same token, we have,

$$\begin{aligned}\|H\Phi u\|_{X(I)} &\le \|HUu_0\|_{X(I)} + C\|u\|^4_{L^{10}(I;L^{10})}\|Hu\|_{X(I)} \\ &\le C\|xu_0\|_2 + C(2\eta_0)^4(2C\|xu_0\|_2) \le 2C\|xu_0\|_2,\end{aligned}$$

$$\|\Phi u\|_{X(I)} \le 2C\|u_0\|_2,$$

for the same η_0. Let $u_1, u_2 \in \mathscr{B}$, then it's easily seen that

$$\|\Phi(u_1) - \Phi(u_2)\|_{X(I)} \leq C(\|u_1\|^4_{L^{10}(I;L^{10})} + \|u_2\|^4_{L^{10}(I;L^{10})})\|u_1 - u_2\|_{X(I)}$$

$$\leq 2C(2\eta_0)^4\|u_1 - u_2\|_{X(I)} \leq \frac{1}{2}\|u_1 - u_2\|_{X(I)}.$$

Applying the fixed point theorem gives a unique solution of (1.1)–(1.2) on the interval I. The norm control (3.3) follows from Strichartz estimate.

Before proving the blow up criterion, let's say a couple of words about how to extend the solution from t_0 forward in time. We need the smallness condition on the linear flow as follows,

$$\|J(\cdot)U(\cdot - t_0)u(t_0)\|_{X([t_0, t_0+T])} \leq \eta_0,$$

which allows us to establish the contraction mapping on the compact set,

$$\mathscr{B} = \{u(x,t); \|Ju\|_{X([t_0, t_0+T])} \leq 2\eta_0;$$

$$\|Hu\|_{X([t_0, t_0+T])} \leq 2C\|H(t_0)u(t_0)\|_2, \|u\|_{X([t_0, t_0+T])} \leq 2C\|u(t_0)\|_2\}$$

by repeating the same proof before.

Now, let's prove the blow up criterion. If otherwise $T^* < \infty$ and $\|u\|_{L^{10}([0,T^*);L^{10})} < \infty$, we aim to get a contradiction by extending the solution beyond T^*. Let t_0 be very close to T^*, then it's enough to find a small $\delta > 0$ such that the linear flow is small, more precisely,

$$\|J(\cdot)U(\cdot - t_0)u(t_0)\|_{X([t_0, T^*+\delta])} \leq \eta_0. \tag{3.4}$$

First we notice that the finite L^{10} norm control implies that

$$\|Au\|_{S^0([0,T^*))} < \infty,$$

by Lemma 3.1.

On the one hand, by Strichartz, we have that

$$\|J(\cdot)U(\cdot - t_0)u(t_0)\|_{X([t_0, T^*))} \lesssim \|Ju\|_{X([t_0, T^*))} + \|Ju\|^5_{X([t_0, T^*))},$$

which $\leq \frac{\eta_0}{2}$ if t_0 is close enough to T^*. On the other hand, once t_0 is fixed, we can choose $\delta > 0$ small enough such that,

$$\|J(\cdot)U(\cdot - t_0)u(t_0)\|_{X([T^*, T^*+\delta])} \leq \frac{\eta_0}{2},$$

by noting that

$$\|J(\cdot)U(\cdot - t_0)u(t_0)\|_{X(\mathbb{R})} \leq \|J(t_0)u(t_0)\|_2 < \infty,$$

$$\lim_{\delta\to 0} \|J(\cdot)U(\cdot - t_0)u(t_0)\|_{X([T^*, T^*+\delta])} = 0.$$

(3.4) then follow from trivial triangle inequality. □

For T sufficiently small (but may depend on the initial profile), we will get a unique solution on $[0, T]$. However, if the initial kinetic energy is small, then by Strichartz estimate,

$$\|JUu_0\|_{X(\mathbb{R})} \leq \|\nabla u_0\|_2 \leq \eta_0,$$

we get immediately the global small solution.

Corollary 3.4. *Let $u_0 \in \Sigma$, then there exists a small constant $\varepsilon_0 > 0$ such that when*

$$\|\nabla u_0\|_2 \leq \varepsilon_0,$$

(1.1)–(1.2) *has a unique solution $u(t, x)$ satisfying*

$$\sup_{A\in\{J,H,I\}} \|Au\|_{S^0(\mathbb{R})} \lesssim C(\|u_0\|_\Sigma).$$

Furthermore, there exists a unique function $u_\pm \in \Sigma$ such that

$$\|U(-t)u(t) - u_\pm\|_\Sigma \to 0, \quad \text{as } t \to \pm\infty.$$

Proof. It's only left to construct the asymptotic state for the global solution u. Let $u_+ = u_0 - i\int_0^\infty U(-s)|u|^4u(s)\,ds$, one sees that

$$U(-t)u(t) - u_+(t) = i\int_t^\infty U(-s)|u|^4u(s)\,ds.$$

Noting that by Lemma 2.3, there holds

$$i\nabla_x(U(-t)u(t) - u_+(t)) = i\int_t^\infty U(-s)J(s)|u|^4u(s)\,ds,$$

$$x(U(-t)u(t) - u_+(t)) = i\int_t^\infty U(-s)H(s)|u|^4u(s)\,ds.$$

We see that

$$\|U(-t)u(t) - u_+(t)\|_\Sigma \le \max_{A\in\{J,H,I\}} \left\| \int_t^\infty U(-s)A(s)|u|^4 u(s)\,ds \right\|_2.$$

Noting that the operator $U(\cdot)$ is unitary on L^2, we can bound the right side of the above equation by

$$\|u\|^4_{L^{10}((t,\infty);L^{10})}\|A(\cdot)u\|_{L^{10/3}((t,\infty);L^{10/3})}$$

which tends to 0 as t tends to ∞. The scattering in the negative direction follows the same way. □

Assume for the moment that the solution is global, then we will show that the decay of the potential energy imply the decay of this global solution, and from which the scattering follows.

Lemma 3.5. *Assume u is a global solution and for any time slab* $I \in \mathbb{R}$, $\|u\|_{L^{10}(I;L^{10})} < C(|I|, \|u_0\|_\Sigma)$, *then u satisfies*

$$\max_{A\in\{J,H,I\}} \|A(\cdot)u\|_{S^0} \le C(\|u_0\|_\Sigma), \tag{3.5}$$

and there is scattering.

Proof. Fixing a small constant ε and taking $T \ge T_0 = \left(\frac{3E_1(0)}{\varepsilon^6}\right)^{1/6}$, we have

$$3E_1(0)\cosh^{-6} T \le \varepsilon^6,$$

thus by the decay estimate Lemma 2.4, one has

$$\|u\|_{L^\infty([T,\infty);L^6)} \le \varepsilon. \tag{3.6}$$

By Duhamel's formula, on $[T,\infty)$, u satisfies the equation

$$u(t) = U(t-T)u(T) - i\int_T^t U(t-s)|u|^4 u(s)\,ds.$$

Applying Strichartz estimate gives,

$$\begin{aligned}\|J(\cdot)u\|_{L^2([T,\infty);L^6)} &\le C\|J(T)u(T)\|_2 + C\|J(\cdot)|u|^4u\|_{L^2([T,\infty);L^{6/5})}\\ &\le C\|J(T)u(T)\|_2 + C\|u\|^4_{L^\infty([T,\infty);L^6)}\|Ju\|_{L^2([T,\infty);L^6)}\\ &\le CE_1(u_0)^{1/2} + C\varepsilon^4\|Ju\|_{L^2([T,\infty);L^6)}\end{aligned}$$

Therefore,

$$\|J(\cdot)u\|_{L^2([T,\infty);L^6)} \leq 2CE_1(u_0)^{1/2}.$$

By time reversing and the assumption, we have

$$\|J(\cdot)u\|_{L^2(\mathbb{R};L^6)} \leq \|J(\cdot)u\|_{L^2((-\infty,T];L^6)} + \|J(\cdot)u\|_{L^2([-T,T];L^6)} + \|J(\cdot)u\|_{L^2([T,\infty);L^6)} \leq C.$$

This implies that

$$\|A(\cdot)u\|_{S^0} \leq C.$$

The scattering result follows from this global spacetime bound as shown in Corollary 3.4. □

On the other hand, since by (2.6) and Lemma 2.4, $E_1(t)$ and $E_2(t)$ grows exponentially in time, in order to prove Theorem 2.5, we need only to show the following "good local wellposedness",

Proposition 3.6. *Let $E_1(u_0) = E_1$, $E_2(u_0) = E_2$, then there exists a small constant η_1 depending only on (E_1, E_2) such that the Cauchy problem of* (1.1), (1.2) *is at least wellposed on $[-\eta_1^4, \eta_1^4]$ and the solution u satisfies*

$$\|u\|_{L^{10}([-\eta_1^4,\eta_1^4];L^{10})} \leq C(E_1, E_2).$$

The remaining part of the paper is devoted to the proof of Proposition 3.6, however, before proceeding to the next section, let's give a sketchy proof of Theorem 2.6.

Proof. We need only to show the integral equation

$$u(t) = U(t)u_+ + i\int_t^\infty U(t-s)|u|^4u(s)\,ds, \tag{3.7}$$

has a unique global solution with global spacetime estimates. First of all, we seek for some local solution. Define the solution map by $\Phi(u)(t) = U(t)u_+ + i\int_t^\infty U(t-s)|u|^4u(s)\,ds$, and denote $R = \|\nabla u_+\|_2$. By choosing $T = T(R)$ large enough, say, $\cosh T \geq CR$, we see that Φ is a contraction map on the set

$$X = \left\{ u(x,t);\quad \begin{array}{l} \|u\|_{L^{10/3}([T,\infty);L^{10/3})} \leq 2C\|u_+\|_2, \\ \|H(\cdot)u\|_{L^{10/3}([T,\infty);L^{10/3})} \leq 2C\|xu_+\|_2, \\ \|J(\cdot)u\|_{L^{10/3}([T,\infty);L^{10/3})\cap L^{10}([T,\infty);L^{30/13})} \leq 2CR \end{array} \right\},$$

endowed with the metric $d(u_1,u_2)=\|u_1-u_2\|_{L^{10/3}([T,\infty);L^{10/3})}$. The proof is routine, except needing to notify that the gain $\cosh^{-1}T$ from the embedding $\|u\|_{L^{10}([T,\infty);L^{10})}\leq C\cosh^{-1}T\|J(\cdot)u\|_{L^{10}([T,\infty);L^{30/13})}$ gives the dependence of T on R. Once we get the local solution, we can find a finite time $T=T(E_1(u_+))$ such that $u(T)\in\Sigma$. At this moment, we can apply Theorem 2.5 to get a unique global solution with global spacetime bound satisfying also (3.7). The scattering part of Theorem 2.6 then follows easily from the global spacetime bound of $u(x,t)$. □

4 Morawetz estimate for solutions of (1.1)

We first give the local mass conservation of u. We'd notice that the Local mass conservation for Schrödinger equation without potential has appeared in [1], [11], and [16].

Taking a smooth function $\chi(x)\in C_0^\infty(\mathbb{R}^3)$ such that $\chi(x)=1$ if $|x|\leq\frac{1}{2}$ and $\chi(x)=0$ if $|x|\geq 1$. Then we have,

Proposition 4.1. *Let u be the smooth solution of* (1.1)*, and define local mass of u to be*

$$Mass(u(t),B(x_0,R))=\left(\int\chi^2\left(\frac{x-x_0}{R}\right)|u(t,x)|^2\,dx\right)^{1/2},$$

then,

$$\partial_t\,Mass(u(t),B(x_0,R))\leq\frac{\|u(t)\|_{\dot{H}^1}}{R},\tag{4.1}$$

and

$$Mass(u(t),B(x_0,R))\leq R\|u(t)\|_{\dot{H}^1}.\tag{4.2}$$

Proof. Noting that u satisfies the equation (1.1), we have

$$\begin{aligned}&\partial_t\,Mass(u(t),B(x_0,R))^2\\&\quad=\int\chi^2\left(\frac{x-x_0}{R}\right)2\,Re\left[\bar{u}\left(\frac{i}{2}\Delta u+\frac{i}{2}|x|^2u-i|u|^4u\right)\right](x)\,dx,\\&\quad=-\int\chi^2\left(\frac{x-x_0}{R}\right)Im(\bar{u}\Delta u)(x)\,dx.\end{aligned}$$

Integrating by parts, we get

$$\partial_t\,Mass(u(t),B(x_0,R))^2=\frac{2}{R}\int\chi\left(\frac{x-x_0}{R}\right)(\nabla\chi)\left(\frac{x-x_0}{R}\right)Im(\bar{u}\nabla u)(x,t)\,dx.$$

By Hölder inequality, the right hand side can be controlled by

$$\frac{2}{R}\,Mass(u(t),B(x_0,R))\|\nabla u(t)\|_2,$$

from which (4.1) follows. Now, let's prove (4.2). Using Hardy's inequality, one has

$$\begin{aligned}Mass(u(t),B(x_0,R))^2 &= \int \chi^2\left(\frac{x-x_0}{R}\right)|u(t,x)|^2\,dx\\ &\le \sup_{x\in\mathbb{R}^3}\chi^2\left(\frac{x-x_0}{R}\right)|x-x_0|^2\int\frac{|u(t,x)|^2}{|x-x_0|^2}dx\\ &\le R^2\|u(t)\|_{\dot H^1}^2.\end{aligned}$$

Thus, we get (4.2). □

Proposition 4.2. *Let u be the solution of* (1.1) *with finite energy. Then we have*

$$(4.3)\qquad \int_I\int_{|x|\le K|I|^{1/2}}\frac{|u(t,x)|^6}{|x|}dx\,dt$$
$$\le CK|I|^{1/2}(\|\nabla u\|^2_{L^\infty(I;L^2)}+\|xu\|^2_{L^\infty(I;L^2)}+\|u\|^6_{L^\infty(I;L^6)})\quad \textit{for all } K\ge 1.$$

Proof. We prove this result by following the idea in [16]. However, we should notice that in the case of NLS without potential, the computation of such localized estimate is first due to [14] and was adapted later by J. Bourgain and M. Grillakis to better suit the critical case. The common approach is differentiating the quantity $\int_{\mathbb{R}^n} Im\left(\frac{x}{|x|}\cdot\nabla u(t,x)\bar u(t,x)\right)dx$.

Assume without loss of generality that u is a smooth solution of (1.1). First, by a direct computation, we get

$$(4.4)\qquad \partial_t\,Im(\partial_k u\bar u)=\frac{1}{4}\partial_k\Delta(|u|^2)-Re\,\partial_j(\bar u_k u_j)-\frac{2}{3}\partial_k(|u|^6)+x_k|u|^2,$$

here, we use $\partial_k f$ or f_k to denote $\frac{\partial f}{\partial x_k}$. Let $a(x)$ be a smooth radial function to be chosen later. Multiplying (4.4) by $a_k(x)$ and integrating on $\mathbb{R}^3$, we get

$$(4.5)\qquad \partial_t\int_{\mathbb{R}^3}Im(\partial_k u\bar u)(x)a_k(x)\,dx=\int a_{jk}(x)\,Re(\bar u_k u_j)(x)\,dx-\frac{1}{4}\int\Delta\Delta a(x)|u|^2(x)\,dx$$
$$+\frac{2}{3}\int\Delta a(x)|u|^6(x)\,dx+\int a_k(x)x_k|u|^2(x)\,dx.$$

Taking $\chi(x) \in C_0^\infty(\mathbb{R}^3)$ satisfying $\chi(x) = 1$ as $|x| \le 1$ and $\chi(x) = 0$ as $|x| \ge 2$. Letting $a(x) = (\varepsilon^2 + |x|^2)^{1/2}\chi(\frac{x}{R})$, we claim that, on $|x| \le R$,

$$a(x) = (\varepsilon^2 + |x|^2)^{1/2}, \quad a_k(x) = \frac{x_k}{(\varepsilon^2 + |x|^2)^{1/2}},$$

$$\Delta a(x) = \frac{2}{(\varepsilon^2 + |x|^2)^{1/2}} + \frac{\varepsilon^2}{(\varepsilon^2 + |x|^2)^{3/2}}, \quad \Delta\Delta a(x) = -\frac{15\varepsilon^2}{(\varepsilon^2 + |x|^2)^{7/2}},$$

$$a_{jk}(x)\, Re(\bar{u}_k u_j)(x) \ge 0, \quad a_k(x)x_k = \frac{|x|^2}{(\varepsilon^2 + |x|^2)^{1/2}} \ge 0.$$

The first four points follow by directly differentiating $a(x)$ on $|x| \le R$. The fifth one follows from

$$a_{jk}(x)\, Re(\bar{u}_k u_j)(x) = \left(\frac{\delta_{jk}}{(\varepsilon^2 + |x|^2)^{1/2}} - \frac{x_j x_k}{(\varepsilon^2 + |x|^2)^{3/2}}\right) Re(\bar{u}_k u_j)(x) = \frac{|\nabla u|^2}{(\varepsilon^2 + |x|^2)^{1/2}} - \frac{Re(x_j u_j x_k \bar{u}_k)}{(\varepsilon^2 + |x|^2)^{3/2}} = \frac{|\nabla u|^2}{(\varepsilon^2 + |x|^2)^{1/2}} - \frac{|x|^2|u_r|^2}{(\varepsilon^2 + |x|^2)^{3/2}}, \tag{4.6}$$

and the simple fact that $|u_r| \le |\nabla u|$. Plugging all the estimates in (4.5) yields that

$$\frac{4}{3}\int_{|x| \le R} \frac{|u|^6}{(\varepsilon^2 + |x|^2)^{1/2}}\, dx \le \partial_t \int_{\mathbb{R}^3} Im(\partial_k u\bar{u})(x)a_k(x)\, dx + \int_{R \le |x| \le 2R} |a_{jk}(x)\, Re(\bar{u}_k u_j)(x)| + \frac{1}{4}|\Delta\Delta a(x)|\, |u|^2(x) + \frac{2}{3}|\Delta a(x)|\, |u|^6(x) + |a_k(x)x_k|\, |u|^2(x)\, dx.$$

Integrating in time on I, we get

$$\frac{4}{3}\int_I \int_{|x| \le R} \frac{|u|^6(x,t)}{(\varepsilon^2 + |x|^2)^{1/2}}\, dx \le \sup_{t \in I}\left|\int_{\mathbb{R}^3} Im(\bar{u}u_k)(x)a_k(x)\, dx\right| + |I| \sup_{t \in I}\left(\int_{R \le |x| \le 2R} |a_{jk}(x)u_k u_j(x,t)| + \frac{1}{4}|\Delta\Delta a(x)|\, |u|^2(x,t)| + \frac{2}{3}|\Delta a(x)|\, |u|^6(x,t)| + |a_k(x)x_k|\, |u|^2(x,t)|\, dx\right). \tag{4.7}$$

Note on $R \le |x| \le 2R$,

$$|a_k(x)| \le C\frac{(\varepsilon^2 + R^2)^{1/2}}{R},$$

$$|a_{jk}(x)| \le C\frac{(\varepsilon^2 + R^2)^{1/2}}{R^2},$$

$$|\Delta\Delta a(x)| \le C\frac{(\varepsilon^2 + R^2)^{1/2}}{R^4},$$

we have

$$\int_I \int_{|x|\le R} \frac{|u|(x,t)}{(\varepsilon^2 + |x|^2)^{1/2}}\,dx\,dt$$
$$\le C\left((\varepsilon^2 + R^2)^{1/2} + |I|\frac{(\varepsilon^2 + R^2)^{1/2}}{R^2}\right)$$
$$\times (\|\nabla u\|^2_{L^\infty(I;L^2)} + \|xu\|^2_{L^\infty(I;L^2)} + \|u\|^6_{L^\infty(I;L^6)}).$$

Choosing $R = K|I|^{1/2}$ and letting $\varepsilon \to 0$ gives

$$\int_I \int_{|x|\le K|I|^{1/2}} \frac{|u|^6(x,t)}{|x|}\,dx\,dt$$
$$\le C(K|I|^{1/2} + K^{-1}|I|^{1/2})(\|\nabla u\|^2_{L^\infty(I;L^2)} + \|xu\|^2_{L^\infty(I;L^2)} + \|u\|^6_{L^\infty(I;L^6)})$$
$$\le CK|I|^{1/2}(\|\nabla u\|^2_{L^\infty(I;L^2)} + \|xu\|^2_{L^\infty(I;L^2)} + \|u\|^6_{L^\infty(I;L^6)}),$$

since $K \ge 1$. This is exactly (4.3). □

As a direct consequence of Proposition 4.2, we have

Corollary 4.3. *Let u be a solution on time slab I satisfying,*

$$E_1(u(t)) \le C_1; \quad E_2(u(t)) \le C_2,\ \forall t \in I,$$

then we have

$$\int_I \int_{|x|\le K|I|^{1/2}} \frac{|u(t,x)|^6}{|x|}\,dx\,dt \le C(C_1, C_2)K|I|^{1/2} \quad \textit{for all } K \ge 1. \tag{4.8}$$

5 Energy localization

Let $\eta_1(E_1, E_2)$ be a small constant that meets all the conditions to appear in the subsequent proof, we aim to show the wellposedness on $[-\eta_1^4, \eta_1^4]$. Thanks to the local theory, we can assume the solution has existed on it and aim to show

$$\|u\|_{L^{10}([-\eta_1^4,\eta_1^4];L^{10})} < C(E_1, E_2).$$

Now, fix this η_1, we divide $[0, \eta_1^4]$ into J_1 subintervals and $[-\eta_1^4, 0]$ into J_2 subintervals such that on each subinterval I_j, $\eta_1 \le \|u\|_{L^{10}(I_j,L^{10})} \le 2\eta_1$. So we are left to control J_1, J_2 by constant $C(E_1, E_2)$. Without loss of generality, we only do analysis in the positive time direction. Following J. Bourgain [1], we classify the subintervals into three components $I^{(1)}$, $I^{(2)}$, $I^{(3)}$, and each contains $\frac{J_1}{3}$ consecutive subintervals. It's on the intermediate component that we do most analysis. Our first aim is to show the localization of the energy. To begin with, we see that

$$\|\nabla u\|_{L^{10/3}(I_j;L^{10/3})} + \|xu\|_{L^{10/3}(I_j;L^{10/3})} \le C(E_1, E_2),$$

which follows from (2.6) and Lemma 3.1.

We have the following localized results.

Proposition 5.1. *Let I_j be one of the subintervals, that is $I_j \in [0, \eta_1^4]$ and $\eta_1 \le \|u\|_{L^{10}(I_j;L^{10})} \le 2\eta_1$. Then there exists $t_j \in I_j$, $x_j \in \mathbb{R}^3$ and $N \ge N_{j0} \approx |I_j|^{-1/2}\eta_1^5$ such that*

$$\|u(t_j)\|_{L^6(|x-x_j|<C(\eta_1)N_j^{-1})} \ge c\eta_1^{3/2}, \tag{5.1}$$

$$\|\nabla u(t_j)\|_{L^2(|x-x_j|<C(\eta_1)N_j^{-1})} \ge c\eta_1^{3/2}, \tag{5.2}$$

$$\|u(t_j)\|_{L^2(|x-x_j|<C(\eta_1)N_j^{-1})} \ge c\eta_1^{3/2}N_j^{-1}. \tag{5.3}$$

Proof. By Bernstein estimate, $\forall N \in 2^Z$, we have

$$\|P_{\le N}u\|_\infty \le N^{1/2}\|P_{\le N}u\|_6 \le CN^{1/2},$$

which allows us to control the L^{10} norm of low frequency by interpolation,

$$\|P_{\le N}u\|_{10} \le \|P_{\le N}u\|_\infty^{4/10}\|P_{\le N}u\|_6^{6/10} \le CN^{1/5},$$

hence, using Hölder inequality in time, we have

$$\|P_{\le N}u\|_{L^{10}(I_j;L^{10})} \le C|I_j|^{1/10}N^{1/5}.$$

Taking $N = N_{j0} = C|I_j|^{-1/2}\eta_1^5$, one sees that

$$\|P_{\le N_{j0}}u\|_{L^{10}(I_j;L^{10})} < \frac{\eta_1}{2},$$

and thus

$$\|P_{\geq N_{j0}} u\|_{L^{10}(I_j; L^{10})} > \frac{\eta_1}{2}.$$

Using Littlewood-Paley theorem, we have

$$\begin{aligned}
\left(\frac{\eta_1}{2}\right)^{10} &\leq \|P_{\geq N_{j0}} u\|^{10}_{L^{10}(I_j; L^{10})} \\
&= \int_{I_j} \|P_{\geq N_{j0}} u(t)\|^{10}_{10}\, dt \\
&= \int_{I_j} \left\| \left(\sum_{N \geq N_{j0}} |P_N u(t)|^2 \right)^{1/2} \right\|^{10}_{10} dt \\
&= C \int_{I_j} \int_{\mathbb{R}^3} \sum_{N_1 \geq \cdots \geq N_5 \geq N_{j0}} |P_{N_1} u(t)|^2 \cdots |P_{N_5} u(t)|^2\, dx\, dt.
\end{aligned}$$

Letting $\sigma_N = N^{1/2} \|P_N u\|_{L^\infty_{xt \in I_j \times \mathbb{R}^3}}$, we see the last line is smaller than

$$\begin{aligned}
(5.4) \qquad & C \sup_{N \geq N_{j0}} \sigma_N^{20/3} \int_{I_j} \int_{\mathbb{R}^3} \sum_{N_1 \geq \cdots \geq N_5 \geq N_{j0}} |P_{N_1} u(t)|^2 |P_{N_2} u(t)|^{4/3} N_2^{1/3} N_3 N_4 N_5\, dx\, dt \\
& \leq C \sup_{N \geq N_{j0}} \sigma_N^{20/3} \int_{I_j} \int_{\mathbb{R}^3} \sum_{N_1 \geq N_2 \geq N_{j0}} N_2^{10/3} |P_{N_1} u(t)|^2 |P_{N_2} u(t)|^{4/3}\, dx\, dt,
\end{aligned}$$

by summing N_5, N_4 and N_3. Using Hölder inequality and Young's inequality, (5.4) can be controlled by

$$\begin{aligned}
& C \sup_{N \geq N_{j0}} \sigma_N^{20/3} \sum_{N_1 \geq N_2 \geq N_{j0}} N_2^{10/3} \|P_{N_1} u\|^2_{L^{10/3}_{xt}} \|P_{N_2} u\|^{4/3}_{L^{10/3}_{xt}} \\
& \leq C \sup_{N \geq N_{j0}} \sigma_N^{20/3} \sum_{N_1 \geq N_2 \geq N_{j0}} N_2^2 N_1^{-2} \|\nabla P_{N_1} u\|^2_{L^{10/3}_{xt}} \|\nabla P_{N_2} u\|^{4/3}_{L^{10/3}_{xt}} \\
& \leq C \sup_{N \geq N_{j0}} \sigma_N^{20/3} \sum_{N \geq N_{j0}} \|\nabla P_N u\|^{10/3}_{L^{10/3}(I_j; L^{10/3})} \\
& \leq C \sup_{N \geq N_{j0}} \sigma_N^{20/3} \|\nabla u\|^{10/3}_{L^{10/3}(I_j; L^{10/3})} \\
& \leq C \sup_{N \geq N_{j0}} \sigma_N^{20/3}.
\end{aligned}$$

This implies that,

$$\sup_{N \geq N_{j0}} \sigma_N \geq c\eta_1^{3/2},$$

thus there exists $t_j \in I_j$, $x_j \in \mathbb{R}^3$ and $N_j \geq N_{j0}$ such that

$$(5.5) \qquad |P_{N_j} u(x_j, t_j)| \geq c\eta_1^{3/2} N_j^{1/2}.$$

Now we deduce (5.1), (5.2), (5.3) from (5.5). By the definition of P_{N_j}, we see that

$$\begin{aligned}
(5.6) \qquad c\eta_1^{3/2} N_j^{1/2} &\leq |P_{N_j} u(x_j, t_j)| \\
&= \left| \int \check{\phi}_{N_j}(x_j - x) u(t_j, x)\, dx \right| \\
&\leq \left| \int_{|x - x_j| < C(\eta_1) N_j^{-1}} \check{\phi}_{N_j}(x_j - x) u(t_j, x)\, dx \right| \\
&\quad + \left| \int_{|x - x_j| > C(\eta_1) N_j^{-1}} \check{\phi}_{N_j}(x_j - x) u(t_j, x)\, dx \right| \\
&\leq \left(\int_{\mathbb{R}^3} |\check{\phi}_{N_j}(x_j - x)|^{6/5}\, dx \right)^{5/6} \left(\int_{|x - x_j| < C(\eta_1) N_j^{-1}} |u(t_j, x)|^6\, dx \right)^{1/6} \\
&\quad + \left(\int_{|x - x_j| > C(\eta_1) N_j^{-1}} |\check{\phi}_N(x_j - x)|^{6/5}\, dx \right)^{5/6} \left(\int_{\mathbb{R}^3} |u(t_j, x)|^6\, dx \right)^{1/6}.
\end{aligned}$$

Noting $\check{\phi}_N(\cdot) = N^3 \check{\phi}\left(\frac{\cdot}{N}\right)$ and $\check{\phi}$ is rapidly decreasing, one obtains

$$(5.6) \leq C N_j^{1/2} \left(\int_{|x - x_j| < C(\eta_1) N_j^{-1}} |u(t_j, x)|^6 \right)^{1/6} + \frac{c}{2} \eta_1^{3/2} N_j^{1/2},$$

by choosing $C(\eta_1)$ sufficiently large and

$$\left(\int_{\mathbb{R}^3} |u(t_j, x)|^6\, dx \right)^{1/6} \leq \|u(t_j)\|_6 \leq C E_1^{1/6}.$$

Thus we obtain (5.1). To see (5.2), we begin with (5.5) that

$$\begin{aligned}
(5.7) \qquad c\eta_1^{3/2} N_j^{1/2} &< |P_{N_j} u(x_j, t_j)| = |(\Delta^{-1} \nabla) P_{N_j} \nabla u(x_j, t_j)| \\
&= |K_{N_j} * \nabla u(x_j, t_j)| = \left| \int K_{N_j}(x_j - x) \nabla u(x, t_j)\, dx \right|,
\end{aligned}$$

where K_{N_j} is the kernel of $(\Delta^{-1}\nabla)P_{N_j}$, $K_{N_j}(x) = \mathscr{F}^{-1}\left(\frac{\cdot}{|\cdot|^2}\phi_{N_j}(\cdot)\right)(\xi)$, and

$$\|K_{N_j}\|_{L^2} = N_j^{1/2}\left\|\mathscr{F}\left(\frac{\cdot}{|\cdot|^2}\phi(\cdot)\right)\right\|_2,$$

$$\|K_{N_j}\|_{L^2(|x|\geq C(\eta_1)N_j^{-1})} = N_j^{1/2}\left\|\mathscr{F}^{-1}\left(\frac{\cdot}{|\cdot|^2}\phi\right)(\cdot)\right\|_{L^2(|x|\geq C(\eta_1))} \leq C\eta_1^2 N_j^{1/2},$$

if $C(\eta_1)$ is large enough. Thus (5.7) has the bound

$$(\textstyle\int|K_{N_j}(x_j-x)|^2\,dx)^{1/2}\left(\int_{|x-x_j|<C(\eta_1)N_j^{-1}}|\nabla u(x,t_j)|^2\,dx\right)^{1/2}$$

$$+\left(\int_{|x-x_j|\geq C(\eta_1)N_j^{-1}}|K_{N_j}(x_j-x)|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^3}|\nabla u(x,t_j)|^2\,dx\right)^{1/2}$$

$$\leq CN_j^{1/2}\left(\int_{|x-x_j|<C(\eta_1)N_j^{-1}}|\nabla u(x,t_j)|^2\,dx\right)^{1/2}+\frac{c}{2}\eta_1^{3/2}N_j^{1/2},$$

and we have

$$\|\nabla u(x,t_j)\|_{L^2(|x-x_j|<C(\eta_1)N_j^{-1})} \geq c\eta_1^{3/2}.$$

The proof of (5.3) is similar. □

Now we use the radial assumption to locate the bubble at origin.

Proposition 5.2. *Let the conditions in Proposition 5.1 be fulfilled. Assume in addition that u is radial, then there holds that*

(5.8) $$\|u(t_j)\|_{L^6(|x|<C(\eta_1)N_j^{-1})} \geq c\eta_1^{3/2},$$

(5.9) $$\|\nabla u(t_j)\|_{L^2(|x|<C(\eta_1)N_j^{-1})} \geq c\eta_1^{3/2},$$

(5.10) $$\|u(t_j)\|_{L^2(|x|<C(\eta_1)N_j^{-1})} \geq c\eta_1^{3/2}N_j^{-1},$$

with t_j, N_j the same with Proposition 5.1.

Proof. We prove (5.8)–(5.10) by showing that

$$|x_j| < C(\eta_1)N_j^{-1},$$

since once this has been done, we can choose a new constant $\tilde{C}(\eta_1)$ large enough such that

$$B(0, \tilde{C}(\eta_1)N_j^{-1}) \supset B(x_j, C(\eta_1)N_j^{-1}),$$

(5.8)–(5.10) then follow from (5.1)–(5.3).

Letting $S(0, |x_j|)$ be a sphere with radius $|x_j|$ and center 0. By geometrical observation, one has $O\left(\frac{|x_j|}{C(\eta_1)N_j^{-1}}\right)$ consecutive balls that have radius $C(\eta_1)N_j^{-1}$ and center at the points on the sphere. By radial assumption and Proposition 5.1, on each ball, $u(t_j)$ has nontrivial L^6 norm. Using the boundedness of L^6 estimate, one has

$$O\left(\frac{|x_j|}{C(\eta_1)N_j^{-1}}\right)(C\eta_1^{3/2})^6 \le \|u(t_j)\|_6^6 \le CE_1.$$

This gives the desired control on $|x_j|$. □

6 Proof of Proposition 3.6: In case of soliton-like solution

Applying Corollary 5.2 on each interval in the middle component $I^{(2)}$, we get a sequence of times $\{t_j\}$, $t_j \in I_j$, $\frac{J_1}{3} + 1 \le j \le \frac{2}{3}J_1$, such that

$$\|\nabla u(t_j)\|_{L^2(|x| \le C(\eta_1)N_j^{-1})} > c\eta_1^{3/2}, \tag{6.1}$$

$$\|u(t_j)\|_{L^2(|x| \le C(\eta_1)N_j^{-1})} > c\eta_1^{3/2}N_j^{-1}, \quad N_j \ge C|I_j|^{-1/2}\eta_1^5. \tag{6.2}$$

(When things like (6.1), (6.2) occur, we describe them as a bubble at the origin.) Now, we discuss two different cases according to the size of the bubble. First, if there exists η_2, $0 < \eta_2 \ll \eta_1$ such that

$$c|I_j|^{-1/2}\eta_1^5 \le N_j \le \frac{C(\eta_1)}{\eta_2}|I_j|^{-1/2}, \quad \frac{J_1}{3} + 1 \le j \le \frac{2}{3}J_1, \tag{6.3}$$

we call the solution solitonlike. Otherwise there must be $j_0 \in \left[\frac{J_1}{3} + 1, \dots, \frac{2}{3}J_1\right]$ such that

$$N_{j_0} \ge \frac{C(\eta_1)}{\eta_2}|I_{j_0}|^{-1/2} \Leftrightarrow C(\eta_1)N_{j_0}^{-1} < \eta_2|I_{j_0}|^{1/2}. \tag{6.4}$$

As a consequence, we have concentration as follows,

$$\|\nabla u(t_0)\|_{L^2(|x| < (1/\sqrt{2})\eta_2|I_{j_0}|^{1/2})} > c\eta_1^{3/2}. \tag{6.5}$$

In this case, we call the solution blow up solution. In this section, we aim to estimate J_1 in case of soliton-like solution. We follow the idea of [16] and begin the proof by showing that (6.2) holds for every $t \in I_j$, and $\frac{J_1}{3} + 1 \le j \le \frac{2}{3}J_1$.

Proposition 6.1. *Assume u satisfies* (6.2), (6.3), *then there exist* $C(\eta_1,\eta_2)$, $c(\eta_1,\eta_2)$ *such that*

$$\|u(t)\|_{L^2(|x|\le C(\eta_1,\eta_2)|I_j|^{1/2})} \ge c(\eta_1,\eta_2)|I_j|^{1/2}, \quad \forall t \in I_j \text{ and } j \in \left[\frac{1}{3}J_1 + 1, \frac{2}{3}J_1\right]. \tag{6.6}$$

Proof. Fix j, from (6.3), we have

$$C\eta_1^{-5}|I_j|^{1/2} \ge N_j^{-1} \ge c(\eta_1)\eta_2|I_j|^{1/2}.$$

Applying this estimate to (6.2), one gets

$$\|u(t_j)\|_{L^2(|x|<C(\eta_1)|I_j|^{1/2})} \ge c(\eta_1)\eta_2|I_j|^{1/2}.$$

From (4.1) and by choosing $C(\eta_1,\eta_2)$ sufficiently large, we have

$$\begin{aligned}\|u(t)\|_{L^2(|x|<C(\eta_1,\eta_2)|I_j|^{1/2})} &\ge \|u(t_j)\|_{L^2(|x|<C(\eta_1,\eta_2)|I_j|^{1/2})} - \frac{|I_j|\,\|u\|_{L^\infty(I;\dot H^1)}}{C(\eta_1,\eta_2)|I_j|^{1/2}}\\ &\ge c(\eta_1)\eta_2|I_j|^{1/2} - c(\eta_1,\eta_2)|I_j|^{1/2}\\ &\ge c(\eta_1,\eta_2)|I_j|^{1/2}.\end{aligned}$$

This is exactly (6.6). □

Once we have gotten (6.6), we can follow the same way in [16] to obtain the finiteness of J_1. Note that

$$\begin{aligned}c(\eta_1,\eta_2)|I_j| &\le \int_{|x|\le C(\eta_1,\eta_2)|I_j|^{1/2}} |u|^2(x,t)\,dx\\ &\le \int_{|x|\le C(\eta_1,\eta_2)|I_j|^{1/2}} |x|^{1/3}\frac{|u|^2(x,t)}{|x|^{1/3}}\,dx\\ &\le \left(\int_{|x|\le C(\eta_1,\eta_2)|I_j|^{1/2}} |x|^{1/2}\,dx\right)^{2/3}\left(\int_{|x|\le C(\eta_1,\eta_2)|I_j|^{1/2}} \frac{|u(x,t)|^6}{|x|}\,dx\right)^{1/3}\\ &\le C(\eta_1,\eta_2)|I_j|^{7/6}\left(\int_{|x|\le C(\eta_1,\eta_2)|I_j|^{1/2}} \frac{|u(x,t)|^6}{|x|}\,dx\right)^{1/3},\end{aligned}$$

we have

$$\int_{|x|\leq C(\eta_1,\eta_2)|I_j|^{1/2}} \frac{|u(x,t)|^6}{|x|}\,dx \geq c(\eta_1,\eta_2)|I_j|^{-1/2}. \tag{6.7}$$

Comparing (6.7) with Morawetz estimate (4.3), one obtains,

Corollary 6.2. *For any $I \subset I^{(2)}$, we have*

$$\sum_{(1/3)J_1+1\leq j\leq(2/3)J_1;\, I_j\subset I} |I_j|^{1/2} \leq C(\eta_1,\eta_2)|I|^{1/2}. \tag{6.8}$$

Proof. Noting $|I_j|^{1/2} < |I|^{1/2}$ and letting $A = C(\eta_1,\eta_2)$, (6.7) becomes

$$\int_{|x|\leq A|I|^{1/2}} \frac{|u|^6(x,t)}{|x|}\,dx \geq c(\eta_1,\eta_2)|I_j|^{-1/2}. \tag{6.9}$$

Integrating (6.9) on I_j and summing together in j, we get,

$$c(\eta_1,\eta_2)\sum_{(1/3)J_1+1\leq j\leq(2/3)J_1;\, I_j\subset I} |I_j|^{1/2}$$

$$\leq \int_I \int_{|x|\leq A|I|^{1/2}} \frac{|u|^6(x,t)}{|x|}\,dx\,dt \leq CA|I|^{1/2} \leq C(\eta_1,\eta_2)|I|^{1/2},$$

this gives (6.8). □

As a direct consequence of Corollary 6.2, we have

Corollary 6.3. *Let $I = \bigcup_{j_1\leq j\leq j_2} I_j$ be a union of consecutive intervals, $\frac{1}{3}J_1 + 1 \leq j_1$, $j_2 \leq \frac{2}{3}J_1$, then there exists $j_1 \leq j \leq j_2$ such that $|I_j| > c(\eta_1,\eta_2)|I|$.*

Proof. From (6.8) we know that

$$C(\eta_1,\eta_2)|I|^{1/2} \geq \sum_{j_1\leq j\leq j_2} |I_j|^{1/2} \geq \sum_{j_1\leq j\leq j_2} |I_j|\left(\sup_{j_1\leq j\leq j_2}|I_j|\right)^{-1/2} = |I|\left(\sup_{j_1\leq j\leq j_2}|I_j|\right)^{-1/2},$$

and hence

$$C(\eta_1,\eta_2)|I|^{-1/2} \geq \left(\sup_{j_1\leq j\leq j_2}|I_j|\right)^{-1/2}, \tag{6.10}$$

therefore, we can find I_j such that

$$|I_j| > c(\eta_1,\eta_2)|I|.$$

Now, we show that the intervals I_j must concentrate at some time t_*. The idea here is originally due to J. Bourgain, while was reproved by T. Tao in [16]. Since in our case, the proof is quite similar, we omit the detailed presentations. $\square$

Proposition 6.4. *There exist* $t_* \in I^{(2)}$ *and distinct intervals* $I_{j_1}, \dots, I_{j_K}$, $j_k \in \left[\frac{J_1}{3} + 1, \dots \frac{2}{3} J_1\right]$, $K > C(\eta_1, \eta_2) \log J_1$ *such that*

$$|I_{j_1}| \geq 2|I_{j_2}| \geq \cdots \geq 2^{K-1}|I_{j_K}|,$$

and $dist(t_*, I_{j_k}) \leq C(\eta_1, \eta_2)|I_{j_k}|$.

Let t_* and $I_{j_1}, \dots, I_{j_k}, \dots, I_{j_K}$ be as in the Proposition 6.4 and for every $t \in I_{j_k}$, there holds

$$Mass(u(t), B(0; C(\eta_1, \eta_2)|I_{j_k}|^{1/2})) \geq c(\eta_1, \eta_2)|I_{j_k}|^{1/2}, \quad \forall t \in I_{j_k}, \quad 1 \leq k \leq K. \tag{6.11}$$

By the local mass conservation, we have

$$\begin{aligned} &Mass(u(t_*), B(0; C(\eta_1, \eta_2)|I_{j_k}|^{1/2})) \\ &\geq c(\eta_1, \eta_2)|I_{j_k}|^{1/2} - \frac{|t_* - t| \, \|u\|_{L^\infty(I_{j_k}, \dot{H}^1)}}{C(\eta_1, \eta_2)|I_{j_k}|^{1/2}}, \\ &\geq c(\eta_1, \eta_2)|I_{j_k}|^{1/2}, \quad \forall 1 \leq k \leq K. \end{aligned}$$

Denote $B_k = B(0; C(\eta_1, \eta_2)|I_{j_k}|^{1/2})$, we rewrite the above estimate as follows,

$$Mass(u(t_*), B_k) \geq c(\eta_1, \eta_2)|I_{j_k}|^{1/2}, \quad 1 \leq k \leq K. \tag{6.12}$$

On the other hand, by the local mass estimate (4.2), we have

$$Mass(u(t_*), B_k) \leq C(\eta_1, \eta_2)|I_{j_k}|^{1/2}.$$

Letting $N := \log\left(\frac{1}{\eta_3}\right)$, then for $k' > k + N$, we have that

$$\begin{aligned} \int_{B'_k} |u(t_*, x)|^2 \, dx &\leq C(\eta_1, \eta_2)|I_{j'_k}| \\ &\leq C(\eta_1, \eta_2) 2^{-(k'-k)}|I_{j_k}| \leq C(\eta_1, \eta_2)\eta_3 2^{-(k'-k-N)}|I_{j_k}|, \end{aligned}$$

and hence,

$$\sum_{k+N \leq k' \leq K} \int_{B_{k'}} |u(t_*, x)|^2 \, dx \leq C(\eta_1, \eta_2)\eta_3 |I_{j_k}| \sum_{k+N \leq k' \leq K} 2^{-(k'-k-N)}. \tag{6.13}$$

By the finiteness of the summation, the assumption on η_1, η_2, η_3, and (6.11), we continue to estimate (6.13) by

$$c(\eta_1,\eta_2)|I_{j_k}| \le \frac{1}{2}\, Mass(u(t_*),B_k)^2 = \frac{1}{2}\int_{B_k} |u(t_*,x)|^2\,dx,$$

and hence

$$\begin{aligned}(6.14)\quad & \int_{B_k\setminus(\bigcup_{k+N\le k'\le K} B_{k'})} |u(t_*,x)|^2\,dx \\ & \ge \int_{B_k} |u(t_*,x)|^2\,dx - \int_{\bigcup_{k+N\le k'\le K} B_{k'}} |u(t_*,x)|^2\,dx \\ & \ge \int_{B_k} |u(t_*,x)|^2\,dx - \sum_{k+N\le k'\le K}\int_{B_{k'}} |u(t_*,x)|^2\,dx \\ & \ge \frac{1}{2}\int_{B_k} |u(t_*,x)|^2\,dx \ge c(\eta_1,\eta_2)|I_{j_k}|.\end{aligned}$$

By Hölder inequality, we further give the upper bounds of the left side as follows,

$$\begin{aligned} & \int_{B_k\setminus(\bigcup_{k+N\le k'\le K} B'_k)} |u(t_*,x)|^2\,dx \\ & \le \left(\int_{B_k\setminus(\bigcup_{k+N\le k'\le K} B'_k)} |u(t_*,x)|^6\,dx\right)^{1/3} mes(B_k)^{2/3} \\ & \le C(\eta_1,\eta_2)|I_{j_k}|\left(\int_{B_k\setminus(\bigcup_{k+N\le k'\le K} B'_k)} |u(t_*,x)|^6\,dx\right)^{1/3},\end{aligned}$$

hence we have

$$(6.15)\quad \int_{B_k\setminus(\bigcup_{k+N\le k'\le K} B'_k)} |u(t_*,x)|^6\,dx \ge c(\eta_1,\eta_2).$$

Summing (6.15) in k, we obtain

$$(6.16)\quad \sum_{k=1}^{K}\int_{B_k\setminus(\bigcup_{k+N\le k'\le K} B'_k)} |u(t_*,x)|^6\,dx \ge c(\eta_1,\eta_2)K.$$

Denoting $P_k := B_k\setminus(\bigcup_{k+N\le k'\le K} B'_k)$, then $\{P_k\}_{k=1}^K$ overlaps at most N times. Thus the left hand side of (6.16) is smaller than

$$N \int_{\mathbb{R}^3} |u(t_*, x)|^6 \, dx.$$

By the definition of η_3, the boundedness of $\|u(t_*)\|_6$, we have an upper bound for K,

$$K \le C(\eta_1, \eta_2, \eta_3, E_1, E_2),$$

and this in turn gives the control of J_1 by Proposition 6.4,

$$J_1 \le C \exp(C(\eta_1, \eta_2, \eta_3, E_1, E_2)).$$

7 In case of blow up solution

Our purpose of this section is to prove the boundedness of J_1 under the condition (6.4) and (6.5). That is, there exists $t_0 \in I_{j_0}$, $j_0 \in \left[\frac{J_1}{3} + 1, \dots, \frac{2}{3} J_2\right]$ such that

$$\|\nabla u(t_0)\|_{L^2(|x|<(1/\sqrt{2})\eta_2 |I_{j_0}|^{1/2})} > c\eta_1^{3/2}. \tag{7.1}$$

If t_0 lies on the left side of I_{j_0}, we take $I = [t_0, b]$ where b is the left end point of I_{j_0}; otherwise we take $I = [a, t_0]$ with a the right end point of I_{j_0}. Then (7.1) becomes

$$\|\nabla u(t_0)\|_{L^2(|x|<\eta_2 |I|^{1/2})} > c\eta_1^{3/2}. \tag{7.2}$$

Assume $I = [t_0, b]$, we aim to re-solve the problem (1.1) forward in time. Otherwise, we do in the reverse direction. First we show that, by removing the small bubble, we remove nontrivial portion of energy.

Let χ be a smooth radial function such that $\chi(x) = 1$ as $|x| \le 1$, and $\chi(x) = 0$ as $|x| \ge 2$. Let $\phi(x) = \chi\left(\frac{x}{N\eta_2 |I|^{1/2}}\right)$ for some $N \ge 1$ to be specified later, and $w(t_0, x) = (1 - \phi(x))u(t_0, x)$, then we have

Lemma 7.1. $E_1(w(t_0)) \le E_1(u(t_0)) - c\eta_1^3$.

Proof. Noting

$$w(t_0) = (1 - \phi)u(t_0),$$

we have

$$\nabla w(t_0) = (1 - \phi)\nabla u(t_0) - \nabla\phi u(t_0),$$

and thus,

$$|\nabla w(t_0)|^2 = |\nabla u(t_0)|^2 + (\phi^2 - 2\phi)|\nabla u(t_0)|^2 + |\nabla\phi|^2 |u(t_0)|^2 - 2\, Re(1 - \phi)\nabla\phi \bar{u}(t_0)\nabla u(t_0).$$

Integrating it on $\mathbb{R}^3$, one gets

$$\|\nabla w(t_0)\|_2^2$$

$$\leq \|\nabla u(t_0)\|_2^2 + \int_{\mathbb{R}^3} (\phi^2 - 2\phi)|\nabla u(t_0, x)|^2 \, dx$$

$$- 2\int_{\mathbb{R}^3} |\nabla\phi(x) u(t_0, x)|^2 \, dx - 2\, Re \int_{\mathbb{R}^3} (1 - \phi)\nabla\phi\bar{u}(t_0)\nabla u(t_0)(x) \, dx.$$

By the trivial inequality: $\phi^2 - 2\phi \leq -\phi$ and (7.2), one can estimate the second term of the right side by

$$-\int_{|x| \leq N\eta_2|I|^{1/2}} |\nabla u(t_0, x)|^2 \, dx \leq -c\eta_1^3.$$

Now, we estimate the remaining two terms. We use Hölder inequality to control them by

$$\text{(7.3)} \qquad C\|\nabla\phi\|_3^2 \|u(t_0)\|^2_{L^6_{N\eta_2|I|^{1/2} \leq |x| \leq 2N\eta_2|I|^{1/2}}} + C\|\nabla\phi\|_3 \|\nabla u(t_0)\|_2 \|u(t_0)\|_{L^6_{N\eta_2|I|^{1/2} \leq |x| \leq 2N\eta_2|I|^{1/2}}}$$

$$\leq C(\|u(t_0)\|^2_{L^6_{N\eta_2|I|^{1/2} \leq |x| \leq 2N\eta_2|I|^{1/2}}} + \|u(t_0)\|_{L^6_{N\eta_2|I|^{1/2} \leq |x| \leq 2N\eta_2|I|^{1/2}}} \|\nabla u(t_0)\|_2).$$

Now, we claim that, there must exist N which depend only on η_1 such that

$$\text{(7.4)} \qquad \|u(t_0)\|_{L^6_{N\eta_2|I|^{1/2} \leq |x| \leq 2N\eta_2|I|^{1/2}}} \leq \eta_1^4.$$

Indeed, if otherwise, we will have N annuluses, on each annulus, $u(t_0)$ has nontrivial L^6 norm. Summing these annuluses together, we obtain

$$N(\eta_1^4)^6 \leq \sum_{N' \leq N \in \mathbf{N}} \|u(t_0)\|^6_{L^6_{N'\eta_2|I|^{1/2} \leq |x| \leq 2N'\eta_2|I|^{1/2}}} \leq C,$$

by the boundedness of L^6 estimate. This will be a contradiction if $N \geq C\eta_1^{-24}$. Hence, one can fix $N = C(\eta_1)$ such that (7.4) holds and

$$(7.3) \leq C\eta_1^4.$$

We finally obtain this Lemma by noting

$$E_1(w(t_0)) = \frac{1}{2}\|\nabla w(t_0)\|_2^2 + \frac{1}{3}\|w(t_0)\|_6^6,$$

and combining the above estimates together. □

Lemma 7.2. *We have that,*

$$E_1(w(t_0)) \le E_1 - c\eta_1^3,$$

$$E_2(w(t_0)) \le E_2 + C\eta_1^4.$$

Proof. Noting Lemma 7.1, it suffices to prove

$$E_1(u(t_0)) \le E_1(u(0)) + C\eta_1^4,$$

$$E_2(u(t_0)) \le E_2(u(0)) + C\eta_1^4.$$

So, Let's compute the increment of $E_i(u(t))$ from 0 to t_0:

$$\int_0^{t_0} \frac{\partial}{\partial t} E_1(u(t))\,dt, \quad \text{and} \quad \int_0^{t_0} \frac{\partial}{\partial t} E_2(u(t))\,dt.$$

From the equation (1.1), we see that

$$\begin{aligned} \frac{\partial}{\partial t} E_2(u(t)) &= \frac{\partial}{\partial t} \|xu(t)\|_2^2 = 2\,Im \int_{\mathbb{R}^3} x\bar{u}\nabla u(t,x)\,dx \\ &\le C\|xu\|_{L^\infty((0,t_0);L^2)} \|\nabla u\|_{L^\infty((0,t_0);L^2)} \le C(E_1, E_2) \quad \forall t \in [0, t_0], \end{aligned}$$

thus, we have

$$\left| \int_0^{t_0} \frac{\partial}{\partial t} E_2(t)\,dt \right| \le C\eta_1^4.$$

By noting that $\frac{\partial}{\partial t} E_1(t) = -\frac{\partial}{\partial t} E_2(t)$, we get

$$\left| \int_0^{t_0} \frac{\partial}{\partial t} E_1(t)\,dt \right| \le C\eta_1^4,$$

hence, Lemma 7.2 follows. □

Let u, v be solutions of the initial data problems

$$\begin{cases} \left(i\partial_t + \frac{\Delta}{2} + \frac{|x|^2}{2}\right) v = |v|^4 v, \\ v(x, t_0) = \phi(x) u(t_0, x). \end{cases} \tag{7.5}$$

(7.6) $$\begin{cases} \left(i\partial_t + \frac{\Delta}{2} + \frac{|x|^2}{2}\right)w = |v+w|^4(v+w) - |v|^4 v, \\ w(x,t_0) = (1-\phi(x))u(t_0,x). \end{cases}$$

then $u = v + w$. Let's first show that

Proposition 7.3. *There exists a unique solution $v(x,t)$ to* (7.5) *satisfying*

$$\|v\|_{L^{10}(I;L^{10})} \le C\eta_1, \quad \|v\|_{L^{10}([b,\infty);L^{10})} \le C\eta_2^{1/5},$$

$$\|A(\cdot)v\|_{L^q(I;L^r)} \le C,$$

where $A \in \{J, H\}$ and (q,r) is admissible pair.

Proof. We begin by computing the L^{10} norm of the linear flow $U(t-t_0)(\phi u(t_0))$. First, we observe that by Duhamel, and Strichartz estimates,

$$\|U(\cdot - t_0)u(t_0)\|_{L^{10}(I;L^{10})} \le \|u\|_{L^{10}(I;L^{10})} + \left\|\int_{t_0}^{t} U(t-s)|u|^4 u(s)\,ds\right\|_{L^{10}(I;L^{10})}$$

$$\le \eta_1 + C\eta_1^4 \|Ju\|_{L^{10/3}(I;L^{10/3})} \le 2\eta_1.$$

Noting $\phi u(t_0)$ is a radial function in space, we have that

$$\begin{aligned} U(t-t_0)(\phi u(t_0))(x) &= \exp\left\{\frac{i(t-t_0)(\Delta + |x|^2)}{2}\right\}(\phi u(t_0))(x) \\ &= \mathscr{F}^{-1}\left(\exp\left\{-\frac{i(t-t_0)(|\xi|^2 + \Delta_\xi)}{2}\right\}\widehat{\phi u(t_0)}(\xi)\right)(x) \\ &= \int_{\mathbb{R}^3} e^{ix\xi} e^{-(i(t-t_0)/2)(|\xi|^2 + \Delta_\xi)} \int_{\mathbb{R}^3} \hat{u}(t_0)(\xi - \xi_1)\hat{\phi}(\xi_1)\,d\xi_1\,d\xi. \end{aligned}$$

Expanding $|\xi|^2 = |\xi - \xi_1|^2 + 2\xi_1(\xi - \xi_1) + |\xi_1|^2$, the above term becomes

$$\iint e^{i(\xi-\xi_1)(x-(t-t_0)\xi_1)} e^{-(i(t-t_0)/2)(|\xi-\xi_1|^2 + \Delta_{\xi-\xi_1})} \hat{u}(t_0)(\xi-\xi_1)\,d\xi e^{ix\xi_1} e^{-(i(t-t_0)/2)|\xi_1|^2}\hat{\phi}(\xi_1)\,d\xi_1,$$

by renaming the variable, one sees that this is exactly

$$\int_{\mathbb{R}^3} U(t-t_0)u(t_0)(x-(t-t_0)\xi_1) e^{ix\xi_1} e^{-(i(t-t_0)/2)|\xi_1|^2}\hat{\phi}(\xi_1)\,d\xi_1,$$

and hence

$$\|U(\cdot - t_0)(\phi u(t_0))\|_{L^{10}(I;L^{10})}$$

$$\le \|U(\cdot - t_0)u(t_0)\|_{L^{10}(I;L^{10})}\|\hat{\phi}\|_1 \le C\|U(\cdot - t_0)u(t_0)\|_{L^{10}(I;L^{10})} \le C\eta_1.$$

The estimate of the linear flow allows us to solve the problem in the following set,

$$X := \{v(x,t) \mid \|v\|_{L^{10}(I;L^{10})} \le C\eta_1, \|A(\cdot)v\|_{L^{10/3}(I;L^{10/3})} \le C, A \in \{J,H\}\},$$

endowed with the metric

$$d(u_1,u_2) = \|u_1 - u_2\|_{L^{10}(I;L^{10})} + \max_{A\in\{J,H\}} \|A(\cdot)(u_1 - u_2)\|_{L^{10/3}(I;L^{10/3})}.$$

We omit the proof of this part since it is routine. Once we have gotten the solution on $I = [t_0, b]$, we extend this solution beyond I. As a consequence, we are left to show a finite apriori spacetime estimate on $[b,\infty)$. Assuming v be a finite energy solution on $[b,\infty)$, we redefine the energy of v by

$$\widetilde{\mathscr{E}_1}(t) = \frac{1}{2}\|J(t-t_0)v(t)\|_2^2 + \frac{1}{3}\cosh^2(t-t_0)\|v(t)\|_6^6;$$

$$\widetilde{\mathscr{E}_2}(t) = \frac{1}{2}\|H(t-t_0)v(t)\|_2^2 + \frac{1}{3}\sinh^2(t-t_0)\|v(t)\|_6^6.$$

Repeating the computations in Lemma 2.4, we find

$$\frac{d\widetilde{\mathscr{E}_1}(t)}{dt} = -\frac{2}{3}\sinh 2(t-t_0)\|v(t)\|_6^6 = \frac{d\widetilde{\mathscr{E}_2(t)}}{dt}.$$

Integrating the second half of the equation, we have

$$\frac{1}{2}\|H(t-t_0)v(t)\|_2^2 + \frac{1}{3}\sinh^2(t-t_0)\|v(t)\|_6^6$$

$$= \frac{1}{2}\|xv(t_0)\|_2^2 - \frac{2}{3}\int_{t_0}^t \sinh(2(\tau - t_0))\|v(\tau)\|_6^6\,d\tau$$

This implies

$$\sinh^2(t-t_0)\|v(t)\|_6^6 \le C\|x\phi u(t_0)\|_2^2. \tag{7.7}$$

By Hölder and direct computation, we continue to estimate the right side as

$$\|x\phi u(t_0)\|_2 \leq \left\| \cdot\chi\left(\frac{\cdot}{C(\eta_1)\eta_2|I|^{1/2}}\right)u(t_0)\right\|_2$$

$$\leq (C(\eta_1)\eta_2)^2|I|\,\|\cdot\chi(\cdot)\|_3\|u(t_0)\|_6$$

$$\leq C(\eta_1)\eta_2^2|I|.$$

$$\sinh^2(t-t_0) \geq |t-t_0|^2 \geq |I|^2, \quad t \geq b,$$

hence, from (7.7), we have

$$\|v(t)\|_6^6 \leq C(\eta_1)\eta_2^4 \leq \eta_2^2, \quad \forall t \geq b. \tag{7.8}$$

On the other hand, noting on $[b, \infty)$, v satisfies,

$$v(t) = U(t-t_0)v(t_0) - i\int_{t_0}^{t} U(t-s)|v|^4 v(s)\,ds, \tag{7.9}$$

we have

$$\|J(\cdot)v\|_{L^6([b,\infty);\,L^{18/7})}$$

$$\leq C\|J(t_0)v(t_0)\|_2 + C\|J(\cdot)|v|^4 v\|_{L^{3/2}([b,\infty);\,L^{18/13})}$$

$$\leq C\|J(t_0)v(t_0)\|_2 + C\|v\|_{L^\infty([b,\infty);\,L^6)}\|v\|^3_{L^6([b,\infty);\,L^{18})}\|J(\cdot)v\|_{L^6([b,\infty);\,L^{18/7})}$$

$$\leq C\|J(t_0)v(t_0)\|_2 + C\eta_2^{1/3}\|J(\cdot)v\|^4_{L^6([b,\infty);\,L^{18/7})}.$$

This implies

$$\|J(\cdot)v\|_{L^6([b,\infty);\,L^{18/7})} \leq C\|J(t_0)v(t_0)\|_2 \leq C, \tag{7.10}$$

where C depends only on E_1, E_2. To see this, we use Lemma 2.3 to expand $J(t_0)v(t_0)$ as

$$\phi(x)J(t_0)u(t_0) + i\cosh t_0 u(t_0)\nabla\phi(x),$$

which can be easily controlled.

Combining the bounds (7.8) and (7.10) together and using interpolation, one obtains

$$(7.11)\quad \|v\|_{L^{10}([b,\infty);L^{10})} \le C\|v\|^{4/10}_{L^{\infty}([b,\infty);L^6)}\|J(\cdot)v\|^{6/10}_{L^6([b,\infty);L^{18/7})} \le C\eta_2^{1/5}.$$

The last bound in Proposition 7.3 follows from Lemma 3.1. □

Now, we are at the position to solve the Cauchy problem (7.6). Before doing this, we list the estimates that follow from Proposition 7.3 and the conditions on u.

$$(7.12)\quad \|v\|_{L^{10}([b,\infty);L^{10})} < C\eta_2^{1/5}$$

$$\|w\|_{L^{10}(I;L^{10})} \le C\eta_1, \quad \|A(\cdot)w\|_{L^{10/3}(I;L^{10/3})} \le C; \quad \|Bw\|_{L^{10/3}(I;L^{10/3})} \le C,$$

$$\|v\|_{L^{10}(I;L^{10})} \le C\eta_1, \quad \|A(\cdot)v\|_{L^{10/3}(I;L^{10/3})} \le C; \quad \|Bv\|_{L^{10/3}(I;L^{10/3})} \le C,$$

$$A \in \{J, H\}, \quad B \in \{i\nabla_x, x\},$$

here, we have used the condition that $I \in [0, \eta_1^4)$ to get the estimate on Bv, Bw. The constants above depend only on E_1, E_2.

For the sake of doing perturbation analysis and applying the induction, it's necessary to introduce the following Lemma.

Lemma 7.4. *We have that*

$$E_1(w(b)) \le E_1 - c\eta_1^3, \quad E_2(w(b)) \le E_2 + C\eta_1^4.$$

Proof. Noting Lemma 7.2, we need only to prove that

$$\left|\int_{t_0}^{b} \frac{\partial}{\partial t} E_1(w(t))\,dt\right| \le C\eta_1^4, \qquad \left|\int_{t_0}^{b} \frac{\partial}{\partial t} E_2(w(t))\,dt\right| \le C\eta_1^4.$$

For simplicity, denote

$$|v+w|^4(v+w) - |v|^4 v = |w|^4 w + F(v,w),$$

hence, w satisfies the equation

$$\left(i\partial_t + \frac{\Delta}{2} + \frac{|x|^2}{2}\right) w = |w|^4 w + F(v,w).$$

By some basic computation, one sees that

$$\frac{\partial}{\partial_t} E_2(w(t)) = 2\,Im \int_{\mathbb{R}^3} x\overline{w}\nabla w\,dx + 2\,Im \int_{\mathbb{R}^3} |x|^2 \overline{w} F(v,w)\,dx,$$

thus, we get

$$\left|\int_{t_0}^{b} \frac{\partial}{\partial t} E_2(w(t))\,dt\right| \le 2|t_0 - b|\,\|xw\|_{L^\infty(I;L^2)}\|\nabla w\|_{L^\infty(I;L^2)}$$

$$+ C\|xw\|^2_{L^{10/3}(I;L^{10/3})}(\|w\|^4_{L^{10}(I;L^{10})} + \|v\|^4_{L^{10}(I;L^{10})}) \le C\eta_1^4.$$

To prove the increment of $E_1(w(t))$ from t_0 to b, we first compute directly that

$$\frac{\partial}{\partial t} E_1(w(t)) = \frac{\partial}{\partial t} E_2(w(t)) + Re\int_{\mathbb{R}^3} F(v,w)\overline{w}_t(x)\,dx$$

$$= \frac{\partial}{\partial t} E_2(w(t)) + Im\int_{\mathbb{R}^3} F(v,w)\left(\frac{1}{2}\Delta\overline{w} + \frac{1}{2}|x|^2\overline{w} - |w|^4\overline{w} - \overline{F(v,w)}\right)(x)\,dx.$$

Integrating over $[t_0, b]$ and using integration by parts, one gets

$$\left|\int_{t_0}^{b} \frac{\partial}{\partial t} E_1(w(t))\right|$$

$$\le C\eta_1^4 + C(\|\nabla w\|^2_{L^{10/3}(I;L^{10/3})} + \|xw\|^2_{L^{10/3}(I;L^{10/3})})$$

$$\times (\|v\|^4_{L^{10}(I;L^{10})} + \|w\|^4_{L^{10}(I;L^{10})}) + C(\|v\|^{10}_{L^{10}(I;L^{10})} + \|w\|^{10}_{L^{10}(I;L^{10})})$$

$$\le C\eta_1^4.$$

This ends Lemma 7.4. □

Now, for the sake of convenience, we make a small adjustment such that, the increment of E_1 and the decrement E_2 take the same value. More precisely, noting Lemma 7.4, we can get

$$E_1(w(b)) \le E_1 - C\eta_1^4, \quad E_2(w(b)) \le E_2 + C\eta_1^4,$$

with the same constant C.

Now, we are at the position to make following inductive assumption.

Assume

$$E_1(u_0) \le E_1 - C\eta_1^4, \quad E_2(u_0) \le E_2 + C\eta_1^4,$$

then the Cauchy problem of (1.1) is wellposed on $[-\eta_1^4, \eta_1^4]$, and the solution u satisfies

$$\|u\|_{L^{10}([-\eta_1^4,\eta_1^4];L^{10})} \le C(E_1 - C\eta_1^4, E_2 + C\eta_1^4).$$

By this assumption and Lemma 7.4, we see that the solution of

$$\begin{cases} iW_t + \frac{\Delta}{2} W = -\frac{|x|^2}{2} W + |W|^4 W, \\ W(b) = w(b) \end{cases}$$

satisfies the estimate

$$\|W\|_{L^{10}([b-\eta_1^4, b+\eta_1^4]; L^{10})} \le C(E_1 - C\eta_1^4, E_2 + C\eta_1^4) \le C(E_1, E_2).$$

Subtracting W from w, we are left to solve the perturbation problem with respect to $\Gamma = w - W$ on $[b, \eta_1^4]$,

$$\begin{cases} \left(i\partial_t + \frac{\Delta}{2} + \frac{|x|^2}{2}\right)\Gamma = |v + W + \Gamma|^4(v + W + \Gamma) - |v|^4 v - |W|^4 W, \\ \Gamma(b) = 0. \end{cases} \tag{7.13}$$

8 Solving the perturbation problem

Our task of this section is to solve (7.13) with the help of (7.12). To insure the smallness of the nonlinear flow, we split $[b, \eta_1^4]$ into finite subintervals such that on each subinterval, W is small, so that we can solve (7.13) on every subinterval. Before doing this, we re-estimate v on $[b, \infty)$.

Lemma 8.1. *In addition to* (7.12), *v satisfies*

$$\|A(\cdot)v\|_{L^{10/3}([b,\infty); L^{10/3})} \le c(\eta_2), \quad A \in \{J, H\}.$$

Proof. Taking J as an example, one sees

$$J(t)v(t) = U(t - t_0)J(t_0)v(t_0) - i\int_{t_0}^{t} U(t - s)J(s)|v|^4 v(s)\, ds.$$

For the linear term, we estimate directly. From decay estimate (2.3),

$$\begin{aligned} &\|U(t - t_0)J(t_0)v(t_0)\|_{L_x^\infty} \\ &\le C|t - t_0|^{-3/2}\|J(t_0)v(t_0)\|_1, \\ &\le C|t - t_0|^{-3/2}(\|J(t_0)u(t_0)\phi\|_1 + \|\cosh t_0 u(t_0)\nabla_x\phi\|_1) \\ &\le C|t - t_0|^{-3/2}(\|\phi\|_2\|J(t_0)u(t_0)\|_2 + \|\cosh t_0 u(t_0)\|_6\|\nabla_x\phi\|_{6/5}) \end{aligned}$$

By noting $\phi(x) = \chi\left(\frac{x}{\eta_2 C(\eta_1)|I|^{1/2}}\right)$, one has

$$\|U(t-t_0)J(t_0)v(t_0)\|_{L_x^\infty} \le C(\eta_1)\left(\frac{\eta_2|I|^{1/2}}{t-t_0}\right)^{3/2}. \tag{8.1}$$

On the other hand,

$$\|U(t-t_0)J(t_0)v(t_0)\|_{L_x^2} \le \|J(t_0)v(t_0)\|_{L^2} \tag{8.2}$$

$$\le \|J(t_0)u(t_0)\|_2\|\phi\|_\infty + \|\nabla\phi\|_3\|\cosh t_0 u(t_0)\|_6 \le C.$$

By interpolation and (8.1) and (8.2), we have that

$$\|U(t-t_0)J(t_0)v(t_0)\|_{L_x^{10/3}}$$

$$\le \|U(t-t_0)J(t_0)v(t_0)\|_\infty^{2/5}\,\|U(t-t_0)J(t_0)v(t_0)\|_2^{3/5} \le C(\eta_1)\left(\frac{|I|^{1/2}\eta_2}{|t-t_0|}\right)^{3/5},$$

and thus

$$\|U(t-t_0)J(t_0)v(t_0)\|_{L^{10/3}([b,\infty);L^{10/3})}^{10/3} \le C(\eta_1)|I|\eta_2^2\int_b^\infty \frac{dt}{|t-t_0|^2} \tag{8.3}$$

$$\le C(\eta_1)\eta_2^2.$$

To estimate the nonlinear term, we denote $t_1 = t_0 + \eta_2|I|$, and split it into two parts,

$$\left\|\int_{t_0}^{t_1} U(t-s)J(s)|v|^4v(s)\,ds\right\|_{L^{10/3}([b,\infty);L^{10/3})} + \left\|\int_{t_1}^{\infty} U(t-s)J(s)|v|^4v(s)\,ds\right\|_{L^{10/3}([b,\infty);L^{10/3})},$$

For the first part, we use $L^p - L^{p'}$ estimate to control it by

$$\left\|\int_{t_0}^{t_1}|t-s|^{-3/5}\|J(s)|v|^4v(s)\|_{L^{10/7}}\,ds\right\|_{L^{10/3}([b,\infty)}.$$

Since for $s\in[t_0,t_1]$, $t>b$, $|t-s|\sim|t-t_0|$, we see the first part is smaller than

$$C\|\,|t-t_0|^{-5/3}\|_{L^{10/3}([b,\infty))}\int_{t_0}^{t_1}\|J(s)|v|^4v(s)\|_{L^{10/7}}\,ds$$

$$\le C|I|^{-3/10}|t_1-t_0|^{3/10}\|v\|_{L^{10}([t_0,t_1];L^{10})}^4\|J(\cdot)v\|_{L^{10/3}([t_0,t_1];L^{10/3})}$$

$$\le C\eta_2^{3/10}.$$

For the second part, we use Strichartz estimate to control it by

$$C\|J(\cdot)|v|^4 v\|_{L^{10/7}([t_1,\infty);L^{10/7})} \le C\|v\|^4_{L^{10}([t_1,\infty);L^{10})}\|J(\cdot)v\|_{L^{10/3}([t_1,\infty);L^{10/3})}. \tag{8.4}$$

At this moment, we follow the same way in proving (7.11) to get

$$\|v\|_{L^{10}([t_1,\infty);L^{10})} \le c(\eta_2),$$

thus finally, (8.4) $\le c(\eta_2)$. □

Now we are at the position to solve the perturbation problem (7.13). By induction assumption, we see that there exists some constant $C = C(E_1, E_2)$ such that

$$\|W\|_{L^{10}([b,\eta_1^4];L^{10})} \le C,$$

$$\|A(\cdot)W\|_{L^{10/3}([b,\eta_1^4];L^{10/3})} \le C, \quad A \subset \{J, H\}.$$

This allows to split $[b, \eta_1^4]$ into finite subintervals

$$[b,\eta_1^4] = \bigcup_{j=1}^{K} I_j = \bigcup_{j=1}^{K} [b_{j-1}, b_j), \quad b_0 = b, \quad b_K = \eta_1^4,$$

and such that

$$\|W\|_{L^{10}(I_j;L^{10})} \sim v, \quad \|A(\cdot)W\|_{L^{10/3}(I_j;L^{10/3})} \sim \varepsilon.$$

Then

$$K \le \max\left(\left(\frac{C}{v}\right)^{10}, \left(\frac{C}{\varepsilon}\right)^{10/3}\right).$$

If (7.13) has been solved on $[b_0, b_{j-1}]$, and

$$\|A(b_{j-1})\Gamma(b_{j-1})\|_2 \le C^{j-1}c(\eta_2)^{1-(j-1)/2K},$$

then we can solve (7.13) on $[b_{j-1}, b_j]$ by proving the solution map

$$\Phi(\Gamma(t)) = U(t-b_{j-1})\Gamma(b_{j-1}) - i\int_{b_{j-1}}^{t} U(t-s)(|v+W+\Gamma|^4(v+W+\Gamma) - |v|^4 v - |W|^4 W)(s)\,ds$$

is contractive on the closed set

$$X := \Big\{ \Gamma \in L^{10}(I_j; L^{10}), A(\cdot)\Gamma \in L^{10/3}(I_j; L^{10/3}), \text{and}$$

$$\|\Gamma\|_X = \|\Gamma\|_{L^{10}(I_j; L^{10})} + \max_{A \in \{J, H\}} \|A(\cdot)\Gamma\|_{L^{10/3}(I_j; L^{10/3})} \le C^j c(\eta_2)^{1-j/2K} \Big\},$$

endowed with the metric

$$d(u_1, u_2) = \|u_1 - u_2\|_X,$$

and complete one step of iteration by estimating $\|A(b_j)u(b_j)\|_2$ from Strichartz estimate. This is feasible since we can choose the absolute constants ε, v, and the constant $c(\eta_2)$ small enough. The proof is routine and is omitted. Now we have a finite energy solution Γ on $[b, \eta_1^4]$ such that

$$\|\Gamma\|_{L^{10}([b,\eta_1^4]; L^{10})}^{10} = \sum_{j=1}^{K} \|\Gamma\|_{L^{10}(I_j; L^{10})}^{10} \le \sum_{j=1}^{K} (C^j c(\eta_2)^{1-j/2K})^{10} \le C.$$

To conclude the proof of Proposition 3.6 in case of blow up solution, we collect all the estimates to get

$$\begin{aligned} \|u\|_{L^{10}(I^{(3)}; L^{10})} &\le \|u\|_{L^{10}([b,\eta_1^4]; L^{10})} \\ &\le \|v\|_{L^{10}([b,\eta_1^4]; L^{10})} + \|W\|_{L^{10}([b,\eta_1^4]; L^{10})} + \|\Gamma\|_{L^{10}([b,\eta_1^4]; L^{10})} \\ &\le C(E_1 E_2, \eta_1, \eta_2). \end{aligned}$$

Thus, J_1 can be controlled by

$$O\left(\frac{C(E_1 E_2, \eta_1, \eta_2, \eta_3)}{\eta_1}\right)^{10}.$$

In the same way, J_2 can also be controlled, and thus

$$\|u\|_{L^{10}([-\eta_1^4, \eta_1^4]; L^{10})} \le C(E_1, E_2, \eta_1, \eta_2, \eta_3).$$

which closes the induction and finally gives Proposition 3.6.

Acknowledgement. The author thanks Professor Ping Zhang for useful discussions and the anonymous referee for numerous corrections and advices on the manuscript.

References

[1] Bourgain J.: Global well-posedness of defocusing 3D critical NLS in the radial case. JAMS **12** (1999), 145–171

[2] Carles R.: Nonlinear Schrödinger equations with repulsive harmonic potential and applications. SIAM J. Math. Anal. **35** (2003), 823–843

[3] Carles R.: Global existence results for nonlinear Schrödinger equations with quadratic potentials. Discrete Contin. Dyn. Syst. **13** (2005), 385–398

[4] Cazenave T.: An introduction to nonlinear Schrödinger equations. Textos de Métodos Matemeáticos 26. Instituto de Matemática UFRJ, 1996

[5] Cazenave T., Weissler F. B.: Critical nonlinear Schrödinger Equation. Non. Anal. TMA, **14** (1990), 807–836

[6] Christ M., Colliander J., Tao T.: Ill-posedness for nonlinear Schrödinger and wave equation; to appear in Annales IHP

[7] Colliander J., Keel M., Staffilani G., Takaoka H., and Tao T.: Global well-posedness and Scattering for the energy-critical nonlinear Schrödinger equation; to appear in Annals Math

[8] Feynman R. P. and Hibbs A. R.: Quantum mechanics and path integrals (International series in Pure and Applied Physics), Maidenhead, Berksh.: McGraw-Hill Publishing Company, Ltd., 365p, 1965

[9] Glassey R. T.: On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations. J. Math. Phys. **18** (1977), 1794–1797

[10] Ginibre J., Velo G.: Scattering theory in the energy space for a class of nonlinear Schrödinger equations. J. Math. Pure. Appl. **64** (1985), 363–401

[11] Grillakis M.: On nonlinear Schrödinger equations. Comm. Partial Differential Equations **25** (2000), no. 9–10, 1827–1844

[12] Grillakis M.: Regularity and asymptotic behavior of the wave equation with a critical nonlinearity. Ann. of Math. **132** (1990), 485–509

[13] Keel M., Tao T.: Endpoint Strichartz estimates. Amer. Math. J. **120** (1998), 955–980

[14] Lin J., Strauss W.: Decay and scattering of solutions of a nonlinear Schrödinger equation. J. Funct. Anal. **30** (1978), 245–263

[15] Ryckman E., Visan M.: Global wellposedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$; preprint math.Ap/0501462

[16] Tao T.: Global well-posedness and scattering for the higher-dimensional energy-critical nonlinear Schrödinger equation for radial data. New York J. Math. **11** (2005), 57–80

[17] Tao T., Visan M.: Stability of energy-critical nonlinear Schrödinger equations in high dimensions; preprint, math.Ap/0507005

[18] Visan M.: The defocusing energy critical nonlinear Schrödinger equation in higher dimensions preprint; math.Ap/0508298

Received November 24, 2004; revised September 27, 2005

Academy of Mathematics and System Sciences, Chinese Academy of Science
xyzhang@amss.ac.cn
zh.xiaoyi@gmail.com